Mathematical Methods in the Applied Sciences, Vol. 20, 31–45 (1997)
MOS subject classification: 73 F 20, 35 B 40, 45 D 05, 45 K 05

Singular Stress Behavior in a Bonded Hereditarily-Elastic Aging Wedge. Part II: General Heredity

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Communicated by E. Meister

0. Introduction

Investigation of stress singularity in a plane problem for bonded isotropic hereditarily elastic (visco-elastic) aging infinite wedge begun in [8] is continued here. The stress asymptotics at the singular point are obtained at initial time $t = +0$; at $t \to \infty$ for loads tending to harmonically oscillating or to constant ones in time. At small times, the power expansion with respect to time and the power-logarithmic asymptotic with respect to radius is presented. At finite times $t \in [0, \infty)$ as well as at $t \to \infty$, the estimates for stresses at the singular point are obtained for sufficiently arbitrary loading behaviour.

We shall use in this paper the notions and notations of [8]. References on the numbers of formulas and sections of [8] will be preceded by the symbol I, reference on Appendix denotes the Appendix of [8].

1. Finite times ($t \in [0, \infty)$)

Consider the general case of aging hereditary Dundurs operators $\underline{\alpha}$, $\underline{\beta}$ whose out-of-integral terms $\alpha^0(\tau), \beta^\circ(\tau) \in C[0, T]$ and integral operators $\underline{\alpha}^*, \underline{\beta}^* \in \mathrm{VC}(0, T)$. Let an action be applied at $\tau = 0$, and $t > 0$ be a time instant under consideration, $0 \leqslant t \leqslant T < \infty$. Let prescribed loading functions $g_i^{(l)}, \tilde{g}_i^{(0)} \in \mathrm{C}\hat{\mathrm{L}}_2\ (\delta_g, 1; 0, T)]$. Then due to [8, Appendix, point 8^0], the right-hand side of system (I.3.17) $\mathbf{G} \in \mathrm{CH}_2^0(S(\delta_g, 1); 0, T)$.

Rewrite system (I.3.17) in the form

$$\mathbf{B}^0(\gamma, t)\mathfrak{F}(\gamma, t) + [\underline{\mathbf{B}}^*(\gamma)\mathfrak{F}(\gamma, \cdot)](t) = \mathbf{G}(\gamma, t).$$

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0170–4214/97/010031–15 *Received 1 February 1996*

For $\gamma \neq \gamma_k(t)$ the inverse matrix $(\mathbf{B}^0)^{-1}(\gamma, t)$ of the out-of-integral terms exists and can be represented by (I.4.3). At the matrix Volterra integral operator $\underline{\mathbf{B}}^*(\gamma)$, only the two last rows differ from zero. Let us represent the matrix operator of this system in the form of multiplication of two operators:

$$\{[\mathbf{I} + \underline{\tilde{\tilde{\mathbf{B}}}}^*(\gamma)]\mathbf{B}^0(\gamma, \cdot)\mathfrak{F}(\gamma, .)\}(t) = \mathbf{G}(\gamma, t), \quad \underline{\tilde{\tilde{\mathbf{B}}}}^*(\gamma) := \underline{\mathbf{B}}^*(\gamma)(\mathbf{B}^0)^{-1}(\gamma, \cdot). \tag{1.1}$$

Here $\mathbf{I}$ is the identity (8×8)-matrix.

After inverting the matrix Volterra operator given by square brackets we arrive at the algebraic system for $\mathfrak{F}$:

$$\mathbf{B}^0(\gamma, t)\mathfrak{F}(\gamma, t) = \tilde{\tilde{\mathbf{G}}}(\gamma, t). \tag{1.2}$$

The right-hand side of this system is the solution of the system of the Volterra integral equations of the second kind with the identity matrix of out-of-integral coefficients:

$$\tilde{\tilde{\mathbf{G}}}(\gamma, t) + [\underline{\tilde{\tilde{\mathbf{B}}}}^*(\gamma)\tilde{\tilde{\mathbf{G}}}(\gamma, \cdot)](t) = \mathbf{G}(\gamma, t). \tag{1.3}$$

Suppose also that $\delta_{gs+}(t) < 1$, where $\delta_{gs+}(t)$ is given by (I.4.11). The kernel matrix $\tilde{\tilde{\mathbf{B}}}^*(\gamma, t, \tau) = \mathbf{B}^*(\gamma, t, \tau)\mathbf{A}(\gamma, \tau)/\Delta(\gamma, \tau)$, and the elements of $\mathbf{B}^*(\gamma, t, \tau)\mathbf{A}(\gamma, \tau)$ are, by Cramer's rule, the determinants of the matrix $\mathbf{B}^0(\gamma, \tau)$ one row of which is replaced by a corresponding row of the matrix $\mathbf{B}^*(\gamma, t, \tau)$. The direct analysis of these determinants (see (I.3.18)), on account of the memberships $\alpha^0(\tau)$, $\beta^0(\tau) \in \mathrm{C}[0, t]$, $\underline{\alpha}^*$, $\underline{\beta}^* \in \mathrm{VC}(0, t)$ and estimate (I.4.6), shows that

$$\tilde{\tilde{\mathbf{B}}}^*(\gamma, t, \tau) = \alpha^*(t, \tau)\tilde{\tilde{\mathbf{B}}}^*_\alpha(\gamma, \tau) \; + \beta^*(t, \tau)\tilde{\tilde{\mathbf{B}}}_\beta(\gamma, \tau),$$
$$|\tilde{\tilde{\mathbf{B}}}^*_\alpha(\gamma, \tau)|, |\tilde{\tilde{\mathbf{B}}}^*_\beta(\gamma, \tau)| < M(\bar{S}'_r(\tau)), \; \gamma \in S'_r(\tau), \; \forall S'_r(\tau) \tag{1.4}$$

where $\tilde{\tilde{\mathbf{B}}}^*_\alpha(\gamma, \tau)$, $\tilde{\tilde{\mathbf{B}}}^*_\beta(\gamma, \tau)$ are meromorphic functions of γ, and $S'_r(\tau)$ is a perforated strip (S'-without r-neighbourhoods of all $\gamma_k(\tau)$). If S' does not include the zeros $\gamma_k(\tau)$, or γ_k are independent of τ, then M is independent of τ, too. Hence, $\underline{\tilde{\tilde{\mathbf{B}}}}^*(\gamma) \in \mathrm{VCH}[S'; 0, t]$ (see point 6^0 of section I.2) for any S' that does not include $\gamma_k(\tau)$, $\tau \in [0, t]$.

Particularly, $\underline{\tilde{\tilde{\mathbf{B}}}}^*(\gamma) \in \mathrm{VCH}[S(\delta_{gs+}(t), 1); 0, t]$. From Appendix, point 10^0, we then obtain that the operator $\mathbf{I} + \underline{\tilde{\tilde{\mathbf{B}}}}^*(\gamma)$ is invertible for every $\gamma \in S(\delta_{gs+}(t), 1)$, and the inverse operator $[\mathbf{I} + \underline{\tilde{\tilde{\mathbf{B}}}}^*(\gamma)]^{-1}$ acts in $\mathrm{CH}_2^0(S(\delta_{gs+}(t), 1); 0, t)$. Hence, the solution $\tilde{\tilde{\mathbf{G}}}(\gamma, t)$ of (1.3) belongs to the same class as its right-hand side $\mathbf{G}(\gamma, t)$.

Since the left-hand sides of the systems (1.2) and (I.4.1) are identical, we obtain by the same reasoning as in section I.4 for the solution of system (1.2) that $\mathfrak{F} \in C\check{\tilde{H}}_2(\Theta_-, \Theta_+; S(\delta_{gs+}(t), 1); 0, t)$ and, consequently (see Appendix, point 2^0), $\Phi_j^{(l)}$, $\Psi_j^{(l)} \in \mathrm{C\tilde{H}}_2(\delta_{gs+}(t), 1; W_j^{(l)}; 0, t)$. It means that a priori supposed membership (I.3.15) holds for $\tilde{\delta}_0 = \delta_{gs+}(t)$. By Appendix, point 6^0 we arrive, as in section I. 4, at the same estimates (I.4.13) for the complex potentials. Consequently, for the stresses and strains we get the same estimate (I.4.18):

$$|\sigma_{ij}^{(l)}(\rho, \theta, \tau)|, |\varepsilon_{ij}^{(l)}(\rho, \theta, \tau)| < \tilde{\tilde{M}}_0(t)\rho^{-\delta_{gs+}(t)-\varepsilon}$$
$$\forall \varepsilon \in (0, 1 - \delta_{gs+}(t)), \quad \forall \theta \in [\theta_-^{(l)'}, \theta_+^{(l)'}) \subset (\theta_-^{(l)}, \theta_+^{(l)}), \; \forall \tau \in [0, t].$$

The parameter $\delta_{gs+}(t)$ is given by (I.4.11), i.e., is independent of the integral operators $\underline{\alpha}$, $\underline{\beta}$ and is determined only by the instantaneous Dundurs parameters $\alpha^0(\tau)$, $\beta^0(\tau)$, $\tau \in [0, t]$ and by the behavior of the prescribed load as $\rho \to 0$. The remark given in section I.4 after (I.4.18) also holds. An analogous estimate for a homogeneous (not bonded) hereditarily-elastic body was obtained in [7].

If the α^0 and β^0 are independent of time, particularly if the materials of the body parts are hereditary but not aging, then the power δ_{gs+} in estimate (I.4.18) is independent of time t and coincides with the main stress singularity power γ_k in the corresponding elastic body whose elastic moduli coincide with the instantaneous elastic moduli $\Lambda^{(l)0}$, $\mu^{(l)0}$ of the hereditary body considered. If $\alpha^0(\tau)$, $\beta^0(\tau)$ depend on time, then the parameter $\delta_{gs+}(t)$ coincides with the supremum on the segment $[0, t]$ of the stress singularity power for α classical elastic body, whose moduli coincide with the momentary moduli $\Lambda^{(l)0}(\tau)$, $\mu^{(l)0}(\tau)$ of the considered body. Thus, if there are no zeros γ_k of $\Delta(\gamma, \tau)$ in the strip $0 \leqslant \operatorname{Re}\gamma < 1$, i.e., stress singularity is absent in the corresponding classical problems with the same elastic moduli for all $\tau > 0$, then the stress singularity in a bonded hereditarily-elastic aging body will not occur at any finite time moment τ (for sufficiently smooth loads).

2. Initial time

Let us consider the solution asymptotics at $t = +0$. Let the boundary loads $g_i^{(l)}(\cdot, 0)$, $f_i^{(0)\prime}(\cdot, 0) \in \hat{L}(\delta_g, 1)$. The hereditary operators $\underline{\alpha}, \underline{\beta}$ in matrix (I.3.18) are reduced in this case to multiplication by the constants $\alpha^0(0)$, $\beta^0(0)$, respectively. Repeating again the reasoning of section I.4, we obtain asymptotics (I.4.16), (I.4.17) for stresses and strains, where $t = 0$ must be set. Thus, the asymptotics in this case are the same as for the non-hereditary elastic body with Lame′ constants $\Lambda_i^{(l)0}(0)$, $\mu_i^{(l)0}(0)$.

3. Small times

Let us now present the dependence of the asymptotics on time for non-zero but small times. This representation will explain, to a certain extent, why we succeeded in obtaining estimate (1.4) only but not asymptotics for finite times.

As before $g_i^{(l)}, \tilde{g}_i^{(0)} \in \mathrm{C}\hat{\mathrm{L}}_2(\delta_g, 1; 0, T)$, $0 \leqslant T < \infty$, then $\mathbf{G} \in \mathrm{CH}_2^0(S(\delta_g, 1); 0, T)$. Let also the kernels of the Dundurs operators $\underline{\alpha}$ and $\underline{\beta}$ be of the Abel type, i.e., $\alpha^*(t, \tau) = \alpha_c^*(t, \tau)\ (t - \tau)^{-\zeta}$, $\beta^*(t, \tau) = \beta_c^*(t, \tau)(t - \tau)^{-\zeta}$, where the functions $\alpha_c^*(t, \tau)$, $\beta_c^*(t, \tau)$ are continuous. (The results for the case of continuous kernels $\alpha^*(t, \tau)$, $\beta^*(t, \tau)$ will be obtained by the substitution $\zeta = 0$.) Then due to (I.3.18), (1.1), (I.4.3), the kernels $\mathbf{B}^*(\gamma, t, \tau)$, $\tilde{\tilde{\mathbf{B}}}^*(\gamma, t, \tau)$ are also of the Abel type: $\mathbf{B}^*(\gamma, t, \tau) = \mathbf{B}_c^*(\gamma, t, \tau)(t - \tau)^{-\zeta}$, $\tilde{\tilde{\mathbf{B}}}^*(\gamma, t, \tau) = \tilde{\tilde{\mathbf{B}}}_c^*(\gamma, t, \tau)\ (t - \tau)^{-\zeta}$, where $\mathbf{B}_c^*(\gamma, t, \tau)$ is continuous with respect to t, τ for any γ. For every t, τ, by (1.1) and (I.4.4), $\tilde{\tilde{\mathbf{B}}}_c^*(\gamma, t, \tau))$ is a meromorphic function with the poles at $\gamma_k(\tau)$, and the pole multiplicity is equal to the rank $P_{k*}(\tau)$ of the eigenvalue $\gamma_k(\tau)$, $P_{k*}(\tau) := \max_{n=1 \div N_k}(P_{kn}(\tau)) \leqslant N_k^0(\tau)$. The kernel $\tilde{\tilde{\mathbf{B}}}_c^*(\gamma, t, \tau)$ is continuous in $t, \tau \in [0, T] \times [0, T]$ uniformly with respect to γ on any strip $\bar{S}'$ which does not include $\gamma_k(\tau)$, and hence $\tilde{\tilde{\mathbf{B}}}_c^* \in \mathrm{VCH}(S'; 0, T)$ for such S'.

Let Q be an arbitrary positive integer. Then from (1.3) and (1.2), we get:

$$\mathfrak{F}(\gamma, t) = (\mathbf{B}^0)^{-1}(\gamma, t)\tilde{\tilde{\mathbf{G}}}(\gamma, t) = \sum_{q=0}^{Q} \mathfrak{F}_q(\gamma, t) + \mathfrak{F}_{Q+1}^{(\mathrm{R})}(\gamma, t), \tag{3.1}$$

$$\mathfrak{F}_q(\gamma, t) := (\mathbf{B}^0)^{-1}(\gamma, t)\{[-\tilde{\tilde{\mathbf{B}}}^*(\gamma)]^q \mathbf{G}(\gamma, \cdot)\}(t), \tag{3.2}$$

$$\mathfrak{F}_{Q+1}^{(\mathrm{R})}(\gamma, t) := (\mathbf{B}^0)^{-1}(\gamma, t)\{[-\tilde{\tilde{\mathbf{B}}}^*(\gamma)]^{Q+1} \tilde{\tilde{\mathbf{G}}}(\gamma, \cdot)\}(t).$$

3.1. The case of constant momentary Dundurs' parameters

Let in this subsection, the out-of-integral terms α^0, β^0 be independent on time. (This particularly holds for non-aging hereditary materials.) Then the matrices $\mathbf{B}^0(\gamma)$, $(\mathbf{B}^0)^{-1}(\gamma)$ and the zeros γ_k are independent of t too.

Taking into account definition (3.2), the memberships for $\mathbf{G}(\gamma, t)$ and $\underline{\mathbf{B}}^*_c$ described at the beginning of section 3, the statements given in points 8°, 9° of Appendix, and estimates (I.4.7)–(I.4.10) for $(\mathbf{B}^0)^{-1}(\gamma)$, we obtain that the functions $\mathfrak{F}_q \in \mathrm{C}\check{\tilde{\mathrm{H}}}_2(\Theta_-, \Theta_+; S'; 0, T)$ in any strip $S' \subset S(\delta_g, 1)$ that does not include γ_k.

Let us denote

$$\mathbf{F}_q(\gamma, t, \tau_1, \ldots, \tau_q) := (-1)^q \tilde{\tilde{\mathbf{B}}}{}^*_c(\gamma, t, \tau_1) \tilde{\tilde{\mathbf{B}}}{}^*_c(\gamma, \tau_1, \tau_2) \ldots \tilde{\tilde{\mathbf{B}}}{}^*_c(\gamma, \tau_{q-1}, \tau_q)$$

If $\gamma \neq \gamma_k$, then using the mean value theorem, we get from (3.2):

$$\mathfrak{F}_q(\gamma, t) := (\mathbf{B}^0)^{-1}(\gamma) \int_0^t (t-\tau_1)^{-\zeta} \mathrm{d}\tau_1 \int_0^{\tau_1} (\tau_1 - \tau_2)^{-\zeta} \mathrm{d}\tau_2 \ldots$$

$$\int_0^{\tau_{q-1}} (\tau_{q-1} - \tau_q)^{-\zeta} \mathrm{d}\tau_q \mathbf{F}_q(\gamma, t, \tau_1, \ldots, \tau_q) \mathbf{G}(\gamma, \tau_q)$$

$$= (\mathbf{B}^0)^{-1}(\gamma) \mathbf{F}_q(\gamma, t, \tilde{\tau}_1, \ldots, \tilde{\tau}_q) \mathbf{G}(\gamma, \tilde{\tau}_q) \int_0^t (t-\tau_1)^{-\zeta} \mathrm{d}\tau_1$$

$$\times \int_0^{\tau_1} (\tau_1 - \tau_2)^{-\zeta} \mathrm{d}\tau_2 \ldots \int_0^{\tau_{q-1}} (\tau_{q-1} - \tau_q)^{-\zeta} \mathrm{d}\tau_q = t^{q(1-\zeta)} \mathfrak{F}^{(t)}_q(\gamma, t), \tag{3.3}$$

$$\mathfrak{F}^{(t)}_q(\gamma, t) := C_q (\mathbf{B}^0)^{-1}(\gamma) \mathbf{F}_q(\gamma, t, \tilde{\tau}_1, \ldots, \tilde{\tau}_q) \mathbf{G}(\gamma, \tilde{\tau}_q), \tag{3.4}$$

$$C_q := \prod_{p=1}^{q} B((p-1)(1-\zeta)+1, 1-\zeta) = [\Gamma(1-\zeta)]^q / \Gamma(q(1-\zeta)+1).$$

Here $0 \leqslant \tilde{\tau}_q(t) \leqslant \cdots \leqslant \tilde{\tau}_1(t) \leqslant t$, B is the Beta-function, and Γ is the Gamma-function.

Comparing the left- and right-hand sides of (3.3) we get that $\mathfrak{F}^{(t)}_q \in \mathrm{C}\check{\tilde{\mathrm{H}}}_2(\Theta_-, \Theta_+; S'; T_1, T)$ for any $T_1 \in (0, T)$ and any strip $S' \subset S(\delta_g, 1)$ that does not include γ_k. Moreover, due to (3.4) and continuity (uniform with respect to $\gamma \in S'$) of $\mathbf{F}_q(\gamma, t, \tilde{\tau}_1(t), \ldots, \tilde{\tau}_q(t))$ at $t = 0$, the functions $\mathfrak{F}^{(t)}_q(\gamma, t)$ are bounded and continuous in t at $t = 0$ (in the sense of definition of CH_2 in point 4^0 of section I.2). Hence, $\mathfrak{F}^{(t)}_q \in \mathrm{C}\check{\tilde{\mathrm{H}}}_2(\Theta_-, \Theta_+; S'; 0, T)$. Besides, $|\mathbf{B}^0(\gamma)\mathfrak{F}_q(\gamma, t)|$, $|\mathbf{B}^0(\gamma)\mathfrak{F}^{(t)}_q(\gamma, t)| < M''(\bar{S}'_r)$, for any perforated strip $S'_r \subset S(\delta_g, 1)$; $\mathfrak{F}_q(\gamma, t)$ and, consequently, $\mathfrak{F}^{(t)}_q(\gamma, t)$ are meromorphic functions of γ with poles of the multiplicity $(q+1)P_{k*}$ at γ_k, and all the coefficients of their Laurent's expansions near γ_k are continuous in t.

Analogously, $\mathfrak{F}^{(\mathrm{R})}_{Q+1}(\gamma, t) = t^{(Q+1)(1-\zeta)} \mathfrak{F}^{(\mathrm{R}t)}_{Q+1}(\gamma, t)$, and the functions $\mathfrak{F}^{(\mathrm{R})}_{Q+1}$, $\mathfrak{F}^{(\mathrm{R}t)}_{Q+1} \in \mathrm{C}\check{\tilde{\mathrm{H}}}_2(\Theta_-, \Theta_+; S(\delta_{gs+}, 1); 0, T)$.

Then we obtain from (3.1)

$$\mathfrak{F}(\gamma, t) = \sum_{q=0}^{Q} t^{q(1-\zeta)} \mathfrak{F}^{(t)}_q(\gamma, t) + t^{(Q+1)(1-\zeta)} \mathfrak{F}^{(\mathrm{R}t)}_{Q+1}(\gamma, t).$$

In this representation the term $\mathfrak{F}^{(t)}_0(\gamma, t)$ coincides with the function $\mathfrak{F}(\gamma, t)$ for the degenerate case, given by (I.4.2).

Applying the inverse Mellin transform, we have

$$\Phi_j^{(l)}(z_j, t) = \frac{1}{2\pi \mathrm{i}} \Bigg[\sum_{q=0}^{Q} t^{q(1-\zeta)} \int_{\delta+\mathrm{i}\infty}^{\delta+\mathrm{i}\infty} [\mathfrak{F}_q^{(t)}(\gamma, t)]_{4l-4+j} z_j^{-\gamma}\, \mathrm{d}\gamma$$

$$+ t^{(Q+1)(1-\zeta)} \int_{\delta+\mathrm{i}\infty}^{\delta+\mathrm{i}\infty} [\mathfrak{F}_{Q+1}^{(\mathrm{R}t)}(\gamma, t)]_{4l-4+j} z_j^{-\gamma}\, \mathrm{d}\gamma \Bigg] \in \mathrm{CH}_2(\delta_{gs+}, 1; W_{jl}; 0, T),$$

$$(j = 1 \div 2)$$

where $\delta_{gs+} < \delta < 1$, and analogous formulas for $\Psi_j^{(l)}(z_j, t)$.

Shifting, as in section I.4, the integration path of the first integral to the left into the strip $S(\delta_g, \delta_-)$, calculating residues of the integrand and using Appendix, points 2^0 and 6^0, for estimates on $S(\delta_g, \delta_-)$, we get representations for the potentials $\Phi_j^{(l)}(z_j, t)$, $\Psi_j^{(l)}(z_j, t)$ and, consequently, for the stresses:

$$\sigma_{ij}^{(l)}(\rho, \theta, t) = \sum_{q=0}^{Q} t^{q(1-\zeta)} \sigma_{ij}^{(lq)}(\rho, \theta, t) + t^{(Q+1)(1-\zeta)} \sigma_{ij}^{(l,Q+1,\mathrm{R})}(\rho, \theta, t), \tag{3.5}$$

$$\sigma_{ij}^{(lq)}(\rho, \theta, t) = \sum_{\delta_g < \mathrm{Re}\,\gamma_k < 1} \rho^{-\gamma_k} \sum_{n=0}^{(q+1)P_{k*}-1} \ln^n \rho \sum_{p=1}^{\tilde{P}_{kn}} \tilde{K}_{knp}^{(q)}(t) \tilde{F}_{ijknp}^{(lq)}(\theta) + \sigma_{*ij}^{(lq)}(\rho, \theta, t),$$

$$|\sigma_{*ij}^{(lq)}(\rho, \theta, t)| < M_* \rho^{-\delta_g - \varepsilon} \quad \forall \varepsilon \in (0, \delta_- - \delta_g),$$

$$|\sigma_{ij}^{(l,Q+1,\mathrm{R})}(\rho, \theta, t)| < M^{(\mathrm{R})} \rho^{-\delta_{gs+} - \varepsilon} \quad \forall \varepsilon \in (0, 1 - \delta_{gs+}),$$

$$\forall \theta \in [\theta_{j-}^{(l)\prime}, \theta_{j+}^{(l)\prime}] \subset (\theta_-^{(l)}, \theta_+^{(l)}), \quad \forall t \in [0, T]; \tilde{K}_{knp}^{(q)}(t) \in C[0, T].$$

Here $Q \geqslant 0$ is an arbitrary integer. The stress singularity powers γ_k are zeros of the function $\Delta(\gamma)$, determined by the instantaneous Dundurs parameters α^0, β^0. The sense and properties of the stress intensity factors $\tilde{K}_{knp}^{(q)}(t)$, eigenfunctions $\tilde{F}_{ijknp}^{(lq)}(\theta)$, and parameters $\tilde{P}_{kn} \geqslant 8$ are close to those (without waves) given in section I.4. The term $\sigma_{ij}^{(l0)}(\rho, \theta, t)$ has the asymptotic representation given by the classical elastic solutions with the moduli equal to the instantaneous moduli of hereditary media at $t = 0$. The other terms $\sigma_{ij}^{(lq)}(\rho, \theta, t)$ have the stress singularity powers γ_k as $\sigma_{ij}^{(l0)}(\rho, \theta, t)$, however, the possible number and the powers of their logarithmic terms grow linearly with q. The remainder term coefficient $\sigma_{ij}^{(l,Q+1,R)}(\rho, \theta, t)$ has the same coarse estimate (1.4) in ρ as the overall solution $\sigma_{ij}^{(l)}(\rho, \theta, t)$.

This means that for a small t and a fixed radius ρ, we can render the remainder term $t^{(Q+1)(1-\zeta)} \sigma_{ij}^{(l,Q+1,\mathrm{R})}(\rho, \theta, t)$ in (3.5), having the principal singularity, arbitrarily small, by choosing Q sufficiently large. But meanwhile, the number and maximal power $(Q+1)P_{k*} - 1$ of logarithms in the other terms will grow, compensating, in a sense, the decrease of the term with the principal singularity.

The representation for the case of continuous kernels is obtained by substituting $\zeta = 0$. Then (3.5) becomes the expansion in integer powers of t.

Note that for the problem for non-aging hereditary materials with some particular continuous ($\zeta = 0$) hereditary kernels for some special loadings, the two-term representation ($Q = 1$), close in a sense to (3.5), was obtained in [2–4] by using the Laplace transform in time.

3.2. A general case of aging materials

1^0 Let now $\alpha^0(t)$, $\beta^0(t)$ depend on time and belong to the space $C^V[0, T]$ of V times continuously differentiable functions, $V \geqslant 0$. Then the functions $\mathbf{B}^0(\gamma, \tau)$, $(\mathbf{B}^0)^{-1}(\gamma, \tau)$, $\in C^V[0, T]$ with respect to τ in the whole γ-plane (at $\gamma \neq \gamma_k(\tau)$ for $(\mathbf{B}^0)^{-1}(\gamma, \tau)$). Consequently, the matrix $(\mathbf{B}^0)^{-1}(\gamma, \tau)$ can be represented in the form of a trancated Tailor series for every $\tau \in [0, T]$, every $\gamma \neq \gamma_k(\tau)$, and for appropriate $\tau_*(\gamma, \tau) \in [0, \tau]$:

$$(\mathbf{B}^0)^{-1}(\gamma, \tau) = \sum_{v=0}^{V} \tau^v \mathbf{B}^{0(v)}(\gamma) + \tau^V \tilde{\mathbf{B}}^{0(V+1)}(\gamma, \tau) = \sum_{v=0}^{V+1} \tau^{w(v,V)} \tilde{\mathbf{B}}^{0(v)}(\gamma, \tau), \tag{3.6}$$

$$\mathbf{B}^{0(v)}(\gamma) := \frac{1}{v!}[\partial^v (\mathbf{B}^0)^{-1}(\gamma, t)/\partial t^v]_{t=0},$$

$$\tilde{\mathbf{B}}^{0(V+1)}(\gamma, \tau) := \frac{1}{V!}[\partial^V (\mathbf{B}^0)^{-1}(\gamma, t)/\partial t^V]_{t=\tau_*(\gamma,\tau)} - \mathbf{B}^{0(V)}(\gamma),$$

$$\tilde{\mathbf{B}}^{0(v)}(\gamma, \tau) := \mathbf{B}^{0(v)}(\gamma) \quad (v = 0 \div V), \quad w(v, V) := \min(v, V).$$

2^0 Differentiating the identity $\mathbf{B}^0(\gamma, \tau)(\mathbf{B}^0)^{-1}(\gamma, \tau) = \mathbf{I}$ and taking into consideration the properties of the matrix $\mathbf{B}^0(\gamma, \tau)$ (see (I.3.18)), it is possible to show that $\mathbf{B}^{0(v)}(\gamma) = (\mathbf{B}^0)^{-1}(\gamma, 0)\check{\mathbf{B}}^{0(v)}(\gamma)$, where $\check{\mathbf{B}}^{0(v)}(\gamma)$ is a meromorphic matrix function in the whole γ-plane with poles at $\gamma_k(0)$ whose multiplicities are not greater than $vP_{k*}(0)$, and

$$|\check{\mathbf{B}}^{0(v)}(\gamma)| < \check{M}(\bar{S}_r'(0)) < \infty, \quad \gamma \in \bar{S}_r'(0), \forall S'. \tag{3.7}$$

The function $\tilde{\mathbf{B}}^{0(V+1)}(\gamma, \tau) = (\mathbf{B}^0)^{-1}(\gamma, 0)\check{\tilde{\mathbf{B}}}^{0(V+1)}(\gamma, \tau)$ where $|\check{\tilde{\mathbf{B}}}^{0(V+1)}(\gamma, \tau)| < \check{\tilde{M}}(\bar{S}') < \infty$, $\gamma \in \bar{S}'$, $\forall \bar{S}' \subset S(\delta_{gs+}(T), 1)$, and $\check{\tilde{\mathbf{B}}}^{0(V+1)}(\gamma, \tau) \to 0$ as $\tau \to 0$ uniformly with respect to γ on any strip $\bar{S}' \subset S(\delta_{gs+}(T), 1)$ at $\tau \neq 0$. The same uniform continuity of $\check{\tilde{\mathbf{B}}}^{0(V+1)}(\gamma, \tau)$, in τ follows from such property for $(\mathbf{B}^0)^{-1}(\gamma, \tau)$ and from (3.6). Consequently, $\check{\tilde{\mathbf{B}}}^{0(V+1)}(\gamma, \tau) \in C[0, T]$ uniformly with respect to γ on any strip $\bar{S}' \subset S(\delta_{gs+}(T), 1)$. For $v \leqslant V$, we also denote $\check{\tilde{\mathbf{B}}}^{0(v)} := \check{\mathbf{B}}^{0(v)}(\gamma)$.

3^0 Let $w_i := \min(v_i, V)$, $\mathfrak{B}^{(v_i)}(\gamma, \tau_i) := \tau_i^{w_i}\tilde{\mathbf{B}}^{0(v_i)}(\gamma, \tau_i)$. Then due to (3.6), the term $\mathfrak{F}_q(\gamma, t)$ given by (3.2) has the form

$$\mathfrak{F}_q(\gamma, t) = (\mathbf{B}^0)^{-1}(\gamma, t)\{[-\underline{\mathbf{B}}^*(\gamma)(\mathbf{B}^0)^{-1}(\gamma, \cdot)]^q \mathbf{G}(\gamma, \cdot)\}(t)$$
$$= \sum_{v_0=0}^{V+1} \sum_{v_1=0}^{V+1} \cdots \sum_{v_q=0}^{V+1} \mathfrak{F}^{(c+)}_{qv_0v_1\ldots v_q}(\gamma, t), \tag{3.7}$$

$$\mathfrak{F}^{(c+)}_{qv_0v_1\ldots v_q}(\gamma, t) := (\mathbf{B}^0)^{-1}(\gamma, 0)\, \mathbf{G}_{qv_0v_1,\ldots,v_q}(\gamma, t),$$

$$\mathscr{G}_{qv_0v_1,\ldots,v_q}(\gamma, t) := (-1)^q t^{w_0} \check{\tilde{\mathbf{B}}}^{0(v_0)}(\gamma, t)\{\underline{\mathbf{B}}^*(\gamma)\mathfrak{B}^{(v_1)}(\gamma, \cdot)\underline{\mathbf{B}}^*(\gamma)\mathfrak{B}^{(v_2)}(\gamma, \cdot) \ldots$$
$$\ldots \underline{\mathbf{B}}^*(\gamma)\mathfrak{B}^{(v_q)}(\gamma, \cdot)\mathbf{G}(\gamma, \cdot)\}(t). \tag{3.8}$$

$$= \tilde{\mathbf{B}}^{0(v_0)}(\gamma, t) t^{w_0} \int_0^t d\tau_1 (t - \tau_1)^{-\zeta} \tau_1^{w_1} \int_0^{\tau_1} d\tau_2 (\tau_1 - \tau_2)^{-\zeta} \tau_2^{w_2}$$
$$\ldots \int_0^{\tau_{q-1}} d\tau_q (\tau_{q-1} - \tau_q)^{-\zeta} \tau_q^{w_q} \mathbf{F}_{qv_1\ldots v_q}(\gamma, t, \tau_1, \ldots, \tau_q)\mathbf{G}(\gamma, \tau_q).$$

Let $v_i \leqslant V$ $(i = 0 \div q)$, then $\tilde{\mathbf{B}}^{0(v_i)}(\gamma, \tau) = \mathbf{B}^{0(v_i)}(\gamma)$, $\check{\tilde{\mathbf{B}}}^{0(v_i)}(\gamma, \tau) = \check{\mathbf{B}}^{0(v_i)}(\gamma)$ are independent of τ, t and

$$\mathbf{B}^*(\gamma, t, \tau)\mathfrak{B}^{(v_i)}(\gamma) = \tau^{v_i}\mathbf{B}^*(\gamma, t, \tau)\tilde{\mathbf{B}}^{0(v_i)}(\gamma) = \tau^{v_i}$$
$$[\mathbf{B}^*(\gamma, t, \tau)(\mathbf{B}^0)^{-1}(\gamma, 0)]\ \check{\mathbf{B}}^{0(v_i)}(\gamma)$$
$$= \tau^{v_i}[\alpha^*(t, \tau)\check{\tilde{\mathbf{B}}}^*_\alpha(\gamma, 0) + \beta^*(t, \tau)\check{\tilde{\mathbf{B}}}^*_\beta(\gamma, 0)]\check{\mathbf{B}}^{0(v_i)}(\gamma).$$

Taking into account estimates (1.4), (3.6′) we get that the operator $\underline{\mathbf{B}}^*(\gamma)\mathfrak{B}^{(v)}(\gamma) \in \mathrm{VCH}(S'; 0, T)$ (see point 6^0 of Section I.2) for every S' that does not contain $\gamma_i(0)$. Hence from the membership $\mathbf{G} \in \mathrm{CH}_2^0(S(\delta_g, 1); 0, T)$, definition (3.8), and point 9^0 of Appendix, we obtain that $\mathscr{G}^{(c+)}_{qv_0\ldots,v_q}(\gamma, t) \in \mathrm{CH}_2^0(S'; 0, T)$ for every S' belonging to $S(\delta_g, 1)$ and not containing $\gamma_k(0)$. Hence due to estimates (I.4.7)–(I.4.10) and point 7^0 of the Appendix, $\mathfrak{F}^{(0+)}_{qv_0\ldots,v_q} \in \check{\widetilde{\mathrm{CH}}}_2(\Theta_-, \Theta_+; S'; 0, T)$ for such strip S'. Summing the pole multiplicities of $(\mathbf{B}^0)^{-1}(\gamma, 0)$ and $\check{\mathbf{B}}^{0(v_i)}(\gamma)$, we get that $\mathfrak{F}^{(c+)}_{qv_0\ldots v_q}(\gamma, t)$ is meromorphic in $S(\delta_g, 1)$ with poles at $\gamma_k(0)$ of multiplicities not greater than $(q + 1 + \sum_{i=0}^q v_i)P_{k^*}(0)$, and all the coefficients of its Laurent's expansions near $\gamma_k(0)$ are continuous in $t \in [0, T]$.

If $v_t = V + 1$ for some $i = 0 \div q$, then by the analogous reasoning and due to the properties of the matrix $\check{\tilde{\mathbf{B}}}^0(\gamma, \tau)$ written above, we obtain that $\mathfrak{F}^{(c+)}_{qv_0\ldots v_q} \in \check{\widetilde{\mathrm{CH}}}_2(\Theta_-, \Theta_+; S(\delta_{gs+}(T), 1); 0, T)$.

4° Let us denote

$$\mathbf{F}_{qv_1\ldots v_q}(\gamma, t, \tau_1, \ldots, \tau_q)$$
$$:= (-1)^q\mathbf{B}^*_c(\gamma, t, \tau_1)\tilde{\mathbf{B}}^{0(v_1)}(\gamma, \tau_1)\mathbf{B}^*_c(\gamma, \tau_1, \tau_2)\tilde{\mathbf{B}}^{0(v_2)}(\gamma, \tau_2)\ldots$$
$$\ldots\mathbf{B}^*_c(\gamma, \tau_{q-1}, \tau_q)\tilde{\mathbf{B}}^{0(v_q)}(\gamma, \tau_q) \quad (v_t = 0 \div V + 1).$$

Taking into account the results of points 1^0 and 2^0 it is possible to draw the following conclusions. Let all $v_i \leqslant V$, then the function $\mathbf{F}_{qv_1\ldots v_q}(\gamma, t, \tau_1, \ldots, \tau_q)$ is meromorphic in the whole γ-plane with poles at $\gamma_k(0)$ multiplicities of not greater than $(q + \sum_{i=1}^q v_i)P_{k^*}(0)$; all the coefficients of its Laurent's expansions near $\gamma_k(0)$ are continuous in t, τ_i; $|\mathbf{F}_{qv_1\ldots v_q}| < M_F(\bar{S}'_r(0)) < \infty$, and $\mathbf{F}_{qv_1\ldots v_q}(\gamma, t, \tau_1, \ldots, \tau_q)$ is continuous in $t, \tau_i \in [0, T]$ uniformly with respect to $\gamma \in \bar{S}'_r(0)$, $\forall S'$. If some $v_i = V + 1$, then $\mathbf{F}_{qv_1\ldots v_q}(\gamma, t, \tau_1, \ldots, \tau_q)$ is holomorphic in $\gamma \in S(\delta_{gs+}(T), 1)$; $|\mathbf{F}_{qv_1\ldots v_q}| < M_F(\bar{S}') < \infty$; $\mathbf{F}_{qv_1\ldots v_q}(\gamma, t, \tau_1, \ldots, \tau_q)$ is continuous in t, τ_i; and $\mathbf{F}_{qv_1\ldots v_q}(\gamma, t, \tau_1, \ldots, \tau_q) \to 0$ as $\{t, \tau_1, \ldots, \tau_q\} \to 0$ uniformly with respect to γ on every strip $\bar{S}' \subset S(\delta_{gs+}(T), 1)$.

Expression (3.8) can be rewritten in the form

$$\mathfrak{F}^{(o+)}_{qv_0\ldots,v_q}(\gamma, t) = \tilde{\mathbf{B}}^{0(v_0)}(\gamma, t)t^{w_0}\int_0^t \mathrm{d}\tau_1(t-\tau_1)^{-\zeta}\tau_1^{w_1}\int_0^{\tau_1}\mathrm{d}\tau_2(\tau_1-\tau_2)^{-\zeta}\tau_2^{w_2}\ldots,$$
$$\ldots\int_0^{\tau_{q-1}}\mathrm{d}\tau_q(\tau_{q-1}-\tau_q)^{-\zeta}\tau_q^{w_q}\mathbf{F}_{qv_1\ldots v_q}(\gamma, t, \tau_{1,\ldots}\tau_q)\mathbf{G}(\gamma, \tau_q).$$

Taking into account that

$$t^{w_0}\int_0^t (t-\tau_1)^{-\zeta}\tau_1^{w_1}\,\mathrm{d}\tau_1\int_0^{\tau_1}(\tau_1-\tau_2)^{-\zeta}\tau_2^{w_2}\,\mathrm{d}\tau_2\ldots\int_0^{\tau_{q-1}}(\tau_{q-1}-\tau_q)^{-\zeta}\tau_q^{w_q}\,\mathrm{d}\tau_q$$
$$= C_{w_0w_1\ldots w_q}t^{q(1-\zeta)+w_0+w_1+\ldots+w_q},$$

where $C_{w_0 w_1 \ldots w_q}$ are constants, and using the mean value theorem, we have

$$\mathfrak{F}^{(c+)}_{qv_0 \ldots v_q}(\gamma, t) = t^{q(1-\zeta)+w_0+\cdots+w_q}\mathfrak{F}^{(tc+)}_{qv_0 \ldots v_q}(\gamma, t), \tag{3.9}$$

$$\mathfrak{F}^{(tc+)}_{qv_0 \ldots v_q}(\gamma, t) := \tilde{\mathbf{B}}^{0(v_0)}(\gamma, t)\mathbf{F}_{qv_1 \ldots v_q}(\gamma, t, \tilde{\tau}_1, \ldots, \tilde{\tau}_q)\mathbf{G}(\gamma, \tilde{\tau}_q)C_{w_1 \ldots w_q}, \tag{3.10}$$

where $0 \leqslant \tilde{\tau}_q(\gamma, t) \leqslant \cdots \leqslant \tilde{\tau}_1(\gamma, t) \leqslant t$.

The functions $\mathfrak{F}^{(tc+)}_{qv_0 \ldots v_q}(\gamma, t)$ have the same properties as described in point 2^0 for the corresponding $\mathfrak{F}^{(c+)}_{qv_0 \ldots v_q}(\gamma, t)$, and moreover, if $v_i = V + 1$ for some $i = 0 \div q$, then $\mathfrak{F}^{(tc+)}_{qv_0 \ldots v_q}(\gamma, 0) = 0$. These properties for $t \neq 0$ follow from (3.9); for $t = 0$, they follow from (3.10) and from the corresponding properties of $\mathbf{F}_{qv_1 \ldots v_q}(\gamma, t, \tau_1, \ldots, \tau_q)$ given above.

5^0 Substituting (3.9) into (3.7) and collecting the addends with the equal powers of t, we obtain

$$\mathfrak{F}_q(\gamma, t) = \sum_{v=0}^{(q+1)V} t^{v+q(1-\zeta)}\mathfrak{F}^{(tc)}_{qv}(\gamma, t) + t^{V+q(1-\zeta)}\mathfrak{F}^{(tc)}_{qV+}(\gamma, t), \tag{3.11}$$

$$\mathfrak{F}^{(tc)}_{qv}(\gamma, t) := \sum_{v_0+\cdots+v_q=v} \mathfrak{F}^{(tc+)}_{qv_0 \ldots v_q}(\gamma, t), \quad v_i \leqslant V,$$

where to $\mathfrak{F}^{(tc)}_{qV+}(\gamma, t)$ all the terms $t^{w_0+\cdots+w_q-V}\mathfrak{F}^{(tc+)}_{qv_1 \ldots v_q}(\gamma, t)$ are collected for which at least one $v_i = V + 1$ (i.e., that were constructed by use of $\tilde{\mathbf{B}}^{0(V+1)}(\gamma, \tau)$). Consequently, $\mathfrak{F}^{(tc)}_{qv}(\gamma, t)$ is meromorphic in $S(\delta_g, 1)$ with poles at $\gamma_k(0)$ of multiplies not greater than $(q + 1 + v)P_{k*}(0)$, and all the coefficients of its Laurent's expansions near $\gamma_k(0)$ are continuous in $t \in [0, T]$. In addition, $\mathfrak{F}^{(tc)}_{qv} \in \mathrm{C}\check{\tilde{\mathrm{H}}}_2(\Theta_-, \Theta_+; S'; 0, T)$ for every S' belonging to $S(\delta_g, 1)$ and not including $\gamma_k(0)$.

Adding to the last term in (3.11) also the terms from the sum, whose powers are greater than $V + q(1 - \zeta)$, we have

$$\mathfrak{F}_q(\gamma, t) = \sum_{v=0}^{V} t^{v+q(1-\zeta)}\, \mathfrak{F}^{(tc)}_{qv}(\gamma, t) + t^{V+q(1-\zeta)}\mathfrak{F}^{(\mathrm{R}tc)}_{qV}(\gamma, t), \tag{3.12}$$

$$\mathfrak{F}^{(\mathrm{R}tc)}_{qV}(\gamma, t) := \mathfrak{F}^{(tc)}_{qV+}(\gamma, t) + \sum_{v=V+1}^{(q+1)V} t^{v-V}\mathfrak{F}^{(tc)}_{qv}(\gamma, t).$$

The functions $\mathfrak{F}^{(\mathrm{R}tc)}_{qV} \in \mathrm{C}\check{\tilde{\mathrm{H}}}_2(\Theta_-, \Theta_+; S(\delta_{gs+}(T), 1); 0, T)$ and $\mathfrak{F}^{(\mathrm{R}tc)}_{qV}(\gamma, 0) = 0$.

By an analogous way it is possible to show that the remaining term of (3.1) is represented in the form $\mathfrak{F}^{(\mathrm{R})}_{Q+1}(\gamma, t) = t^{(Q+1)(1-\zeta)}\ \mathfrak{F}^{(\mathrm{R}t)}_{Q+1}(\gamma, t)$, where $\mathfrak{F}^{(\mathrm{R}t)}_{Q+1} \in \mathrm{C}\check{\tilde{\mathrm{H}}}_2(\Theta_-, \Theta_+; S(\delta_{gs+}(T), 1); 0, T)$.

After substituting (3.12) into (3.1), we get:

$$\mathfrak{F}(\gamma, t) = \sum_{q=0}^{Q}\sum_{v=0}^{V} t^{v+q(1-\zeta)}\mathfrak{F}^{(tc)}_{qv}(\gamma, t) + t^{V}\mathfrak{F}^{(\mathrm{R}tc)}_{QV*}(\gamma, t) + t^{(Q+1)(1-\zeta)}\mathfrak{F}^{(\mathrm{R}t)}_{Q+1}(\gamma, t), \tag{3.13}$$

$$\mathfrak{F}^{(\mathrm{R}tc)}_{QV*}(\gamma, t) := \sum_{q=0}^{Q} t^{q(1-\zeta)}\mathfrak{F}^{(\mathrm{R}tc)}_{qV}(\gamma, t). \tag{3.14}$$

Let the function $\mathrm{int}(x)$ be the integer part of x. Denote

$$Q_+(v) := \mathrm{int}[(V - v)/(1 - \zeta)]. \tag{3.15}$$

To get in the sum only those terms having powers of t not greater than r^V (power in the first remainder term), let us set $Q = Q_+(0)$, interchange the order of summation in (3.13), and retain in the sum only the terms with $q \leqslant Q_+(v)$ including the other ones to the remainder term. Taking into account that $(Q_+(0)+1)(1-\zeta) > V$, we can write:

$$\mathfrak{F}(\gamma, t) = \sum_{v=0}^{V} \sum_{q=0}^{Q_+(v)} t^{v+q(1-\zeta)} \mathfrak{F}^{(tc)}_{qv}(\gamma, t) + t^V \mathfrak{F}^{(\mathrm{R}c)}_{V+}(\gamma, t), \tag{3.16}$$

$$\mathfrak{F}^{(\mathrm{R}c)}_{V+}(\gamma, t) := \mathfrak{F}^{(\mathrm{R}tc)}_{Q_+(0)V*}(\gamma, t) + t^{(Q_+(0)+1)(1-\zeta)-V} \mathfrak{F}^{(\mathrm{R}t)}_{Q_+(0)+1}(\gamma, t) + \sum_{v=1}^{V} \sum_{q=Q_+(v)+1}^{Q_+(0)} t^{v+q(1-\zeta)} \mathfrak{F}^{(tc)}_{qv}(\gamma, t). \tag{3.17}$$

Due to the properties of the functions $\mathfrak{F}^{(\mathrm{R}tc)}_{qV}$, $\mathfrak{F}^{(\mathrm{R})}_{Q+1}$, and $\mathfrak{F}^{(tc)}_{qv}$, we obtain from (3.14) and (3.17) that $\mathfrak{F}^{(\mathrm{R}c)}_{V+} \in \mathrm{C}\breve{\tilde{\mathrm{H}}}_2(\Theta_-, \Theta_+; S(\delta_{gs+}(T), 1); 0, T)$ and $\mathfrak{F}^{(\mathrm{R}c)}_{V+}(\gamma, 0) = 0$.

6^0 We apply then the inverse Mellin transform to (3.16), shift the integration path of the transform for the terms in the sum to the left into the strip $S(\delta_g, \delta_-(0))$, and evaluate residues as in section 3.1 (it is possible to see that the conditions as $\operatorname{Im} r \to \pm\infty$ sufficient for the shift are met). Using the estimates of Appendix, point 6^0, and the Kolosov–Muskhelishvili formulas (I.1.3), we obtain the stress representation:

$$\sigma^{(l)}_{ij}(\rho, \theta, \mathrm{t}) = \sum_{v=0}^{V} \sum_{q=0}^{Q_+(v)} t^{v+q(1-\zeta)} \sigma^{(lqv)}_{ij}(\rho, \theta, t) + t^V \sigma^{(lV+)}_{ij}(\rho, \theta, t), \tag{3.18}$$

$$\sigma^{(lqv)}_{ij}(\rho, \theta, t) = \sum_{\delta_g < \operatorname{Re}\gamma_k < 1} \rho^{-\gamma_k(0)} \sum_{n=0}^{P_{kvq}} \ln^n \rho \sum_{p=1}^{\tilde{P}_{kn}} \tilde{K}^{(qv)}_{knp}(t) \tilde{F}^{(lqv)}_{ijknp}(\theta) + \sigma^{(lqv)}_{*ij}(\rho, \theta, t), \tag{3.19}$$

$$P_{kvq} := (v+q+1)P_{k*}(0) - 1, \; |\sigma^{(lqv)}_{*ij}(\rho, \theta, t)| < M_* \rho^{-\delta_g - \varepsilon} \quad \forall \varepsilon \in (0, \delta_-(0) - \delta_g),$$

$$\sigma^{(lV+)}_{ij}(\rho, \theta, t) < M^{(\mathrm{R})} \rho^{-\delta_{gs+}(T)+\varepsilon} \quad \forall \varepsilon \in (0, 1 - \delta_{gs+}(T)),$$

$$\forall \theta \in [\theta^{(l)\prime}_-, \theta^{(l)\prime}_+] \subset (\theta^{(l)}_-, \theta^{(l)}_+), \quad \forall t \in [0, T], \quad \tilde{K}^{(q)}_{knp}(t) \in C[0, T]. \tag{3.20}$$

The stress singularity powers $\gamma_k(0)$ are zeros for the function $\Delta(\gamma, 0)$, determined by the instantaneous Dundurs parameters $\alpha^0(0)$, $\beta^0(0)$ at the initial time instant $t = 0$. The sense and properties of the stress intensity factors $\tilde{K}^{(qv)}_{knp}(t)$, eigenfunctions $\tilde{F}^{(lqv)}_{ijknp}(\theta)$, and parameters $\tilde{P}_{kn}$ are analogous to those given in section I.4. The functions $\sigma^{(lqv)}_{*ij}(\rho, \theta, t)$, $\sigma^{(lV+)}_{ij}(\rho, \theta, t)$ are continuous with respect to $t \in [0, T]$ for any $\rho, \theta \in W^{(l)}$ and $\sigma^{(lV+)}_{ij}(\rho, \theta, 0) = 0$. The term $\sigma^{(l00)}_{ij}(\rho, \theta, t)$ has the asymptotics given by the classical elastic solutions with the moduli equal to the instantaneous moduli of hereditary media at $t = 0$. The other terms $\sigma^{(lqv)}_{ij}(\rho, \theta, t)$ have the same stress singularity powers $\gamma_k(0)$ as $\sigma^{(l00)}_{ij}(\rho, \theta, t)$, however, the possible number of their logarithmic terms grows linear with $v + q$. The residual term $\sigma^{(lV+)}_{ij}(\rho, \theta, t)$ has the same rough estimate (1.4) with respect to ρ as the total solution $\sigma^{(l)}_{ij}(\rho, \theta, t)$.

The corresponding representation for the case of continuous kernels $\alpha^*(t, \tau)$, $\beta^*(t, \tau)$ is obtained from (3.15) and (3.18)–(3.20) by substituting $\zeta = 0$. Then we can collect the terms with equal powers of t and obtain from (3.18) the representation with the single summation:

$$\sigma^{(l)}_{ij}(\rho, \theta, t) = \sum_{v=0}^{V} t^v \sigma^{(lv)}_{ij}(\rho, \theta, t) + t^V \sigma^{(lV+)}_{ij}(\rho, \theta, t). \tag{3.21}$$

The form of $\sigma^{(lv)}_{ij}(\rho, \theta, t)$ coincides with the form of $\sigma^{(l0v)}_{ij}(\rho, \theta, t)$ from (3.19).

On the other hand, if $V = 1$, i.e. the instantaneous Dundurs parameters $\alpha^0(t)$, $\beta^0(t)$ are only continuous, then only one term remains in the sum in (3.18), and for this case we have

$$\sigma_{ij}^{(l)}(\rho, \theta, t) = \sigma_{ij}^{(l00)}(\rho, \theta, t) + \sigma_{ij}^{(l0+)}(\rho, \theta, t), \tag{3.22}$$

where the asymptotic for $\sigma_{ij}^{(l00)}(\rho, \theta, t)$ is given by (3.19), the estimate for $\sigma_{ij}^{(l0+)}(\rho, \theta, t)$ by (3.20) and $\sigma_{ij}^{(l0+)}(\rho, \theta, 0) = 0$.

Choosing the considered time moment $t = T$ at the asymptotic representations (3.5), (3.18)–(3.22), we obtain that the power $\delta_{gs+}(T)$ of the remainder term estimates depends due to (I.4.11) only on the instantaneous Dundurs parameters $\alpha^0(\tau)$, $\beta^0(\tau)$ at times $\tau \leqslant T$, i.e., at times preceding (and coinciding) the considered one T.

4. Long behaviour time ($t \to \infty$)

Let $\alpha^0(\tau)$, $\beta^0(\tau)$ be continuous and bounded on $[0, \infty)$, $\underline{\alpha}^*$, $\underline{\beta}^* \in \mathrm{VC}(0, \infty)$. Suppose $g_i^{(l)}$, $\tilde{g}_i^{(0)}$ belong to $\mathrm{C}\hat{\mathrm{L}}_2(\delta_g, 1; \infty)$, i.e., these functions have bounded norms in the sense of point 1^0 of section I.2 but may have no limits as $t \to \infty$. Then, the Mellin transform $\mathbf{G} \in \mathrm{CH}_2^0(S(\delta_g, 1); 0, \infty)$. We will investigate in this section the solution behaviour on the whole half-axis $[0, \infty) \ni t$ and for $t \to \infty$. When necessary, we will consider further some more narrow classes of the hereditary operators $\underline{\alpha}$, $\underline{\beta}$ and of the boundary loads $g_i^{(l)}$, $\tilde{g}_i^{(0)}$.

In the degenerate case all the asymptotics and estimates of section I.4 are true for $T = \infty$ and $t, \tau \in [0, \infty)$.

Let us now consider the non-degenerate heredity for three cases of load behaviour when $t \to \infty$: the loads belonging to $\mathrm{C}\hat{\mathrm{L}}_2(\delta_g, 1; 0, \infty)$ only, the loads belonging to $\mathrm{C}\hat{\mathrm{L}}_2(\delta_g, 1; 0, \infty)$ and tending to the oscillating ones in time, and the loads belonging to $\mathrm{C}\hat{\mathrm{L}}_2(\delta_g, 1; 0, \infty)$ and stabilizing in time.

4.1. Loads bounded in a sense as $t \to \infty$

Like in section 1, we arrive in this case to equations (1.2) and (1.3), where $\tilde{\tilde{B}}(\gamma) \in \mathrm{VCH}\,[S(\delta_{gs+}(\infty); 0, \infty]$. Let the Dundurs operators $\underline{\alpha}$, $\underline{\beta}$ tend to $\underline{\alpha}^-$, $\underline{\beta}^-$ of the convolution type, i.e. these exist constats $\alpha^0(\infty)$, $\beta^0(\infty)$ and functions $\alpha^{*-}(\tau)$, $\beta^{*-}(\tau) \in L_1(0, \infty)$ such that $\alpha^0(\tau) \to \alpha^0(\infty)$, $\beta^0(\tau) \to \beta^0(\infty)$ as $\tau \to \infty$ and

$$\sup_{T_0 \leqslant t < \infty} \int_{T_0}^{t} |\alpha^*(t, \tau) - \alpha^{*-}(t - \tau)|\,\mathrm{d}\tau \to 0,$$

$$\sup_{T_0 \leqslant t < \infty} \int_{T_0}^{t} |\beta^*(t, \tau) - \beta^{*-}(t - \tau)|\,\mathrm{d}\tau \to 0, \quad (T_0 \to \infty).$$

Such property is quite natural for aging materials (see e.g. [1]).

Let us introduce notations

$$\hat{\alpha}^*(\omega) := \int_0^{\infty} \alpha^{*-}(\tau) \mathrm{e}^{-\omega\tau}\,\mathrm{d}\tau, \qquad \hat{\beta}^*(\omega) := \int_0^{\infty} \beta^{*-}(\tau) \mathrm{e}^{-\omega\tau}\,\mathrm{d}\tau,$$

$$\hat{\alpha}(\omega) := \alpha^0(\infty) + \hat{\alpha}^*(\omega), \qquad \hat{\beta}(\omega) := \beta^0(\infty) + \hat{\beta}(\omega), \tag{4.1}$$

and let $\hat{\mathbf{B}}(\gamma, \omega)$ be obtained from the operator matrix $\underline{\mathbf{B}}(\gamma)$ (see (I.3.18)) after replacing $\underline{\alpha}, \underline{\beta}$ by $\hat{\alpha}(\omega), \hat{\beta}(\omega)$. Let $\hat{\gamma}_k(\omega)$ be zeros of the function $\det[\hat{\mathbf{B}}(\gamma, \omega)]$ and

$$\hat{\delta} := \sup_{k,\omega} \operatorname{Re}\hat{\gamma}_k(\omega)\ (\operatorname{Re}\hat{\gamma}_k(\omega) < 1, \operatorname{Re}\omega \geqslant 0), \qquad \delta^\infty := \max[\delta_{gs+}(\infty), \hat{\delta}]. \tag{4.2}$$

Suppose $\delta^\infty < 1$ (it will be the case at least for the operators $\underline{\alpha}, \underline{\beta}$ with sufficiently small norms $|||\alpha^*; 0, \infty|||$ and $|||\beta^*; 0, \infty|||$ defined in section I.2, point 5⁰). Then according to [6, Theorem 3.19], which is a generalization of the half-line Paley–Wiener theorem, the operator $\mathbf{I} + \underline{\tilde{\tilde{\mathbf{B}}}}^*(\gamma)$ is invertible on the half-axis $[0, \infty) \ni t$ for $\gamma \in S(\delta^\infty, 1)$, and the inverse operator $[\mathbf{I} + \underline{\tilde{\tilde{\mathbf{B}}}}^*(\gamma)]^{-1}$ acts in $\mathrm{CH}_2^0(S(\delta^\infty, 1); 0, \infty)$. Hence, the solution $\tilde{\tilde{\mathbf{G}}}$ of (1.3) belongs to $\mathrm{CH}_2^0(S(\delta^\infty, 1); 0, \infty)$.

Then we obtain by the same reasoning as in sections I.4 and 1 that the solution of the system (1.2) $\mathfrak{F} \in C\check{\tilde{H}}_2(\Theta_-, \Theta_+; S(\delta^\infty, 1); 0, \infty)$ and the same estimates (I.4.13) hold for the complex potentials $\Phi_j^{(l)}(z_j, t), \Phi_j^{(l)}(z_j, t)$, where the parameter $\delta_{gs+}(t)$ must be replaced by δ^∞. Consequently, for the stresses and strains we have:

$$\begin{aligned} &|\sigma_{ij}^{(l)}(\rho, \theta, \tau)|, |\varepsilon_{ij}^{(l)}(\rho, \theta, \tau)| < \tilde{\tilde{M}}_\infty \rho^{-\delta^\infty - \varepsilon} \\ &\forall \varepsilon \in (0, 1 - \delta^\infty), \forall \theta \in [\theta_-^{(l)\prime}, \theta_+^{(l)\prime}] \subset (\theta_-^{(l)}, \theta_+^{(l)}), \forall \tau \in [0, \infty), \end{aligned} \tag{4.3}$$

where the constant $\tilde{\tilde{M}}_\infty$ is independent of ρ, θ, τ; δ^∞ is given by (4.2) and determined by the behaviour of the momentary Dundurs parameters $\alpha^0(\tau), \beta^0(\tau)$ and of the limit convolution kernels $\alpha^{*-}(\tau), \beta^{*-}(\tau)$ on the whole half-axis $[0, \infty) \ni \tau$ (and certainly by smoothness of given loads at $\rho \to 0$). Estimates (4.3) are more coarse than (1.4) but they are uniform with respect to τ on the half-axis $[0, \infty)$.

4.2. *Harmonic oscillations* ($t \to \infty$)

Let us consider an important particular case of loads. Suppose $g_i^{(l)}(\rho, t), \tilde{g}_i^{(0)}(\rho, t)$ belong to $\mathrm{C}\hat{\mathrm{L}}_2(\delta_g, 1; 0, \infty)$ and tend to time harmonic functions $g_{i\Omega}^{(l)}(\rho)e^{i\Omega t}, \tilde{g}_{i\Omega}^{(0)}(\rho)e^{i\Omega t}$ as $t \to \infty$, i.e.,

$$\begin{aligned} &\sup_{T_0 \leqslant t < \infty} \|g_i^{(l)}(\cdot, t) - g_{i\Omega}^{(l)}(\cdot)e^{i\Omega t}\|_2 \to 0, \quad T_0 \to \infty, \\ &\sup_{T_0 \leqslant t < \infty} \|\tilde{g}_i^{(0)}(\cdot, t) - \tilde{g}_{i\Omega}^{(0)}(\cdot)e^{i\Omega t}\|_2 \to 0, \quad T_0 \to \infty, \end{aligned} \tag{4.4}$$

where $g_{i\Omega}^{(l)}(\rho), \tilde{g}_{i\Omega}^{(0)}(\rho) \in \hat{L}_2(\delta_g, 1)$ (see point 1⁰ of section I.2).

Then (see [6, Theorem 2.16]) also their Mellin transform $\mathbf{G}(\gamma, t)$ tends to $\mathbf{G}_\Omega(\gamma)e^{i\Omega t}$ as $t \to \infty$, where $\mathbf{G}_\Omega \in H_2^0(S(\delta_g, 1))$, and

$$\sup_{T_0 \leqslant t < \infty} M_2^0[\mathbf{G}(\cdot, t) - \mathbf{G}_\Omega(\cdot)e^{i\Omega t}; \delta] \to 0, \quad T_0 \to \infty \tag{4.5}$$

uniformly with respect to δ on every $[\delta_0, \delta_\infty] \subset (\delta_g, 1)$ (see definition 3⁰ of section I.2).

Let the operators $\underline{\alpha}, \underline{\beta}$ tend to operators $\underline{\alpha}^-, \underline{\beta}^-$ of the convolution type and, in addition, $\underline{\alpha}^*, \underline{\beta}^*$ are operators with fading memory, i.e.,

$$\int_0^{T_0} |\alpha^*(t, \tau)|\, d\tau \to 0, \int_0^{T_0} |\beta^*(t, \tau)|\, d\tau \to 0, \quad (t \to \infty),$$

for any $T_0 \in [0, \infty)$.

Suppose $\delta^\infty < 1$, where δ^∞ is defined by (4.2); then (see (I.3.18)) the operator $\tilde{\tilde{\mathbf{B}}}(\gamma)$ in (1.3) belongs to VCH$[S(\delta^\infty, 1); 0, \infty]$, tends to a convolution operator, and is the operator with fading memory on $S(\delta^\infty, 1)$, i.e.,

$$\int_0^{T_0} \sup_{\gamma \in S'} |\tilde{\tilde{\mathbf{B}}}(\gamma, t, \tau)| \, \mathrm{d}\tau \to 0, \quad (T_0 \to \infty), \quad \forall \bar{S}' \in S(\delta^\infty, 1).$$

Under these conditions, it follows from [6, Lemma 3.28, Theorem 3.29, and Theorem 3.25] that the solution $\tilde{\tilde{\mathbf{G}}}(\gamma, t)$ of (1.3) tends to $\tilde{\tilde{\mathbf{G}}}_\Omega(\gamma)\mathrm{e}^{\mathrm{i}\Omega t}$ in the sense of (4.5) as $t \to \infty$, where $\tilde{\tilde{\mathbf{G}}}_\Omega(\gamma) = [I + \tilde{\tilde{B}}(\gamma, \mathrm{i}\Omega)]^{-1}\mathbf{G}_\Omega(\gamma) \in H_2^0(S(\delta^\infty, 1))$.

The solution of (1.2) is $\mathfrak{F}(r, t) = (\mathbf{B}^0)^{-1}(r, t)\tilde{\tilde{\mathbf{G}}}(r, t)$. At the same time

$$(\mathbf{B}^0)^{-1}(\gamma, t) = (B^0)^{-1}(\gamma, \infty) + (\mathbf{B}^0)^{-1}(\gamma, t)\check{\mathbf{B}}_-(\gamma, t),$$

$$\check{\mathbf{B}}_-(\gamma, t) := [\mathbf{B}^0(\gamma, t) - \mathbf{B}^0(\gamma, \infty)](\mathbf{B}^0)^{-1}(\gamma, \infty).$$

Taking into account the form of matrix $\mathbf{B}^0$ (see (I.3.18)), we obtain that $\sup_{\gamma \in S'}[\check{\mathbf{B}}_-(\gamma, t)| \to 0$ $(t \to \infty)$, $\forall \bar{S}' \subset S(\delta^\infty, 1)$. Using the estimates (I.4.7)–(I.4.10) for $(\mathbf{B}^0)^{-1}$ and [6, Lemma 2.17], we obtain that $\mathfrak{F}(\gamma, \tau) \to \mathfrak{F}_\Omega(\gamma)\mathrm{e}^{\mathrm{i}\Omega t}$ as $t \to \infty$, and

$$\sup_{T_0 \leqslant t < \infty} \check{M}(\mathfrak{F}(\cdot, t) - \mathfrak{F}_\Omega(\cdot)\mathrm{e}^{\mathrm{i}\Omega t}; \Theta_-, \Theta_+; \delta) \to 0, \quad T_0 \to \infty \tag{4.6}$$

uniformly with respect to δ on every $[\delta_0, \delta_\infty] \subset (\delta_g, 1)$ (see definition 4^0 of section I.2). Moreover, $\mathfrak{F}_\Omega(\gamma) = (\mathbf{B}^0)^{-1}(\gamma, \infty)\tilde{\tilde{\mathbf{G}}}_\Omega(\gamma) = \hat{\mathbf{B}}^{-1}(\gamma, \mathrm{i}\Omega)\mathbf{G}_\Omega(\gamma) \in \check{\tilde{H}}_2(\Theta_-, \Theta_+; S(\delta^\infty, 1))$.

As above, the matrix $\hat{\mathbf{B}}(\gamma, \mathrm{i}\Omega)$ is obtained from $\mathbf{B}(\gamma)$ after replacing the Dundurs operators $\underline{\alpha}$, $\underline{\beta}$ by their complex counterparts $\hat{\alpha}(\mathrm{i}\Omega)$, $\hat{\beta}(\mathrm{i}\Omega)$ given by (4.1). The latter, in turn, we also be obtained by replacing of the elastic hereditary operators $\underline{\mu}^{(l)}$, $\underline{\kappa}^{(l)}$ in (I.3.12) by their complex counterparts $\hat{\mu}^{(l)}(\mathrm{i}\Omega)$, $\hat{\kappa}^{(l)}(\mathrm{i}\Omega)$ given by the formulas of type (4.1).

From [6, Theorems 2.16, 1.15], after applying the inverse Mellin transform, we have that $\Phi_j^{(l)}(z_j, t) \to \Phi_{j\Omega}^{(l)}(z_j)\mathrm{e}^{\mathrm{i}\Omega t}$, $\Psi_j^{(l)}(z_j, t) \to \Psi_{j\Omega}^{(l)}(z_j)\mathrm{e}^{\mathrm{i}\Omega t}(t \to \infty)$ and

$$\sup_{T_0 \leqslant t < \infty} M_2[\Phi_j^{(l)}(\cdot, t) - \Phi_{j\Omega}^{(l)}(\cdot)\mathrm{e}^{\mathrm{i}\Omega t}; W_j^{(l)}; \delta] \to 0, \quad T_0 \to \infty,$$

$$\sup_{T_0 \leqslant t < \infty} M_2[\Psi_j^{(l)}(\cdot, t) - \Psi_{j\Omega}^{(l)}(\cdot)\mathrm{e}^{\mathrm{i}\Omega t}; W_{j\mp}^{(l)'}; \delta] \to 0, \quad T_0 \to \infty \tag{4.7}$$

for every $\delta \in [\delta_0, \delta_\infty] \subset (\delta_g, 1)$ (see definition 3^0 of section I.2), and $W_{j\mp}^{(l)'}$ is like in definition of $\mathrm{C}\tilde{\mathrm{H}}_2$ in section I.3. Moreover,

$$\Phi_{j\Omega}^{(l)}(z_j) = \frac{1}{2\pi \mathrm{i}} \int_{\delta - \mathrm{i}\infty}^{\delta + \mathrm{i}\infty} [\mathfrak{F}_\Omega(\gamma)]_{4l-4+j} z_j^{-\gamma} \, \mathrm{d}\gamma,$$

$$\Psi_{j\Omega}^{(l)}(z_j) = \frac{1}{2\pi \mathrm{i}} \int_{\delta - \mathrm{i}\infty}^{\delta + \mathrm{i}\infty} [\mathfrak{F}_\Omega(\gamma)]_{4l-2+j} z_j^{-\gamma} \, \mathrm{d}\gamma, \tag{4.8}$$

$$\delta^\infty < \delta < 1, \qquad \Phi_{j\Omega}^{(l)}(z_j), \Psi_{j\Omega}^{(l)}(z_j) \in \tilde{H}_2(\delta^\infty, 1; W(\theta_{j-}^{(l)}, \theta_{j+}^{(l)})).$$

The class $\tilde{H}_2$ is defined analogously as to e.g. $\mathrm{C}\tilde{\mathrm{H}}_2$ given in section I.3.

For any real Ω, $\hat{\mathbf{B}}^{-1}(\gamma, \mathrm{i}\Omega)$ is a meromorphic function with poles of a finite multiplicity at the zeros $\hat{\gamma}_k(\mathrm{i}\Omega)$ of the function $\det[\hat{\mathbf{B}}(\gamma, \mathrm{i}\Omega)]$ and has the properties analogous to those described in section I.4 for the matrix $\mathbf{B}^0(\gamma, t)$. Due to holomorphy of $\mathbf{G}_\Omega(\gamma)$ in $S(\delta_g, 1)$, we can continue the function $\mathfrak{F}_\Omega(\gamma) = \hat{\mathbf{B}}^{-1}(\gamma, \mathrm{i}\Omega)\mathbf{G}_\Omega(\gamma)$ analytically from

$S(\delta^{\infty}, 1)$ into $S(\delta_g, 1)$. Shifting, as usual, the integration path in (4.8) to the left and evaluating the residues of the integrands at $\hat{\gamma}_k(\mathrm{i}\Omega)$, we obtain for the amplitudes $\sigma_{i,j\Omega}(\rho, \theta)$, $\varepsilon_{ij\Omega}(\rho, \theta)$ the same classical asymptotics (I.4.16), (I.4.17), where all the parameters depend on the frequency Ω instead of time t and $\gamma_k(t)$ must be replaced by $\hat{\gamma}_k(\mathrm{i}\Omega)$. Moreover,

$$\sigma_{ij}(\rho, \theta, t) \to \sigma_{ij\Omega}(\rho, \theta)\,\mathrm{e}^{\mathrm{i}\Omega t}, \quad \varepsilon_{ij}(\rho, \theta, t) \to \varepsilon_{ij\Omega}(\rho, \theta)\,\mathrm{e}^{\mathrm{i}\Omega t} \qquad (t \to \infty).$$

Thus if the loads, tend to those oscillating harmonically in time, an aging hereditarily-elastic body possesses fading memory, and its hereditary operators tend to operators of the convolution type, then the stress and strain asymptotics at large times are the same as for classical elastic body, whose elastic moduli coincide with complex moduli of the body considered.

4.3. Stabilizing loads

Let $g_i^{(l)}(\rho, t)$, $\tilde{g}_i^{(0)}(\rho, t)$ belong to $\mathrm{C}\hat{\mathrm{L}}_2(\delta_g, 1; 0, \infty)$ and tend to functions $g_{i0}^{(l)}(\rho) = g_i^{(l)}(\rho, \infty)$, $\tilde{g}_{i0}^{(0)}(\rho) = \tilde{g}_i^{(0)}(\rho, \infty)$ (in the sense of (4.4) for $\Omega = 0$) as $t \to \infty$. Let $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$ have the same properties as in subsection 4.2. Then we obtain a special case of section 4.2 at $\Omega = 0$.

Thus, we again come for $t = \infty$ to classical elasticity asymptotics (I.4.16), (I.4.17) for the limiting values $\sigma_{ij}(\rho, \theta, \infty)$, $\varepsilon_{ij}(\rho, \theta, \infty)$, where $\gamma_k(t)$ must be replaced by $\gamma_{k\infty} = \hat{\gamma}_k(0)$—zeros of the determinant of matrix $\hat{\mathbf{B}}(\gamma, 0)$ obtained from $\mathbf{B}(\gamma)$ after replacing the Dundurs operators $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$ by their long-time counterparts $\hat{\alpha}(0)$, $\hat{\beta}(0)$ given by (4.1), and

$$\sigma_{ij}(\rho, \theta, t) \to \sigma_{ij}(\rho, \theta, \infty), \quad \varepsilon_{ij}(\rho, \theta, t) \to \varepsilon_{ij}(\rho, \theta, \infty) \quad (t \to \infty).$$

That means, stress and strain asymptotics at large times and stabilizing loads for an aging hereditarily-elastic body with fading memory and with hereditary operators tending to operators of the convolution type, are the same as for classical elastic body, whose elastic moduli coincide with long-time moduli of the body considered.

Let us remark that, since estimates (1.4) are not uniform with respect to t on the half-infinite interval, then they, generally speaking, do not correspond as $t \to \infty$ to the asymptotics obtained for the case when loads $g_i^{(l)}(\rho, t)$, $\tilde{g}_t^{(0)}(\rho, t)$ tend to finite limits $g_i^{(l)}(\rho, \infty)$, $\tilde{g}_i^{(0)}(\rho, \infty)$ or to oscillating loads $g_{i\Omega}^{(l)}(\rho)$, $\tilde{g}_{i\Omega}^{(0)}(\rho)$. The last asymptotics may not correspond to each other (for different frequencies Ω). However, all the asymptotics and estimates given in the paper correspond to uniform estimate (4.3) and refine it for these special cases.

5. Solution in other function classes

For the sake of brevity, we have considered here and in Part I only the case of continuous in time (in a sense) loads, i.e., $g_i^{(l)}, \tilde{g}_i^{(0)} \in C\hat{L}_2(\delta_g, 1; 0, T)$; then the solution that is Kolosov-Muskhelishvili potentials $\Phi_j^{(l)}$, $\Psi_j^{(l)}$ belong to $\mathrm{C}\tilde{\mathrm{H}}_2(\delta_{gs+}, 1; W_j^{(l)}; 0, T)$. If the loads belong to the more wide class $L_\infty\hat{L}_2(\delta_g, 1; 0, T)$ of functions $g(\rho, t)$ such that $\operatorname{ess\,sup}_{0 \leqslant t \leqslant T} \|g(.\,, t); \delta\|_2 < \infty$ for any $\delta \in (\delta_0, \delta_\infty)$ or to the class $\mathrm{B}\hat{\mathrm{L}}_2(\delta_g, 1; 0, T)$ of functions $g(\rho, t)$ such that $\sup_{0 \leqslant t \leqslant T} \|g(.\,, t); \delta\|_2 < \infty$, for any $\delta \in (\delta_0, \delta_\infty)$ (cf. Section I.2, point 1^0), then the solution in the corresponding classes $L_\infty\tilde{H}_2(\delta_{ge+}, 1; W_j^{(l)}; 0, T)$ or $\mathrm{B}\tilde{\mathrm{H}}_2(\delta_{ge+}, 1; W_j^{(l)}; 0, T)$ is obtained by the analogous procedure. The last two classes are respectively defined as in Section I.3 by use of the class $L_\infty H_2(\delta_0, \delta_\infty; W(\theta_-, \theta_+); 0, T)$ of functions $h(z, t) \in H_2(\delta_0, \delta_\infty; W(\theta_-, \theta_+))$ with respect to z for almost every $t \in [0, T]$ and such that $\operatorname{ess\,sup}_{0 \leqslant t \leqslant T} M_2(h(., t); \theta_-, \theta_+; \delta) < \infty$ for any $\delta \in (\delta_0, \delta_\infty)$, or of the class $\mathrm{BH}_2(\delta_0, \delta_\infty; W(\theta_-, \theta_+); 0, T)$ of functions $h(z, t) \in H_2(\delta_0, \delta_\infty; W(\theta_-, \theta_+))$ with respect to z for every $t \in [0, T]$ and such that $\sup_{0 \leqslant t \leqslant T} M_2(h(., t); \theta_-, \theta_+; \delta) < \infty$ for any $\delta \in (\delta_0, \delta_\infty)$ (cf. Section I.2. point 3). Moreover, the results concerning the asymptotics and estimates hold with account of the following remarks.

The asymptotics (I.4.16) as well as estimates (I.4.18), (4.1) and (4.3) hold for all $t \in [0, T]$ if $g_i^{(l)}, \tilde{g}_i^{(0)} \in \mathrm{B}\hat{\mathrm{L}}_2(\delta_g, 1; 0, T)$ and only for almost all $t \in [0, T]$ if $g_i^{(l)}, \tilde{g}_i^{(0)} \in L_\infty \hat{L}_2(\delta_g, 1; 0, T)$.

For the degenerate and non-aging parameters $\underline{\alpha} = \alpha^0 = \text{const.}$, $\underline{\beta} = \beta^0 = \text{const.}$, the stress intensity factors $\mathrm{K}_{knp}(t)$ are bounded on $[0, T]$ if $g_i^{(l)}, \tilde{g}_i^{(0)} \in \mathrm{B}\hat{\mathrm{L}}_2(\delta_g, 1; 0, T)$ and are only essentially bounded on $[0, T]$ if $g_i^{(l)}, \tilde{g}_i^{(0)} \in L_\infty \hat{L}_2(\delta_g, 1; 0, T)$.

Results of Section 3 are proved only for $g_i^{(l)}, \tilde{g}_i^{(0)} \in C\hat{L}_2(\delta_g, 1; 0, T)$.

6. Bonded body of general geometry

For the present, we discussed problems for a bonded infinite wedge without mass forces. However, by use of the Kondrat'ev method [5] it seems possible to show that the results obtained here concerning principal (singular) asymptotic terms (with $0 \leqslant \gamma_k < 1$) are extendible to the problems for a finite or infinite body with mass forces and curvilinear boundaries of the body having the same local geometry (see [7]).

By repeating practically the same reasoning, it is easy to show that the given approach is also applicable for intersection of a few interfaces and for other boundary conditions on the external boundary and on the interface.

7. Conclusion

Thus, singular asymptotics and estimates of stresses and strains were considered near the corner point in the plane problems of hereditary elasticity for a bonded aged wedge. We succeeded in obtaining the following results.

1. If the Dundurs parameters α, β are not hereditary operators (degenerate case) then in any time instant the stress asymptotics is the same as for a classical elastic body, whose elastic moduli coincide with the instantaneous moduli of the body considered at the same time instant.
2. For finite times $0 \leqslant t < \infty$, estimate (1.4) for stresses and strains is obtained, whose power for the case of sufficiently smooth loads coincides with the supremum over the segment $[0, t]$ of the stress singularity power in a classical elastic body, whose moduli coincide with the instantaneous moduli of the body considered. Particularly for a non-aging hereditary body, if the instantaneous elastic moduli are such that any stress singularity is impossible at the initial time instant, then stress singularity cannot arise for any other finite time instant (for sufficiently smooth loads).
3. For initial time $t = 0$, i.e., immediately after applying a load, the asymptotics of stresses and strains are the same as for a classical elastic body, whose

elastic moduli coincide with the instantaneous moduli of the body considered for $t = 0$.

4. For small times $t > 0$, the solution is represented by an expansion with respect to powers of t. Asymptotic of each term of the expansion includes the same stress singularities as for a classical elastic body, whose elastic moduli coincide with the instantaneous moduli of the body considered, but the power and the number of logarithm multiplicands grow linearly with the number of the expansion term.
5. For large times ($t \to \infty$) for an aging hereditarily-elastic body with fading-memory and with hereditary operators tending to operators of the convolution type, stress and strain asymptotics are the same as for a classical elastic body, whose elastic moduli coincide (i) with the complex moduli of the body considered for loads tending to those oscillating harmonically in time; and (ii) with the long-time moduli of the body considered for loads tending to those constant in time.

Acknowledgement

This work was carried out under the support of the author by an Alexander von Humboldt Foundation fellowship as a guest professor at the University of Stuttgart.

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