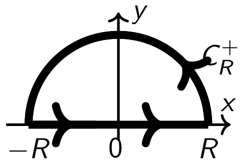
Integrals on $(-\infty, \infty)$ evaluated using residue theory

With $P(z)$ and $Q(z)$ being polynomials we consider

$$f(z) = \frac{P(z)}{Q(z)} \quad \text{(week 24)} \quad \text{and} \quad f(z) = \frac{P(z)}{Q(z)} e^{imz}. \quad \text{(week 25)}$$



Suppose that $f(z)$ has poles at points $z_1, \ldots, z_n$ in the upper half plane. Suppose that $Q(z)$ has no zeros on the real axis.

With $\Gamma_R = [-R, R] \cup C_R^+$ denoting the closed contour

$$\oint_{\Gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R^+} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f,\, z_k).$$

When the integral involving C_R^+ tends to 0 as $R \to \infty$ we get

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \text{or} \quad \text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$$

Examples in the lectures

In week 24.

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \pi.$$

$$I = \int_{-\infty}^{\infty} \frac{1}{x^4 + 16}\,dx = \frac{\pi\sqrt{2}}{16}.$$

In week 25. Let $a > 0$.

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{1 + x^2}\,dx = \pi e^{-a}.$$

$$\int_{-\infty}^{\infty} \frac{x \sin(x)}{1 + x^2}\,dx = \pi e^{-1}.$$

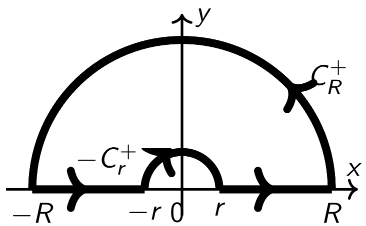
The last example needed Jordan's lemma to justify that the contribution from C_R^+ tends to 0 as $R \to \infty$.

Singularities on $\mathbb{R}$ and Cauchy principal values

Suppose $f(z)$ has a simple pole on $\mathbb{R}$ and we want to evaluate

$$\int_{-\infty}^{\infty} f(x)\,dx.$$

The integrals need to be considered in a principal valued sense. In the case of a pole at $z = 0$ we need an indented contour as illustrated below.



The knowledge of the Laurent series enables us to determine the contribution from the smaller half circle in the limit $r \to 0$.

The principal value for a singularity on $\mathbb{R}$

When we have a singularity of $f(z)$ at $x_0 \in [-R, R]$ the principal value means

$$\text{p.v.} \int_{-R}^{R} f(x)dx = \lim_{r\to 0} \left(\int_{-R}^{x_0 - r} f(x)dx + \int_{x_0 + r}^{R} f(x)dx \right)$$

In the above the part of the real line can be described as $[-R, R] \setminus (x_0 - r, x_0 + r)$. The part of $[-R, R]$ that we are excluding has x_0 exactly in the middle.

The C_r^+ contribution as $r \to 0$

When $f(z)$ has a simple pole at 0 it has a Laurent series of the following form for z sufficiently close to 0.

$$f(z) = \frac{a_{-1}}{z} + g(z) \quad \text{where } g(z) = \text{analytic function.}$$

$$\int_{C_r^+} f(z)\,\mathrm{d}z = a_{-1} \int_{C_r^+} \frac{\mathrm{d}z}{z} + \int_{C_r^+} g(z)\,\mathrm{d}z.$$

$z(\theta) = r\mathrm{e}^{i\theta}$, $0 \le \theta \le \pi$ describes C_r^+ and the length of C_r^+ is πr.

$$\int_{C_r^+} \frac{\mathrm{d}z}{z} = \int_0^\pi \frac{ir\mathrm{e}^{i\theta}}{r\mathrm{e}^{i\theta}}\,\mathrm{d}\theta = i \int_0^\pi \mathrm{d}\theta = \pi i.$$

Now a function $g(z)$ which is analytic on and near C_r^+ has the property of being bounded, i.e. there exists K such that $|g(z)| \le K$ in the region. ($K = 2|g(0)|$ will do if $g(0) \ne 0$ when r is sufficiently small.) Using the ML inequality we have

$$\left| \int_{C_r^+} g(z)\,\mathrm{d}z \right| \le K\pi r \to 0 \quad \text{as } r \to 0. \qquad \lim_{r \to 0} \int_{C_r^+} f(z)\,\mathrm{d}z = \pi i \mathrm{Res}(f, 0).$$

Examples which use indented contours

In week 25 we showed the following.

$$I_1 = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,\mathrm{d}x = \pi, \quad I_2 = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\,\mathrm{d}x = \pi.$$

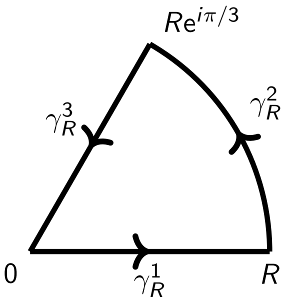
We do these by using an indented contour and using the following expressions.

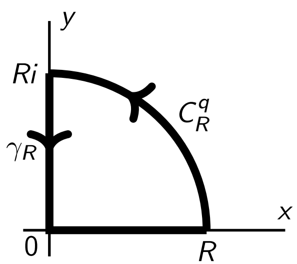
$$I_1 = \mathrm{Im}\left\{ \text{p.v} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \right\}.$$

$$I_2 = \mathrm{Re}\left\{ \text{p.v} \int_{-\infty}^{\infty} \frac{1 - \mathrm{e}^{2ix}}{2x^2}\,\mathrm{d}x \right\}.$$

I_1 and I_2 exist in the usual sense, it is just intermediate quantities which need the principal value meaning.

Other loops in the exercises





$$f_1(z) = \frac{64}{z^6 + 64}, \qquad f_2(z) = \frac{1}{z^4 + 16}.$$

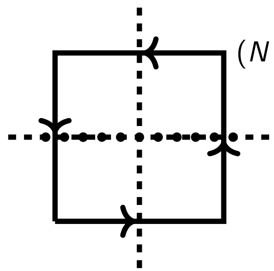
When $R > 2$ the function $f_1(z)$ has one simple pole at $2\mathrm{e}^{\pi i/6}$ inside this loop. In the case of $f_2(z)$ it has one simple pole at $2\mathrm{e}^{\pi i/4}$. Both problems could have also been done using an upper half circle which would have involved more than one pole inside the loop in each case.

A square as a loop in the exercises

In the context of the sum of a series

$$\sum_{n=1}^{N} f(n), \quad f(z) \text{ being even,}$$

the following loop Γ_N, which is a square, is used.



This has length $L_N = 4(2N + 1)$. $M_N = \max\{|f(z)| : z \in \Gamma_N\}$. We need $M_N L_N \to 0$ as $N \to \infty$.

Semester 1 exercises involving p_n'/p_n, q'/q

Let $z_1, z_2, \ldots, z_n$ be points in the complex plane and let

$$p_n(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

Prove by induction on n that

$$\frac{p_n'(z)}{p_n(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n}.$$

Let

$$q(z) = (z - z_1)^{r_1}(z - z_2)^{r_2}\cdots(z - z_n)^{r_n}$$

where $z_1, \ldots, z_n$ are distinct points. What can you say about the multiplicity of the zeros of $q'(z)$ at the points $z_1, \ldots, z_n$? Using a derivation based on partial fractions show that

$$\frac{q'(z)}{q(z)} = \frac{r_1}{z - z_1} + \frac{r_2}{z - z_2} + \cdots + \frac{r_n}{z - z_n}.$$

In both cases it is simple poles and the residues are integers. We generalise this next.

Counting zeros and poles

Suppose that $f(z)$ is analytic in a domain except for a finite number of poles. Let

$$G(z) = \frac{f'(z)}{f(z)}.$$

Let z_0 be a zero of $f(z)$ of multiplicity m and let z_p be a pole of $f(z)$ of order n. It can quickly be shown that

$$\mathrm{Res}(G,\, z_0) = m, \quad \text{and} \quad \mathrm{Res}(G,\, z_p) = -n.$$

Let $f(z)$ be analytic inside a loop Γ and let $N_0(f)$ be the number of zeros of $f(z)$ inside Γ. By the residue theorem

$$N_0(f) = \frac{1}{2\pi i}\oint_\Gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

If $g(z)$ is also analytic inside C and $|g(z)| < |f(z)|$ on Γ then

$$N_0(f + g) = N_0(f).$$

This is Rouche's theorem. A smaller enough change to $f(z)$ on Γ does not change the integer.

The fundamental theorem of algebra

Let

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \ne 0$$

denote a polynomial of degree n. Let

$$f(z) = a_n z^n, \quad g(z) = a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

For R sufficiently large $|f(z)| > |g(z)|$ on the circle $|z| = R$. As $f(z)$ has a zero at $z = 0$ of multiplicity n the use of Rouche's theorem implies that $p(z) = f(z) + g(z)$ also has n zeros inside $|z| = R$. This is the fundamental theorem of algebra and the proof here is independent of the proof given in chapter 6.

Another example using Rouche's theorem

Let

$$\begin{aligned} h(z) &= z^5 + 3z^3 - 1 = z^5\left(1 + \frac{3}{z^2} - \frac{1}{z^5}\right) \\ &= z^5\tilde{h}(w), \quad \tilde{h}(w) = 1 + 3w^2 - w^5, \quad w = \frac{1}{z}. \end{aligned}$$

$$h(z) = f(z) + g(z), \quad \text{with} \quad f(z) = z^5, \quad g(z) = 3z^3 - 1.$$

On the circle $|z| = 2$ we have
$|g(z)| \le 3(8) + 1 = 25 < 32 = |f(z)|$. $f(z)$ has one zero of multiplicity 5 at 0. Thus by Rouche's theorem $h(z)$ has 5 zeros inside the circle $|z| = 2$.

Similary by considering $\tilde{h}(w)$ with $\tilde{f}(w) = -w^5$, $\tilde{g}(w) = 1 + w^2$ and the circle $|w| = 2$ we get all the roots of $\tilde{h}(w)$ satisfy $|w| < 2$.

Conclusion: All the roots of $f(z)$ satisfy $1/2 < |z| < 2$.