Exercises on Sequences and Series of Real Numbers

1. *This was about half of question 1 of the June 2004* `MA2930` *paper.*

Let (x_n) denote a sequence of real numbers.

(a) (i) Define what it means for the sequence (x_n) to converge, using the usual ϵ and N notation.

[1 mark]

(ii) Define what it means for the sequence (x_n) to be strictly increasing.

[1 mark]

(iii) If the sequence is bounded above then define the least upper bound (i.e. the supremum) of (x_n).

[1 mark]

(iv) If a sequence (x_n) is both increasing and bounded above then state what you can deduce about the convergence or divergence of the sequence?

[1 mark]

(b) Explain why each of the following sequences converges and in the case of (i) and (ii) determine the limits.

(i)
$$x_n := \frac{103n^2 - 8}{4n^2 + 99n - 3}.$$
[2 marks]

(ii)
$$x_n := -n + \sqrt{n^2 + 3n}.$$
[2 marks]

(iii)
$$x_n := \sum_{k=1}^{n} \frac{3k^2 + 2k}{2^k}.$$
[2 marks]

ANSWER

(a) (i) (x_n) converges to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there exists a N such that

$$|x_n - x| < \epsilon \quad \text{for all } n \geq N.$$

1 mark

(ii) (x_n) is strictly increasing if $x_{n+1} > x_n$ for $n = 1, 2, \cdots$.

1 mark

(iii) $u \in \mathbb{R}$ is a least upper bound or supremum of (x_n) if it is an upper bound of (x_n) and no number smaller than u is an upper bound.

1 mark

(iv) A sequence (x_n) which is increasing and bounded above converges. It converges to the least upper bound of the set $\{x_n\}$.

1 mark

(b) (i) The sequence converges because it is a combination of standard convergent sequences. We have

$$x_n = \frac{103n^2 - 8}{4n^2 + 99n - 3} = \frac{103 - 8/n^2}{4 + 99/n - 3/n^2} \to \frac{103}{4} \quad \text{as } n \to \infty.$$

2 marks

(ii)

$$\begin{aligned} x_n &= -n + \sqrt{n^2 + 3n} \\ &= \frac{(n^2 + 3n) - n^2}{\sqrt{(n^2 + 3n)} + n} \\ &= \frac{3n}{n\sqrt{(1 + 3/n)} + n} \\ &= \frac{3}{\sqrt{(1 + 3/n)} + 1} \to \frac{3}{2} \quad \text{as } n \to \infty. \end{aligned}$$

2 marks

(iii) (x_n) is a sequence of partial sums (i.e. a series). With $a_k = (3k^2 + 2k)/2^k$ the ratio test gives

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \left(\frac{1}{2}\right) \frac{3(k+1)^2 + 2(k+1)}{3k^2 + 2k} \\ &= \left(\frac{1}{2}\right) \frac{3(1 + 1/k)^2 + 2(1/k + 1/k^2)}{3 + 2/k} \to \frac{1}{2} \quad \text{as } k \to \infty. \end{aligned}$$

As this limit is less than 1 in magnitude the series converges.

2 marks

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2. *This was question 1 of the January 2003* MA2034A *paper. Cauchy sequences will be covered later in the module this year and thus you will not be able to answer (a)(ii) and (b) yet.*

   Let (x_n) denote a sequence of real numbers.

   (a) (i) Define what it means for the sequence (x_n) to converge using the usual ϵ and N notation.

   [1 mark]

   (ii) Define what it means for the sequence (x_n) to be a Cauchy sequence.

   [2 marks]

(iii) Define what it means for the sequence (x_n) to be strictly increasing.

[1 mark]

(b) Prove that if a sequence (x_n) converges then it is a Cauchy sequence.

[4 marks]

(c) Determine the limits of the following sequences (x_n) whose nth term x_n is given below.

(i)
$$x_n := \frac{5n^3 + 3n + 1}{15n^3 + n^2 + 2}.$$
[2 marks]

(ii)
$$x_n := \frac{\sin(n^2 + 1)}{n^2 + 1}.$$
[2 marks]

(iii)
$$x_n := \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1} - \sqrt{n}}.$$
[2 marks]

(d) Let
$$a_k = \frac{1}{k2^k}, \quad b_k = \frac{k}{2^k}, \quad s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad t_n = \sum_{k=1}^{n} b_k \ .$$

(i) Find the limits of the sequences (a_{k+1}/a_k) and (b_{k+1}/b_k).

[2 marks]

(ii) Given that
$$a_k \le \frac{1}{2^k} \le b_k \quad \text{and} \quad b_k \le \left(\frac{3}{4}\right)^{k-2}, \quad k \ge 3,$$
explain why (s_n) and (t_n) both converge with
$$\lim_{n\to\infty} s_n \le 1 \le \lim_{n\to\infty} t_n \le 4.$$

[4 marks]

ANSWER

(a) (i) (x_n) converges to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there exists a N such that

$$|x_n - x| < \epsilon \quad \text{for all } n \geq N.$$

1 mark

(ii) (x_n) is a Cauchy sequence if for every $\epsilon > 0$ there exists a N such that

$$|x_n - x_m| < \epsilon \quad \text{for all } n, m \text{ satisfying } n \geq N \text{ and } m \geq N.$$

2 marks

(iii) (x_n) is strictly increasing if $x_{n+1} > x_n$ for $n = 1, 2, \cdots$.

1 mark

(b) From the definition of convergence of (x_n) to x there exists a $N = N(\epsilon/2)$ such that

$$|x_n - x| < \epsilon/2 \quad \text{for all } n \geq N.$$

If both $n \geq N$ and $m \geq N$ then

$$|x_n - x_m| = |(x_n - x) - (x_m - x)| \leq |(x_n - x)| + |(x_m - x)| < \epsilon/2 + \epsilon/2 = \epsilon$$

by the triangle inequality and the above bound. Thus (x_n) is a Cauchy sequence.

4 marks

(c) (i)

$$x_n := \frac{5n^3 + 3n + 1}{15n^3 + n^2 + 2} = \frac{5 + 3/n^2 + 1/n^3}{15 + 1/n + 2/n^3} \to \frac{5}{15} = \frac{1}{3} \quad \text{as } n \to \infty.$$

In the above we have used the result that $1/n \to 0$ as $n \to \infty$ and a result about combining convergent sequences and noting that the denominator converges to a non-zero value.

2 marks

(ii) $|\sin(n^2 + 1)| \leq 1$ and hence

$$|x_n| \leq \frac{1}{n^2 + 1} = \frac{1/n^2}{1 + 1/n^2} \to 0 \quad \text{as } n \to \infty.$$

Thus $x_n \to 0$ as $n \to \infty$.

2 marks

(iii) Applying the identity $a - b = (a^2 - b^2)/(a + b)$ to the numerator and the denominator gives

$$\begin{aligned} x_n &:= \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1} - \sqrt{n}} = \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n+1}} \\ &= \frac{\sqrt{1 + 1/n} + 1}{\sqrt{1 + 2/n} + \sqrt{1 + 1/n}} \to \frac{1+1}{1+1} = 1 \quad \text{as } n \to \infty. \end{aligned}$$

2 marks

(d) (i)

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{k2^k}{(k+1)2^{(k+1)}} = \frac{k}{2(k+1)}\\ &= \frac{1}{2(1+1/k)} \to \frac{1}{2} \quad \text{as } k \to \infty.\\ \frac{b_{k+1}}{b_k} &= \frac{(k+1)2^k}{k2^{(k+1)}} = \frac{k+1}{2k}\\ &= \frac{1+1/k}{2} \to \frac{1}{2} \quad \text{as } k \to \infty.\end{aligned}$$

2 marks

(ii) $s_n - s_{n-1} = a_n > 0$ and $t_n - t_{n-1} = b_n > 0$ and thus both (s_n) and (t_n) are strictly increasing. They both converge if they are bounded.

$$s_n = \sum_1^n a_k \le \sum_1^n \frac{1}{2^k} = \frac{1}{2}\left(\sum_0^{n-1} \frac{1}{2^k}\right) = \frac{1}{2}\left(\frac{1-(1/2)^n}{1-(1/2)}\right) = 1 - \frac{1}{2^n} < 1.$$

Thus $\lim s_n \le 1$.

$$\begin{aligned}t_n &= \sum_1^n b_k \le b_1 + b_2 + \sum_3^n \left(\frac{3}{4}\right)^{k-2} = \frac{1}{2} + \frac{1}{2} + \left(\frac{3}{4}\right)\sum_{k=0}^{n-3}\left(\frac{3}{4}\right)^k\\ &< 1 + \left(\frac{3}{4}\right)\frac{1}{1-(3/4)} = 4.\end{aligned}$$

Thus $\lim t_n \le 4$.
Finally as

$$t_n \ge 1 - \frac{1}{2^n}$$

for all n it follows that $t_n \ge 1$.

4 marks

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3. *These are parts of the question 1 of the January 1999–2002 examination papers. The marks shown in bold and in square brackets next to the question are the part marks that were shown on the January 2001 and 2002 examination papers. The mark breakdown was not indicated on* `MA2034A` *examination papers before January 2001.*

Let (x_n) denote a sequence of real numbers.

(i) Define what it means for the sequence (x_n) to be bounded.

[1 mark]

ANSWER

(x_n) is bounded if there exists a M such that $|x_n| \le M$ for all n.

1 mark

(ii) Prove that if a sequence (x_n) is strictly increasing and bounded above then it converges to u where u is the least upper bound of (x_n).

[4 marks]

ANSWER

As u is the least upper bound then no smaller number $u - \epsilon$ is an upper bound for any $\epsilon > 0$. Thus there must be an x_N with

$$u - \epsilon < x_N \le u \ .$$

The increasing property of the sequence then implies that

$$u - \epsilon < x_N \le x_{N+1} \le \cdots \le u$$

and we satisfy the convergence definition for all $n \ge N$.

4 marks

(iii) Determine the limits of the following sequences (x_n) whose nth term x_n is given below.

(a)

$$x_n := \frac{7n^4 + n^2 - 2}{14n^4 + 5n - 4} \ . \qquad \textbf{[2 marks]}$$

ANSWER

$$x_n = \frac{7n^4 + n^2 - 2}{14n^4 + 5n - 4} = \frac{7 + (1/n^2) - (2/n^4)}{14 + (5/n^3) - (4/n^4)} \to \frac{7}{14} = \frac{1}{2} \quad \text{as } n \to \infty.$$

In the above we have used the result that $1/n \to 0$ as $n \to \infty$ and a result about combining convergent sequences and noting that the denominator converges to a non-zero value.

2 marks

(b)

$$x_n := \frac{n^3 + 2n^2 + 1}{6n^3 + n + 4} \ . \qquad \textbf{[2 marks]}$$

ANSWER

$$x_n = \frac{n^3 + 2n^2 + 1}{6n^3 + n + 4} = \frac{1 + (2/n) + (1/n^3)}{6 + (1/n^2) + (4/n^3)} \to \frac{1}{6} \quad \text{as } n \to \infty.$$

In the above we have used the result that $1/n \to 0$ as $n \to \infty$ and a result about combining convergent sequences and noting that the denominator converges to a non-zero value.

2 marks

(c)

$$x_n := \frac{n^2 + n + 1}{3n^2 + 4} .$$

ANSWER

$$x_n = \frac{n^2 + n + 1}{3n^2 + 4} = \frac{1 + (1/n) + (1/n^2)}{3 + (4/n^2)} \to \frac{1}{3} \quad \text{as } n \to \infty.$$

In the above we have used the result that $1/n \to 0$ as $n \to \infty$ and a result about combining convergent sequences. The denominator converges and it is at least 3 for all n.

2 marks

(d)

$$x_n := \sqrt{n^4 + n^2} - n^2 .$$ **[2 marks]**

ANSWER

$$\begin{aligned} x_n = \sqrt{n^4 + n^2} - n^2 &= \frac{n^2}{\sqrt{n^4 + n^2} + n^2} \\ &= \frac{1}{\sqrt{1 + 1/n^2} + 1} \to \frac{1}{1+1} = \frac{1}{2} \quad \text{as } n \to \infty. \end{aligned}$$

2 marks

It is acceptable here to use the general binomial expansion to give

$$\begin{aligned} x_n &= n^2 \left((1 + 1/n^2)^{1/2} - 1 \right) = n^2((1/2)(1/n^2) + \mathcal{O}(1/n^4)) \\ &= 1/2 + \mathcal{O}(1/n^2) \to 1/2 \quad \text{as } n \to \infty. \end{aligned}$$

(e)

$$x_n := -n + \sqrt{n^2 + n} .$$ **[2 marks]**

ANSWER

$$\begin{aligned} x_n = \sqrt{n^2 + n} - n &= \frac{n}{\sqrt{n^2 + n} + n} \\ &= \frac{1}{\sqrt{1 + 1/n} + 1} \to \frac{1}{1+1} = \frac{1}{2} \quad \text{as } n \to \infty. \end{aligned}$$

2 marks

(f)

$$x_n := \frac{\sin n}{n} + (\sqrt{n+1} - \sqrt{n}) .$$

ANSWER

Since $|\sin n| \le 1$ we have

$$\left|\frac{\sin n}{n}\right| \le \frac{1}{n} \to 0 \quad \text{as } n \to \infty \ .$$

Also,

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0 \quad \text{as } n \to \infty.$$

Thus $x_n \to 0$ as $n \to \infty$.

2 marks

(iv) (a) Show that the sequence (x_n) whose nth term is

$$x_n := \frac{n^3+3n^2}{n+1} - n^2$$

is unbounded.

[2 marks]

ANSWER

$$\begin{aligned} x_n &= \frac{n^3+3n^2}{n+1} - n^2 = \frac{n^3+3n^2-(n^3+n^2)}{n+1} \\ &= \frac{2n^2}{n+1} \ge \frac{2n^2}{2n} = n \end{aligned}$$

(x_n) is unbounded by comparison with n.

(b) Show that the sequence (x_n) whose nth term is

$$x_n := (n+1/n)^3 - n^3$$

is unbounded.

ANSWER

$$x_n = (n^3+3n+3/n+1/n^3) - n^3 = 3n+3/n+1/n^3 > 3n.$$

(x_n) is unbounded by comparison with $3n$.

(c) Show that the sequence (x_n) whose nth term is

$$x_n := \left(n+1/n^2\right)^4 - n^4$$

[3 marks]

is unbounded.

ANSWER

$$\begin{aligned} x_n &= \left(n + 1/n^2\right)^4 - n^4 \\ &= (n^4 + 4n + 6/n^2 + 4/n^5 + 1/n^8) - n^4 \\ &= 4n + 6/n^2 + 4/n^5 + 1/n^8 > 4n \ . \end{aligned}$$

The sequence (n) is unbounded and hence by comparison (x_n) is also unbounded.

3 marks

(v) (a) Given that $k! \geq 2^{k-1}$ for all $k \geq 1$, show that the sequence (x_n) whose nth term is

$$x_n := \sum_{k=0}^{n} \frac{1}{k!}$$

is bounded above by 3. Explain why you can deduce that it converges.

ANSWER

We are given that $k! \geq 2^{k-1}$ for $k \geq 1$ and thus

$$\frac{1}{k!} \leq \frac{1}{2^{k-1}} \quad \text{for } k \geq 1.$$

Thus

$$\begin{aligned} x_n &= 1 + \sum_{k=1}^{n} \frac{1}{k!} \\ &\leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} \\ &= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \qquad \text{(by summing the geometric series)} \\ &= 1 + 2\left(1 - \frac{1}{2^n}\right) < 3 \ . \end{aligned}$$

Since $x_{n+1} - x_n = 1/(n+1)! > 0$ the sequence is strictly increasing and as it is also bounded above it converges by the monotone convergence theorem.

(b) Given that $k^k \geq 2^k$ for all $k \geq 2$, show that the sequence (x_n) whose nth term is

$$x_n := \sum_{k=1}^{n} \frac{1}{k^k}$$

is bounded above by $3/2$. Explain why you can deduce that it converges.

ANSWER

For $k \geq 2$,

$$k^k \geq 2^k \quad \text{implies} \quad \frac{1}{k^k} \leq \frac{1}{2^k}.$$

Thus

$$\begin{aligned} x_n - 1 &= \sum_{k=2}^{n} \frac{1}{k^k} \\ &\leq \sum_{k=2}^{n} \frac{1}{2^k} = \frac{1}{4} \sum_{j=0}^{n-2} \frac{1}{2^j} \\ &= \frac{1}{4} \left(\frac{1 - (1/2)^{n-1}}{1 - (1/2)} \right) \\ &< \frac{1}{4} \left(\frac{1}{1 - (1/2)} \right) = \frac{1}{2} \end{aligned}$$

by summing the geometric series. Hence $x_n \leq 3/2$ for all n.

4 marks

Since $x_n - x_{n-1} = 1/n^n > 0$ the sequence is strictly increasing and as it is also bounded above it converges by the monotone convergence theorem.

(c) Let

$$a_k = \frac{1}{k2^k}, \quad b_k = \frac{k}{2^k}, \quad s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad t_n = \sum_{k=1}^{n} b_k \ .$$

Given that

$$a_k \leq \frac{1}{2^k} \leq b_k \quad \text{and} \quad b_k \leq \left(\frac{3}{4} \right)^{k-2}, \quad k \geq 3,$$

explain why (s_n) and (t_n) both converge with

$$\lim_{n\to\infty} s_n \leq 1 \leq \lim_{n\to\infty} t_n \leq 4.$$

[4 marks]

ANSWER

This repeats the last part of question 1. See the answer to the last part of question 1.

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4. If the sequence (x_n) converges then show that its limit is unique.

ANSWER

This is a proof by contradiction. Hence you start by assuming that the converse is true which in this case means that you assume that the sequence converges to 2 distinct limits $x \neq y$. We need now to show that this always leads to a contradiction.

All we know about the sequence is that it converges and thus we start by using this. We have from the definition of convergence that for every $\epsilon > 0$ there exists N_x and N_y such that

$$|x_n - x| < \epsilon \quad \text{for all } n \geq N_x \quad \text{and} \quad |x_n - y| < \epsilon \quad \text{for all } n \geq N_y.$$

As we are aiming to show that it is impossible for x and y to be different the next step is to consider the difference $x - y$. If $n > \max\{N_x, N_y\}$ then by the triangle inequality we have

$$\begin{aligned} 0 \neq x - y &= (x - x_n) + (x_n - y) \\ 0 < |x - y| &= |(x - x_n) + (x_n - y)| \leq |x_n - x| + |x_n - y| < \epsilon + \epsilon = 2\epsilon \ . \end{aligned}$$

This is true for all ϵ and by taking ϵ sufficiently small we get our contradiction, specifically we get a contradiction if we take

$$\epsilon = \frac{|x - y|}{4} > 0 \ .$$

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5. Show that if (x_n) is a sequence of real numbers which converges to x then the sequence (s_n) where

$$s_n := \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to x.

ANSWER

(x_n) converges to x implies that for every $\epsilon > 0$ there exists an $\tilde{N}$ such that

$$|x_n - x| < \epsilon \quad \text{for all } n \geq \tilde{N}.$$

We now divide the sum into those terms before $\tilde{N}$ and those after $\tilde{N}$. We have

$$\begin{aligned} |s_n - x| = \frac{1}{n}\left|\sum_{1}^{n}(x_i - x)\right| &\leq \frac{1}{n}\left(\sum_{1}^{\tilde{N}-1}|x_i - x|\right) + \frac{1}{n}\left(\sum_{\tilde{N}}^{n}|x_i - x|\right) \\ &\leq \frac{C}{n} + \epsilon, \quad \text{where} \quad C = \sum_{1}^{\tilde{N}-1}|x_i - x| \ . \end{aligned}$$

Now

$$\frac{C}{n} + \epsilon < 2\epsilon \quad \text{provided} \quad \frac{C}{n} < \epsilon, \quad \text{which requires that } n > C/\epsilon.$$

Thus if we let $N = \max\{\tilde{N}, C/\epsilon\}$ then for all $n \geq N$ we have $|s_n - x| < 2\epsilon$ which proves that the sequence (s_n) converges to x.

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6. Given that $(1 + 1/n)^n \to \mathrm{e} = 2.718\cdots$ as $n \to \infty$ and for $c > 0$, $c^{1/n} \to 1$ as $n \to \infty$, show the following.

   (i)
   $$\left(1 + \frac{1}{n^2}\right)^n \to 1 \quad \text{as } n \to \infty.$$

   (ii) The sequence (x_n) defined by
   $$x_n := \left(1 + \frac{1}{\sqrt{n}}\right)^n$$
   is unbounded.

   (iii) If $r = p/q \in \mathbb{Q}$ is a rational number (i.e. $p, q \in \mathbb{N}$ with $q \neq 0$) and assuming that the sequence (t_n) defined by
   $$t_n := \left(1 + \frac{r}{n}\right)^n$$
   converges then show that the subsequence $(t_{np}) = (t_p, t_{2p}, \cdots)$ converges to e^r.

ANSWER

The key here is to introduce the convergent sequence (y_n) where

$$y_n := \left(1 + \frac{1}{n}\right)^n \to \mathrm{e} = 2.718\cdots \text{ as } n \to \infty$$

and to express the given sequences in terms of (y_n).

(i) Let
$$x_n := \left(1 + \frac{1}{n^2}\right)^n = (y_{n^2})^{1/n} \ .$$
Now $y_n \to \mathrm{e}$ implies that y_n is near e for sufficiently large n. In particular, $y_n < 3$ for all sufficiently large n. (A closer inspection would reveal that this is true for all n but this amount of detail is not required to establish that the given sequence is convergent.) Thus
$$1 \le x_n = (y_{n^2})^{1/n} < 3^{1/n} \ .$$
The right hand side converges to 1 as $n \to \infty$ and thus the squeeze theorem implies that $x_n \to 1$ as $n \to \infty$.

(ii) Now let
$$x_n := \left(1 + \frac{1}{\sqrt{n}}\right)^n \ .$$
Observe that
$$x_{n^2} = \left(1 + \frac{1}{n}\right)^{n^2} = y_n^n \ .$$
$y_n \to \mathrm{e}$ implies that y_n is near e for sufficiently large n. In particular, $y_n \ge 2$ for sufficiently large n. (A closer inspection would reveal that this is true for all n but,

as in part (i), this amount of detail is not required to establish that the sequence is unbounded.) Thus

$$x_{n^2} \geq 2^n$$

and hence the subsequence (x_{n^2}) is unbounded. This in turn implies that (x_n) is unbounded.

(iii) We are told that the sequences (t_n) converges and we know that the any subsequence of a convergent sequence also converges with the same limit. Now

$$t_{np} := \left(1 + \frac{p/q}{np}\right)^{np} = \left(1 + \frac{1}{nq}\right)^{np} .$$

Observe that

$$y_{nq} := \left(1 + \frac{1}{nq}\right)^{nq} .$$

Thus

$$t_{np} = y_{nq}^r .$$

As $y_n \to \mathrm{e}$ as $n \to \infty$ we have for the subsequence that $y_{nq} \to \mathrm{e}$ as $n \to \infty$. From this it follows that $y_{nq}^r \to \mathrm{e}^r$ as $n \to \infty$ by a result about this type of function of a convergent sequence. Hence we have shown that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} t_{np} = \lim_{n\to\infty} y_{nq}^r = \mathrm{e}^r .$$

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7. Let (s_n) be the sequence given by

$$s_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Show that the sequence is increasing. Does it converge?

By noting that

$$0 < \int_k^{k+1} \frac{1}{x}\,dx - \frac{1}{k+1} < \frac{1}{k} - \frac{1}{k+1}$$

show that

$$0 < \ln 2 - s_n < \frac{1}{2n}.$$

ANSWER

$$\begin{aligned} s_{n+1} - s_n &= \left(\frac{1}{n+2} + \cdots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right) \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{(4n+3)}{(2n+1)(2n+2)} - \frac{2}{2n+2} \\ &= \frac{(4n+3) - 2(2n+1)}{(2n+1)(2n+2)} = \frac{1}{(2n+1)(2n+2)} > 0 . \end{aligned}$$

Thus (s_n) is increasing. (s_n) is the sum of n terms the largest of which is $\dfrac{1}{n+1}$. Thus

$$s_n \le \frac{n}{n+1} < 1\ .$$

(s_n) is hence increasing and bounded and it converges to a limit $s \le 1$.

$$\frac{1}{k+1} \le \frac{1}{x} \le \frac{1}{k} \quad \text{for} \quad x \in (k, k+1).$$

Now

$$\frac{1}{k+1} = \int_k^{k+1} \frac{dx}{k+1}, \quad \text{and hence} \quad \int_k^{k+1} \left(\frac{1}{x} - \frac{1}{k+1}\right) dx > 0$$

from which it follows that

$$0 < \sum_{k=n}^{2n-1} \left(\int_k^{k+1} \frac{1}{x}\,dx - \frac{1}{k+1} \right) = \int_n^{2n} \frac{1}{x}\,dx - s_n = [\ln x]_n^{2n} - s_n = \ln 2 - s_n\ .$$

Thus $\ln 2$ bounds the sequence. Also note that

$$\begin{aligned} 0 < \ln 2 - s_n &= \sum_{k=n}^{2n-1} \left(\int_k^{k+1} \frac{1}{x}\,dx - \frac{1}{k+1} \right) \le \sum_{k=n}^{2n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) \\ &= \left(\frac{1}{n} - \frac{1}{2n}\right) = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus we have shown that $s_n \to \ln 2$ as $n \to \infty$.

---

8. In each of the following cases determine whether or not the series converges.

(a)

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}.$$

ANSWER

We could show convergence here by using the ratio or root test or more simply by using the comparison test by noting that

$$0 \le \frac{1}{2^n + 1} \le \frac{1}{2^n}.$$

The upper bound is a term from a convergent geometric series.

(b)

$$\sum_{n=1}^{\infty} \frac{4n^2 - n + 3}{n^3 + 2n}.$$

ANSWER

This is divergent.

$$a_n = \frac{4n^2 - n + 3}{n^3 + 2n} = \frac{1}{n} c_n, \quad c_n = \frac{4 - 1/n + 3/n^2}{1 + 2/n^2} \to 4 \quad \text{as } n \to \infty.$$

$c_n \to 4$ implies that there exists N such that $c_n > 3$ for $n \ge N$. Hence for $n \ge N$ we have $a_n \ge 3/n$ and since $\sum 1/n$ diverges we have by comparison that $\sum a_n$ diverges.

(c)

$$\sum_{n=1}^{\infty} \frac{n + \sqrt{n}}{2n^3 - 1}.$$

ANSWER

This converges.

$$a_n = \frac{n + \sqrt{n}}{2n^3 - 1} = \frac{1}{n^2} c_n, \quad c_n = \frac{1 + 1/\sqrt{n}}{2 - 1/n^3} \to \frac{1}{2} \quad \text{as } n \to \infty.$$

$c_n \to 1/2$ implies that there exists N such that $c_n < 1$ for $n \ge N$. Hence for $n \ge N$ we have $a_n \le 1/n^2$ and since $\sum 1/n^2$ converges we have by comparison that $\sum a_n$ diverges.

(d)

$$\sum_{n=1}^{\infty} n^4 \mathrm{e}^{-n^2}.$$

ANSWER

By the root test

$$a_n = n^4 \mathrm{e}^{-n^2}, \quad a_n^{1/n} = (n^{1/n})^4 \left(\mathrm{e}^{-n^2}\right)^{1/n} = (n^{1/n})^4 \mathrm{e}^{-n} \to 0 \quad \text{as } n \to \infty.$$

Here the results is as a consequence of $n^{1/n} \to 1$ and $\mathrm{e}^{-n} \to 0$. By the root test the series converges.

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9. You will see questions like this in the section on series of functions.

   For each of the following series determine the values of $x \in \mathbb{R}$ such that the given series converges.

   (a)

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

ANSWER

Let $a_k = x^k/k!$ and use the ratio test. We have

$$\frac{a_{k+1}}{a_k} = \frac{x^{k+1}/(k+1)!}{x^k/k!} = \frac{x}{k+1} \to 0 \quad \text{as } k \to \infty.$$

By the ratio test the series converges (absolutely) for all $x \in \mathbb{R}$.

(b) In the following $\alpha \in \mathbb{R}$ is not an integer.

$$\sum_{k=0}^{\infty} \left(\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \right) x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \cdots$$

ANSWER

Let $a_k = \alpha(\alpha-1)\cdots(\alpha-k+1)x^k/k!$. Using the ratio test

$$\frac{a_{k+1}}{a_k} = \frac{\alpha-k}{k+1} x = \frac{\alpha/k - 1}{1 + 1/k} x \to x \quad \text{as } k \to \infty.$$

Thus the series $\sum a_k$ converges absolutely if $|x| < 1$ which in turn implies that the series converges for $|x| < 1$.

If $|x| > 1$ then the terms of the series are unbounded and thus the series diverges.

What happens when $x = -1$ or $x = 1$ needs more refined tests to determine if the series converges or diverges and the outcome depends on α. This will not be considered further here.

(c)

$$\sum_{k=0}^{\infty} \frac{k^3 x^k}{3^k}.$$

ANSWER

The root test is the easiest test to use here. With $a_k = k^3 x^k / 3^k$ we have

$$|a_k|^{1/k} = \frac{(k^{1/k})^3 |x|}{3} \to \frac{|x|}{3} \quad \text{as } k \to \infty.$$

By the root test the series converges (absolutely) if $|x| < 3$, it diverges if $|x| > 3$. If $|x| = 3$ then $|a_k| = k^3$ and since these terms become unbounded it follows that the series diverges when $|x| = 3$.

(d)

$$\sum_{k=0}^{\infty} k^k x^k.$$

ANSWER

The root test is the easiest test to use here. With $a_k = k^k x^k$ we have

$$|a_k^{1/k}| = |kx|.$$

This only converges if $x = 0$ and is unbounded for $x \neq 0$. Hence the series only converges when $x = 0$.

(e)

$$\sum_{k=0}^{\infty} a_k x^k = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots ,$$

i.e. with $a_{2k} = 1$ and $a_{2k+1} = 2$ for $k = 0, 1, 2, \cdots$.

ANSWER

Let $b_k = a_k x^k$. The ratio test does not give any information here as a_{k+1}/a_k does not have a limit as $k \to \infty$. However we can still use the root test. Since

$$1 \le a_k \le 2, \quad 1 \le a_k^{1/k} \le 2^{1/k} \to 1 \quad \text{as } k \to \infty.$$

Thus

$$|b_k|^{1/k} = a_k^{1/k}|x| \to |x| \quad \text{as } k \to \infty.$$

The series converges (absolutely) if $|x| < 1$ and diverges if $|x| > 1$. By inspection the series diverges if $x = 1$ as the terms of the series do not tend to 0 as $k \to \infty$. It can be shown that the series also diverges when $x = -1$.

(f)

$$\sum_{k=1}^{\infty} \frac{\sqrt{x^2 + k} - |x|}{k^2} .$$

ANSWER

Let

$$a_k = \frac{\sqrt{x^2 + k} - |x|}{k^2} = \frac{(x^2 + k) - x^2}{(\sqrt{x^2 + k} + |x|)k^2} = \frac{1}{(\sqrt{x^2 + k} + |x|)k} \le \frac{1}{k^{3/2}} \quad \text{for all } x$$

since $x^2 \ge 0$. Since $0 \le a_k \le 1/k^{3/2}$ the series $\sum a_k$ converges by comparison with the convergent series $\sum 1/k^{3/2}$.

(g)

$$\sum_{k=1}^{\infty} \left(\frac{\cos kx}{k^3} + 3\frac{\sin kx}{k^2} \right) .$$

ANSWER

We can test for absolute convergence. If a_k denotes the kth term then by the triangle inequality and that $|\sin kx| \le 1$ and $|\cos kx| \le 1$ we have

$$|a_k| \le \frac{1}{k^3} + 3\frac{1}{k^2}$$

for all $x \in \mathbb{R}$. Since $\sum 1/k^3$ and $\sum 1/k^2$ are standard convergent series it follows that $\sum |a_k|$ converges by the comparison test. Hence the original series converges for all x.

Exercises on continuity and the contraction mapping theorem

1. *This was the first part of question 2 of the June 2004* `MA2930` *exam paper.*

   (a) Let $f : [a, b] \to \mathbb{R}$ denote a function. Define what it means for f to be Lipschitz on $[a, b]$ and explain when such a function is also a contraction on $[a, b]$.

   [2 marks]

   (b) Show that the following function is Lipschitz on $[-1, 1]$ and give the smallest Lipschitz constant.

   $$f : [-1, 1] \to \mathbb{R}, \quad f(x) := \begin{cases} x^2, & 0 \le x \le 1, \\ -3x, & -1 \le x < 0. \end{cases}$$

   [3 marks]

   (c) If $f : [a, b] \to \mathbb{R}$ is continuously differentiable on $[a, b]$ then it can be shown that f is Lipschitz on $[a, b]$ with the smallest Lipschitz constant L given by

   $$L = \max_{x \in [a,b]} |f'(x)| \ .$$

   Use this to obtain the smallest Lipschitz constant for the following functions on their domains of definition.

   (i) $f : [-1, 1] \to \mathbb{R}, \quad f(x) := 2x - 2x^3.$ **[3 marks]**

   (ii) $f : [-2, 2] \to \mathbb{R}, \quad f(x) := (x^2 - 4)^2.$ **[4 marks]**

ANSWER

(a) f is Lipschitz on $[a, b]$ if there exists a constant $L \ge 0$ such that

$$|f(x) - f(y)| \le L|x - y|$$

for all $x, y \in [a, b]$. f is a contraction if $0 \le L < 1$.

2 marks

(b) If $x, y \in [-1, 0)$ then

$$|f(x) - f(y)| = 3|x - y|.$$

For $x > 0$, $f'(x) = 2x$ and thus if $x, y \in [0, 1]$ then

$$|f(x) - f(y)| \le f'(1)|x - y| = 2|x - y|.$$

If $x > 0$, $y < 0$ then $x - y = |x| + |y|$ and

$$\begin{aligned} |f(x) - f(0)| &\le 2|x|, \\ |f(0) - f(y)| &\le 3|y|, \\ |f(x) - f(y)| &\le 2|x| + 3|y| < 3(|x| + |y|) = 3|x - y|. \end{aligned}$$

3 is the smallest Lipschitz constant.

3 marks

(c) (i)

$$f'(x) = 2 - 6x^2.$$

$f''(x) = -12x = 0$ at $x = 0$. Possible values for the maximum of $|f'(x)|$ are $x = 0$ and the end points $x = \pm 1$. Thus the smallest constant is

$$L = |f'(1)| = 4.$$

3 marks

(ii)

$$\begin{aligned} f'(x) &= 2(x^2-4)(2x) = 4x(x^2-4), \\ f''(x) &= 4(3x^2-4). \end{aligned}$$

$f''(x) = 0$ when $x^2 = 4/3$ which gives two points in $[-2, 2]$.

$$f'(\pm 2) = 0 \quad \text{and} \quad |f'(\pm 2/\sqrt{3})| = 4\left(\frac{2}{\sqrt{3}}\right)\left(4 - \frac{4}{3}\right) = \frac{64}{3\sqrt{3}}.$$

Thus the smallest constant is

$$L = \frac{64}{3\sqrt{3}}.$$

4 marks

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2. *This was question 2 of the Jan 2003* `MA2034A` *exam paper.*

(a) Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a function.

(i) Give the definition of the continuity of the function f at the point $c \in I$ using the usual ϵ, δ notation.

[2 marks]

(ii) Define what it means for f to be Lipschitz on I and show that such a function is continuous on I.

[3 marks]

(b) Let $f : [0, 1] \to \mathbb{R}$, $f(x) := \sqrt{x}$. Show that f is continuous at $x = 0$ but that f is not Lipschitz on $[0, 1]$.

[4 marks]

(c) If $f : [a, b] \to \mathbb{R}$ is continuously differentiable on $[a, b]$ then it can be shown that f is Lipschitz on $[a, b]$ with the smallest Lipschitz constant L given by

$$L = \max_{x \in [a,b]} |f'(x)| \ .$$

Use this to obtain the smallest Lipschitz constant for the following functions on their domains of definition.

(i) $f : [5, 13] \to \mathbb{R}, \quad f(x) := 2x^2 - 17.$

[3 marks]

(ii) $f : [-1,1] \to \mathbb{R}, \quad f(x) := (1-x^2)^2.$

[4 marks]

(iii) $f : [0,4] \to \mathbb{R}, \quad f(x) := (x+1)e^{-x}.$

[4 marks]

ANSWER

(a) (i) f is continuous at $c \in I$ if for every $\epsilon > 0$ there exists a δ such that

$$|f(x) - f(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta \ .$$

2 marks

(ii) f is Lipschitz on I if there exists a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in I$.

$$|f(x) - f(y)| \leq L|x - y| < \epsilon \quad \text{provided} \quad |x - y| < \epsilon/L \ .$$

Hence we satisfy the continuity requirement by taking $\delta = \epsilon/L$.

3 marks

(b) For $x \geq 0$

$$|f(x) - f(0)| = |f(x)| = \sqrt{x} < \epsilon \quad \text{provided} \quad x < \epsilon^2.$$

Thus we satisfy the ϵ–δ definition with $\delta = \epsilon^2$.

2 marks

For $x > 0$, $y > 0$ and $x \neq y$ we have

$$\frac{f(x) - f(y)}{x - y} = \frac{\sqrt{x} - \sqrt{y}}{x - y} = \frac{1}{\sqrt{x} + \sqrt{y}}.$$

As $x \to 0$ and $y \to 0$ this becomes unbounded and hence no constant $L \geq 0$ can exist as the Lipschitz constant of f on $[0,1]$.

2 marks

(c) (i)

$$f'(x) = 4x \quad \text{and} \quad f''(x) = 4.$$

There are no turning points of f' and

$$L = \max\{|f'(5)|, |f'(13)|\} = f'(13) = 52.$$

3 marks

(ii)

$$f'(x) = 2(1-x^2)(-2x) = 4x(x^2-1) = 4(x^3-x) \quad \text{and} \quad f''(x) = 4(3x^2-1) \; .$$

There are turning points of f' at $\pm 1/\sqrt{3}$.

$$f'(-1) = f'(1) = 0 \quad \text{and} \quad |f'(\pm 1/\sqrt{3})| = (4/\sqrt{3})(1-1/3) = \frac{8}{3\sqrt{3}} \; .$$

Thus

$$L = \frac{8}{3\sqrt{3}} \; .$$

4 marks

(iii)

$$\begin{aligned} f'(x) &= e^{-x} - (x+1)e^{-x} = -xe^{-x} \\ f''(x) &= xe^{-x} - e^{-x} = (x-1)e^{-x}. \end{aligned}$$

f' has a local turning point at $x = 1 \in [0,4]$. Thus

$$L = \max\{|f'(0)|, |f'(1)|, |f'(4)|\} = \max\{0, e^{-1}, 4e^{-4}\} = e^{-1}.$$

4 marks

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3. *The following were parts of question 2 from the Jan 2000, 2001 and 2002* `MA2034A` *exam papers.*

(a) Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a function.
Give the definition of the continuity of the function f at the point $c \in I$ which involves convergent sequences.

[1 mark]

ANSWER

f is continuous at $c \in I$ if for every sequence (x_n) in I which converges to c then the corresponding sequence $(f(x_n))$ converges to $f(c)$.

1 mark

(b) Show from the definition that the function $f : [-1,1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x & \text{if } 0 \le x \le 1 \\ -2x & \text{if } -1 \le x < 0 \end{cases}$$

is Lipschitz on $[-1,1]$.

[3 marks]

ANSWER

Let $x, y \in [-1, 1]$. In the case $x, y \in [-1, 0]$ we have

$$|f(x) - f(y)| = |x - y|.$$

In the case $x, y \in [0, 2]$ we have

$$|f(x) - f(y)| = 2|x - y|.$$

If $x \in [-1, 0]$ and $y \in [0, 1]$ then $y - x = y + |x|$ and we have

$$|f(x) - f(y)| = |x + 2y| \le |x| + 2y \le 2(|x| + y) = 2|x - y|.$$

f is Lipschitz on $[-1, 1]$ with a Lipschitz constant of 2.

3 marks

(c) If $f : [a, b] \to \mathbb{R}$ is continuously differentiable on $[a, b]$ then it can be shown that f is Lipschitz on $[a, b]$ with the smallest Lipschitz constant L given by

$$L = \max_{x \in [a,b]} |f'(x)| \ .$$

Use this to obtain the smallest Lipschitz constant for the following functions on their domains of definition.

(i) $f : [-5, 5] \to \mathbb{R}, \quad f(x) := x^3.$

[3 marks]

ANSWER

$$f : [-5, 5] \to \mathbb{R}, \quad f(x) := x^3, \quad f'(x) := 3x^2, \quad f''(x) := 6x \ .$$

$f''(x) = 0$ when $x = 0$. This is the only turning value of f'. Thus

$$\max_{[-5,5]} |f'(x)| = \max\{|f'(-5)|, |f'(0)|, |f'(5)|\} = 3(5)^2 = 75 \ .$$

The smallest constant is 75.

3 marks

(ii) $f : [-2, 0] \to \mathbb{R}, \quad f(x) := x^4 + 6x^3 + 12x^2 - x.$

[4 marks]

ANSWER

$$\begin{aligned} f:[-2,0]\to\mathbb{R},\quad f(x) &:= x^4+6x^3+12x^2-x,\\ f'(x) &:= 4x^3+18x^2+24x-1,\\ f''(x) &:= 12x^2+36x+24 = 12(x^2+3x+2)\\ &= 12(x+1)(x+2)\ . \end{aligned}$$

$f''(x)=0$ when $x=-1$ and when $x=-2$.

$$\max_{[-2,0]}|f'(x)| = \max\{|f'(-2)|,|f'(-1)|,|f'(0)|\}\ .$$

$|f'(0)|=1$, $|f'(-1)| = |-4+18-24-1| = 11$ and $|f'(-2)| = |-32+72-48-1| = 9$. The smallest constant is 11.

4 marks

(iii) $f:[10^{-6},1]\to\mathbb{R},\quad f(x):=\sqrt{x}$.

[3 marks]

ANSWER

$$f:[10^{-6},1]\to\mathbb{R},\quad f(x):=\sqrt{x},\quad f'(x):=\frac{1}{2\sqrt{x}}\ .$$

f' is decreasing and positive on the interval and hence the smallest constant is $f'(10^{-6}) = 10^3/2 = 500$.

3 marks

(iv) $f:[0,1]\to\mathbb{R},\quad f(x):=x^2/2-x^3/3$.

[4 marks]

ANSWER

$$\begin{aligned} f:[0,1]\to\mathbb{R},\quad f(x) &:= x^2/2-x^3/3,\\ f'(x) &:= x-x^2 = x(1-x),\\ f''(x) &:= 1-2x\ . \end{aligned}$$

$f''(x)=0$ when $x=1/2$.

$$\max_{[0,1]}|f'(x)| = \max\{|f'(0)|,|f'(1)|,|f'(1/2)|\}\ .$$

$f'(0)=f'(1)=0$ and $f'(1/2)=1/4$. The smallest constant is $1/4$.

4 marks

(v) $f : [0, 2] \to \mathbb{R}, \quad f(x) := x^4 - 6x^3 + 12x^2 + 4x.$

[4 marks]

ANSWER

$$\begin{aligned} f : [0,2] \to \mathbb{R}, \quad f(x) &:= x^4 - 6x^3 + 12x^2 + 4x, \\ f'(x) &:= 4x^3 - 18x^2 + 24x + 4, \\ f''(x) &:= 12x^2 - 36x + 24, \\ &= 12(x^2 - 3x + 2) = 12(x-1)(x-2) \ . \end{aligned}$$

f' has two turning values at $x = 1$ and at $x = 2$ (which is also an end point). Now $f'(0) = 4$, $f'(1) = 14$ and $f'(2) = 32 - 72 + 48 + 4 = 12$. The smallest Lipschitz constant is

$$L = \max\{4, 14, 12\} = 14 \ .$$

4 marks

(vi) $f : [0, 1/2] \to \mathbb{R}, \quad f(x) := x^3 - x^2.$

ANSWER

$$f'(x) = 3x^2 - 2x = x(3x - 2) \quad \text{and} \quad f''(x) = 6x - 2 \ .$$

f' has a turning point at $x = 1/3$. To determine the maximum of $|f'|$ on $[0, 1/2]$ we need to consider the end points and any intermediate turning points.

$$f'(0) = 0, \quad f'(1/3) = -1/3 \quad \text{and} \quad f'(1/2) = -1/4.$$

Thus

$$L = \max\{0, 1/3, 1/4\} = 1/3.$$

3 marks

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4. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \ln(x^2 + k^2)$ with $k > 0$. Show that the smallest Lipschitz constant is $1/k$.

ANSWER

$$f'(x) = \frac{2x}{x^2 + k^2} \quad \text{and} \quad f''(x) = \frac{(x^2 + k^2)2 - (2x)(2x)}{(x^2 + k^2)^2} = \frac{2(k^2 - x^2)}{(x^2 + k^2)^2}.$$

f' has turning points when $f''(x) = 0$, i.e. when $x = \pm k$. As $f'(0) = 0$ and $f'(x) \to 0$ as $|x| \to \infty$ these local turning points are where f' attains its maximum in magnitude. Thus the smallest Lipschitz constant is

$$L := f'(k) = 1/k \ .$$

---

5. The following all involve making use of the (Bolzano) Intermediate Value Theorem for continuous functions.

   (a) Show that every polynomial of odd degree with real coefficients has at least one real root.

ANSWER

Let $n \geq 1$ be an odd number and let

$$p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = x^n \left(a_n + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n\right)$$

where $a_n \neq 0$. Observe that

$$a_n + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n \to a_n \quad \text{as } |x| \to \infty.$$

Thus for sufficiently large $|x|$, $p(x)/x^n$ has the same sign as a_n. As n is odd this implies that if $A > 0$ then $p(-A)$ and $p(A)$ have opposite sign when A is sufficiently large. This in turn implies by the intermediate value theorem that p has a root in $[-A, A]$ as the polynomial p is continuous on $\mathbb{R}$.

   (b) Let $p : \mathbb{R} \to \mathbb{R}$ be the polynomial

$$p(x) := a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad a_i \in \mathbb{R}.$$

Show that if $n \geq 2$, n is even, $a_n = 1$ and $a_0 < 0$ then p has at least two real roots.

ANSWER

$p(x)/x^n \to 1$ as $x \to \infty$. Thus if $M > 0$ is sufficiently large then $p(-M) > 0$ and $p(M) > 0$. Also $p(0) = a_0 < 0$. Thus by the intermediate value theorem p has a root in $[-M, 0]$ and in $[0, M]$ as the polynomial p is continuous on $\mathbb{R}$.

   (c) Show that if $\phi : [a, b] \to [a, b]$ is continuous then there exists at least one point $x^* \in [a, b]$ with $\phi(x^*) = x^*$.

ANSWER

This follows by applying the intermediate value theorem to the continuous function

$$g : [a, b] \to \mathbb{R}, \quad g(x) := \phi(x) - x\ .$$

Since $\phi : [a, b] \to [a, b]$ we have $\phi(a)$, $\phi(b) \in [a, b]$ and in particular $\phi(a) \geq a$ and $\phi(b) \leq b$ giving $g(a) \geq 0$ and $g(b) \leq 0$. Thus either g has a root at $x = a$ or at $x = b$ or by the intermediate value theorem at some point between a and b.

(d) This was part of question 2 of the January 1999 paper.

Let $[a,b]$ denote a closed bounded interval and let $f : [a,b] \to \mathbb{R}$ be a function which is continuous on $[a,b]$. Also let $c, d \in [a,b]$ be such that

$$f(c) = \min_{[a,b]} f(x) \quad \text{and} \quad f(d) = \max_{[a,b]} f(x) \ .$$

The intermediate value theorem implies that f maps $[a,b]$ onto $[f(c), f(d)]$. In the case $c < d$ and for a function f for which

$$f(c) < f(a) = f(b) < f(d)$$

show that for each $y \in (f(c), f(d))$ there is at least 2 distinct values of x in $[a,b]$ for which $f(x) = y$.

ANSWER

Consider $g : [a,b] \to \mathbb{R}$, $g(x) := f(x) - y$ which is continuous on $[a,b]$.

If $y \in (f(c), f(a))$ then $g(a) = f(a) - y > 0$ and $g(c) = f(c) - y < 0$. Hence g changes sign on (a,c) and has a root in (a,c) by the intermediate value theorem. Similarly by considering the interval (c,d), $g(d) = f(d) - y > 0$ and g changes sign on (c,d) and has a root in (c,d) by the intermediate value theorem. Thus we have at least 2 points at which $f(x) = y$.

If $y = f(a) = f(b)$ then $x = a$ and $x = b$ are 2 points at which $f(x) = y$.

If $y \in (f(a), f(d))$ then $g(c) = f(c) - y < 0$ and $g(d) = f(d) - y > 0$. Hence g changes sign on (c,d) and has a root in (c,d) by the intermediate value theorem. Similarly by considering the interval (d,b), $g(b) = f(b) - y = f(a) - y < 0$ and g changes sign on (d,b) and has a root in (d,b) by the intermediate value theorem. Thus we have at least 2 points at which $f(x) = y$.

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6. *This was question 3 of the Jan 2003* `MA2034A` *exam paper.*

Let $\phi : [0,1] \to [0,1]$ be a function.

(a) (i) Explain what it means for ϕ to be a contraction on $[0,1]$.

[2 marks]

(ii) Explain the term onto in the statement ϕ maps $[0,1]$ onto $[m, M]$ where $0 \le m \le M \le 1$.

[1 mark]

(iii) Given that a contraction mapping $\phi : [0,1] \to [0,1]$ has a fixed point, use a proof by contradiction to show that the fixed point is unique.

[3 marks]

(iv) If $L \ge 0$ is the smallest Lipschitz constant of $\phi : [0,1] \to [m, M]$ given in (ii) then explain why $M - m \le L$.

[2 marks]

(b) (i) Let $\phi_1 : [0,1] \to \mathbb{R}$, $\phi_1(x) = (x^3+3)/5$. Determine m and M such that $\phi_1 : [0,1] \to [m, M]$ is onto and show that ϕ_1 satisfies the conditions of the contraction mapping theorem.

[3 marks]

(ii) Let $\phi_2 : [0,1] \to \mathbb{R}$, $\phi_2(x) = 4x(1-x)$. Determine m and M such that $\phi_2 : [0,1] \to [m, M]$ is onto and show that ϕ_2 is not a contraction on $[0,1]$. Determine the fixed points of ϕ_2 and classify them as stable or unstable.

[5 marks]

(iii) Let $a > 1$ and let $\phi_3 : [\sqrt{a}, a] \to [\sqrt{a}, (1+a)/2]$, $\phi_3(x) = \frac{1}{2}(x + a/x)$. Explain why ϕ_3 maps $[\sqrt{a}, a]$ onto $[\sqrt{a}, (1+a)/2]$.
For what values of $a > 1$ is ϕ_3 a contraction on $[\sqrt{a}, a]$?
Determine the fixed point or points of ϕ_3 and classify any fixed point found as stable or unstable.

[4 marks]

ANSWER

(a) (i) ϕ is a contraction on $[0,1]$ if there exists a constant L, $0 \le L < 1$ such that

$$|\phi(x) - \phi(y)| \le L|x - y|$$

for all $x, y \in [0,1]$.

2 marks

(ii) The statement ϕ maps $[0,1]$ onto $[m, M]$ means that for all $y \in [m, M]$ there exists an $x \in [0,1]$ such that $y = \phi(x)$.

1 mark

(iii) Suppose x_1^* and x_2^* are two different fixed points, i.e.

$$x_1^* \ne x_2^* \quad \text{with} \qquad \phi(x_1^*) = x_1^* \quad \text{and} \quad \phi(x_2^*) = x_2^* \, .$$

Thus

$$0 < |x_1^* - x_2^*| = |\phi(x_1^*) - \phi(x_2^*)| \le L|x_1^* - x_2^*| < |x_1^* - x_2^*|$$

by the contraction property with $0 \le L < 1$. This is a contradiction and hence there is only one fixed point.

3 marks

(iv) If $\alpha, \beta \in [0,1]$ are such that $\phi(\alpha) = m$ and $\phi(\beta) = M$ then the Lipschitz condition is

$$L \ge \left| \frac{\phi(\beta) - \phi(\alpha)}{\beta - \alpha} \right| = \frac{M - m}{|\beta - \alpha|} \ge M - m$$

because $|\beta - \alpha| \le 1$.

2 marks

(b) (i)

$$\phi_1'(x) = \frac{3x^2}{5} \geq 0 \quad \text{and} \quad \phi_1''(x) = \frac{6x}{5}.$$

$\phi_1'(x) \geq 0$ implies that $\phi_1(x)$ is increasing and thus $m = \phi_1(0) = 3/5$ and $M = \phi_1(1) = 4/5$.
$\phi_1''(x) = 0$ when $x = 0$, an edge point, $\phi_1'(0) = 0$ and thus the smallest Lipschitz constant of ϕ_1 on $[0, 1]$ is

$$L = \phi_1'(1) = \frac{3}{5} < 1.$$

As $[m, M] \subset [0, 1]$ and $L < 1$ the function ϕ_1 satsifies the condition of the contraction mapping theorem.

3 marks

(ii)

$$\phi_2(x) = 4x(1 - x) = 4x - 4x^2 \quad \text{and} \quad \phi_2'(x) = 4(1 - 2x)\ .$$

$\phi_2'(x) > 0$ for $0 \leq x < 1/2$ and $\phi_2'(x) < 0$ for $1/2 < x \leq 1$. Thus ϕ_2 is increasing in $[0, 1/2)$ and decreasing in $(1/2, 1]$. $\phi_2(0) = \phi_2(1) = 0$ and $\phi_2(1/2) = 1$. Thus

$$\phi_2 : [0, 1] \rightarrow [0, 1]$$

is onto.
ϕ_2' has no turning points on $[0, 1]$ and hence the maximum of $|\phi_2'(x)|$ is attained at the end points. The smallest Lipschitz constant for the region is hence

$$L = \max_{x\in[0,1]} \{|\phi_2'(x)|\} = \max\{|\phi_2'(0)|, |\phi_2'(1)|\} = 4\ .$$

The function is hence not contractive on $[0, 1]$.
A fixed point of ϕ_2 satisfies

$$x = 4x(1 - x) \quad \text{thus } x = 0 \text{ or } 1 = 4(1 - x), \text{ i.e. } x = 3/4.$$

$\phi_2'(0) = 4$ and $\phi_2'(3/4) = -2$. In both cases $|\phi_2'(x^*)| > 1$ and thus $x^* = 0$ and $x^* = 3/4$ are both unstable fixed points.

5 marks

(iii)

$$\phi_3'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right) \geq 0 \quad \text{for } x^2 \geq a.$$

ϕ_3 increases in $[\sqrt{a}, a]$ and ϕ_3 hence maps the interval onto $[\sqrt{a}, (1 + a)/2]$.

$$\phi_3''(x) = \frac{1}{2}\left(\frac{2a}{x^3}\right) > 0.$$

ϕ_3'' has no local turning points, $\phi_3'(\sqrt{a}) = 0$ and thus the smallest Lipschitz constant is

$$L = |\phi_3'(a)| = \frac{1 - 1/a}{2} \leq \frac{1}{2}.$$

Hence the mapping is a contraction for all $a > 1$.
The fixed point at $x = \sqrt{a}$ is stable as $\phi_3'(\sqrt{a}) = 0$.

4 marks

7. *This was question 3 of the Jan 2002* `MA2034A` *exam paper.*

   Let $\phi : [0, 1] \to [0, 1]$ be a function.

   (i) Explain what it means for ϕ to be a contraction on $[0, 1]$.

   [2 marks]

   (ii) If ϕ is a contraction on $[0, 1]$ and x^* is the unique fixed point of ϕ then show that if $x^* \in [0, 1]$ and $x_{n+1} = \phi(x_n)$, $n = 0, 1, 2, \cdots$ then

   $$|x_n - x^*| \le L^n |x_0 - x^*|, \quad n = 1, 2, \cdots$$

   where L is the smallest Lipschitz constant of ϕ on $[0, 1]$.

   [3 marks]

   (iii) In the case of the function

   $$\phi(x) = \frac{3 + \mathrm{e}^{-x}}{5}$$

   explain why

   $$\phi : [0, 1] \to [\phi(1), \phi(0)] \subset [0, 1]$$

   and determine the smallest Lipschitz constant of ϕ on $[0, 1]$.

   [5 marks]

   State any conclusion you can make about fixed points of ϕ in $[0, 1]$.

   [2 marks]

   (iv) Let $\phi_1 : [0, 1] \to \mathbb{R}$, $\phi_1(x) := x((2 - 4a)x + (4a - 1))$ where $a \in \mathbb{R}$, $a \ne 1/2$.

   (a) Determine the fixed points of ϕ_1 and classify them as stable or unstable depending on the value of a.

   [4 marks]

   (b) Explain why for $0 \le x \le 1$ we have

   $$\min\{4a - 1, 3 - 4a\} \le \phi'(x) \le \max\{4a - 1, 3 - 4a\}.$$

   [2 marks]

   (c) By using part (b), or otherwise, explain why $\phi_1 : [0, 1] \to [0, 1]$ is one-to-one and onto when $1/4 \le a \le 3/4$.

   [2 marks]

ANSWER

(i) ϕ is a contraction on $[0,1]$ if there exists a constant L, $0 \le L < 1$ such that

$$|\phi(x) - \phi(y)| \le L|x-y|$$

for all $x, y \in [0,1]$.

2 marks

(ii)

$$|x_n - \phi(x^*)| = |\phi(x_{n-1}) - \phi(x^*)| \le L|x_{n-1} - x^*| \le \cdots \le L^n|x_0 - x^*|$$

by repeated use of the contraction property.

3 marks

(iii)

$$\phi'(x) = \frac{-\mathrm{e}^{-x}}{5} < 0$$

and hence ϕ is decreasing. From this it follows that

$$\phi : [0,1] \to [\phi(1), \phi(0)] \ .$$

Now $\phi(0) = 4/5 < 1$ and $\phi(1) = (3+1/\mathrm{e})/5 > 3/5 > 0$ and thus $[\phi(1), \phi(0)] \subset [0,1]$.

$$\phi''(x) = \frac{+\mathrm{e}^{-x}}{5} > 0.$$

As the second derivative is positive, the maximum of $|\phi'|$ on $[0,1]$ is at an end point. The smallest Lipschitz constant is hence $L = |\phi'(0)| = 1/5$.

5 marks

All the conditions of the contraction mapping theorem are satisfied and hence ϕ has a unique fixed point in $[0,1]$.

2 marks

(iv) (a)

$$\phi_1(x) = x((2-4a)x + (4a-1)) = x$$

when

$$x = 0 \quad \text{and} \quad (2-4a)x + (4a-1) = 1.$$

Thus we have fixed points at

$$x = 0 \quad \text{and at} \quad x = \frac{2-4a}{2-4a} = 1.$$

To classify the fixed points we need $\phi'(x)$.

$$\phi'(x) = 2(2-4a)x + (4a-1), \quad \phi'(0) = 4a-1, \quad \phi'(1) = 3-4a.$$

The fixed point at $x = 0$ is stable if $-1 < 4a-1 < 1$, $0 < a < 1/2$. It is unstable if $a < 0$ or $a > 1/2$.
The fixed point at $x = 1$ is stable if $-1 < 3-4a < 1$, $-1 < 4a-3 < 1$, i.e. $1/2 < a < 1$. It is unstable if $a < 1/2$ or $a > 1$.

4 marks

(b) As ϕ' is linear the minimum and maximum on $[0,1]$ occur at the end points. Hence for all $x \in [0,1]$.

$$\min\{\phi'(0), \phi'(1)\} \le \phi'(x) \le \max\{\phi'(0), \phi'(1)\}.$$

2 marks

(c) If $a \ge 1/4$ then $4a - 1 \ge 0$ and $3 - 4a \le 2$. If $a \le 3/4$ then $4a - 1 \le 2$ and $3 - 4a \ge 0$. In all cases the result in part (b) shows that $\phi'(x) \ge 0$. Thus $\phi'(x)$ is increasing and as $\phi(0) = 0$ and $\phi(1) = 1$ it follows that $\phi : [0,1] \to [0,1]$ is one-to-one and onto.

2 marks

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8. The following were parts of question 3 of the Jan 2001 and Jan 2000 papers.

   Let $\phi : [0,1] \to [0,1]$ be a function.

   (a) In the the case of the functions ϕ_1 and ϕ_2 defined below, show that both functions map $[0,1]$ onto $[0,1]$ and show that both functions are not contractive on $[0,1]$.

$$\phi_1 : [0,1] \to [0,1], \quad \phi_1(x) := 4x(1-x).$$

$$\phi_2 : [0,1] \to [0,1], \quad \phi_2(x) := \frac{1}{2}(1 - \cos 2\pi x).$$

[9 marks]

In the case of ϕ_1 determine the fixed points and classify each fixed point as stable or unstable.

[2 marks]

ANSWER

$$\begin{aligned} \phi_1 : [0,1] \to [0,1], \quad \phi_1(x) &:= 4x(1-x) = 4x - 4x^2, \\ \phi_1'(x) &= 4(1-2x), \\ \phi_1''(x) &= -8\ . \end{aligned}$$

ϕ_1 increases in $[0,1/2)$ and decreases in $(1/2,1]$ Since $\phi_1(0) = \phi_1(1) = 0$ and $\phi_1(1/2) = 1$, ϕ_1 maps $[0,1]$ onto $[0,1]$.

3 marks

ϕ_1 is continuously differentiable on $[0,1]$. As ϕ_1'' does not change sign, the smallest Lipschitz constant on $[0,1]$ is $L = \max\{|\phi_1'(0)|, |\phi_1'(1)|\} = 4$. As this constant is greater than 1 the function is not contractive on $[0,1]$.

2 marks

Because of properties of the cosine function, $\cos 2\pi x$ takes all values between -1 and $+1$ as x varies between 0 and 1. Thus $1 - \cos 2\pi x$ takes all values between 0 and 2 and ϕ_2 takes all values between 0 and 1.

2 marks

$$\phi_2 : [0,1] \to [0,1], \quad \phi_2(x) := \frac{1}{2}(1 - \cos 2\pi x),$$
$$\phi_2'(x) = \pi \sin 2\pi x \ .$$

As $\phi_2'(1/2) = \pi \sin \pi/2 = \pi > 1$ the smallest Lipschitz constant is greater than 1 and the function is not contractive on $[0,1]$.

2 marks

A fixed point of ϕ_1 satisfies

$$x = 4x(1-x) \quad \text{thus } x = 0 \text{ or } 1 = 4(1-x), \text{ i.e. } x = 3/4.$$

$\phi_1'(0) = 4$ and $\phi_1'(3/4) = -2$. In both cases $|\phi'(x^*)| > 1$ and thus $x^* = 0$ and $x^* = 3/4$ are both unstable fixed points.

2 marks

(b) Let ϕ denote the function

$$\phi : [0,2] \to \mathbb{R}, \quad \phi(x) = \frac{x^2 - 2x + 5}{4}.$$

Show that the conditions of the contraction mapping theorem are satisfied and give the fixed point.

[5 marks]

ANSWER

$$\phi : [0,2] \to \mathbb{R}, \quad \phi(x) = \frac{x^2 - 2x + 5}{4},$$
$$\phi'(x) = \frac{2x - 2}{4} = \frac{1}{2}(x-1),$$
$$\phi''(x) = \frac{1}{2}.$$

As $\phi'(x) < 0$ in $(0,1)$ and $\phi'(x) > 0$ in $(1,2)$ the function ϕ decreases in $(0,1)$ and increases in $(1,2)$. $\phi(0) = 5/4$, $\phi(1) = 1$ and $\phi(2) = 5/4$. Thus $\phi : [0,2] \to [1, 5/4] \subset [0,2]$.
As $\phi''(x)$ does not change sign on $[0,2]$ the smallest Lipschitz constant is

$$L = \max\{|\phi'(0)|, |\phi'(2)|\} = \frac{1}{2} < 1 \ .$$

Thus ϕ is contractive on $[0,2]$ and satisfies all the conditions of the contraction mapping theorem.

5 marks

(c) Determine which of the following is a contraction on their domain of definition.

(i)

$$\phi_1 : [-\pi/8, \pi/8] \to [-\pi/8, \pi/8], \quad \phi_1(x) := \frac{\pi}{8}\sin(2x)\ .$$

ANSWER

$$\phi_1'(x) = \frac{\pi}{4}\cos(2x) \quad \text{and hence} \quad |\phi_1'(x)| \le \frac{\pi}{4} < 1\ .$$

ϕ_1 is a contraction on $[-\pi/8, \pi/8]$ with the constant

$$L = \max_{[-\pi/8,\pi/8]} |\phi_1'(x)| = \frac{\pi}{4} < 1\ .$$

3 marks

(ii)

$$\phi_2 : [-\pi/4, \pi/4] \to [-\pi/4, \pi/4], \quad \phi_2(x) := \frac{\pi}{4}\sin(2x)\ .$$

Classify, as stable or unstable, the three fixed points $x = -\pi/4$, $x = 0$ and $x = \pi/4$ in this case.

ANSWER

$$\phi_2'(x) = \frac{\pi}{2}\cos(2x)\ .$$

The smallest Lipschitz constant for the domain is

$$L = \max_{[-\pi/4,\pi/4]} |\phi_2'(x)| = \frac{\pi}{2} > 1\ .$$

Thus ϕ_2 is not contractive on $[-\pi/4, \pi/4]$.

3 marks

(d) In the case of the function

$$\phi(x) := \frac{1 + \mathrm{e}^x}{4}$$

determine the smallest Lipschitz constant L on $[0, 1]$ and explain why

$$\phi : [0,1] \to \ [\phi(0), \phi(1)] \subset [0,1]\ .$$

ANSWER

$$\phi'(x) = \mathrm{e}^x/4 > 0 \quad \text{and} \quad \phi''(x) = \mathrm{e}^x/4 > 0 \ .$$

Thus ϕ' is increasing on $[0, 1]$ and the maximum of $|\phi'(x)|$ is attained at $x = 1$ and the smallest Lipschitz constant L is given by

$$L = \phi'(1) = \mathrm{e}/4 < 1 \ .$$

3 marks

Since ϕ is increasing on $[0, 1]$ we have

$$\phi : [0, 1] \to [\phi(0), \phi(1)] \ .$$

$\phi(0) = 1/2 > 0$ and $\phi(1) = (1 + \mathrm{e})/4 < 1$ and hence $[\phi(0), \phi(1)] \subset [0, 1]$.

2 marks

Exercises 8A: Cauchy sequences and subsequences in $\mathbb{R}$

1. Prove that if a sequence (x_n) converges then it is a Cauchy sequence. (*This has been asked in previous* MA2034A *exam papers.*)

ANSWER

From the definition of convergence of (x_n) to x there exists a $N = N(\epsilon/2)$ such that

$$|x_n - x| < \epsilon/2 \quad \text{for all } n \ge N.$$

If both $n \ge N$ and $m \ge N$ then

$$|x_n - x_m| = |(x_n - x) - (x_m - x)| \le |(x_n - x)| + |(x_m - x)| < \epsilon/2 + \epsilon/2 = \epsilon$$

by the triangle inequality and the above bound. Thus (x_n) is a Cauchy sequence.

---

2. Show that if (x_n) satisfies the property

$$|x_{n+1} - x_n| \le \frac{1}{2^n}, \quad n = 1, 2, \cdots$$

then it is a Cauchy sequence.

Given that a Cauchy sequence converges show that if x denotes the limit of the sequence then

$$|x_n - x| \le \frac{1}{2^{n-1}}.$$

With S denoting the set

$$S := \{x_n : \ n = 1, 2, \cdots\}$$

also show that

$$\sup S - \inf S \le \frac{3}{2}\ .$$

This was the last part of the Jan 1999 question 1. Most people found this difficult.

ANSWER

We are given $|x_{n+1} - x_n| < \frac{1}{2^n}$. Thus

$$\begin{aligned} x_{n+j} - x_n &= (x_{n+j} - x_{n+j-1}) + (x_{n+j-1} - x_{n+j-2}) \\ &\quad + \cdots + (x_{n+1} - x_n) \end{aligned}$$

and by the triangle inequality

$$\begin{aligned} |x_{n+j}-x_n| &\le |x_{n+j}-x_{n+j-1}| + |x_{n+j-1}-x_{n+j-2}| \\ &\quad + \cdots + |x_{n+1}-x_n| \\ &\le \left(\frac{1}{2}\right)^{n+j-1} + \cdots + \left(\frac{1}{2}\right)^{n} \\ &= \left(\frac{1}{2}\right)^{n}\left(1+\frac{1}{2}+\cdots+\left(\frac{1}{2}\right)^{j-1}\right) \\ &\le 2\left(\frac{1}{2}\right)^{n} = \left(\frac{1}{2}\right)^{n-1} . \end{aligned}$$

Since the right hand side does not involve j and tends to 0 as $n \to \infty$ the sequence is a Cauchy sequence and hence converges.

With

$$x = \lim_{m\to\infty} x_m,$$

and letting $j \to \infty$ in the above we get

$$|x - x_n| \le \left(\frac{1}{2}\right)^{n-1} .$$

As a consequence of this inequality the set $\{x_2, x_3, \cdots\}$ is contained in the interval $[x-1/2, x+1/2]$. Only x_1 of the set $s = \{x_1, x_2, x_3, \cdots\}$ may lie outside of this interval. The complete set is thus contained in $[x-1, x+1/2]$ or $[x-1/2, x+1]$ from which it follows that

$$\sup S - \inf S \le \frac{3}{2} .$$

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3. (*This is similar to question 2.*)

   Let (x_n) be a sequence of vectors in $\mathbb{R}^p$ which are such that $\underline{x}_0 = \underline{0}$ and

$$\|\underline{x}_{n+1} - \underline{x}_n\|_2 \le r^r, \quad n = 0, 1, 2, \cdots \quad \text{for some } r \in [0, 1).$$

   Show that the sequence converges to some $\underline{x} \in \mathbb{R}^p$ and that $\underline{x}$ satisfies

$$\|\underline{x}\|_2 \le \frac{1}{1-r}.$$

ANSWER

To show that the sequence converges we show that the sequence is a Cauchy sequence.

$$\begin{aligned} \underline{x}_{n+q} - \underline{x}_n &= (\underline{x}_{n+q} - \underline{x}_{n+q-1}) + \cdots + (\underline{x}_{n+1} - \underline{x}_n). \\ \|\underline{x}_{n+q} - \underline{x}_n\|_2 &\le \|(\underline{x}_{n+q} - \underline{x}_{n+q-1}\|_2 + \cdots + \|\underline{x}_{n+1} - \underline{x}_n\|_2 \\ &\le r^{n+q-1} + \cdots + r^n \le \sum_n^\infty r^k = \frac{r^n}{1-r}. \end{aligned}$$

Since $|r| < 1$ the right hand side tends to 0 as $n \to \infty$ for all $q > 0$ which is sufficient to show that the sequence is a Cauchy sequence.

Since $(\underline{x}_n)$ is a Cauchy sequence it converges to some $\underline{x} \in \mathbb{R}^p$.

Since $\underline{x}_0 = \underline{0}$ we have

$$\underline{x}_n = \underline{x}_0 + \sum_{k=1}^n (\underline{x}_k - \underline{x}_{k-1}) = \sum_{k=1}^n (\underline{x}_k - \underline{x}_{k-1}).$$

By the triangle inequality we have

$$\begin{aligned} \|\underline{x}_n\|_2 &\le \sum_{k=1}^n \|\underline{x}_k - \underline{x}_{k-1}\|_2 \\ &\le \sum_{k=1}^n r^{k-1} \le \sum_{k=0}^\infty r^k = \frac{1}{1-r}. \end{aligned}$$

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4. *The following question relates to the Bolzano-Weierstrass theorem concerning sequences in* $\mathbb{R}$.

   Let (x_n) denote the bounded sequence given by

$$x_n := \sin \sqrt{n}.$$

   Observe that since the sine function $\sin x$ is increasing in $[-\pi/2, \pi/2]$ then for all $y \in [-1, 1]$ there is unique $x \in [-\pi/2, \pi/2]$ with $\sin x = y$ and from the periodicity of the sine function

$$\sin(x + 2k\pi) = y, \quad \text{for } k = \pm 1, \pm 2, \cdots$$

   By defining $n_k \in \mathbb{N}$ to be the integer part of $(x + 2k\pi)^2$ and by considering the subsequence (x_{n_k}) of (x_n), show that $x_{n_k} \to y$ as $k \to \infty$.

   (In your answer you can assume all the usual properties and identities of the sine function such as

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \quad \text{etc.}$$

   and

$$|\sin x| \le |x| \quad \text{for all } x \in \mathbb{R}.)$$

ANSWER

The key point here is to note that y is given by

$$y = \sin(x) = \sin(x + 2k\pi) \quad \text{for all } k.$$

Thus

$$\begin{aligned} x_{n_k} - y &= \sin\sqrt{\lfloor (x+2k\pi)^2 \rfloor} - \sin(x + 2k\pi), \\ &\quad \text{where} \quad \lfloor z \rfloor = \text{integer part of } z \in \mathbb{R}. \end{aligned}$$

To rewrite this in a form where we can establish that this difference is small we note the trigonometric identities

$$\begin{aligned} \sin(a+b) &= \sin a \cos b + \cos a \sin b \\ \sin(a-b) &= \sin a \cos b - \cos a \sin b \\ \sin(a+b) - \sin(a-b) &= 2\cos a \sin b \end{aligned}$$

which with $c = a + b$ and $d = a - b$ gives

$$\sin c - \sin d = 2\cos\left(\frac{c+d}{2}\right)\sin\left(\frac{c-d}{2}\right).$$

Using the properties of the cosine and sine functions (which can be established from a geometric definition of these functions) that

$$|\cos u| \le 1 \quad \text{and} \quad |\sin u| \le |u| \quad \text{for all } u \in \mathbb{R}$$

we have

$$|\sin c - \sin d| \le |c - d| \ .$$

In our case with

$$c = \sqrt{\lfloor (x+2k\pi)^2 \rfloor} \quad \text{and} \quad d = x + 2k\pi$$

we have

$$c - d = \frac{c^2 - d^2}{c + d} = \frac{\lfloor (x+2k\pi)^2 \rfloor - (x+2k\pi)^2}{\sqrt{\lfloor (x+2k\pi)^2 \rfloor} + (x+2k\pi)}$$

and by noting that the integer part of a number can differ by at most 1 from the number we get

$$|x_{n_k} - y| \le \left| \frac{1}{\sqrt{\lfloor (x+2k\pi)^2 \rfloor} + (x+2k\pi)} \right| \to 0 \quad \text{as } k \to \infty.$$

Exercises 8B: Sequences and series of functions

1. *The following was the last part of question 2 of the 2004* `MA2930` *exam paper.*

   Let (f_n), $f_n : [a, b] \to \mathbb{R}$ denote a sequence of functions. Define what it means for (f_n) to converge pointwise on $[a, b]$.

   [2 marks]

   (i) Determine the pointwise limit function in the case

   $$f_n : (-1, 1] \to \mathbb{R}, \quad f_n(x) := x^n, \quad n = 1, 2, \cdots .$$

   Explain why the convergence is not uniform.

   [2 marks]

   (ii) Show that the pointwise limit function of the sequence

   $$f_n : [0, 1] \to \mathbb{R}, \quad f_n(x) := \frac{nx}{1 + n^2x^2}, \quad n = 1, 2, \cdots$$

   is the zero function but that

   $$\max\{|f_n(x)| : \ 0 \le x \le 1\} \not\to 0 \quad \text{as } n \to \infty.$$ **[4 marks]**

<u>ANSWER</u>

(f_n) converges pointwise on $[a, b]$ if the sequence of numbers $(f_n(x))$ converges for all $x \in [a, b]$.

2 marks

(i) If $|x| < 1$ then $x^n \to 0$ as $n \to \infty$. If $x = 1$ then $x^n = 1$ for all n. The pointwise limit function is $f : (-1, 1] \to \mathbb{R}$,

$$f(x) := \begin{cases} 0, & \text{if } -1 < x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

Each function f_n is continuous on $(-1, 1]$ and the limit function is f is discontinuous at $x = 1$ which implies that the convergence is not uniform.

2 marks

(ii) For $x = 0$ we have $f_n(0) = 0$.
For $x > 0$ we have

$$f_n(x) = \frac{x/n}{1/(n^2) + x^2} \to \frac{0}{0 + x^2} = 0 \quad \text{as } n \to \infty.$$

Thus for all $x \in [0, 1]$ we have $f_n(x) \to 0$ as $n \to \infty$.
By inspection when $x = 1/n$ we have $f_n(1/n) = 1/(1 + 1) = 1/2$ for all n. Thus

$$\max\{|f_n(x)| : \ 0 \le x \le 1\} \ge f_n(1/n) = \frac{1}{2}$$

and we hence do not have convergence to 0.

4 marks

2. *The following was the last part of question 1 of the 2004* `MA2930` *exam paper.*

Let (f_n), $f_n : [a,b] \to \mathbb{R}$ denote a sequence of continuous functions.

Define the uniform norm of (f_n) and explain what it means for (f_n) to converge to $f : [a,b] \to \mathbb{R}$ uniformly on $[a,b]$.

[2 marks]

The Weierstrass M-test gives a sufficient condition for the series $\sum f_n$ to converge uniformly on $[a,b]$. State the conditions of this test.

[2 marks]

Use the test to show that the following series converge uniformly on the given domain.

(i) $$\sum_0^\infty \frac{x^n}{n^n} \quad \text{on } |x| \le r < \infty.$$ **[2 marks]**

(ii) $$\sum_0^\infty \frac{(n!)^2}{(2n)!} x^n \quad \text{on } |x| \le r < 4.$$ **[2 marks]**

(iii) $$\sum_0^\infty \frac{1}{x^2 + 2^n} \quad \text{on } \mathbb{R}.$$ **[2 marks]**

ANSWER

The uniform norm of the continuous function f_n is

$$\|f_n\|_\infty = \sup\{|f_n(x)| : \; a \le x \le b\} = \max\{|f_n(x)| : \; a \le x \le b\}.$$

(f_n) converges to f uniformly means that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

2 marks

The conditions of the Weierstrass M-test are satisfied if $\|f_n\|_\infty \le M_n$ with the series $\sum M_n$ converging.

2 marks

(i) In this case $f_n(x) = x^n/n^n$. On the domain $\{x : \; |x| \le r\}$ we have

$$\|f_n\|_\infty = \frac{r^n}{n^n} =: M_n.$$

We test for the convergence of the series $\sum M_n$ by using the root test.

$$M_n^{1/n} = \frac{r}{n} \to 0 \quad \text{as } n \to \infty.$$

As the limit is less than 1 the series converges and by the M-test the power series converges uniformly.

2 marks

(ii) In this case $f_n(x) = (n!)^2 x^n/(2n)!$. On the domain $\{x : |x| \le r\}$ we have

$$\|f_n\|_\infty = \frac{(n!)^2}{(2n)!} r^n =: M_n.$$

We test for the convergence of the series $\sum M_n$ by using the ratio test.

$$\begin{aligned} \frac{M_{n+1}}{M_n} &= \frac{(n+1)^2}{(2n+2)(2n+1)} r \\ &= \frac{(1+1/n)^2}{(2+2/n)(2+1/n)} r \to \frac{r}{4} \quad \text{as } n \to \infty. \end{aligned}$$

As the limit is less than 1 when $r < 4$ the series converges and by the M-test the power series converges uniformly.

2 marks

(iii) In this case $f_n(x) = 1/(x^2 + 2^n)$ and on the domain $\mathbb{R}$ we have by inspection that $\|f_n\|_\infty = f_n(0) = 1/2^n$. The geometric series $\sum_{n=0}^\infty 1/2^n$ is convergent and by the M-test the series converges uniformly on $\mathbb{R}$.

2 marks

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3. *This was question 4 of the Jan 2003* `MA2034A` *exam paper.*

   (a) Let $I \subset \mathbb{R}$ and let f be a bounded function on I. Define the uniform norm $\|f\|$ of f on I.

   [1 mark]

   (b) Let $I \subset \mathbb{R}$ and let $f_n : I \to \mathbb{R}$, $n = 1, 2, \cdots$ and $f : I \to \mathbb{R}$ be functions.

   (i) Define what it means for (f_n) to converge pointwise on I.

   [2 marks]

   (ii) Define what it means for (f_n) to converge to f uniformly on I.

   [2 marks]

   (c) In each of the following cases of sequences of functions, determine the pointwise limit function and determine whether or not the convergence is uniform.

   (i)

$$f_n : [0, \infty) \to \mathbb{R}, \quad f_n(x) := \frac{x^n}{1 + x^n}.$$

   [4 marks]

   (ii)

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) := \frac{\cos(nx)}{n}.$$

   [3 marks]

(iii)

$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) := x^n(1-x).$$

[4 marks]

(iv)

$$f_n : [0,\infty) \to \mathbb{R}, \quad f_n(x) := x\mathrm{e}^{-nx}.$$

[4 marks]

ANSWER

(a)

$$\|f\| := \sup\{|f(x)| \ : \ x \in I\} \ .$$

1 mark

(b) (i) (f_n) converges pointwise on I if the sequence of numbers $(f_n(x))$ converges for every $x \in I$.

2 marks

(ii) (f_n) converges uniformly on I to f, $f : I \to \mathbb{R}$, if

$$\|f_n - f\| \to 0 \quad \text{as } n \to \infty.$$

2 marks

(c) (i) If $0 \le x < 1$ then $x^n \to 0$ as $n \to \infty$ and we get $f_n(x) \to 0$ as $n \to \infty$.
If $x = 1$ then $f_n(1) = 1/2 \to 1/2$ as $n \to \infty$.
If $x > 1$ then

$$f_n(x) = \frac{1}{(1/x^n) + 1} \to \frac{1}{0+1} = 1 \quad \text{as } n \to \infty.$$

Hence the pointwise limit function f is the discontinuous function

$$f : [0,\infty) \to \mathbb{R}, \quad f(x) := \begin{cases} 0, & 0 \le x < 1, \\ 1/2, & x = 1, \\ 1, & x > 1. \end{cases}$$

As each f_n is continuous and the pointwise limit is discontinuous the convergence is not uniform.

4 marks

(ii) For all $x \in \mathbb{R}$

$$\left| \frac{\cos(nx)}{n} \right| \le \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

Thus (f_n) converges uniformly to the zero function on $\mathbb{R}$ which is thus the pointwise limit.

3 marks

(iii) If $x = 1$ then $f_n(1) = 0$ and $(f_n(1))$ is a constant sequence. If $0 \le x < 1$ then $x^n \to 0$ as $n \to \infty$. Thus (f_n) converges pointwise to the zero function on $[0, 1]$. To test for uniform convergence we determine the uniform norm of each f_n.
As $f_n(x) \ge 0$ with $f_n(0) = f_n(1) = 0$ we need to find the maximum of $f_n(x)$, $0 < x < 1$.

$$f_n'(x) = nx^{n-1} - (n+1)x^n = x^{n-1}(n - (n+1)x).$$

The maximum occurs at $x = n/(n+1)$. Since for this x, $x^n < 1$ we have

$$|f_n(n/(n+1))| \le 1 - \frac{n}{n+1} = \frac{1}{n+1} \to 0 \quad \text{as } n \to \infty.$$

Hence $f_n \to 0$ uniformly on $[0, 1]$.

4 marks

(iv) If $x = 0$ then $f_n(0) = 0$ and $(f_n(0))$ is a constant sequence. For $x > 0$ the ratio test gives

$$\frac{f_{n+1}(x)}{f_n(x)} = \frac{\mathrm{e}^{-(n+1)x}}{\mathrm{e}^{-nx}} = \mathrm{e}^{-x} < 1$$

and hence $f_n(x) \to 0$ as $n \to \infty$. Thus (f_n) converges pointwise to the zero function on $[0, 1]$.
To test for uniform convergence we determine the uniform norm of each f_n.

$$f_n'(x) = \mathrm{e}^{-nx}(1 - nx) = 0 \quad \text{when } x = \frac{1}{n}.$$

As $f_n'(x) > 0$ in $[0, 1/n)$ and $f_n'(x) < 0$ in $(1/n, \infty)$ this is a local maximum. We have

$$\|f_n\| = f_n(1/n) = \frac{\mathrm{e}^{-1}}{n} \to 0 \quad \text{as } n \to \infty.$$

The sequence converges uniformly to the zero function.

4 marks

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4. *This was question 4 of the Jan 2002* `MA2034A` *exam paper.*

(i) Let $I \subset \mathbb{R}$ and let f be a bounded function on I. Define the uniform norm $\|f\|$ of f on I.

[1 mark]

(ii) Let $I \subset \mathbb{R}$ and let $f_n : I \to \mathbb{R}$, $n = 1, 2, \cdots$ and $f : I \to \mathbb{R}$ be functions.

(a) Define what is means by saying that $f_n \to f$ pointwise on I.

[2 marks]

(b) Define what is means by saying that $f_n \to f$ uniformly on I.

[2 marks]

(iii) In the following $I \subset \mathbb{R}$ and (f_n) is a sequence of functions such that $f_n : I \to \mathbb{R}$.

(a) Explain why when $I = (0, 1)$ and $f_n(x) := x^n$, $\|f_n\| = 1$ for all n. Hence show that the sequence (f_n) converges pointwise on I but not uniformly on I.

[3 marks]

(b) Show that when $I = [0, 1]$ and

$$f_n(x) := \frac{nx}{1 + n^2x^2},$$

$\|f_n\| = 1/2$ for all n. Hence show that the sequence (f_n) converges pointwise on I but not uniformly on I.

[5 marks]

(c) Suppose that (f_n) is a sequence of continuous functions which converges pointwise on I to f. What can you conclude about the uniformity of the convergence in the following cases: (1) when f is continuous on I, (2) when f is discontinuous on I.

[2 marks]

(iv) Let $f_n : I \to \mathbb{R}$, $n = 1, 2, \cdots$ be a sequence of bounded functions. Define what it means for (f_n) to be a Cauchy sequence in the uniform norm and show that if $f_n \to f$ uniformly then (f_n) is a Cauchy sequence. (In your answer you can assume that the uniform norm satisfies all the norm axioms, e.g. the triangle inequality.)

[5 marks]

ANSWER

(i)

$$\|f\| := \sup\{|f(x)| \ : \ x \in I\} .$$

1 mark

(ii) (a) (f_n) converges pointwise on I if the sequence of numbers $(f_n(x))$ converges for every $x \in I$.

2 marks

(b) (f_n) converges uniformly on I to f, $f : I \to \mathbb{R}$, if

$$\|f_n - f\| \to 0 \quad \text{as } n \to \infty.$$

2 marks

(iii) (a) For all $x \in I$, $x^n < 1$ and hence 1 is an upper bound. Also, $\lim_{x\to 1} x^n = 1$. Given any $\epsilon > 0$, the definition of the limit implies that $1 - \epsilon$ is exceeded by some $x \in I$ and thus no number less than 1 can be an upper bound. Hence the least upper bound is 1 and $\|f_n\| = 1$.

For all $x \in I$ we have $|x| < 1$ and consequently $x^n \to 0$ as $n \to \infty$. The sequence converges pointwise to the zero function but as $\|f_n\| \not\to 0$ the sequence does not converge uniformly.

3 marks

(b) As f_n is continuous on $[0,1]$ it attains its maximum on $[0,1]$. $f_n(0) = 0$ and $f_n(1) = n/(1+n^2) = 1/((1/n)+n)$. $f_1(1) = 1/2$ and for $n \geq 2$ $f_n(1) < 1/n \leq 1/2$. To find the maximum we consider turning points of f_n.

$$f_n'(x) = \frac{(1+n^2x^2)n - (nx)(2n^2x)}{(1+n^2x^2)^2} = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} = 0$$

when $x = 1/n$. f_n increases in $[0, 1/n)$ and decreases in $(1/n, \infty)$. Thus

$$\|f_n\| = f_n(1/n) = \frac{1}{2}.$$

For the pointwise convergence observe that for $x = 0$, $f_n(x) = 0$ for all n. For $x > 0$,

$$f_n(x) = \frac{1/(nx)}{(1/nx)^2 + 1} \to \frac{0}{0+1} = 0 \quad \text{as } n \to \infty.$$

Thus the sequence converges pointwise to the zero function. As $\|f_n\| \not\to 0$ the sequence does not converge uniformly.

5 marks

(c) (1) If the limit function is discontinuous then this is sufficient to prove that the convergence is not uniform. (2) If the limit function is continuous then nothing can be concluded about whether or not the convergence is uniform.

2 marks

(iv) (f_n) is a Cauchy sequence if for every $\epsilon > 0$ there exists an N such that

$$\|f_n - f_m\| < \epsilon, \quad \text{for all } n \geq N \text{ and } m \geq N.$$

(f_n) converges uniformly to f means that for every $\epsilon > 0$ there exists a N such that

$$\|f_n - f\| < \epsilon/2 \quad \text{for all } n \geq N.$$

Then for all $m, n \geq N$ and for all $x \in I$ we compare both f_m and f_n with the limit f to give

$$\begin{aligned} |f_m(x) - f_n(x)| &= |(f_m(x) - f(x)) + (f(x) - f_n(x))| \\ &\leq |f_m(x) - f(x)| + |f(x) - f_n(x)| \\ &\leq \|f_m - f\| + \|f - f_n\| < \epsilon/2 + \epsilon/2 = \epsilon\ . \end{aligned}$$

As this is true for for all $x \in I$ we have $\|f_m - f_n\| < \epsilon$ as required.

5 marks

5. *This was question 5 of the Jan 2003* `MA2034A` *exam paper.*

(a) State the Weierstrass M-test.

[3 marks]

(b) In the following you may assume in your answer that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges for all $p > 1$ and diverges for all $p \le 1$.

Let

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) := \sum_{k=1}^{n} \frac{\cos(kx)}{k^3}.$$

Use the the Weierstrass M-test to explain why (f_n) and (f_n') converge uniformly on $\mathbb{R}$.

[4 marks]

Does the sequence (f_n'') converge pointwise on $\mathbb{R}$?

[1 mark]

(c) Determine the radius of convergence of the following power series and state regions in which the series converge uniformly.

(i)

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

[3 marks]

(ii) In the following $\alpha \in \mathbb{R}$ is not an integer.

$$\sum_{k=0}^{\infty} \left(\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \right) x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \cdots$$

[4 marks]

(iii)

$$\sum_{k=0}^{\infty} \frac{k^3 x^k}{3^k}.$$

[3 marks]

(iv)

$$\sum_{k=0}^{\infty} k^k x^k.$$

[2 marks]

ANSWER

(a) The Weierstrass M-test relates to series of functions (f_k). Let $f_k : I \to \mathbb{R}$ and suppose that

$$\|f_k\| \le M_k,$$

where $\|f_k\|$ is the uniform norm of f_k on I. If the series of numbers $\sum M_k$ converges then the series of functions $\sum f_k$ converges uniformly on I.

3 marks

(b) To apply the M-test to the series for $f_n(x)$ we note that $|\cos(kx)| \le 1$ so that the kth component function is bounded by

$$M_k = \frac{1}{k^3}.$$

The series $\sum 1/k^3$ converges and thus the series for f_n converges uniformly on $\mathbb{R}$. f_n' is given by

$$f_n'(x) = -\sum_{k=1}^{n} \frac{\sin(kx)}{k^2}.$$

The kth component function is bounded by

$$M_k = \frac{1}{k^2}.$$

The series $\sum 1/k^2$ converges and thus the series for f_n' converges uniformly on $\mathbb{R}$.

4 marks

f_n'' is given by

$$f_n''(x) = -\sum_{k=1}^{n} \frac{\cos(kx)}{k}.$$

When $x = 0$, $\cos(0) = 1$ and $-f_n''(0)$ is the nth partial sum of the divergent harmonic series. Hence the series does not converge pointwise on $\mathbb{R}$.

1 mark

(c) (i) Let $f_k(x) = x^k/k!$. On $|x| \le r$

$$|f_k(x)| \le \frac{r^k}{k!} =: M_k.$$

By the ratio test

$$\frac{M_{k+1}}{M_k} = \frac{r^{k+1}/(k+1)!}{r^k/k!} = \frac{r}{k+1} \to 0 \quad \text{as } k \to \infty.$$

The series $\sum M_k$ converges for all r and hence the radius of convergence is ∞. The series converges uniformly in any region of the form $|x| \le r$.

3 marks

(ii) Let $f_k(x) = a_k x^k$ where

$$a_k := \left(\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \right) .$$

On $|x| \le r$

$$|f_k(x)| \le |a_k| r^k =: M_k.$$

Using the ratio test

$$\frac{M_{k+1}}{M_k} = \left| \frac{a_{k+1}}{a_k} \right| r = \left| \frac{\alpha - k}{k+1} \right| r = \left| \frac{\alpha/k - 1}{1 + 1/k} \right| r \to r \quad \text{as } k \to \infty.$$

Thus the series converges absolutely if $r < 1$ and diverges for $r > 1$. The radius of convergence is $R = 1$ and the series converges uniformly in $[-r, r]$ for all r satisfying $0 \le r < 1$.

4 marks

(iii) Let $f_k(x) = a_k x^k$ where

$$a_k := \frac{k^3}{3^k} .$$

On $|x| \le r$

$$|f_k(x)| \le |a_k| r^k =: M_k.$$

Using the ratio test

$$\frac{M_{k+1}}{M_k} = \frac{a_{k+1}}{a_k} r = \frac{(k+1)^3}{3k^3} r = \frac{(1+1/k)^3}{3} r \to \frac{r}{3}.$$

The series converges if $r < 3$ and diverges for $r > 3$. The radius of convergence is $R = 3$. The series converges uniformly in $|x| \le r$ for all $r < 3$.

3 marks

(iv) Let $f_k(x) = k^k x^k$. On $|x| \le r$

$$|f_k(x)| \le (kr)^k =: M_k.$$

Using the root test

$$M_k^{1/k} = kr.$$

The sequence $(M_k^{1/k})$ only converges when $r = 0$ The radius of convergence is $R = 0$.

2 marks

6. *This was question 5 of the Jan 2002* `MA2034A` *exam paper.*

(i) State the Weierstrass M-test.

[3 marks]

(ii) In the following you may assume in your answer that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges for all $p > 1$.

(a) Let

$$f_k(x) := \frac{1}{x^2 + k^2}.$$

Show that

$$|f_k'(x)| \le \frac{C}{k^3}, \quad C = \frac{9}{8\sqrt{3}}, \quad \text{for all } x \in \mathbb{R}.$$

[4 marks]

Hence use the Weierstrass M-test to show that

$$\sum_{k=1}^{\infty} f_k(x) \quad \text{and} \quad \sum_{k=1}^{\infty} f_k'(x)$$

both converge uniformly on $\mathbb{R}$.

[2 marks]

(b) Show that the following series converges uniformly on $[0, \infty)$:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \left(\frac{1-x}{1+x} \right)^k.$$

[Hint: make the substitution $y = (1-x)/(1+x)$.]

[3 marks]

(iii) Determine the radius of convergence of the following power series and state regions in which the series converge uniformly.

(a)

$$\sum_{k=0}^{\infty} \frac{2^k x^k}{k!}.$$

[3 marks]

(b)

$$\sum_{k=0}^{\infty} \frac{k^3 x^k}{3^k}$$

[3 marks]

(c)

$$\sum_{k=0}^{\infty} k! x^k.$$

[2 marks]

ANSWER

(i) The Weierstrass M-test relates to series of functions (f_k). Let $f_k : I \to \mathbb{R}$ and suppose that

$$\|f_k\| \le M_k,$$

where $\|f_k\|$ is the uniform norm of f_k on I. If the series of numbers $\sum M_k$ converges then the series of functions $\sum f_k$ converges uniformly on I.

3 marks

(ii) (a)

$$f_k'(x) = \frac{-2x}{(x^2+k^2)^2}.$$

To determine the maximum of $|f_k'(x)|$ we need to consider turning points of f_k' which correspond to points at which $f_k''(x) = 0$.

$$f_k''(x) = \frac{(x^2+k^2)^2(-2) - (-2x)2(x^2+k^2)2x}{(x^2+k^2)^4}$$

and hence $f_k''(x) = 0$ when

$$(x^2+k^2)^2(-2) = -8x^2(x^2+k^2),$$
$$\text{i.e. when } x^2+k^2 = 4x^2, \quad 3x^2 = k^2 \ .$$

At $x = \pm k/\sqrt{3}$, $x^2+k^2 = 4k^2/3$ and

$$|f_k'(\pm k/\sqrt{3})| = \frac{C}{k^3} \quad \text{where } C = \frac{2/\sqrt{3}}{(4/3)^2} = \frac{9}{8\sqrt{3}} \ .$$

As we only have 2 turning values we obtain the bound

$$|f_k'(x)| \le \frac{C}{k^3} \quad \text{for all } x \in \mathbb{R}.$$

4 marks

As for all $x \in \mathbb{R}$,

$$|f_k(x)| \le \frac{1}{k^2} \quad \text{and} \quad |f_k'(x)| \le \frac{C}{k^3}$$

and $\sum 1/k^2$ and $\sum 1/k^3$ are standard convergent series the series converges uniformly by the Weierstrass M-test.

2 marks

(b) Let

$$y = \frac{1-x}{1+x} \quad \text{and} \quad f_k(x) = \frac{y^k}{k^2}.$$

For $0 \le x \le 1$, $|1-x| \le 1 \le 1+x$. For $1 < x$, $|1-x| = x-1 \le 1+x$. That is for all $x \ge 0$, $|1-x| \le |1+x|$. Thus $|y| \le 1$ for all $x \ge 0$ and

$$|f_k(x)| \le \frac{1}{k^2}.$$

As $\sum 1/k^2$ is a standard convergent series it follows that the series converges uniformly when $x \geq 0$ by the Weierstrass M-test.

3 marks

(iii) (a) Let $f_k(x) = 2^k x^k/k!$. As $(k+1)! = (k+1)k!$ etc. we have for $x \neq 0$,

$$\left|\frac{f_{k+1}(x)}{f_k(x)}\right| = \left|\frac{2x}{k+1}\right| \to 0 \quad \text{as } k \to \infty.$$

By the ratio test the series converges for all $x \in \mathbb{R}$ and by the M-test the series converges uniformly in all regions of the form $[-R, R]$, $R > 0$. The radius of convergence is ∞.

3 marks

(b) Let $f_k(x) = k^3 x^k/3^k$. We have for $x \neq 0$,

$$\left|\frac{f_{k+1}(x)}{f_k(x)}\right| = \left|\frac{(k+1)^3 x}{3k^3}\right| = \left|\frac{(1+1/k)^3 x}{3}\right| \to \frac{|x|}{3} \quad \text{as } k \to \infty.$$

By the ratio test the series converges for $|x| < 3$ and by the M-test the series converges uniformly in all regions of the form $[-r, r]$, $r < 3$. The radius of convergence is 3.

3 marks

(c) Let $f_k(x) = k!x^k$. We have for $x \neq 0$,

$$\left|\frac{f_{k+1}(x)}{f_k(x)}\right| = (k+1)|x|.$$

This is unbounded for all $x \neq 0$ and by the ratio test the series diverges. Hence the series only converges at $x = 0$. The radius of convergence is 0.

2 marks

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7. *These were part of question 4 of the Jan 2000 and Jan 2001* `MA2034A` *exam papers.*

(a) Let (f_n), $f_n : I \to \mathbb{R}$, denote a sequence of continuous functions defined on I. If (f_n) converges to f uniformly on I then what properties will the limit function f have?

(b) In each of the following cases of sequences of functions, determine whether or not the sequence converges pointwise on its given domain. If the sequence does converge pointwise then give the pointwise limit function and determine whether or not the convergence is uniform.

(i)

$$f_n : (-1, 1] \to \mathbb{R}, \quad f_n(x) := x^n, \quad n = 1, 2, \cdots .$$ **[2 marks]**

(ii)

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) := \frac{\sin nx}{n} \quad n = 1, 2, \cdots .$$ **[3 marks]**

(iii)

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) := \cos nx .$$ **[3 marks]**

(iv)

$$f_n : (-1, 1] \to \mathbb{R}, \quad f_n(x) := \frac{x^n}{(1 + x^n)^2}, \quad n = 1, 2, \cdots .$$

(v)

$$f_n : [0, \infty) \to \mathbb{R}, \quad f_n(x) := \frac{x^n}{(1 + x^n)^n}, \quad n = 1, 2, \cdots .$$

ANSWER

(a) As each f_n is continuous and (f_n) converges to f uniformly on I then f is continuous on I.

1 mark

(b) (i)

$$x^n \to 0 \quad \text{as } n \to \infty \text{ if } |x| < 1.$$

Thus for $x \in (-1, 1)$, $f_n(x) \to 0$ as $n \to \infty$. Also, $f_n(1) = 1$ for all n. Thus $(f_n(x))$ converges for all x in $(-1, 1]$ and hence the sequence of functions does converge pointwise. The pointwise limit function is

$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } x \in (-1, 1), \\ 1, & \text{if } x = 1. \end{cases}$$

As each f_n is continuous and f is discontinuous this indicates that the convergence is not uniform.

2 marks

(ii) By the properties of the sine function we have for all $x \in \mathbb{R}$ that

$$|f_n(x)| \le \frac{1}{n}$$

and thus

$$\|f_n\| \le \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

Thus the sequence converges uniformly to the function f

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) := 0 .$$

Uniform convergence implies pointwise convergence and hence f is also the pointwise limit.

3 marks

(iii) If we let $x = \pi$ then $f_n(\pi) = \cos n\pi = (-1)^n$. The sequence of numbers $(f_n(\pi))$ does not converge and thus the sequence does not converge pointwise and as a consequence it does not converge uniformly.

3 marks

(iv)
$$x^n \to 0 \quad \text{as } n \to \infty \text{ if } |x| < 1.$$
Thus for $x \in (-1, 1)$, $f_n(x) \to 0/(1+0)^2 = 0$ as $n \to \infty$. Also, $f_n(1) = 1/4$ for all n. Thus
$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } x \in (-1,1), \\ 1/4, & \text{if } x = 1. \end{cases}$$
As each f_n is continuous and f is discontinuous this indicates that the convergence is not uniform.

4 marks

(v) Let $g_n : \mathbb{R} \to \mathbb{R}$,
$$g_n(x) := \frac{x}{1+x^n} = \frac{(1/x)^{n-1}}{(1/x)^n + 1} \ .$$
If $0 \le x < 1$ then
$$f_n(x) = \left(\frac{x}{1+x^n}\right)^n \le x^n \to 0 \quad \text{as } n \to \infty.$$
If $x = 1$ then
$$f_n(1) = (1/2)^n \to 0 \quad \text{as } n \to \infty.$$
If $x > 1$ then $(1/x)^n \to 0$ as $n \to \infty$ and
$$f_n(x) = \left(\frac{(1/x)^{n-1}}{(1/x)^n+1}\right)^n < \left(\frac{1}{x}\right)^{n(n-1)} \to 0 \quad \text{as } n \to \infty.$$
The pointwise limit function is $f : \mathbb{R} \to \mathbb{R}$, $f(x) := 0$.

4 marks

To establish that the convergence is uniform we need to bound g_n and hence f_n. We note that $g_n(0) = 0$ and $g_n(x) \to 0$ as $|x| \to \infty$. For turning values we have
$$g_n'(x) = \frac{(1+x^n) - x(nx^{n-1})}{(1+x^n)^2} = 0 \quad \text{when} \quad 1 = (n-1)x^n \ .$$
As there is only one turning value it is the point where g_n has a global maximum. Thus
$$\max_{\mathbb{R}} |g_n(x)| \le \frac{(1/(n-1))^{1/n}}{1 + 1/(n-1)} < \left(\frac{1}{(n-1)}\right)^{1/n}$$
and
$$\|f_n\| = \max_{\mathbb{R}} |f_n(x)| \le \frac{1}{n-1} \to 0 \quad \text{as } n \to \infty.$$

6 marks

8. *This was most of question 5 of the January 2001* MA2034A *exam paper.*

(i) State the Weierstrass M-test.

[3 marks]

Use the Weierstrass M-test to show that the following series converge uniformly on $\mathbb{R}$.

(a)

$$\sum_{k=1}^{\infty} \left(\frac{\cos kx}{k^3} + 3\frac{\sin kx}{k^2} \right) .$$

[3 marks]

(b)

$$\sum_{k=1}^{\infty} \frac{\sqrt{x^2+k} - |x|}{k^2} .$$

[4 marks]

In your answer you may assume that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges for all $p > 1$.

(ii) Determine the radius of convergence of the following power series and state regions in which the series converge uniformly.

(a)

$$\sum_{k=0}^{\infty} \left(\frac{k^2}{5^k} \right) x^k = \left(\frac{1}{5} \right) x + \left(\frac{2^2}{5^2} \right) x^2 + \left(\frac{3^2}{5^3} \right) x^3 + \left(\frac{4^2}{5^4} \right) x^4 + \left(\frac{5^2}{5^5} \right) x^5 + \cdots .$$

[4 marks]

(b)

$$\sum_{k=0}^{\infty} \left(\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \right) \left(\frac{x}{2} \right)^k = 1 + \alpha\frac{x}{2} + \frac{\alpha(\alpha-1)}{2} \left(\frac{x}{2} \right)^2 + \cdots .$$

[4 marks]

ANSWER

(i) The Weierstrass M-test relates to series of functions (f_k). Let $f_k : I \to \mathbb{R}$ and suppose that

$$\|f_k\| \le M_k,$$

where $\|f_k\|$ is the uniform norm of f_k on I. If the series of numbers $\sum M_k$ converges then the series of functions $\sum f_k$ converges uniformly on I.

3 marks

(a) With

$$f_k(x) := \frac{\cos kx}{k^3} + 3\frac{\sin kx}{k^2}$$

it follows that for all $x \in \mathbb{R}$

$$|f_k(x)| \le M_k := \frac{1}{k^3} + \frac{3}{k^2} \ .$$

$\sum 1/k^2$ and $\sum 1/k^3$ are standard convergent series and thus $\sum M_k$ converges. Thus $\sum f_k$ converges uniformly on $\mathbb{R}$.

3 marks

(b) Let

$$\begin{aligned} f_k(x) &:= \frac{\sqrt{x^2+k} - |x|}{k^2} \\ &= \frac{k}{k^2(\sqrt{x^2+k} + |x|)} = \frac{1}{k(\sqrt{x^2+k} + |x|)} \le \frac{1}{k^{3/2}} =: M_k \end{aligned}$$

where the bound was obtained by observing that the denominator takes its smallest value in $\mathbb{R}$ when $x = 0$.
$\sum M_k$ is a standard convergent series and thus the series $\sum f_k$ converges uniformly on $\mathbb{R}$.

4 marks

(ii) (a) Let

$$f_k(x) = \frac{k^2}{5^k} x^k \ .$$

On $|x| \le r$

$$|f_k(x)| \le \frac{k^2}{5^k} r^k =: M_k \ .$$

Using the ratio test

$$\frac{M_{k+1}}{M_k} = \frac{(k+1)^2}{k^2}\frac{r}{5} = \left(1 + \frac{1}{k}\right)^2 \frac{r}{5} \to \frac{r}{5} \quad \text{as } k \to \infty.$$

Thus the series $\sum M_k$ converges if $r < 5$ and the series $\sum f_k$ converges uniformly in $[-r, r]$ for all r satisfying $0 \le r < 5$. The radius of convergence is $R = 5$.

4 marks

(b) Let

$$f_k(x) := \left(\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\right)\left(\frac{x}{2}\right)^k .$$

On $|x| \le r$

$$|f_k(x)| \le \left|\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\right|\left(\frac{r}{2}\right)^k =: M_k .$$

Using the ratio test

$$\frac{M_{k+1}}{M_k} = \left|\frac{\alpha-k}{k+1}\right|\left(\frac{r}{2}\right) = \left|\frac{\alpha/k-1}{1+1/k}\right|\left(\frac{r}{2}\right) \to \frac{r}{2} \quad \text{as } k \to \infty.$$

Thus the series $\sum M_k$ converges if $r < 2$ and the series $\sum f_k$ converges uniformly in $[-r, r]$ for all r satisfying $0 \le r < 2$. The radius of convergence is $R = 2$.

4 marks

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9. *This was part of question 5 of the January 2000 paper.*

Use the Weierstrass M-test to show that the following series converge uniformly on $\mathbb{R}$.

(a)

$$\sum_{k=1}^{\infty} \frac{1}{x^2+k^2} .$$

(b)

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k^{3/2}} .$$

(c) Determine the radius of convergence of the following power series and state the region in which the series converge uniformly.

(i)

$$\sum_{k=0}^{\infty} \left(\frac{k}{2^k}\right) x^k = \left(\frac{1}{2}\right) x + \left(\frac{2}{2^2}\right) x^2 + \left(\frac{3}{2^3}\right) x^3 + \cdots .$$

(ii)

$$\sum_{k=0}^{\infty} \left(\frac{1}{k^k}\right) x^k = 1 + x + \left(\frac{1}{2^2}\right) x^2 + \left(\frac{1}{3^3}\right) x^3 + \cdots .$$

ANSWER

(a) The denominator takes its smallest value when $x = 0$ and hence

$$|f_k(x)| \le \frac{1}{k^2} .$$

The series $\sum 1/k^2$ is a standard convergent series and hence the series of functions converges uniformly on $\mathbb{R}$.

3 marks

(b)

$$\left|\frac{\sin kx}{k^{3/2}}\right| \le \frac{1}{k^{3/2}}$$

and $\sum 1/(k^{3/2})$ is a standard convergent series. Hence the series of functions converges uniformly on $\mathbb{R}$.

2 marks

(c) (i) Let

$$a_k := \frac{k}{2^k} x^k .$$

Using the ratio test

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)}{k}\frac{x}{2} = (1+1/k)\frac{x}{2} \to \frac{x}{2} \quad \text{as } k \to \infty.$$

Thus the series converges absolutely if $|x| < 2$ and converges uniformly in $[-r, r]$ for all r satisfying $0 \le r < 2$.

4 marks

(ii) Let

$$a_k := \left(\frac{x}{k}\right)^k .$$

Using the root test

$$|a_k|^{1/k} = \frac{|x|}{k} \to 0 \quad \text{as } k \to \infty.$$

Thus the series converges absolutely for all $x \in \mathbb{R}$ and converges uniformly in $[-r, r]$ for all r satisfying $0 \le r < \infty$.

4 marks

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10. *Book work question.*

Let $I \subset \mathbb{R}$ and let (f_n) be a sequence of continuous functions such that $f_n : I \to \mathbb{R}$. Suppose that (f_n) converges uniformly to f on I and note that

$$\begin{aligned} |f(x) - f(c)| &= |(f(x) - f_n(x)) + (f_n(x) - f_n(c)) + (f_n(c) - f(c))| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \\ &\le \left(2 \sup_{x \in I} |f(x) - f_n(x)|\right) + |f_n(x) - f_n(c)| . \end{aligned}$$

Show that f is also continuous on I.

ANSWER

Let $\epsilon > 0$. By the uniform convergence there exists N such that

$$\sup_{x \in I} |f(x) - f_n(x)| < \epsilon$$

for $n \geq N$. The continuity of f_N gives the existence of δ such that

$$|f_N(x) - f_N(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta \ .$$

Thus for $|x - c| < \delta$ we have

$$|f(x) - f(c)| < 3\epsilon \quad \text{whenever} \quad |x - c| < \delta$$

which is sufficient to establish the continuity of f at the arbitrary point c.

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11. *The following is a miscellaneous collection of questions relating to sequences and series of functions.*

   (a) Show that if $f_n(x) := x + 1/n$ and $f(x) := x$ for all $x \in \mathbb{R}$ then $f_n \to f$ uniformly on $\mathbb{R}$ but that (f_n^2) does not converge uniformly on $\mathbb{R}$.

   (b) Let $C^{(1)}[a, b]$ denote the set of continuously differentiable functions on a finite interval $[a, b]$. For each f in $C^{(1)}[a, b]$, define

   $$\|f\|_{C^1} := \|f\| + \|f'\| \ .$$

   Show that $\|.\|_{C^1}$ satisfies the norm requirements of non-negativity, linearity and the triangle inequality. (Remark: It can be shown that the linear space $C^{(1)}[a, b]$ is complete in this norm.)

   (c) Use the Weierstrass M-test to deduce that if $\sum |a_n|$ and $\sum |b_n|$ converge then the Fourier series

   $$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

   converges uniformly on $\mathbb{R}$.

   (d) Let

   $$s_n(x) := \sum_{k=1}^{n} \frac{\cos(kx)}{k^5}.$$

   Explain why (s_n), (s_n'), (s_n'') and (s_n''') all converge uniformly on $\mathbb{R}$ but that $(s_n''''(0))$ does not converge.

ANSWER

(a) $f_n(x) = x + 1/n$ and $f(x) = x$ and hence $f_n(x) - f(x) = 1/n$. Thus

$$\|f_n - f\|_\infty = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

$$f_n^2 - f^2 = (f_n + f)(f_n - f)$$

and thus in this case

$$f_n(x)^2 - f(x)^2 = \frac{1}{n}\left(2x + \frac{1}{n}\right)$$

and

$$\sup_{x\in\mathbb{R}} \left|f_n(x)^2 - f(x)^2\right| = \frac{1}{n} \sup_{x\in\mathbb{R}} \left|2x + \frac{1}{n}\right| = \infty \quad \text{for all } n.$$

(b) Nonnegativity: Clearly $\|f\|_{C^1} \geq 0$ since $\|f\| \geq 0$ and $\|f'\| \geq 0$. If $\|f\|_{C^1} = 0$ then $\|f\| = 0$ and hence $f(x) = 0$ for all $x \in [a, b]$.

Linearity: By using the linearity of the uniform norm

$$\|\alpha f\|_{C^1} = \|\alpha f\| + \|\alpha f'\| = |\alpha|\,\|f\| + |\alpha|\,\|f'\| = |\alpha|\,\|f\|_{C^1}\ .$$

Triangle inequality: By the triangle inequality for the uniform norm

$$\begin{aligned} \|f + g\|_{C^1} &= \|f + g\| + \|f' + g'\| \\ &\leq (\|f\| + \|g\|) + (\|f'\| + \|g'\|) \\ &= \|f\|_{C^1} + \|g\|_{C^1}\ . \end{aligned}$$

(c) With $f_n(x) := a_n \cos nx + b_n \sin nx$ we have

$$|f_n(x)| \leq |a_n| + |b_n|$$

by the triangle inequality and that $|\cos nx| \leq 1$ and $|\sin nx| \leq 1$. As $\sum |a_n|$ and $\sum |b_n|$ both converge we have that $\sum(|a_n| + |b_n|)$ also converges. Hence the conditions of the Weierstrass M-test apply with $M_n = |a_n| + |b_n|$ and the series converges uniformly on $\mathbb{R}$.

(d)

$$\begin{aligned} s_n(x) &:= \sum_{k=1}^{n} \frac{\cos(kx)}{k^5}, \\ s_n'(x) &:= -\sum_{k=1}^{n} \frac{\sin(kx)}{k^4}, \\ s_n''(x) &:= -\sum_{k=1}^{n} \frac{\cos(kx)}{k^3}, \\ s_n'''(x) &:= \sum_{k=1}^{n} \frac{\sin(kx)}{k^2}, \\ s_n''''(x) &:= \sum_{k=1}^{n} \frac{\cos(kx)}{k}. \end{aligned}$$

The kth term in the series of s_n, s_n', s_n'' and s_n''' are bounded on $\mathbb{R}$ by respectively $1/k^5$, $1/k^4$, $1/k^3$ and $1/k^2$ and as the series of the bounds converge the sequence of functions converge uniformly on $\mathbb{R}$. However

$$s_n''''(0) = \sum_{k=1}^{n} \frac{1}{k}.$$

are the partial sums of the harmonic series which is divergent.