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SAMPLING TYPE METHODS FOR AN INVERSE WAVEGUIDE PROBLEM

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Abstract. We consider the problem of locating a penetrable obstacle in an acoustic waveguide from measurements of pressure waves due to point sources inside the waveguide. More precisely, we assume that we are given the scattered field and its normal derivative for any source point and receiver placed on a pair of surfaces known as the source and the measurement surfaces, respectively. A novel feature of this work is that the obstacle is allowed to touch the boundary of the pipe.

We first analyze the associated interior transmission problem. Then, we adapt and analyze the Reciprocity Gap Method (RGM) and the Linear Sampling Method (LSM) to deal with the inverse problem. We also study the relationship between these two methods and provide numerical results.

1. **Introduction.** We consider the problem of detecting bounded inhomogeneous obstacles in an infinite tubular waveguide. We have in mind the application of acoustic techniques for the inspection of underground pipes such as sewers (see for example [11, 14]). In this application a loud-speaker and microphone are lowered into a man-hole. Sound pulses are created in the sewer pipe, and the acoustic field reflected by obstructions in the pipe is measured. From these data it is desired to determine the size and position of the blockages. Data from a single microphone and single source is not sufficient for the qualitative algorithms we have in mind; nevertheless, it seems likely that the required data could, in principle, be measured and therefore it is worthwhile to consider if qualitative algorithms could be used in this application. Their principal advantage is that they are easy to implement and rapid.

In particular we will examine, both theoretically and numerically, the use of the Linear Sampling Method (LSM) and the related Reciprocity Gap Method (RGM). The LSM is due to Colton and Kirsch [8] while the RGM for inverse scattering

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is due to Colton and Haddar [7]. Generally, most applications of the LSM and RGM have been to the detection of bounded scatterers in an infinite background medium. However, Bourgeois and Lunéville [3] have considered the use of the LSM for detecting sound soft obstacles in infinite sound hard tubular pipes. The assumption of a sound hard pipe is in accordance with engineering practice for hard plastic and clay pipes [11]. But the computational examples in [3] are all of scatterers (e.g. balls) away from the boundary of the pipe, whereas in the application we have in mind, the scatterers are perturbations to the boundary of the pipe. Nevertheless our paper is strongly motivated by [3], particularly the highly successful numerical results therein.

We seek to extend [3] in two ways. Firstly, porous sediments can support acoustic waves, so we will analyze the RGM and LSM for detecting penetrable scatterers in the pipe allowing these scatterers to be anisotropic and touch the wall of the pipe. This involves the analysis of a new interior transmission problem in which the usual interior transmission conditions are present on part of the obstacle, while the sound hard boundary condition applies on other parts where the blockage touches the pipe walls.

More importantly, inverse scattering algorithms for this problem face the difficulty that the manhole is a significant perturbation of the pipe. The application of the LSM to a realistic sewer would require the calculation of the fundamental solution for the sewer and manhole (as proposed for the technique in [17]). Using the RGM, which in principle requires to measure more data (the field and its derivative along the pipe), we can use the fundamental solution for the pipe alone in order to apply the algorithm to the pipe away from the manhole. Thus, no modeling of the manhole is required which could perhaps outweigh the usual disadvantage of the LSM or RGM: the need for many measurements and many sources (i.e. multistatic data).

Of course the use of the RGM or LSM in a waveguide maintains the usual positive points for these methods: speed of reconstruction and independence of the nature of the blockage. The same RGM or LSM is applied independently of whether there is a hard or penetrable blockage in the pipe, since only the mathematical justification changes.

Other work on detecting objects in a tubular waveguide includes the time-reversal technique [13] which is generally analyzed for small obstacles (whereas we may be wanting to detect substantial blockages). In addition, there has been much more work on inverse problems for layered waveguides (i.e. infinite in two directions, rather than infinite in just one direction as we consider). Examples of such works include qualitative methods [18] and time-reversal [16, 15, 1].

In this paper, we shall analyze a particular RGM method (and related LSM) using the single layer ansatz for the near field inverse waveguide problem. We extend the analysis of [5, 12, 6] to the waveguide problem. Moreover, as in [12], we prove the theoretical equivalence of this RGM and a generalized LSM in which the source and measurement domains are possibly disjoint. We can thus apply the RGM or the LSM depending on the data available.

The paper is organized as follows: In Section 2 and the Appendix we present some results on the forward waveguide problem so that we can formulate the inverse problem. Then, in Section 3 we discuss a related interior transmission problem that is used in the analysis of the inversion schemes. In Section 4 we present the inverse problem and propose the RGM to deal with this problem. We also see that

the RGM possesses the properties needed for regularization, and prove standard theorems about the method, now in the waveguide context. This work is based on a reformulation of the equation of the RGM, and in particular we show a new result stating that the RGM is equivalent to a generalized LSM in which the measurement and source surfaces can be distinct. Thus either method is appropriate for the problem in hand, although, as we have said, the RGM has the advantage that background scatterers (e.g. the manhole) outside the region enclosed by the source curve do not need to be explicitly modeled. Moreover, as part of the analysis, we also prove that the single layer operator for waveguides is an injective and surjective operator on suitable function spaces, even when the single layer operator is defined on open arcs or surfaces (for closed arcs or surfaces see for example [3]). In Section 5 we provide numerical results. In Section 6 we draw some conclusions.

The numerical results contain a surprising example: the RGM fails to detect a complete blockage of the pipe. This example, which is not covered by the theory here or the theory in [3], is obviously troubling. Characterizing the blockages that cannot be detected by the LSM or RGM and finding other methods to process the data in order to detect such blockages is important future work. To our knowledge this is the first case in which the RGM completely fails to detect an obstacle.

Throughout this paper, we will distinguish vectors by means of boldface. Moreover, we will denote the divergence and the curl of a regular enough vector field with div and **curl**, respectively.

In addition to usual Sobolev spaces, we will make use of spaces defined on open arcs or surfaces. More precisely, let Γ denote an open subset of a Lipschitz closed arc ($d = 2$) or surface ($d = 3$) $\tilde{\Gamma}$.

- For any $s \in [0, 1)$, we consider

$$\begin{aligned} H^s(\Gamma) &:= \left\{ g|_\Gamma \,;\; g \in H^s(\tilde{\Gamma}) \right\}, \\ \tilde{H}^s(\Gamma) &:= \left\{ g \in L^2(\Gamma) \,;\; g^0 \in H^s(\tilde{\Gamma}) \right\}, \end{aligned}$$

  where g^0 is the extension of g by 0 from Γ to $\tilde{\Gamma}$, that is,

$$g^0 := \begin{cases} g & \text{in } \Gamma\,, \\ 0 & \text{in } \tilde{\Gamma} \setminus \Gamma\,. \end{cases}$$

  These are endowed with the natural norms

$$\begin{aligned} ||g||_{H^s(\Gamma)} &:= \inf\{||\tilde{g}||_{H^s(\tilde{\Gamma})};\; \tilde{g} \in H^s(\tilde{\Gamma})\,,\; \tilde{g}|_\Gamma = g\}\,, \\ ||g||_{\tilde{H}^s(\Gamma)} &:= ||g^0||_{H^s(\tilde{\Gamma})}\,. \end{aligned}$$

- Given any $s \in (-1, 0)$, we take $H^s(\Gamma)$ and $\tilde{H}^s(\Gamma)$ the dual spaces of $\tilde{H}^{-s}(\Gamma)$ and $H^{-s}(\Gamma)$, respectively, with pivot space $L^2(\Gamma)$.

Notice that, for any $s \in [0, 1)$,

$$\tilde{H}^s(\Gamma) \hookrightarrow H^s(\Gamma) \hookrightarrow L^2(\Gamma) \hookrightarrow \tilde{H}^{-s}(\Gamma) \hookrightarrow H^{-s}(\Gamma)\,,$$

all of these inclusions being dense. Moreover, $\tilde{H}^{-1/2}(\Gamma)$ can be identified with

$$\left\{ g \in H^{-1/2}(\tilde{\Gamma}) \,;\; \mathrm{supp}(g) \subseteq \overline{\Gamma} \right\},$$

where the overbar denotes closure.

2. **The direct waveguide problem.** Let Σ denote a bounded smooth domain in $\mathbb{R}^{d-1}$, $d = 2$ or $d = 3$, having connected complement. This is the cross-section of the waveguide. In particular, we consider an infinite tubular waveguide $\mathbb{R}\times\Sigma \subset \mathbb{R}^d$ with a penetrable obstacle occupying a Lipschitz smooth bounded domain $D \subset \mathbb{R} \times \Sigma$ that can lay on or touch the boundary of the waveguide. Let ∂D and $\partial\Sigma$ denote the boundary of D and Σ respectively.

We identify each point $\mathbf{x} \in \mathbb{R}^d$ with $(x_1, \hat{\mathbf{x}}) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Moreover, we take the unit vectors $\boldsymbol{\nu}_0 := (1, \hat{\mathbf{0}})$, $\boldsymbol{\nu} := (0, \hat{\boldsymbol{\nu}})$ and $\boldsymbol{\nu}_D$ that are normal to $\{0\} \times \Sigma$, $\mathbb{R} \times \partial\Sigma$ and ∂D, and directed to the right and outwards of $\mathbb{R} \times \Sigma$ and D, respectively; see Figure 1.

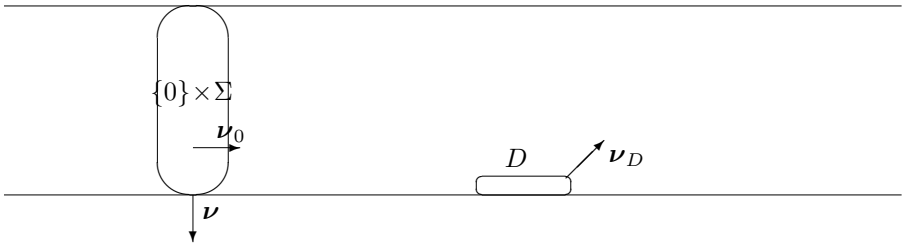


FIGURE 1. A schematic of the problem geometry: The penetrable obstacle occupies an unknown region D that may touch or even lay on the boundary of the waveguide $\mathbb{R} \times \Sigma$.

In applications, an incident field due to an acoustic point source located in the waveguide and outside the tubular neighborhood of D is assumed known. By this we mean that the given incident field u^i is a smooth solution of the Helmholtz equation $\triangle u^i + k^2 u^i = 0$ in a section of the waveguide, $(-R_S, R_S) \times \overline{\Sigma}$ for some $R_S > 0$, that contains $\overline{D}$. Here $k > 0$ is the wave number of the incident field (for theoretical purposes we also need to consider the case when $k \in \mathbb{C}$, in which case $\mathrm{Re}(k) > 0$ and $\mathrm{Im}(k) \geq 0$). In addition we assume that u^i satisfies the sound hard boundary condition $\partial_{\boldsymbol{\nu}} u^i = 0$ on $\mathbb{R} \times \partial\Sigma$. Then, we denote by $F = u^i|_{\partial D}$ and $f = \partial u^i/\partial\nu_D$ on ∂D.

In terms of the scattered field u^s outside D and the total field v inside D, the forward problem is to find u^s and v such that

$$\begin{cases} \triangle u^s + k^2 u^s &= 0 & \text{in } (\mathbb{R}\times\Sigma)\setminus\overline{D}\,, \\ \mathrm{div}(A\nabla v) + k^2\,\mathrm{n}\,v &= 0 & \text{in } D\,, \\ v - u^s &= F & \text{across } \partial D \cap (\mathbb{R}\times\Sigma)\,, \\ \partial_{\boldsymbol{\nu}_A} v - \partial_{\boldsymbol{\nu}_D} u^s &= f & \text{across } \partial D \cap (\mathbb{R}\times\Sigma)\,, \\ \partial_{\boldsymbol{\nu}} u^s &= 0 & \text{on } (\mathbb{R}\times\partial\Sigma)\setminus\partial D\,, \\ \partial_{\boldsymbol{\nu}_A} v &= 0 & \text{on } (\mathbb{R}\times\partial\Sigma)\cap\partial D\,, \end{cases}$$

where we denote $\partial_{\boldsymbol{\nu}_A} v := (A\nabla v)\cdot\boldsymbol{\nu}_D$ on ∂D. Notice that the anisotropy and inhomogeneity of the obstacle D are represented by the coefficients $A \in \mathcal{C}(\overline{D})^{d\times d}$ and $\mathrm{n} \in \mathcal{C}(\overline{D})$, respectively. We assume that A is a uniformly positive definite and real-valued symmetric matrix function, whereas $\mathrm{Re}(\mathrm{n}) > 0$ and $\mathrm{Im}(\mathrm{n}) \geq 0$ in D. The total field outside D is then given by $u = u^i + u^s$. These equations are closed with a suitable radiation condition for u^s that represents the fact that this field is a superposition of outgoing guided modes (i.e. guided modes propagating away from

the obstacle D or decaying exponentially with respect to the distance from such obstacle D). In order to formalize this radiation condition, we proceed as in [3]:

1. We take
   - any $R > R_S$ and denote $B_R := (-R, R) \times \Sigma$;
   - $k_n \in (0, +\infty)$, where k_n^2 is an eigenvalue of the Neumann problem for the negative Laplacian on Σ, sorted in such way that $k_n \nearrow +\infty$;
   - θ_n an eigenfunction associated to the eigenvalue k_n^2 and with $\{\theta_n\}_{n\in\mathbb{N}}$ an orthonormal basis of $L^2(\Sigma)$.
2. Let us denote by $g_n^{\pm}(x_1, \hat{\mathbf{x}}) := \theta_n(\hat{\mathbf{x}})\, e^{\pm \imath \beta_n x_1}$, where $\beta_n := \sqrt{k^2 - k_n^2} \in \mathbb{C}$ and we choose $\mathrm{Re}(\beta_n) \geq 0$ and $\mathrm{Im}(\beta_n) \geq 0$. More generally, if $\mathrm{Im}(k) > 0$ we follow [2] and choose the branch cut for the square root such that $\arg(\beta_n) \in (0, \pi/2)$ and assume $|\beta_n| > 0$ for all n.
3. Then,
   - the solutions of
   $$\begin{cases} \triangle u + k^2 u = 0 & \text{in } \mathbb{R} \times \Sigma\,, \\ \partial_{\boldsymbol{\nu}} u = 0 & \text{on } \mathbb{R} \times \partial\Sigma\,, \end{cases}$$
   are linear combinations of $\{g_n^{\pm}\}_{n\in\mathbb{N}}$;
   - when k is real there exists $N_p \in \mathbb{N}$ s.t.
   $$k_n \leq k \quad \text{for } n \leq N_p \qquad \text{and} \qquad k_n > k \quad \text{for } n > N_p\,,$$
   so that
   $$\begin{cases} \mathrm{Re}(\beta_n) \geq 0 \\ \mathrm{Im}(\beta_n) = 0 \end{cases} \text{for } n \leq N_p\,, \quad \text{and} \quad \begin{cases} \mathrm{Re}(\beta_n) = 0 \\ \mathrm{Im}(\beta_n) > 0 \end{cases} \text{for } n > N_p\,;$$
   this condition means that g_n^+ (respectively g_n^-) propagates from left to right (respectively from right to left) when $n \leq N_p$, whereas it decays exponentially from left to right (respectively from right to left) if $n > N_p$.
   - the Dirichlet–to–Neumann operator $T^{\pm R} : H^{1/2}(\Sigma_{\pm R}) \to \tilde{H}^{-1/2}(\Sigma_{\pm R})$ is given by
   $$T^{\pm R} g = \pm \imath \sum_{n\in\mathbb{N}} \beta_n \int_{\Sigma_{\pm R}} g\, \overline{\theta}_n \, dS \; \theta_n\,, \tag{1}$$
   where $\Sigma_{\pm R} := \{\pm R\} \times \Sigma$;
   - the radiation condition can be written as
   $$T^{\pm R} u^s = \partial_{\boldsymbol{\nu}_0} u^s \qquad \text{on } \Sigma_{\pm R}\,.$$

Summing up, the direct problem consists of finding $u \equiv u^s \in H^1_{loc}(B_R \setminus \overline{D})$ (we drop the superscript s for simplicity) and $v \in H^1(D)$ such that

$$\begin{cases} \triangle u + k^2\, u &= 0 & \text{in } B_R \setminus \overline{D}\,, \\ \mathrm{div}(A \nabla v) + k^2\, \mathrm{n}\, v &= 0 & \text{in } D\,, \\ v - u &= F & \text{across } \partial D \cap B_R\,, \\ \partial_{\boldsymbol{\nu}_A} v - \partial_{\boldsymbol{\nu}_D} u &= f & \text{across } \partial D \cap B_R\,, \\ \partial_{\boldsymbol{\nu}} u &= 0 & \text{on } ((-R, R) \times \partial\Sigma) \setminus \partial D\,, \\ \partial_{\boldsymbol{\nu}_A} v &= 0 & \text{on } ((-R, R) \times \partial\Sigma) \cap \partial D\,, \\ \partial_{\boldsymbol{\nu}_0} u &= T^{\pm R} u & \text{on } \Sigma_{\pm R}\,, \end{cases} \tag{2}$$

where $F = u^i \in H^{1/2}(\partial D \cap B_R)$ and $f = \partial_{\boldsymbol{\nu}_D} u^i \in \tilde{H}^{-1/2}(\partial D \cap B_R)$.

We have the following characterization of the Dirichlet-to-Neumann map:

Lemma 2.1. *For any $g_1, g_2 \in H^{1/2}(\Sigma_{\pm R})$, the following holds in the sense of the duality product $\tilde{H}^{-1/2}(\Sigma_{\pm R}) \times H^{1/2}(\Sigma_{\pm R})$:*

$$\int_{\Sigma_{\pm R}} T^{\pm R} g_1 \, \overline{g}_2 \, dS = \int_{\Sigma_{\pm R}} T^{\pm R} \overline{g}_2 \, g_1 \, dS = \pm \imath \sum_{n \in \mathbb{N}} \beta_n \int_{\Sigma_{\pm R}} g_1 \overline{\theta}_n dS \int_{\Sigma_{\pm R}} \overline{g}_2 \theta_n dS \, .$$

Proof. Since $\{\theta_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Sigma)$, we can write g_1 and g_2 as

$$g_1 = \sum_{m \in \mathbb{N}} \left(\int_{\Sigma_{\pm R}} g_1 \, \overline{\theta}_m \, dS \right) \theta_m \quad \text{and} \quad g_2 = \sum_{m \in \mathbb{N}} \left(\int_{\Sigma_{\pm R}} g_2 \, \overline{\theta}_m \, dS \right) \theta_m \qquad \text{in } \Sigma_{\pm R} \, .$$

Thus, we conclude recalling the expression (1) of the Dirichlet–to–Neumann operator $T^{\pm R}$ and that $\{\theta_n\}_{n\in\mathbb{N}}$ is an orthonormal set in $L^2(\Sigma)$. □

2.1. **Uniqueness of the solution for the direct problem.** In order to study the uniqueness of solutions to problem (2), let us consider $u \in H^1(B_R \setminus \overline{D})$ and $v \in H^1(D)$ a weak solution of its homogeneous counterpart, that is, when $F = f = 0$ in $\partial D \cap B_R$. Multiplying the first equation of (2) by $\overline{u}$ and integrating by parts in $B_R \setminus \overline{D}$, then multiplying the second equation of (2) by $\overline{v}$ and integrating by parts in D, and adding the results, we obtain

$$\int_{B_R \setminus \overline{D}} \left(-|\nabla u|^2 + k^2 |u|^2 \right) d\mathbf{x} + \int_D \left(-(A \nabla v) \cdot \nabla \overline{v} + k^2 \mathrm{n} |v|^2 \right) d\mathbf{x} \pm \int_{\Sigma_{\pm R}} T^{\pm R} u \, \overline{u} \, dS = 0 \, . \quad (3)$$

Here we have also taken into account the radiation and boundary conditions, as well as the homogeneous transmission conditions from (2).

Let us first assume that k is real. Substituting the expression of Lemma 2.1 when $g_1 := g_2 := u|_{\Sigma_{\pm R}}$ in (3) and taking the imaginary part gives

$$k^2 \int_D \mathrm{Im}(\mathrm{n}) \, |v|^2 \, d\mathbf{x} + \sum_{n \in \mathbb{N}} \mathrm{Re}(\beta_n) \, | \int_{\Sigma_{\pm R}} u|_{\Sigma_{\pm R}} \, \overline{\theta_n} \, dS|^2 = 0 \, ,$$

thanks to our assumption on A being real-valued. Since $\mathrm{Im}(\mathrm{n}) \geq 0$ in D and $\mathrm{Re}(\beta_n) \geq 0$ for all $n \in \mathbb{N}$, we deduce that

$$\int_D \mathrm{Im}(\mathrm{n}) \, |v|^2 \, d\mathbf{x} = 0 \qquad \text{and} \qquad \mathrm{Re}(\beta_n) \, | \int_{\Sigma_{\pm R}} u|_{\Sigma_{\pm R}} \, \overline{\theta_n} \, dS|^2 = 0 \quad \forall n \in \mathbb{N} \, . \quad (4)$$

Therefore, if $\mathrm{Im}(\mathrm{n})$ is strictly positive in D, then $v = 0$ in D and by unique continuation $u = 0$.

Alternatively suppose $\mathrm{Im}(\mathrm{n}) = 0$ in D and $\mathrm{Im}(k) = \sigma > 0$. Then, multiplying (3) by $\overline{k}$ and taking imaginary parts, we obtain

$$\sigma \int_{B_R \setminus \overline{D}} \left(|\nabla u|^2 + |k|^2 |u|^2 \right) d\mathbf{x} + \sigma \int_D \left((A \nabla v) \cdot \nabla \overline{v} + |k|^2 \mathrm{n} |v|^2 \right) d\mathbf{x}$$
$$+ \sum \mathrm{Re}(\overline{k} \beta_n) \, | \int_{\Sigma_{\pm R}} u|_{\Sigma_{\pm R}} \, \overline{\theta_n} \, dS|^2 = 0 \, .$$

But since $\arg(\beta_n) \in (0, \pi/2)$ we have that $\mathrm{Re}(\overline{k}\beta_n) \geq 0$, so $u = 0$ and $v = 0$ verifying uniqueness in this case, and we have proved the following result.

Proposition 1. *If $\mathrm{Im}(\mathrm{n})$ is strictly positive in D and k is real, or if $\mathrm{Im}(\mathrm{n}) = 0$ in D and $\mathrm{Im}(k) > 0$, then the forward problem (2) admits at most one solution.*

Remark 1. The first hypothesis of this proposition can be weakened: All that is needed is that $\mathrm{Im}(\mathrm{n}) > 0$ on a sufficiently regular domain in D such that we can appeal to the unique continuation principle to ensure that the solution vanishes in D. The same holds for later results, but for simplicity we do not give the more general case.

2.2. **Existence of a solution for the forward problem.** In order to analyze the existence of a solution to the forward problem (2), we first write down a weak formulation. To do so, let us consider a test function $w \in H^1(B_R)$ and proceed as usual along the lines of our derivation of (3) to derive the following weak formulation for (2): we seek $u \in H^1(B_R \setminus \overline{D})$ and $v \in H^1(D)$ that satisfy

$$\begin{aligned} \int_{B_R\setminus\overline{D}} \left(-\nabla u\cdot\nabla\overline{w} + k^2\, u\,\overline{w}\right) d\mathbf{x} + \int_D \left(-(A\nabla v)\cdot\nabla\overline{w} + k^2\,\mathrm{n}\,v\,\overline{w}\right) d\mathbf{x} \\ \pm\int_{\Sigma_{\pm R}} T^{\pm R}u\,\overline{w}\,dS = -\int_{\partial D\cap B_R} f\,\overline{w}\,dS\,, \end{aligned} \tag{5}$$

for all $w \in H^1(B_R)$ together with the continuity condition $v = u + F$ on $\partial D \cap B_R$.

Since we assume that the boundary data F of problem of (2) belongs to $H^{1/2}(\partial D\cap B_R)$, there exists an $\tilde{F} \in H^{1/2}(\partial D)$ such that $F = \tilde{F}$ on $\partial D \cap B_R$. Now let $U_{\tilde{F}} \in H^1(D)$ satisfy $\triangle U_{\tilde{F}} - U_{\tilde{F}} = 0$ in D together with $U_{\tilde{F}} = \tilde{F}$ on ∂D and define

$$U_0 := \begin{cases} u & \text{in } B_R\setminus\overline{D}\,, \\ v - U_{\tilde{F}} & \text{in } D\,. \end{cases}$$

Note that U_0 is continuous across $\partial D \cap B_R$ so $U_0 \in H^1(B_R)$ (for the scattering problem where $F = u^i$ we could alternatively simply use u^i as the extension of F, but the construction given here allows data F not arising from an incident field). Define also

$$A_0 = \begin{cases} I & \text{in } B_R\setminus\overline{D}\,, \\ A & \text{in } D\,, \end{cases} \qquad \text{and} \qquad N = \begin{cases} 1 & \text{in } B_R\setminus\overline{D}\,, \\ \mathrm{n} & \text{in } D\,. \end{cases}$$

Using these definitions in (5) we obtain the problem of finding $U_0 \in H^1(B_R)$ such that

$$\begin{aligned} \int_{B_R} \left(-(A_0\nabla U_0)\cdot\nabla\overline{w} + k^2\,N\,U_0\,\overline{w}\right) d\mathbf{x} \pm \int_{\Sigma_{\pm R}} T^{\pm R}U_0\,\overline{w}\,dS \\ = -\int_D \left(-(A\nabla U_{\tilde{F}})\cdot\nabla\overline{w} + k^2\,n\,U_{\tilde{F}}\,\overline{w}\right) d\mathbf{x} - \int_{\partial D\cap B_R} f\,\overline{w}\,dS\,, \end{aligned}$$

for any $w \in H^1(B_R)$, The above formulation suggests introducing the sesquilinear continuous form $a : H^1(B_R)\times H^1(B_R) \to \mathbb{C}$ and the conjugate linear continuous form $l : H^1(B_R) \to \mathbb{C}$ given by

$$a(w_1, w_2) := \int_{B_R} \left(-(A_0\nabla w_1)\cdot\nabla\overline{w}_2 + k^2\,N\,w_1\,\overline{w}_2\right) d\mathbf{x} \pm \int_{\Sigma_{\pm R}} T^{\pm R}w_1\,\overline{w}_2\,dS$$

and

$$l(w) := -\int_{\partial D\cap B_R} f\,\overline{w}\,dS + \int_D \left((A\nabla U_{\tilde{F}})\cdot\nabla\overline{w} - k^2\,n\,U_{\tilde{F}}\,\overline{w}\right) d\mathbf{x}.$$

Then problem (2) is equivalent to finding $U_0 \in H^1(B_R)$ such that

$$a(U_0, w) = l(w) \qquad \forall w \in H^1(B_R)\,. \tag{6}$$

Indeed, $u := U_0|_{B_R\setminus\overline{D}}$ and $v := (U_0 + U_{\tilde{F}})|_D$ is a weak solution of problem (2) if and only if U_0 solves (6).

We will now use the analytic Fredholm theory to conclude the existence of a solution to the forward problem except for at most a discrete set of wave numbers k. A straight forward application of this theory is thwarted by the fact that the Dirichlet-to-Neumann operators $T_R^{\pm}$ are defined in terms of $\beta_n = \sqrt{k^2 - k_n^2}$. A similar problem arises in considering waveguides between two flat planes and in [2] it is shown how to circumvent this. The Dirichlet-to-Neumann problem in our case is simpler than the one studied in [2] which involves Hankel functions, but the eigenvalues for the cross-section are no longer explicit. We provide some details in the appendix where we prove the following result.

Proposition 2. *Let us consider a real wave number k. If $\mathrm{Im(n)} > 0$ in D, then the forward problem (2) has a unique solution and is well posed. If $\mathrm{Im(n)} = 0$ in D, then (2) is well posed except for, at most, a discrete set of k-values.*

In the sequel, we assume that $k \in \mathbb{R}$, $k > 0$, is chosen such that the direct problem (2) is well posed. Recall that we have also supposed that $k \neq k_n$ for every $n \in \mathbb{N}$.

3. **The interior transmission problem.** To deal with the inverse problem for a penetrable obstacle, we need to know how the following interior transmission problem behaves: Find $v_1, v_2 \in H^1(D)$ such that

$$\begin{cases} \triangle v_1 + k^2\, v_1 &= 0 & \text{in } D\,, \\ \mathrm{div}(A\nabla v_2) + k^2\, \mathrm{n}\, v_2 &= 0 & \text{in } D\,, \\ v_1 - v_2 &= G & \text{across } \partial D \cap B_R\,, \\ \partial_{\boldsymbol{\nu}_D} v_1 - \partial_{\boldsymbol{\nu}_A} v_2 &= g & \text{across } \partial D \cap B_R\,, \\ \partial_{\boldsymbol{\nu}} v_1 \,=\, \partial_{\boldsymbol{\nu}_A} v_2 &= 0 & \text{on } ((-R,R)\times\partial\Sigma)\cap\partial D\,. \end{cases} \tag{7}$$

Although this problem is new, due to the mixed boundary and transmission condition, it can be studied by reasoning as in [4, Sec. 6.2] and [6]. More precisely, we can first show that, whenever $\mathrm{Im(n)} > 0$, the solution of this problem is unique.

Proposition 3. *When $\mathrm{Im(n)} > 0$ in D, the interior transmission problem (7) admits at most one solution.*

Proof. Suppose $v_1, v_2 \in H^1(D)$ is a solution of the homogeneous counterpart of (7), that is, for $G = g = 0$ on $\partial D \cap B_R$. Mulitplying the first equation in (7) by $\overline{v}_1$ and the second by $\overline{v}_2$, then integrating by parts and subtracting the results we obtain

$$\int_D \left(|\nabla v_1|^2 - k^2\,|v_1|^2\right)\, d\mathbf{x} = \int_D \left((A\nabla v_2)\cdot\nabla\overline{v}_2 - k^2\,\mathrm{n}\,|v_2|^2\right)\, d\mathbf{x},$$

where we have also used the transmission and boundary conditions,

In particular, taking the imaginary part and using the assumptions on A,

$$\int_D \mathrm{Im(n)}\,|v_2|^2\, d\mathbf{x} = 0\,.$$

Thus, if $\mathrm{Im(n)} > 0$ in D, then $v_2 = 0$ in D. In that case, we can rewrite (7) as

$$\begin{cases} \triangle v_1 + k^2\, v_1 &= 0 & \text{in } D\,, \\ v_1 &= 0 & \text{on } \partial D \cap B_R\,, \\ \partial_{\boldsymbol{\nu}_D} v_1 &= 0 & \text{on } \partial D\,. \end{cases}$$

Notice that

$$V_1 := \begin{cases} v_1 & \text{in } D\,, \\ 0 & \text{in } B_R \setminus \overline{D}\,, \end{cases}$$

belongs to $H^1(B_R)$, solves Helmholtz equation $\triangle V_1 + k^2\, V_1 = 0$ in B_R and vanishes in $B_R \setminus \overline{D}$. The unique continuation principle guarantees that $V_1 = 0$ in B_R and, in particular, $v_1 = 0$ in D. □

We now analyze the existence of a solution of the interior transmission problem (7) by means of the following modified version: Find $w_1, w_2 \in H^1(D)$ such that

$$\begin{cases} \triangle w_1 - w_1 &= f_1 & \text{in } D\,, \\ \operatorname{div}(A\nabla w_2) - \mathrm{n}\, w_2 &= f_2 & \text{in } D\,, \\ w_1 - w_2 &= G & \text{across } \partial D \cap B_R\,, \\ \partial_{\boldsymbol{\nu}_D} w_1 - \partial_{\boldsymbol{\nu}_A} w_2 &= g & \text{across } \partial D \cap B_R\,, \\ \partial_{\boldsymbol{\nu}} w_1 \,=\, \partial_{\boldsymbol{\nu}_A} w_2 &= 0 & \text{on } ((-R,R)\times\partial\Sigma)\cap\partial D. \end{cases} \tag{8}$$

In order to give a variational formulation of this problem, we define

- the function space

$$\mathbf{W}(D) := \big\{ \mathbf{w} \in L^2(D)^d; \quad \operatorname{div}\mathbf{w} \in L^2(D),\ \mathbf{curl}(A^{-1}\mathbf{w}) = \mathbf{0} \text{ in } D, \\ \mathbf{w}\cdot\boldsymbol{\nu}_D = 0 \text{ on } ((-R,R)\times\partial\Sigma)\cap\partial D\big\}\,,$$

that is a Hilbert space with its natural norm

$$||\mathbf{w}||^2_{\mathbf{W}(D)} := ||\mathbf{w}||^2_{(L^2(D))^d} + ||\operatorname{div}\mathbf{w}||^2_{L^2(D)} + ||\mathbf{curl}(A^{-1}\mathbf{w})||^2_{(L^2(D))^d};$$

- the sesquilinear continuous form $\tilde{a}:\ (H^1(D)\times\mathbf{W}(D))\times(H^1(D)\times\mathbf{W}(D)) \to \mathbb{C}$ given by

$$\begin{aligned} \tilde{a}((w,\mathbf{w}),(\xi,\boldsymbol{\eta})) &:= \int_D \big(\nabla w\cdot\nabla\overline{\xi} + w\,\overline{\xi}\big)\,d\mathbf{x} \\ &+ \int_D \left((A^{-1}\mathbf{w})\cdot\overline{\boldsymbol{\eta}} + \frac{1}{n}\operatorname{div}\mathbf{w}\,\operatorname{div}\overline{\boldsymbol{\eta}}\right) d\mathbf{x} \\ &- \int_{\partial D} \big(w\,\overline{\boldsymbol{\eta}}\cdot\boldsymbol{\nu}_D + \overline{\xi}\,\mathbf{w}\cdot\boldsymbol{\nu}_D\big)\,dS\,; \end{aligned} \tag{9}$$

- the conjugate linear continuous form $\tilde{l}:\ H^1(D)\times\mathbf{W}(D)\to\mathbb{C}$ defined by

$$\tilde{l}(\xi,\boldsymbol{\eta}) := \int_D \left(\frac{1}{n} f_2 \operatorname{div}\overline{\boldsymbol{\eta}} - f_1\,\overline{\xi}\right) d\mathbf{x} + \int_{B_R\cap\partial D} \big(-G\,\overline{\boldsymbol{\eta}}\cdot\boldsymbol{\nu}_D + g\,\overline{\xi}\big)\,dS\,.$$

Now, we consider the following weak formulation of (8): Find $w \in H^1(D)$ and $\mathbf{w} \in \mathbf{W}(D)$ such that

$$\tilde{a}((w,\mathbf{w}),(\xi,\boldsymbol{\eta}) = \tilde{l}(\xi,\boldsymbol{\eta}) \qquad \forall \xi \in H^1(D),\ \boldsymbol{\eta} \in \mathbf{W}(D)\,. \tag{10}$$

Proposition 4. *Suppose D is simply connected. If a pair of functions $w_1, w_2 \in H^1(D)$ solves (8) and if we set $w := w_1 \in H^1(D)$ and $\mathbf{w} := A\nabla w_2 \in \mathbf{W}(D)$ we have that $(w,\mathbf{w})$ satisfies (10). If $(w,\mathbf{w})$ satisfies (10), then we may choose $(w_1,w_2) \in H^1(D)^2$ such that $w = w_1$ and $\mathbf{w} = A\nabla w_2$ and such that (w_1,w_2) satisfies (8).*

Proof. First, let us assume that $w_1, w_2 \in H^1(D)$ solves (8) and show that if $w := w_1$ and $\mathbf{w} := A\nabla w_2$ then w and $\mathbf{w}$ satisfy (10).
To start with, notice that $\mathbf{w} := A\nabla w_2 \in \mathbf{W}(D)$:

$$\begin{aligned}
&\mathrm{div}\mathbf{w} = \mathrm{div}(A\nabla w_2) = \mathrm{n}\, w_2 + f_2 \in L^2(D)\,,\\
&\mathbf{curl}(A^{-1}\mathbf{w}) = \mathbf{curl}(\nabla w_2) = \mathbf{0} \quad \text{in } D\,,\\
&\mathbf{w}\cdot\boldsymbol{\nu}_D = \partial_{\boldsymbol{\nu}_A} w_2 = 0 \quad \text{on } ((-R,R)\times\partial\Sigma)\cap\partial D\,.
\end{aligned}$$

In addition, for any $\xi \in H^1(D)$ and $\boldsymbol{\eta} \in \mathbf{W}(D)$, by the definition of $\tilde{a}$,

$$\begin{aligned}
\tilde{a}((w_1, A\nabla w_2),(\xi,\boldsymbol{\eta})) = &\int_D \left(\nabla w_1\cdot\nabla\overline{\xi} + w_1\,\overline{\xi}\right) d\mathbf{x}\\
&+\int_D \left(\nabla w_2\cdot\overline{\boldsymbol{\eta}} + \frac{1}{n}\mathrm{div}(A\nabla w_2)\,\mathrm{div}\overline{\boldsymbol{\eta}}\right) d\mathbf{x}\\
&-\int_{\partial D}\left(w_1\,\overline{\boldsymbol{\eta}}\cdot\boldsymbol{\nu}_D + \overline{\xi}\,\partial_{\boldsymbol{\nu}_A} w_2\right) dS\,;
\end{aligned}$$

so that, integrating by parts in D and using the equations of (8) as well as the definition of $\tilde{l}$, we see that indeed

$$\tilde{a}((w_1, A\nabla w_2),(\xi,\boldsymbol{\eta})) = \tilde{l}(\xi,\boldsymbol{\eta}) \qquad \text{for all } (\xi,\boldsymbol{\eta}) \in H^1(D)\times\mathbf{W}(D).$$

Now suppose $(w,\mathbf{w}) \in H^1(D)\times\mathbf{W}(D)$ solves (10). By definition, any $\mathbf{w}\in\mathbf{W}(D)$ is such that that $A^{-1}\mathbf{w}$ is curl free in D, so that, when D is simply connected,

$$\mathbf{w} = A\nabla w_2 \qquad \text{for some } w_2 \in H^1(D)\,.$$

Therefore, setting $w_1 = w$ we see that there exist $w_1, w_2 \in H^1(D)$ such that $(w_1, A\nabla w_2) \in H^1(D)\times\mathbf{W}(D)$ solves (10). We now show that there is a constant $C\in\mathbb{C}$ such that $w_1, w_2 + C \in H^1(D)$ satisfies (8).

1. Since $A\nabla w_2 \in \mathbf{W}(D)$, the following boundary condition holds:
$$\partial_{\boldsymbol{\nu}_A} w_2 = (A\nabla w_2)\cdot\boldsymbol{\nu}_D = 0 \quad \text{on } ((-R,R)\times\partial\Sigma)\cap\partial D\,.$$
2. For any $\xi\in H^1(D)$, we can test (10) with $(\xi,\mathbf{0})\in H^1(D)\times\mathbf{W}(D)$:
$$\begin{aligned}
\int_D \left(\nabla w_1\cdot\nabla\overline{\xi} + w_1\,\overline{\xi}\right) d\mathbf{x} - \int_{\partial D}\overline{\xi}\,\partial_{\boldsymbol{\nu}_A} w_2\, dS =\\
= -\int_D f_1\,\overline{\xi}\, d\mathbf{x} + \int_{B_R\cap\partial D} g\,\overline{\xi}\, dS.
\end{aligned}$$
In particular, this equation holds for any $\xi\in\mathcal{C}_0^\infty(D)\subset H^1(D)$, so that in distributional sense
$$\triangle w_1 - w_1 = f_1 \qquad \text{in } D.$$
Now, considering again $\xi\in H^1(D)$ and integrating by parts, we deduce that
$$\begin{aligned}
&\partial_{\boldsymbol{\nu}_D} w_1 - \partial_{\boldsymbol{\nu}_A} w_2 = g \qquad &&\text{on } B_R\cap\partial D\,,\\
&\partial_{\boldsymbol{\nu}_D} w_1 = \partial_{\boldsymbol{\nu}_A} w_2 = 0 \qquad &&\text{on } ((-R,R)\times\partial\Sigma)\cap\partial D\,.
\end{aligned}$$
3. For any $\boldsymbol{\eta}\in\mathbf{W}(D)$, we may test (10) with $(0,\boldsymbol{\eta})\in H^1(D)\times\mathbf{W}(D)$:
$$\begin{aligned}
\int_D \left(\nabla w_2\cdot\overline{\boldsymbol{\eta}} + \frac{1}{n}\mathrm{div}(A\nabla w_2)\,\mathrm{div}\overline{\boldsymbol{\eta}}\right) d\mathbf{x} - \int_{\partial D} w_1\,\overline{\boldsymbol{\eta}}\cdot\boldsymbol{\nu}_D\, dS =\\
= \int_D \frac{1}{n} f_2\,\mathrm{div}\overline{\boldsymbol{\eta}}\, d\mathbf{x} - \int_{B_R\cap\partial D} G\,\overline{\boldsymbol{\eta}}\cdot\boldsymbol{\nu}_D\, dS\,.
\end{aligned} \tag{11}$$

- First, for any $\phi \in L^2_0(D)$ we can take $\Phi \in H^1(D)$ a weak solution (unique up to an additive constant) of

$$\begin{cases} \operatorname{div}(A\nabla\Phi) = \phi & \text{in } D\,, \\ \partial_{\boldsymbol{\nu}_A}\Phi = 0 & \text{on } \partial D\,; \end{cases}$$

notice that $\boldsymbol{\eta} := A\nabla\Phi \in \mathbf{W}(D)$ and thus, substituting in (11),

$$\int_D \left(\nabla w_2 \cdot (A\nabla\overline{\Phi}) + \frac{1}{n}\operatorname{div}(A\nabla w_2)\,\overline{\phi}\right) d\mathbf{x} = \int_D \frac{1}{n} f_2\,\overline{\phi}\, d\mathbf{x}.$$

Integrating by parts,

$$\int_D \left(-w_2 + \frac{1}{n}\operatorname{div}(A\nabla w_2)\right)\overline{\phi}\, d\mathbf{x} = \int_D \frac{1}{n} f_2\,\overline{\phi}\, d\mathbf{x}\,.$$

Since this holds for any $\phi \in L^2_0(D)$, we deduce that there exists a constant $C_1 \in \mathbb{C}$ such that

$$\operatorname{div}(A\nabla w_2) - \mathrm{n}\, w_2 = f_2 + nC_1 \qquad \text{in } D\,.$$

- Next, for any $\psi \in L^2_0(\partial D)$ such that $\psi = 0$ on $((-R,R)\times\partial\Sigma)\cap\partial D$, we select a solution (here again, unique up to an additive constant) of

$$\begin{cases} \operatorname{div}(A\nabla\Psi) = 0 & \text{in } D\,, \\ \partial_{\boldsymbol{\nu}_A}\Psi = \psi & \text{on } \partial D\,. \end{cases}$$

Notice that $\boldsymbol{\eta} := A\nabla\Psi \in \mathbf{W}(D)$ and hence, substituting in (11) and integrating by parts,

$$\int_{\partial D} (w_1 - w_2)\,\overline{\psi}\, dS = \int_{B_R\cap\partial D} G\,\overline{\psi}\, dS\,;$$

since this holds for any $\psi \in L^2_0(\partial D)$ with $\psi = 0$ on $((-R,R)\times\partial\Sigma)\cap\partial D$, there is a constant $C_2 \in \mathbb{C}$ such that

$$w_1 - w_2 = G + C_2 \qquad \text{on } B_R\cap\partial D\,.$$

- Finally, let us consider any $\boldsymbol{\eta} \in \mathbf{W}(D)$ and integrate by parts in (11) as follows:

$$\int_D \left(-w_2\operatorname{div}\overline{\boldsymbol{\eta}} + \frac{1}{n}\operatorname{div}(A\nabla w_2)\operatorname{div}\overline{\boldsymbol{\eta}}\right) d\mathbf{x} - \int_{\partial D} (w_1 - w_2)\,\overline{\boldsymbol{\eta}}\cdot\boldsymbol{\nu}_D\, dS = \\ = \int_D \frac{1}{n} f_2 \operatorname{div}\overline{\boldsymbol{\eta}}\, d\mathbf{x} - \int_{B_R\cap\partial D} G\,\overline{\boldsymbol{\eta}}\cdot\boldsymbol{\nu}_D\, dS\,.$$

Taking into account the previous results, we can rewrite this as

$$C_1 \int_D \operatorname{div}\overline{\boldsymbol{\eta}}\, d\mathbf{x} + C_2 \int_{B_R\cap\partial D} \overline{\boldsymbol{\eta}}\cdot\boldsymbol{\nu}_D\, dS = 0\,.$$

Integrating by parts here again, we obtain

$$(C_1 + C_2) \int_{B_R\cap\partial D} \overline{\boldsymbol{\eta}}\cdot\boldsymbol{\nu}_D\, dS = 0\,,$$

for any $\boldsymbol{\eta} \in \mathbf{W}(D)$, so that $C_1 + C_2 = 0$.

Thus, $w_1, (w_2 - C_1) \in H^1(D)$ solve (8). □

Hence, the following result is a consequence of the previous proposition.

Proposition 5. *If D is simply connected, then the modified interior transmission problem (8) has a unique solution if and only if (10) does.*

Proof. First, let us assume that there is a unique solution to problem (8), that we denote $w_1, w_2 \in H^1(D)$.

- On the one hand, by Proposition 4, we know that $w_1 \in H^1(D)$ and $\mathbf{w} := A\nabla w_2 \in \mathbf{W}(D)$ solve problem (10).
- On the other hand, consider $w \in H^1(D)$ and $\mathbf{w} \in \mathbf{W}(D)$ a solution to problem (10). As we remarked before, when D is simply connected, there is some $w_2' \in H^1(D)$ such that $\mathbf{w} = A\nabla w_2'$. In that case, we have shown in Proposition 4 that $w_1, w_2' + C_1 \in H^1(D)$ satisfy problem (8) for some $C_1 \in \mathbb{C}$. Then, the hypothesis of uniqueness of solution to (8) leads to

$$w_1 = w \quad \text{and} \quad w_2 = w_2' + C_1 \qquad \text{in } D\,,$$

and hence $w = w_1$ and $\mathbf{w} = A\nabla w_2' = A\nabla w_2$.

Now, let us suppose that problem (10) has a unique solution given by $w \in H^1(D)$ and $\mathbf{w} \in \mathbf{W}(D)$.

- Since D is simply connected, there is some $w_2' \in H^1(D)$ such that $\mathbf{w} = A\nabla w_2'$ and then, by Proposition 4, there also exists some $C_1 \in \mathbb{C}$ such that $w_1 := w, w_2 := w_2' - C_1 \in H^1(D)$ solve problem (8).
- Besides, if $w_1, w_2 \in H^1(D)$ satisfy (8) then $w' := w_1 \in H^1(D)$ and $\mathbf{w}' := A\nabla w_2 \in \mathbf{W}(D)$ solve (10); and, by uniqueness of solution for (10),

$$w_1 = w \quad \text{and} \quad A\nabla w_2 = \mathbf{w} \qquad \text{in } D\,.$$

In particular, $w_2 - w_2' \in H^1(D)$ solves

$$\begin{cases} \operatorname{div}(A\nabla(w_2 - w_2')) - n(w_2 - w_2') = 0 & \text{in } D\,, \\ w_2 - w_2' = 0 & \text{on } B_R \cap \partial D\,, \\ \partial_{\boldsymbol{\nu}_A}(w_2 - w_2') = 0 & \text{on } \partial D\,; \end{cases}$$

and, thus, $w_2 - w_2' = 0$ in D.

□

Now we verify that, under suitable restrictions on A and n, the sesquilinear form $\tilde{a}$ is coercive. The assumptions on the coefficients are appropriate for our later application. In particular let $\lambda_{\max}(A)$ denote the largest eigenvalue of A in D, and

$$\frac{1}{n^*} = \min_{\mathbf{x}\in D} \operatorname{Re} \frac{1}{\mathrm{n}(\mathbf{x})}\,.$$

Lemma 3.1. *Assuming there are some constants $a_0 < 1$ and $n_0 < 1$ such that $\lambda_{\max}(A) < a_0$ and $n^* < n_0$ in D, then $\tilde{a}$ is coercive.*

Proof. Setting $\xi = w$ and $\boldsymbol{\eta} = \mathbf{w}$ in (9) we have

$$\begin{aligned} \operatorname{Re}(\tilde{a}((w,\mathbf{w}),(w,\mathbf{w})) &:= \int_D \left(|\nabla w|^2 + |w|^2\right) d\mathbf{x} \\ &+ \int_D \left((A^{-1}\mathbf{w}) \cdot \overline{\mathbf{w}} + \operatorname{Re}\left(\frac{1}{\mathrm{n}}\right) |\operatorname{div}\mathbf{w}|^2 \right) d\mathbf{x} \\ &- \operatorname{Re} \int_{\partial D} \left(w\,\overline{\mathbf{w}} \cdot \boldsymbol{\nu}_D + \overline{w}\,\mathbf{w} \cdot \boldsymbol{\nu}_D\right) dS\,; \end{aligned}$$

Using the definition of $\lambda_{\max}(A)$ and n^* we have

$$\begin{aligned}\mathrm{Re}(\tilde{a}((w,\mathbf{w}),(w,\mathbf{w})) \geq & \int_D \left(|\nabla w|^2 + |w|^2\right) d\mathbf{x} \\ & + \int_D \left(\frac{1}{\lambda_{\max}(A)}|\mathbf{w}|^2 + \frac{1}{n^*}|\mathrm{div}\mathbf{w}|^2\right) d\mathbf{x} \\ & -\mathrm{Re}\int_{\partial D} \left(w\,\overline{\mathbf{w}}\cdot\boldsymbol{\nu}_D + \overline{w}\,\mathbf{w}\cdot\boldsymbol{\nu}_D\right) dS\,.\end{aligned}$$

Recall that the divergence theorem shows that, for any $\delta_1 > 0$ and $\delta_2 > 0$,

$$\begin{aligned}\left|\int_{\partial D} w\,\overline{\mathbf{w}}\cdot\boldsymbol{\nu}_D\, dS\right| = \left|\int_D \mathrm{div}(w\overline{\mathbf{w}})\, dV\right| \leq \\ \leq \frac{\delta_1}{2}\|w\|^2_{L^2(D)} + \frac{1}{2\delta_1}\|\mathrm{div}\mathbf{w}\|^2_{L^2(D)} + \frac{\delta_2}{2}\|\nabla w\|^2_{(L^2(D))^d} + \frac{1}{2\delta_2}\|\mathbf{w}\|^2_{(L^2(D))^d}.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Re}(\tilde{a}((w,\mathbf{w}),(w,\mathbf{w})) \geq & \int_D \left(|\nabla w|^2 + |w|^2\right) d\mathbf{x} \\ & + \int_D \left(\frac{1}{\lambda_{\max}(A)}|\mathbf{w}|^2 + \frac{1}{n^*}|\mathrm{div}\mathbf{w}|^2\right) d\mathbf{x} \\ & - \left(\delta_1\|w\|^2_{L^2(D)} + \frac{1}{\delta_1}\|\mathrm{div}\mathbf{w}\|^2_{L^2(D)} + \delta_2\|\nabla w\|^2_{(L^2(D))^d} + \frac{1}{\delta_2}\|\mathbf{w}\|^2_{(L^2(D))^d}\right) \\ = (1-\delta_2)\|\nabla w\|^2_{(L^2(D))^d} + (1-\delta_1)\|w\|^2_{L^2(D)} & + \left(\frac{1}{\lambda_{\max}(A)} - \frac{1}{\delta_2}\right)\|\mathbf{w}\|^2_{(L^2(D))^d} \\ & + \left(\frac{1}{n^*} - \frac{1}{\delta_1}\right)\|\mathrm{div}\mathbf{w}\|^2_{L^2(D)}.\end{aligned}$$

Given our assumptions we can choose for example $\delta_1 = (1+n_0)/2 < 1$ and $\delta_2 = (a_0+1)/2 < 1$, and conclude that

$$\left(\frac{1}{\lambda_{\max}(A)} - \frac{1}{\delta_2}\right) > \frac{1-a_0}{a_0(1+a_0)} > 0 \quad \text{and} \quad \left(\frac{1}{n^*} - \frac{1}{\delta_1}\right) > \frac{1-n_0}{n_0(1+n_0)} > 0.$$

This verifies coercivity. □

Since the sesquilinear continuous form $\tilde{a}$ is $(H^1(D)\times\mathbf{W}(D))$–elliptic, the Lax Milgram Lemma guarantees that problem (10) is well posed. By Proposition 5, we deduce that problem (8) also has a unique solution; in addition, taking into account the form of the solution with respect to that of (10) and applying Poincaré's inequality, we deduce that the solution to (8) depends continuously on the data of the problem (see [4, Th. 6.7] for similar reasoning); in other words, problem (8) is well posed.

Proposition 6. *Suppose A and* n *satisfy the conditions of Lemma 3.1. If D is simply connected and we consider data $f_1, f_2 \in L^2(D)$, $G \in H^{1/2}(\partial D \cap B_R)$ and $g \in \tilde{H}^{-1/2}(\partial D \cap B_R)$, then problem (8) is well posed.*

Let us consider

$$H^1_{\triangle}(D) := \left\{w \in H^1(D);\ \triangle w \in L^2(D)\right\}$$

and

$$H^1_A(D) := \left\{w \in H^1(D);\ \mathrm{div}(A\nabla w) \in L^2(D)\right\},$$

that are Hilbert spaces with the natural norms

$$||w||_{H^1_\triangle(D)} := \left(||w||^2_{H^1(D)} + ||\triangle w||^2_{L^2(D)}\right)^{1/2}$$

and

$$||w||_{H^1_A(D)} := \left(||w||^2_{H^1(D)} + ||\mathrm{div}(A\nabla w)||^2_{L^2(D)}\right)^{1/2},$$

respectively. In order to study the behaviour of problem (7), we consider the function space

$$\boldsymbol{\chi}(D) := \Big\{(w_1, w_2) \in H^1_\triangle(D) \times H^1_A(D)\,; \\ \partial_{\boldsymbol{\nu}_D} w_1 = \partial_{\boldsymbol{\nu}_A} w_2 = 0 \quad \text{on } ((-R,R)\times\partial\Sigma)\cap\partial D\Big\},$$

that is a closed subspace of $H^1_\triangle(D) \times H^1_A(D)$.

Let $\mathcal{G} : \boldsymbol{\chi}(D) \to L^2(D) \times L^2(D) \times H^{1/2}(B_R \cap \partial D) \times \tilde{H}^{-1/2}(B_R \cap \partial D)$ be defined by

$$\mathcal{G}(w_1, w_2) := (\triangle w_1 - w_1, \mathrm{div}(A\nabla w_2) - \mathrm{n} w_2, (w_1 - w_2)|_{B_R\cap\partial D}, \\ (\partial_{\boldsymbol{\nu}_D} w_1 - \partial_{\boldsymbol{\nu}_A} w_2))|_{B_R\cap\partial D}).$$

This linear form is well defined and bounded. Moreover, under the conditions of Lemma 3.1, if D is simply connected, then $\mathcal{G}$ is invertible and its inverse $\mathcal{G}^{-1}$ is linear and continuous (see the previous Proposition 6).

The operator $\mathcal{T} : \boldsymbol{\chi}(D) \to L^2(D) \times L^2(D) \times H^{1/2}(B_R \cap \partial D) \times \tilde{H}^{-1/2}(B_R \cap \partial D)$ defined by

$$\mathcal{T}(w_1, w_2) := ((k^2+1)\, w_1, (k^2+1)\, \mathrm{n}\, w_2, 0, 0),$$

is linear, continuous and also compact.

Recalling that $\mathcal{G} + \mathcal{T}$ is the operator associated to problem (7), the Fredholm alternative guarantees that this problem is well posed if it admits at most one solution; thus, by Proposition 3 we deduce the following result.

Proposition 7. *Suppose D is simply connected, the conditions on A and* n *in Lemma 3.1 are satisfied, and* $\mathrm{Im}(\mathrm{n}) > 0$ *in D. Then the interior transmission problem (7) is well posed for data $G \in H^{1/2}(B_R \cap \partial D)$ and $g \in \tilde{H}^{-1/2}(B_R \cap \partial D)$.*

This result does not give information about problem (7) when Im(n) vanishes in D. To study that case, we again take the linear operators $\mathcal{G}$ and $\mathcal{T}$ defined above. Again both operators are bounded and, in addition, $\mathcal{T}$ is compact. Also recall that $\mathcal{G}$ is the operator associated to (8) and, when $\mathrm{Re}(\mathrm{n}) > 0$ in D, we know that $\tilde{a}$ is $(H^1(D) \times \mathbf{W}(D))$–elliptic; so that $\mathcal{G}$ is invertible with continuous inverse. Since $\mathcal{G}+(1+k^2)\mathcal{T}$ is the operator associated to problem (7), we deduce that this problem is well posed except for at most a discrete set of k values (cf. [4, Th. 1.22] and apply it to the operator $\mathcal{I} + (1+k^2)\mathcal{G}^{-1}\mathcal{T}$, where $\mathcal{I}$ is the identity operator in $\boldsymbol{\chi}(D)$).

Proposition 8. *Suppose the conditions on A and* n *in Lemma 3.1 are satisfied. If D is simply connected and* $\mathrm{Im}(\mathrm{n}) = 0$ *in D, problem (7) is well posed for any k except for, at most, a discrete set of k values.*

In the above proposition, if such k values exist, they are known as *transmission eigenvalues* of the problem (2). Existence of these eigenvalues has yet to be verified.

4. **The inverse obstacle problem.** We consider two surfaces $\Sigma^S := \Sigma_{a_S}$ and $\Sigma^M := \Sigma_{a_M}$ where we place sources and take measurements, respectively. We assume that

$$-R < a_S < a_M < R \qquad \text{and} \qquad D \subset (a_M, R) \times \overline{\Sigma}\,, \tag{12}$$

as in the Figure 2 below. In the sequel, we denote by $B_R^S := (a_S, R) \times \Sigma$ and $B_R^M := (a_M, R) \times \Sigma$, so that the above assumption (12) means that $D \subset B_R^M \subset B_R^S \subset B_R$.

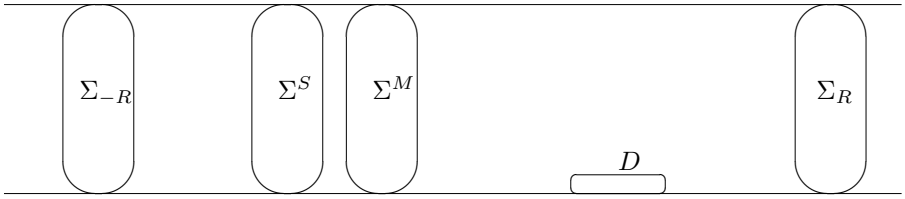


FIGURE 2. A schematic of the geometry for the inverse problem: The penetrable obstacle occupies an unknown region D that may touch or even lay on the boundary of the waveguide $\mathbb{R} \times \Sigma$. Measurements are made on Σ^M using sources on Σ^S.

For each source point $\mathbf{x}_0 \in \Sigma^S$, the incident field $u^i_{\mathbf{x}_0}$ associated to this source is given by

$$u^i_{\mathbf{x}_0}(\mathbf{x}) := \Phi(\mathbf{x}_0, \mathbf{x}) \qquad \text{for } \mathbf{x}_0 \in \Sigma^S, \mathbf{x} \in \mathbb{R} \times \Sigma \quad \text{with } \mathbf{x} \neq \mathbf{x}_0\,,$$

where we denote by Φ the *fundamental solution of the waveguide* (see [3]):

$$\Phi(\mathbf{x}, \mathbf{y}) := \Phi_{\mathbf{x}}(\mathbf{y}) := -\sum_{n \in \mathbb{N}} \frac{e^{\imath \beta_n |x_1 - y_1|}}{2\imath\beta_n}\, \theta_n(\hat{\mathbf{x}})\, \theta_n(\hat{\mathbf{y}})$$

for $\mathbf{x} = (x_1, \hat{\mathbf{x}}), \mathbf{y} = (y_1, \hat{\mathbf{y}}) \in \mathbb{R} \times \Sigma$ with $\mathbf{x} \neq \mathbf{y}$. For this incident field $u^i_{\mathbf{x}_0}$ we consider the corresponding total field

$$u_{\mathbf{x}_0} = u^s_{\mathbf{x}_0} + u^i_{\mathbf{x}_0} \text{ in } B_R \setminus \overline{D} \qquad \text{and} \qquad v_{\mathbf{x}_0} \text{ in } D\,,$$

where $(u^s_{\mathbf{x}_0}, v_{\mathbf{x}_0})$ is the solution of the direct problem (2).

Remark 2. The fundamental solution Φ satisfies, for any $\mathbf{x} \in \mathbb{R} \times \Sigma$,

$$\begin{cases} \triangle \Phi_{\mathbf{x}} + k^2\, \Phi_{\mathbf{x}} &= -\delta_{\mathbf{x}} & \text{in } B_R\,, \\ \partial_{\boldsymbol{\nu}} \Phi_{\mathbf{x}} &= 0 & \text{on } (-R, R) \times \partial\Sigma\,, \\ \partial_{\boldsymbol{\nu}_0} \Phi_{\mathbf{x}} &= T^{\pm R} \Phi_{\mathbf{x}} & \text{on } \Sigma_{\pm R}\,, \end{cases}$$

provided $R > |x_1|$.

4.1. **The Reciprocity Gap operator.** For any regular enough subdomain $B \subseteq B_R$, we consider the function space

$$\mathbb{H}(B) := \left\{ u \in H^1(B)\,;\ \triangle u + k^2\, u = 0 \text{ in } B,\ \partial_{\boldsymbol{\nu}} u = 0 \text{ on } ((-R, R) \times \partial\Sigma) \cap \partial B \right\}\,,$$

as well as its subspaces

$$\begin{aligned} \mathbb{H}_R(B) &:= \{ u \in \mathbb{H}(B)\,;\ \partial_{\boldsymbol{\nu}_0} u = T^R u \text{ on } \Sigma_R \} && \text{if } \Sigma_R \subset \partial B, \\ \mathbb{H}_{-R}(B) &:= \{ u \in \mathbb{H}(B)\,;\ \partial_{\boldsymbol{\nu}_0} u = T^{-R} u \text{ on } \Sigma_{-R} \} && \text{if } \Sigma_{-R} \subset \partial B, \\ \mathbb{H}_{\pm R}(B) &:= \mathbb{H}_R(B) \cap \mathbb{H}_{-R}(B) && \text{if } \Sigma_R \cup \Sigma_{-R} \subset \partial B\,. \end{aligned}$$

We also define

$$\mathbb{U} := \{u_{\mathbf{x}_0} = u^s_{\mathbf{x}_0} + u^i_{\mathbf{x}_0}\,; \quad \mathbf{x}_0 \in \Sigma^S\}\,.$$

For sources placed on $\Sigma^S = \Sigma_{a_S}$ and measurements made on $\Sigma^M = \Sigma_{a_M}$, we introduce the Reciprocity Gap (RG) functional

$$\mathcal{R}(u_1, u_2) := \int_{\Sigma^M} (u_1\, \partial_{\boldsymbol{\nu}_0} u_2 - \partial_{\boldsymbol{\nu}_0} u_1\, u_2)\, dS\,, \tag{13}$$

for any $u_1 \in \mathbb{U}$ and $u_2 \in \mathbb{H}_R(B_R^S)$. We also define the RG operator $\mathrm{R} : \mathbb{H}_R(B_R^S) \to L^2(\Sigma^S)$ by

$$\mathrm{R}u(\mathbf{x}_0) := \mathcal{R}(u_{\mathbf{x}_0}, u)\,, \tag{14}$$

for a.e. $\mathbf{x}_0 \in \Sigma^S$ and any $u \in \mathbb{H}_R(B_R^S)$.

Lemma 4.1. *If k is not a transmission eigenvalue, the RG operator $\mathrm{R} : \mathbb{H}_R(B_R^S) \to L^2(\Sigma^S)$ is injective.*

Proof. Let us consider $u \in \mathbb{H}_R(B_R^S)$ such that

$$\mathrm{R}u(\mathbf{x}_0) := \int_{\Sigma^M} (u_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_0} u - \partial_{\boldsymbol{\nu}_0} u_{\mathbf{x}_0}\, u)\, dS = 0 \quad \text{for a.e. } \mathbf{x}_0 \in \Sigma^S\,.$$

Integrating by parts in $B_R^M \setminus \overline{D}$ and taking into account that $\triangle u_{\mathbf{x}_0} + k^2\, u_{\mathbf{x}_0} = \triangle u + k^2\, u = 0$ in $B_R^M \setminus \overline{D}$ and $\partial_{\boldsymbol{\nu}} u_{\mathbf{x}_0} = \partial_{\boldsymbol{\nu}} u = 0$ on $((a_M, R) \times \partial\Sigma) \setminus \partial D$,

$$\int_{\Sigma_R \cup (\partial D \cap B_R)} (u\, \partial_{\boldsymbol{\nu}} u_{\mathbf{x}_0} - \partial_{\boldsymbol{\nu}} u\, u_{\mathbf{x}_0})\, dS = 0\,.$$

Since $\partial_{\boldsymbol{\nu}} u_{\mathbf{x}_0} = T^R u_{\mathbf{x}_0}$ and $\partial_{\boldsymbol{\nu}} u = T^R u$ on Σ_R, by Lemma 2.1 we know that the above integral cancels on Σ_R. Therefore, using the transmission conditions on $\partial D \cap B_R$ satisfied by $u_{\mathbf{x}_0}$ and $v_{\mathbf{x}_0}$, and integrating by parts in D,

$$\int_D (\mathrm{div}(A\nabla u) + k^2 \mathrm{n} u)\, v_{\mathbf{x}_0}\, d\mathbf{x} = \int_{\partial D} \partial_{\boldsymbol{\nu}_{(A-I)}} u\, v_{\mathbf{x}_0}\, dS\,. \tag{15}$$

Let us take $w \in H^1(B_R \setminus D)$ and $z \in H^1(D)$ the unique solution to

$$\begin{cases} \triangle w + k^2 w = 0 & \text{in } B_R \setminus \overline{D}\,, \\ \mathrm{div}(A\nabla z) + k^2 \mathrm{n} z = \mathrm{div}(A\nabla u) + k^2 \mathrm{n} u & \text{in } D\,, \\ w = z & \text{on } B_R \cap \partial D\,, \\ \partial_{\boldsymbol{\nu}} w = \partial_{\boldsymbol{\nu}_A} z + \partial_{\boldsymbol{\nu}_{(A-I)}} u & \text{on } B_R \cap \partial D\,, \\ \partial_{\boldsymbol{\nu}} w = 0 & \text{on } ((-R, R) \times \partial\Sigma) \setminus \partial D\,, \\ \partial_{\boldsymbol{\nu}_A} z = 0 & \text{on } ((-R, R) \times \partial\Sigma) \cap \partial D\,, \\ T^{\pm R} w = w & \text{on } \Sigma_{\pm R}\,. \end{cases}$$

Then, we rewrite (15) as

$$\int_D (\mathrm{div}(A\nabla z) + k^2 \mathrm{n} z)\, v_{\mathbf{x}_0}\, d\mathbf{x} = \int_{\partial D} \partial_{\boldsymbol{\nu}_{(A-I)}} u\, v_{\mathbf{x}_0}\, dS\,.$$

Integrating by parts in D and using that $\mathrm{div}(A\nabla v_{\mathbf{x}_0}) + k^2 \mathrm{n} v_{\mathbf{x}_0} = 0$ in D as well as the transmission conditions for $u_{\mathbf{x}_0}$ and $v_{\mathbf{x}_0}$ on $\partial D \cap \partial B_R$,

$$\int_{\partial D} (\partial_{\boldsymbol{\nu}_D} w\, u_{\mathbf{x}_0} - w\, \partial_{\boldsymbol{\nu}_D} u_{\mathbf{x}_0})\, dS = 0\,.$$

Recalling that $u_{\mathbf{x}_0} = u^s_{\mathbf{x}_0} + u^i_{\mathbf{x}_0}$ with $u^i_{\mathbf{x}_0} = \Phi_{\mathbf{x}_0}$,

$$\int_{\partial D} \left(w\, \partial_{\boldsymbol{\nu}_D} u^s_{\mathbf{x}_0} - \partial_{\boldsymbol{\nu}_D} w\, u^s_{\mathbf{x}_0} \right) dS + \int_{\partial D} \left(w\, \partial_{\boldsymbol{\nu}_D} \Phi_{\mathbf{x}_0} - \partial_{\boldsymbol{\nu}_D} w\, \Phi_{\mathbf{x}_0} \right) dS = 0\,.$$

On the one hand, we can see that the first term in the above expression cancels if we integrate by parts in $B_R \setminus \overline{D}$ because $w|_{B_R\setminus\overline{D}}, u^s_{\mathbf{x}_0}|_{B_R\setminus\overline{D}} \in \mathbb{H}_{\pm R}(B_R \setminus \overline{D})$. On the other hand, $w \in \mathbb{H}_{\pm R}(B_R \setminus \overline{D})$ so that, in particular, it admits the integral representation

$$w(\mathbf{x}) = -\int_{\partial D} \left(w(\mathbf{x}_0)\, \partial_{\boldsymbol{\nu}_D} \Phi_{\mathbf{x}}(\mathbf{x}_0) - \partial_{\boldsymbol{\nu}_D} w\, \Phi_{\mathbf{x}}(\mathbf{x}_0) \right) dS_{(\mathbf{x}_0)} \quad \text{for a.e. } \mathbf{x} \in B^S_R \setminus \overline{D}\,;$$

cf. [3, Lemma 2] and Remark 3 below. Since $\Phi_{\mathbf{x}_0}(\mathbf{x}) = \Phi_{\mathbf{x}}(\mathbf{x}_0)$, we deduce that $w = 0$ on Σ^S; and then [3, Lemma 1] guarantees that $w = 0$ in B_R. Therefore, $(w-u)|_D \in H^1(D)$ and $u|_D \in H^1(D)$ solve the homogeneous counterpart of the interior transmission problem (7); and, under our assumption on k, we have that $u = 0$ in D, so that the unique continuation principle for Helmholtz equation guarantees that $u = 0$ in B_R. □

Remark 3. In [3, Lemma 2], it is proven that any $u \in \mathbb{H}_R(B^S_R \setminus \overline{D})$ admits the integral representation

$$u(\mathbf{x}) = -\int_{\partial D} \left(u(\mathbf{y})\, \partial_{\boldsymbol{\nu}_D} \Phi_{\mathbf{x}}(\mathbf{y}) - \partial_{\boldsymbol{\nu}_D} u(\mathbf{y})\, \Phi_{\mathbf{x}}(\mathbf{y}) \right) dS_{\mathbf{y}} \quad \text{for a.e. } \mathbf{x} \in B^S_R \setminus \overline{D}\,,$$

when $\overline{D} \subset B_R$. Nevertheless, this representation formula can be generalized for $D \subset B_R$ (in fact, the proof given there remains valid for $D \subset B_R$).

4.2. **The Single Layer operator.** Our next objective is to introduce a suitable single layer operator that we can use in combination with the RG operator R. To this end, we first consider a bounded domain $\mathcal{O} \subseteq B_R$; notice that, in particular, $\mathcal{O}$ can be B_R or touch its boundary. We then take the *single layer potential* defined on $\partial\mathcal{O}$ as

$$(\mathcal{S}_{\partial\mathcal{O}} g)(\mathbf{x}) := \int_{\partial\mathcal{O}} \Phi(\mathbf{x},\mathbf{y})\, g(\mathbf{y})\, dS_{\mathbf{y}} \qquad \text{for a.e. } \mathbf{x} \in B_R \setminus \partial\mathcal{O}\,.$$

In [3, Section 4.2] it is shown that $\mathcal{S}_{\partial\mathcal{O}}$ is well defined for any $g \in H^{-1/2}(\partial\mathcal{O})$ and a.e. $\mathbf{x} \in B_R \setminus \partial\mathcal{O}$. Moreover, for any $g \in H^{-1/2}(\partial\mathcal{O})$, it holds that $\mathcal{S}_{\partial\mathcal{O}} g$ is continuous across $\partial\mathcal{O}$ in the sense that

$$\left((\mathcal{S}_{\partial\mathcal{O}} g)\,|_{B_R\setminus\overline{\mathcal{O}}} \right)|_{\partial\mathcal{O}} = ((\mathcal{S}_{\partial\mathcal{O}} g)\,|_{\mathcal{O}})\,|_{\partial\mathcal{O}}\,;$$

in consequence, we can extend its definition to $\partial\mathcal{O}$ by

$$(S_{\partial\mathcal{O}} g)(\mathbf{x}) := \int_{\partial\mathcal{O}} \Phi(\mathbf{x},\mathbf{y})\, g(\mathbf{y})\, dS_{\mathbf{y}} \qquad \text{for a.e. } \mathbf{x} \in \partial\mathcal{O}\,.$$

This is known as the *single layer operator* on $\partial\mathcal{O}$ and defines a linear continuous operator $S_{\partial\mathcal{O}} : H^{-1/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$. Moreover, when k^2 is not an eigenvalue of the negative Laplacian operator in $\mathcal{O}$ with Dirichlet boundary conditions, we know that $S_{\partial\mathcal{O}} : H^{-1/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$ is an isomorphism.

We want to exploit the above results for single layer potentials on closed arcs ($d = 2$) or surfaces ($d = 3$) to deduce properties of the corresponding operators on open arcs or surfaces of the form

$$\Sigma^{SL} := \Sigma_{a_{SL}} \qquad \text{with } a_{SL} \in (-R, R)\,.$$

So we first consider a bounded domain $\mathcal{O} := (a_{SL}, a_{SL}+\varepsilon)\times\Sigma \subset B_R$, where $\varepsilon > 0$ is small enough. Then, for any $g \in \tilde{H}^{-1/2}(\Sigma^{SL})$, we can consider its extension by 0 to $\partial\mathcal{O}$, $g^0 \in H^{-1/2}(\partial\mathcal{O})$, and introduce the single layer potential and operator on Σ^{SL} as

$$(\mathcal{S}_{\Sigma^{SL}} g)(\mathbf{x}) := (\mathcal{S}_{\partial\mathcal{O}} g^0)(\mathbf{x}) = \int_{\Sigma^{SL}} \Phi(\mathbf{x},\mathbf{y})\, g(\mathbf{y})\, dS_{\mathbf{y}} \qquad \text{for a.e. } \mathbf{x} \in B_R \setminus \Sigma^{SL}\,,$$
$$(S_{\Sigma^{SL}} g)(\mathbf{x}) := (S_{\partial\mathcal{O}} g^0)(\mathbf{x}) = \int_{\Sigma^{SL}} \Phi(\mathbf{x},\mathbf{y})\, g(\mathbf{y})\, dS_{\mathbf{y}} \qquad \text{for a.e. } \mathbf{x} \in \Sigma^{SL}\,.$$

So defined, $\mathcal{S}_{\Sigma^{SL}} : \tilde{H}^{-1/2}(\Sigma^{SL}) \to \mathbb{H}_{\pm R}(B_R \setminus \Sigma^{SL})$ and $S_{\Sigma^{SL}} : \tilde{H}^{-1/2}(\Sigma^{SL}) \to H^{1/2}(\Sigma^{SL})$ are linear continuous operators.

Lemma 4.2. *The single layer operator $S_{\Sigma^{SL}} : \tilde{H}^{-1/2}(\Sigma^{SL}) \to H^{1/2}(\Sigma^{SL})$ is surjective.*

Proof. We take $g \in H^{1/2}(\Sigma^{SL})$, and consider $u^{\pm}$ such that

$$\begin{cases} \triangle u^- + k^2\, u^- = 0 & \text{in } (-R, a_{SL})\times\Sigma\,, \\ \triangle u^+ + k^2\, u^+ = 0 & \text{in } (a_{SL}, R)\times\Sigma\,, \\ u^{\pm} = g & \text{on } \Sigma^{SL}\,, \\ T^{\pm R} u^{\pm} = \partial_{\boldsymbol{\nu}_0} u^{\pm} & \text{on } \Sigma_{\pm R}\,, \\ \partial_{\boldsymbol{\nu}} u^- = 0 & \text{on } (-R, a_{SL})\times\partial\Sigma\,, \\ \partial_{\boldsymbol{\nu}} u^+ = 0 & \text{on } (a_{SL}, R)\times\partial\Sigma\,. \end{cases}$$

Notice that [3, Lemma 1] assures the existence and uniqueness of $u^{\pm}$. If we now denote $h := (\partial_{\boldsymbol{\nu}} u^+)|_{\Sigma^{SL}} - (\partial_{\boldsymbol{\nu}} u^-)|_{\Sigma^{SL}} \in \tilde{H}^{-1/2}(\Sigma^{SL})$, then $u' := \mathcal{S}_{\Sigma^{SL}} h \in \mathbb{H}_{\pm R}(B_R \setminus \Sigma^{SL})$ is continuous across Σ^{SL}, with

$$[\partial_{\boldsymbol{\nu}_0} u']_{\Sigma^{SL}} := \partial_{\boldsymbol{\nu}_0}(u'|_{(a_{SL},R)\times\Sigma}) - \partial_{\boldsymbol{\nu}_0}(u'|_{(-R,a_{SL})\times\Sigma}) = h \quad \text{on } \Sigma^{SL}\,.$$

Therefore, $w := u' - u^{\pm}$ satisfies the homogeneous forward problem

$$\begin{cases} \triangle w + k^2\, w = 0 & \text{in } B_R\,, \\ T^{\pm R} w = \partial_{\boldsymbol{\nu}_0} w & \text{on } \Sigma_{\pm R}\,, \\ \partial_{\boldsymbol{\nu}} w = 0 & \text{on } (-R, R)\times\partial\Sigma\,, \end{cases}$$

This problem is uniquely solvable [3, Lemma 1] so we deduce that $w = 0$ in $B_R \setminus \Sigma^{SL}$ and conclude that

$$g = u^{\pm}|_{\Sigma^{SL}} = u'|_{\Sigma^{SL}} = (\mathcal{S}_{\Sigma^{SL}} h)|_{\Sigma^{SL}} = S_{\Sigma^{SL}} h\,.$$

□

Lemma 4.3. *The single layer operator $S_{\Sigma^{SL}} : \tilde{H}^{-1/2}(\Sigma^{SL}) \to H^{1/2}(\Sigma^{SL})$ is injective.*

Proof. Let us consider $h \in \tilde{H}^{-1/2}(\Sigma^{SL})$ such that $S_{\Sigma^{SL}} h = 0$ on Σ^{SL}. In this case,

$$u := \mathcal{S}_{\Sigma^{SL}} h \in \mathbb{H}_{\pm R}(B_R \setminus \Sigma^{SL}) \quad \text{with } [u]_{\Sigma^{SL}} = 0\,,\ u = 0 \text{ on } \Sigma^{SL}\,.$$

In particular, u solves

$$\begin{cases} \triangle u + k^2\, u = 0 & \text{in } (a_{SL}, R) \times \Sigma\,, \\ u = 0 & \text{on } \Sigma^{SL}\,, \\ \partial_{\boldsymbol{\nu}_0} u = T^{+R} u & \text{on } \Sigma_{+R}\,, \\ \partial_{\boldsymbol{\nu}} u = 0 & \text{on } (a_{SL}, R) \times \partial\Sigma\,; \end{cases}$$

then, [3, Lemma 1] guarantees that $u = 0$ in $(a_{SL}, R) \times \Sigma$. With a similar argument, we deduce that $u = 0$ in $(-R, a_{SL}) \times \Sigma$. Therefore, $h = [\partial_{\boldsymbol{\nu}_0} u]_{\Sigma^{SL}} = 0$ on Σ^{SL}. □

Summing up, we have shown that the single layer operator

$$S_{\Sigma^{SL}} : \tilde{H}^{-1/2}(\Sigma^{SL}) \to H^{1/2}(\Sigma^{SL})$$

is linear and continuous; moreover, it defines an isomorphism.

We will also make use of the following property of the single layer potential.

Proposition 9. *Assume that k^2 is not an eigenvalue of the negative Laplacian in D with mixed boundary conditions (those of Dirichlet on $\partial D \cap B_R$ and Neumann on the other part $\partial D \cap \partial B_R$). Then, for any $a_{SL} \in (-R, R)$ such that $\overline{D} \subset (a_{SL}, R) \times \Sigma$, the space*

$$\left\{ (\mathcal{S}_{\Sigma^{SL}} f)|_D ; f \in L^2(\Sigma^{SL}) \right\}$$

is dense in $\mathbb{H}(D)$.

Proof. First notice that, for any $f \in L^2(\Sigma^{SL})$ and $g \in L^2(\partial D \cap B_R)$, we can interchange the order of integration so that

$$\langle g, \mathcal{S}_{\Sigma^{SL}} f \rangle_{\partial D \cap B_R} = \langle f, \mathcal{S}_{\partial D \cap B_R} g \rangle_{\Sigma^{SL}}\,.$$

By density, it must also hold for any $f \in \tilde{H}^{-1/2}(\Sigma^{SL})$ and $g \in \tilde{H}^{-1/2}(\partial D \cap B_R)$; thus, the adjoint operator of $f \in \tilde{H}^{-1/2}(\Sigma^{SL}) \mapsto (\mathcal{S}_{\Sigma^{SL}} f)|_{\partial D \cap B_R} \in H^{1/2}(\partial D \cap B_R)$ is $g \in \tilde{H}^{-1/2}(\partial D \cap B_R) \mapsto (\mathcal{S}_{\partial D \cap B_R} g)|_{\Sigma^{SL}} \in H^{1/2}(\Sigma^{SL})$.

Next, notice that the range of $f \in \tilde{H}^{-1/2}(\Sigma^{SL}) \mapsto (\mathcal{S}_{\Sigma^{SL}} f)|_{\partial D \cap B_R} \in H^{1/2}(\partial D \cap B_R)$ is dense if and only if its adjoint operator, $g \in \tilde{H}^{-1/2}(\partial D \cap B_R) \mapsto (\mathcal{S}_{\partial D \cap B_R} g)|_{\Sigma^{SL}} \in H^{1/2}(\Sigma^{SL})$, is injective. This suggests considering $h \in \tilde{H}^{-1/2}(\partial D \cap B_R)$ such that $\mathcal{S}_{\partial D \cap B_R} h = 0$ on Σ^{SL}. Let us denote $u := \mathcal{S}_{\partial D \cap B_R} h \in \mathbb{H}_{\pm R}(B_R \setminus \partial D)$. On the one hand,

$$\begin{cases} \triangle u + k^2\, u = 0 & \text{in } (-R, a_{SL}) \times \Sigma\,, \\ \partial_{\boldsymbol{\nu}} u = 0 & \text{on } (-R, a_{SL}) \times \partial\Sigma\,, \\ \partial_{\boldsymbol{\nu}_0} u = T^{-R} u & \text{on } \Sigma_{-R}\,, \\ u = 0 & \text{on } \Sigma^{SL}\,, \end{cases}$$

so by uniqueness of solution to this problem (see [3, Lemma 1]) we deduce that $u = 0$ in $(-R, a_{SL}) \times \Sigma$; and then the unique continuation principle for Helmholtz equation guarantees that $u = 0$ in $B_R \setminus \overline{D}$. Now, by the continuity of the trace of the single layer potential, $u = 0$ on $\partial D \cap B_R$. Thus,

$$\begin{cases} \triangle u + k^2\, u = 0 & \text{in } D\,, \\ \partial_{\boldsymbol{\nu}} u = 0 & \text{on } \partial D \cap \partial B_R\,, \\ u = 0 & \text{on } \partial D \cap B_R\,. \end{cases}$$

Assuming that k^2 is not an eigenvalue of the negative Laplacian in D with the above mixed boundary conditions (i.e. Dirichlet type in $\partial D \cap B_R$ and Neumann in the other part $\partial D \cap \partial B_R$), we have that $u = 0$ in D. Therefore,

$$h = [\partial_{\boldsymbol{\nu}_D}(\mathcal{S}_{\partial D\cap B_R} h)]_{\partial D\cap B_R} = [\partial_{\boldsymbol{\nu}_D} u]_{\partial D\cap B_R} = 0 \qquad \text{on } \partial D \cap B_R\,,$$

and we deduce that $g \in \tilde{H}^{-1/2}(\partial D \cap B_R) \mapsto (\mathcal{S}_{\partial D\cap B_R} g)|_{\Sigma^{SL}} \in H^{1/2}(\Sigma^{SL})$ is injective; i.e. the range of $f \in \tilde{H}^{-1/2}(\Sigma^{SL}) \mapsto (\mathcal{S}_{\Sigma^{SL}} f)|_{\partial D\cap B_R} \in H^{1/2}(\partial D \cap B_R)$ is dense. Furthermore, since $L^2(\Sigma^{SL})$ is dense in $\tilde{H}^{-1/2}(\Sigma^{SL})$ and $f \in \tilde{H}^{-1/2}(\Sigma^{SL}) \mapsto (\mathcal{S}_{\Sigma^{SL}} f)|_{\partial D\cap B_R} \in H^{1/2}(\partial D \cap B_R)$ is continuous, we deduce that $\{(\mathcal{S}_{\Sigma^{SL}} f)|_{\partial D\cap B_R}\,;\ f \in L^2(\Sigma^{SL})\}$ is dense in $H^{1/2}(\partial D \cap B_R)$.

Let us finally consider any $v \in \mathbb{H}(D)$. We have just shown that there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^2(\Sigma^{SL})$ such that

$$(\mathcal{S}_{\Sigma^{SL}} f_n)|_{\partial D\cap B_R} \to v|_{\partial D\cap B_R} \quad \text{in } H^{1/2}(\partial D \cap B_R)\,. \tag{16}$$

Denoting by $v_n := \mathcal{S}_{\Sigma^{SL}} f_n \in \mathbb{H}_{\pm R}(B_R \setminus \Sigma^{SL})$, we have

$$\begin{cases} \triangle v_n + k^2\, v_n = 0 & \text{in } D\,, \\ \partial_{\boldsymbol{\nu}} v_n = 0 & \text{on } \partial D \cap \partial B_R\,, \\ v_n = (\mathcal{S}_{\Sigma^{SL}} f_n)|_{\partial D\cap B_R} & \text{on } \partial D \cap B_R\,. \end{cases}$$

Thus, the convergence property (16) and the well–posedness of this problem (when k^2 is not an eigenvalue of the negative Laplacian in D with mixed boundary conditions) leads to

$$v_n = \mathcal{S}_{\Sigma^{SL}} f_n \to v \quad \text{in } H^1(D)\,.$$

□

4.3. **Properties of the Reciprocity Gap operator.** We have already seen that the RG operator $\mathrm{R} : \mathbb{H}_R(B_R^S) \to L^2(\Sigma^S)$ is injective. We now verify that it has dense range.

Lemma 4.4. *Suppose k is not a transmission eigenvalue. Then for any $-R < a_{SL} \le a_S \le a_M < R$, the operator*

$$\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}} : L^2(\Sigma^{SL}) \to L^2(\Sigma^S)$$

has dense range.

Proof. First notice that we take $-R < a_{SL} \le a_S$ so that $\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}$ is well–defined:

$$L^2(\Sigma^{SL}) \hookrightarrow \tilde{H}^{-1/2}(\Sigma^{SL}) \xrightarrow{\mathcal{S}_{\Sigma^{SL}}} \mathbb{H}_{\pm R}(B_R \setminus \Sigma^{SL}) \hookrightarrow \mathbb{H}_R(B_R^S) \xrightarrow{\mathrm{R}} L^2(\Sigma^S)\,.$$

Recall that $(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}})(L^2(\Sigma^{SL}))$ is dense in $L^2(\Sigma^S)$ if and only if its annihilator is trivial in $L^2(\Sigma^S)$. This suggests considering $g \in L^2(\Sigma^S)$ with

$$\int_{\Sigma^S} g(\mathbf{x}_0)\, (\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) h(\mathbf{x}_0)\, dS_{\mathbf{x}_0} = 0 \quad \forall h \in L^2(\Sigma^{SL})\,.$$

Using the definition of the RG operator, R, and interchanging the order of integration, we can rewrite this property as

$$\begin{aligned} &\int_{\Sigma^M} \partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} h)(\mathbf{x}) \left(\int_{\Sigma^S} g(\mathbf{x}_0)\, u_{\mathbf{x}_0}(\mathbf{x})\, dS_{\mathbf{x}_0} \right) dS_{\mathbf{x}} \\ &\quad - \int_{\Sigma^M} (\mathcal{S}_{\Sigma^{SL}} h)(\mathbf{x}) \left(\int_{\Sigma^S} g(\mathbf{x}_0)\, \partial_{\boldsymbol{\nu}_0} u_{\mathbf{x}_0}(\mathbf{x})\, dS_{\mathbf{x}_0} \right) dS_{\mathbf{x}} = 0 \quad \forall h \in L^2(\Sigma^{SL})\,. \end{aligned}$$

Let us define, for any $\mathbf{x} \in B_R \setminus \overline{D}$,

$$U_g(\mathbf{x}) := \int_{\Sigma^S} g(\mathbf{x}_0)\, u_{\mathbf{x}_0}(\mathbf{x}) dS_{\mathbf{x}_0}\,,$$

and also

$$U_g^i(\mathbf{x}) := \int_{\Sigma^S} g(\mathbf{x}_0)\, \Phi_{\mathbf{x}_0}(\mathbf{x}) dS_{\mathbf{x}_0} \qquad \text{and} \qquad U_g^s(\mathbf{x}) := \int_{\Sigma^S} g(\mathbf{x}_0)\, u^s_{\mathbf{x}_0}(\mathbf{x}) dS_{\mathbf{x}_0}\,.$$

Besides, for $\mathbf{x} \in D$, we take

$$V_g(\mathbf{x}) := \int_{\Sigma^S} g(\mathbf{x}_0)\, v_{\mathbf{x}_0}(\mathbf{x}) dS_{\mathbf{x}_0}\,.$$

Then, by superposition, we know that U_g^s and V_g solve the direct problem (2) for the incident field U_g^i. Moreover, the above condition on g reads

$$\int_{\Sigma^M} (\partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} h)(\mathbf{x})\, U_g(\mathbf{x}) - (\mathcal{S}_{\Sigma^{SL}} h)(\mathbf{x})\, \partial_{\boldsymbol{\nu}_0} U_g(\mathbf{x}))\, dS = 0 \quad \forall h \in L^2(\Sigma^{SL})\,.$$

But, integrating by parts in B_R^M for $(\mathcal{S}_{\Sigma^{SL}} h)|_{B_R^M}, U_g^i|_{B_R^M} \in \mathbb{H}_R(B_R^M)$, using as well as Lemma 2.1, we can see that

$$\begin{aligned}\int_{\Sigma^M} \left(\partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} h)\, U_g^i - (\mathcal{S}_{\Sigma^{SL}} h)\, \partial_{\boldsymbol{\nu}_0} U_g^i\right) dS = \\ = \int_{\Sigma_R} \left(T^R(\mathcal{S}_{\Sigma^{SL}} h)\, U_g^i - (\mathcal{S}_{\Sigma^{SL}} h)\, T^R U_g^i\right) dS = 0\,.\end{aligned}$$

Thus,

$$\int_{\Sigma^M} \left(\partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} h)\, U_g^s - (\mathcal{S}_{\Sigma^{SL}} h)\, \partial_{\boldsymbol{\nu}_0} U_g^s\right) dS = 0 \qquad \forall h \in L^2(\Sigma^{SL})\,.$$

Since U_g^s, $\mathcal{S}_{\Sigma^{SL}} h \in \mathbb{H}((a_{SL}, a_M) \times \Sigma)$, we can integrate by parts in $(a_{SL}, a_M) \times \Sigma$ to deduce that

$$\int_{\Sigma^{SL}} \left(\partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} h)\, U_g^s - (\mathcal{S}_{\Sigma^{SL}} h)\, \partial_{\boldsymbol{\nu}_0} U_g^s\right) dS = 0 \qquad \forall h \in L^2(\Sigma^{SL})\,. \tag{17}$$

Moreover, recall that the single layer potential is continuous across Σ^{SL} whereas its normal derivative has the jump h. If we define

$$(\mathcal{S}_{\Sigma^{SL}} h)^+ := (\mathcal{S}_{\Sigma^{SL}} h)|_{(a_{SL}, R)\times\Sigma} \quad \text{and} \quad (\mathcal{S}_{\Sigma^{SL}} h)^- := (\mathcal{S}_{\Sigma^{SL}} h)|_{(-R, a_{SL})\times\Sigma}\,,$$

the following holds:

$$[\mathcal{S}_{\Sigma^{SL}} h]_{\Sigma^{SL}} := (\mathcal{S}_{\Sigma^{SL}} h)^+|_{\Sigma^{SL}} - (\mathcal{S}_{\Sigma^{SL}} h)^-|_{\Sigma^{SL}} = 0 \qquad \text{on } \Sigma^{SL}\,,$$

and also

$$[\partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} h)]_{\Sigma^{SL}} := \partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} h)^+ - \partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} h)^- = h \qquad \text{on } \Sigma^{SL}\,.$$

Therefore, the above condition (17) implies that

$$\int_{\Sigma^{SL}} \left(\partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} h)^-\, U_g^s - (\mathcal{S}_{\Sigma^{SL}} h)^-\, \partial_{\boldsymbol{\nu}_0} U_g^s\right) dS + \int_{\Sigma^{SL}} h\, U_g^s\, dS = 0 \qquad \forall h \in L^2(\Sigma^{SL})\,.$$

Since

$$U_g^s,\, \mathcal{S}_{\Sigma^{SL}} h \in \mathbb{H}_{-R}((-R, a_{SL}) \times \Sigma)\,,$$

we can integrate by parts in $(-R, a_{SL}) \times \Sigma$ to deduce that the first term in the above expression cancels, so that

$$\int_{\Sigma^{SL}} h\, U_g^s\, dS = 0 \qquad \forall h \in L^2(\Sigma^{SL})\,.$$

In other words, $U_g^s = 0$ on Σ^{SL}. Since $U_g^s \in \mathbb{H}_{-R}((-R, a_{SL}) \times \Sigma)$ with $U_g^s = 0$ on $\Sigma^{SL} = \Sigma_{a_{SL}}$, [3, Lemma 1] guarantees that $U_g^s = 0$ in $(-R, a_{SL}) \times \Sigma$; and the unique continuation principle for Helmholtz equation applied to U_g^s in $B_R \setminus \overline{D}$ leads to $U_g^s = 0$ in $B_R \setminus \overline{D}$. In consequence, $v_1 := U_g^i|_D$ and $v_2 := V_g$ solve the homogeneous interior transmission problem (7); since k is not a transmission eigenvalue, it follows that $U_g^i = V_g = 0$ in D. In particular, by the unique continuation principle for Helmholtz equation applied to U_g^i in B_R^S, we deduce that $U_g^i = 0$ in B_R^S. Thus, $0 = U_g^i|_{\Sigma^S} = S_{\Sigma^S} g$ and, by Lemma 4.3, we conclude that $g = 0$ on Σ^S. □

Summing up, we have proved the following result.

Proposition 10. *Suppose k is not a transmission eigenvalue. Then for any $-R < a_{SL} \le a_S \le a_M < R$,*

$$\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}} : L^2(\Sigma^{SL}) \to L^2(\Sigma^S)$$

defines a linear continuous operator that is compact and injective, and has a dense range.

4.4. **A reciprocity relation.** In the sequel, we will make use of a reciprocity relation for the direct problem (2), that we now state and prove. The same result, but for sound hard rather than a penetrable obstacle was proven in [3].

Lemma 4.5. *For almost every $\mathbf{x}, \mathbf{y} \in B_R \setminus \overline{D}$, it holds that*

$$u_{\mathbf{x}}^s(\mathbf{y}) = u_{\mathbf{y}}^s(\mathbf{x})\,.$$

Proof. Let us consider $\mathbf{x}$, $\mathbf{y} \in B_R \setminus \overline{D}$. Since $\operatorname{div}(A\nabla v_{\mathbf{x}}) + k^2\,\mathrm{n}\,v_{\mathbf{x}} = \operatorname{div}(A\nabla v_{\mathbf{y}}) + k^2\,\mathrm{n}\,v_{\mathbf{y}} = 0$ in D, we can integrate by parts in D as follows:

$$\begin{aligned} 0 = \int_D \left(\operatorname{div}(A\nabla v_{\mathbf{x}})(\mathbf{z})\, v_{\mathbf{y}}(\mathbf{z}) - v_{\mathbf{x}}(\mathbf{z})\operatorname{div}(A\nabla v_{\mathbf{y}})(\mathbf{z})\right) d\mathbf{z} = \\ = \int_{\partial D} \left(\partial_{\boldsymbol{\nu}_A} v_{\mathbf{x}}(\mathbf{z})\, v_{\mathbf{y}}(\mathbf{z}) - v_{\mathbf{x}}(\mathbf{z})\,\partial_{\boldsymbol{\nu}_A} v_{\mathbf{y}}(\mathbf{z})\right) dS_{\mathbf{z}}\,. \end{aligned}$$

By the transmission conditions on ∂D, and decomposing $u_{\mathbf{x}}$ and $u_{\mathbf{y}}$ into their incident and scattered parts,

$$\begin{aligned} &\int_{\partial D} \left(\partial_{\boldsymbol{\nu}_D} u_{\mathbf{x}}^s\, u_{\mathbf{y}}^s - u_{\mathbf{x}}^s\,\partial_{\boldsymbol{\nu}_D} u_{\mathbf{y}}^s\right) dS + \int_{\partial D} \left(\partial_{\boldsymbol{\nu}_D} u_{\mathbf{x}}^i\, u_{\mathbf{y}}^i - u_{\mathbf{x}}^i\,\partial_{\boldsymbol{\nu}_D} u_{\mathbf{y}}^i\right) dS \\ &\quad + \int_{\partial D} \left(\partial_{\boldsymbol{\nu}_D} u_{\mathbf{x}}^s\, u_{\mathbf{y}}^i - u_{\mathbf{x}}^s\,\partial_{\boldsymbol{\nu}_D} u_{\mathbf{y}}^i\right) dS + \int_{\partial D} \left(\partial_{\boldsymbol{\nu}_D} u_{\mathbf{x}}^i\, u_{\mathbf{y}}^s - u_{\mathbf{x}}^i\,\partial_{\boldsymbol{\nu}_D} u_{\mathbf{y}}^s\right) dS \;=\; 0\,. \end{aligned}$$

Notice that

- on the one hand, integrating by parts in $B_R \setminus \overline{D}$ and using the equations (2) satisfied by these functions,

  $$\int_{\partial D} \left(\partial_{\boldsymbol{\nu}_D} u_{\mathbf{x}}^s\, u_{\mathbf{y}}^s - u_{\mathbf{x}}^s\,\partial_{\boldsymbol{\nu}_D} u_{\mathbf{y}}^s\right) dS = \pm \int_{\Sigma_{\pm R}} \left(T^{\pm R} u_{\mathbf{x}}^s\, u_{\mathbf{y}}^s - u_{\mathbf{x}}^s\, T^{\pm R} u_{\mathbf{y}}^s\right) = 0\,,$$

  where the last identity follows by Lemma 2.1;
- on the other hand, integrating by parts in D for $u_{\mathbf{x}}^i, u_{\mathbf{y}}^i \in \mathbb{H}(D)$,

  $$\int_{\partial D} \left(\partial_{\boldsymbol{\nu}_D} u_{\mathbf{x}}^i\, u_{\mathbf{y}}^i - u_{\mathbf{x}}^i\,\partial_{\boldsymbol{\nu}_D} u_{\mathbf{y}}^i\right) dS = \int_D \left(\triangle u_{\mathbf{x}}^i\, u_{\mathbf{y}}^i - u_{\mathbf{x}}^i\,\triangle u_{\mathbf{y}}^i\right) d\mathbf{z} = 0\,;$$

Thus,

$$\int_{\partial D} (\partial_{\boldsymbol{\nu}_D} u^s_{\mathbf{x}}\, \Phi_{\mathbf{y}} - u^s_{\mathbf{x}}\, \partial_{\boldsymbol{\nu}_D} \Phi_{\mathbf{y}})\, dS + \int_{\partial D} \left(\partial_{\boldsymbol{\nu}_D} \Phi_{\mathbf{x}}\, u^s_{\mathbf{y}} - \Phi_{\mathbf{x}}\, \partial_{\boldsymbol{\nu}_D} u^s_{\mathbf{y}}\right) dS \,=\, 0\,. \tag{18}$$

Recall that $u^s_{\mathbf{x}}, u^s_{\mathbf{y}} \in H^1(B_R \setminus \overline{D})$ satisfy, in weak sense,

$$\begin{cases} \triangle u^s_{\mathbf{x}} + k^2\, u^s_{\mathbf{x}} = \triangle u^s_{\mathbf{y}} + k^2\, u^s_{\mathbf{y}} = 0 & \text{in } B_R \setminus \overline{D}\,, \\ \partial_{\boldsymbol{\nu}} u^s_{\mathbf{x}} = \partial_{\boldsymbol{\nu}} u^s_{\mathbf{y}} = 0 & \text{on } ((-R,R)\times\partial\Sigma)\setminus\partial D\,, \\ \partial_{\boldsymbol{\nu}_0} u^s_{\mathbf{x}} = \pm T^{\pm R} u^s_{\mathbf{x}},\ \partial_{\boldsymbol{\nu}_0} u^s_{\mathbf{y}} = \pm T^{\pm R} u^s_{\mathbf{y}} & \text{on } \Sigma_{\pm R}\,; \end{cases}$$

hence, they admit the integral representation

$$u^s_{\mathbf{x}}(\mathbf{y}) = \int_{\partial D} (\partial_{\boldsymbol{\nu}_D} u^s_{\mathbf{x}}\, \Phi_{\mathbf{y}} - u^s_{\mathbf{x}}\, \partial_{\boldsymbol{\nu}_D} \Phi_{\mathbf{y}})\, dS \quad \text{and} \quad u^s_{\mathbf{y}}(\mathbf{x}) = \int_{\partial D} \left(\partial_{\boldsymbol{\nu}_D} u^s_{\mathbf{y}}\, \Phi_{\mathbf{x}} - u^s_{\mathbf{y}}\, \partial_{\boldsymbol{\nu}_D} \Phi_{\mathbf{x}}\right) dS$$

for a.e. $\mathbf{x}, \mathbf{y} \in B_R \setminus \overline{D}$; cf. Remark 3. In consequence, the above property (18) means that $u^s_{\mathbf{x}}(\mathbf{y}) - u^s_{\mathbf{x}}(\mathbf{y}) = 0$ for a.e. $\mathbf{x}, \mathbf{y} \in B_R \setminus \overline{D}$. □

4.5. **A Reciprocity Gap method.** We propose locating and reconstructing D by means of the Reciprocity Gap Method (RGM): Given $\mathbf{z} \in B^M_R$, we decide whether

$$\mathbf{z} \in D\,, \quad \mathbf{z} \in \partial D \quad \text{or} \quad \mathbf{z} \in B^M_R \setminus \overline{D}\,,$$

by studying a regularized solution $f_{\mathbf{z}} \in L^2(\Sigma^{SL})$ of the ill–posed integral equation

$$(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f_{\mathbf{z}} = \mathrm{R}\Phi_{\mathbf{z}} \quad \text{on } \Sigma^S\,. \tag{19}$$

In order to justify this method, we rewrite (19) in a simpler way.

(i) First consider $(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f(\mathbf{x}_0)$ for $f \in L^2(\Sigma^{SL})$ and $\mathbf{x}_0 \in \Sigma^S$:
Using the definitions of the RG operator and functional, and decomposing $u_{\mathbf{x}_0}$ in its incident and scattered part $u_{\mathbf{x}_0} = \Phi_{\mathbf{x}_0} + u^s_{\mathbf{x}_0}$, we have

$$\begin{aligned}(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f(\mathbf{x}_0) = &\int_{\Sigma^M} (\Phi_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} f) - \partial_{\boldsymbol{\nu}_0} \Phi_{\mathbf{x}_0}\, (\mathcal{S}_{\Sigma^{SL}} f))\, dS \\ &+ \int_{\Sigma^M} \left(u^s_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} f) - \partial_{\boldsymbol{\nu}_0} u^s_{\mathbf{x}_0}\, (\mathcal{S}_{\Sigma^{SL}} f)\right) dS\,.\end{aligned}$$

Since $\Phi_{\mathbf{x}_0}, \mathcal{S}_{\Sigma^{SL}} f \in \mathbb{H}_R((a_M, R)\times\Sigma)$, we can integrate by parts in $(a_M, R)\times\Sigma$ and apply Lemma 2.1 to deduce that the first term in the right hand side of the above relation vanishes, so that

$$(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f(\mathbf{x}_0) = \int_{\Sigma^M} \left(u^s_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} f) - \partial_{\boldsymbol{\nu}_0} u^s_{\mathbf{x}_0}\, (\mathcal{S}_{\Sigma^{SL}} f)\right) dS\,.$$

Moreover $u^s_{\mathbf{x}_0}, \mathcal{S}_{\Sigma^{SL}} f \in \mathbb{H}((a_{SL}, a_M)\times\Sigma)$, and hence we can integrate by parts in $(a_{SL}, a_M)\times\Sigma$ to deduce

$$(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f(\mathbf{x}_0) = \int_{\Sigma^{SL}} \left(u^s_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} f)^+ - \partial_{\boldsymbol{\nu}_0} u^s_{\mathbf{x}_0}\, (\mathcal{S}_{\Sigma^{SL}} f)^+\right) dS\,,$$

where the superindexes $\pm$ are used as in the proof of Lemma 4.4. Recalling the jump conditions of the single layer operator on Σ^{SL} (also mentioned in the proof of Lemma 4.4), we have

$$\begin{aligned}(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f(\mathbf{x}_0) = &\int_{\Sigma^{SL}} \left(u^s_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_0}(\mathcal{S}_{\Sigma^{SL}} f)^- - \partial_{\boldsymbol{\nu}_0} u^s_{\mathbf{x}_0}\, (\mathcal{S}_{\Sigma^{SL}} f)^-\right) dS \\ &+ \int_{\Sigma^{SL}} u^s_{\mathbf{x}_0}\, f\, dS\,.\end{aligned}$$

Integrating by parts for $u^s_{\mathbf{x}_0}, (\mathcal{S}_{\Sigma^{SL}} f)^- \in \mathbb{H}_{-R}((-R, a_{SL}) \times \Sigma)$ and using Lemma 2.1, we deduce that the first term in the right hand side vanishes. Thus,

$$(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f(\mathbf{x}_0) := \int_{\Sigma^{SL}} u^s_{\mathbf{x}_0}\, f\, dS\,.$$

(ii) Next consider $\mathrm{R}\Phi_{\mathbf{z}}(\mathbf{x}_0)$ for $\mathbf{z} \in B^M_R$ and $\mathbf{x}_0 \in \Sigma^S$:
Once again, using the definitions of the RG operator and functional, and decomposing $u_{\mathbf{x}_0}$ in its incident and scattered parts,

$$\begin{aligned}\mathrm{R}\Phi_{\mathbf{z}}(\mathbf{x}_0) = &\int_{\Sigma^M} (\Phi_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{z}} - \partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{x}_0}\, \Phi_{\mathbf{z}})\, dS \\ &+ \int_{\Sigma^M} \left(u^s_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{z}} - \partial_{\boldsymbol{\nu}_0} u^s_{\mathbf{x}_0}\, \Phi_{\mathbf{z}}\right) dS\,.\end{aligned}$$

Integrating by parts in $(-R, a_M) \times \Sigma$ and applying Lemma 2.1, we deduce that the last term in this relation cancels and then

$$\mathrm{R}\Phi_{\mathbf{z}}(\mathbf{x}_0) = \int_{\Sigma^M} (\Phi_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{z}} - \partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{x}_0}\, \Phi_{\mathbf{z}})\, dS\,.$$

Besides, integrating by parts in $(a_S, a_M) \times \Sigma$, we have

$$\mathrm{R}\Phi_{\mathbf{z}}(\mathbf{x}_0) = \int_{\Sigma^S} \left(\Phi^+_{\mathbf{x}_0}\, (\partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{z}})^+ - (\partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{x}_0})^+\, \Phi^+_{\mathbf{z}}\right) dS\,.$$

In terms of the single and double layer operators on Σ^S, denoted by $\mathcal{S}_{\Sigma^S}$ and $\mathcal{D}_{\Sigma^S}$ respectively, the above relation means that

$$\mathrm{R}\Phi_{\mathbf{z}}(\mathbf{x}_0) = (\mathcal{S}_{\Sigma^S}(\partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{z}}))^+(\mathbf{x}_0) - (\mathcal{D}_{\Sigma^S}(\Phi_{\mathbf{z}}))^+(\mathbf{x}_0)\,.$$

Taking into account the continuity of the single layer potential across Σ^S, as well as the jump of the trace of the double layer potential,

$$(\mathcal{D}_{\Sigma^S}(\Phi_{\mathbf{z}}))^+ - (\mathcal{D}_{\Sigma^S}(\Phi_{\mathbf{z}}))^- = -\Phi_{\mathbf{z}} \qquad \text{on } \Sigma^S\,,$$

we deduce

$$\begin{aligned}\mathrm{R}\Phi_{\mathbf{z}}(\mathbf{x}_0) &= (\mathcal{S}_{\Sigma^S}(\partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{z}}))^-(\mathbf{x}_0) - (\mathcal{D}_{\Sigma^S}(\Phi_{\mathbf{z}}))^-(\mathbf{x}_0) + \Phi_{\mathbf{z}}(\mathbf{x}_0) \\ &= \int_{\Sigma^S} \left(\Phi^-_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{z}} - \partial_{\boldsymbol{\nu}_0}\Phi_{\mathbf{x}_0}\, \Phi_{\mathbf{z}}\right) dS + \Phi_{\mathbf{z}}(\mathbf{x}_0)\,.\end{aligned}$$

But, integrating by parts in $(-R, a_S) \times \Sigma$, we deduce that the first term in the right hand side vanishes and, hence,

$$\mathrm{R}\Phi_{\mathbf{z}}(\mathbf{x}_0) = \Phi_{\mathbf{z}}(\mathbf{x}_0) \qquad \text{for a.e. } \mathbf{x}_0 \in \Sigma^S,\ \mathbf{z} \in B^M_R\,.$$

Lemma 4.6. *Given $\mathbf{z} \in B^M_R$, $f_{\mathbf{z}} \in L^2(\Sigma^{SL})$ is an approximate solution of the RGM equation (19) if and only if it is an approximate solution of the generalized LSM equation*

$$\int_{\Sigma^{SL}} u^s_{\mathbf{x}_0}\, f_{\mathbf{z}}\, dS = \Phi_{\mathbf{z}}(\mathbf{x}_0) \qquad \textit{for a.e. } \mathbf{x}_0 \in \Sigma^S\,.$$

We now consider each case for $\mathbf{z}$.

4.5.1. **The case when $\mathbf{z} \in D$.** Let us consider $\mathbf{z} \in D$. Assuming that k is not a transmission eigenvalue, we can take $v_1, v_2 \in H^1(D)$ that solve (8) with $f_1 = f_2 = 0$, $G := \Phi_{\mathbf{z}}$ and $g := \partial_{\boldsymbol{\nu}_D} \Phi_{\mathbf{z}}$. Then, by the definition of the RG operator and the transmission conditions of (8),

$$\begin{aligned} \mathrm{R}\Phi_{\mathbf{z}}(\mathbf{x}_0) = & \int_D (u_{\mathbf{x}_0}\, \partial_{\boldsymbol{\nu}_D} \Phi_{\mathbf{z}} - \partial_{\boldsymbol{\nu}_D} u_{\mathbf{x}_0}\, \Phi_{\mathbf{z}})\, d\mathbf{x} = \\ & = \int_D (u_{\mathbf{x}_0}\, (\partial_{\boldsymbol{\nu}_D} v_1 - \partial_{\boldsymbol{\nu}_A} v_2) - \partial_{\boldsymbol{\nu}_D} u_{\mathbf{x}_0}\, (v_1 - v_2))\, d\mathbf{x}\,. \end{aligned}$$

Integrating by parts in D and using the equations satisfied by $u_{\mathbf{x}_0}$, v_1 and v_2, we deduce that

$$\mathrm{R}\Phi_{\mathbf{z}} = \mathrm{R}v_1 \qquad \text{a.e. on } \Sigma^S\,.$$

Besides, by Proposition 9 and density of $H(D)$ in $L^2(D)$, there is a sequence of functions $\{f_{\mathbf{z}}^n\}_{n\in\mathbb{N}} \subset L^2(\Sigma^{SL})$ with

$$\mathcal{S}_{\Sigma^{SL}} f_{\mathbf{z}}^n \to v_1 \quad \text{in } H^1(D)\,.$$

In consequence, the sequence $\{f_{\mathbf{z}}^n\}_{n\in\mathbb{N}} \subset L^2(\Sigma^{SL})$ satisfies

$$(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}})(f_{\mathbf{z}}^n)(\mathbf{x}_0) \to \mathrm{R}\Phi_{\mathbf{z}}(\mathbf{x}_0) \quad \text{for a.e. } \mathbf{x}_0 \in \Sigma^S\,.$$

4.5.2. **The case when $\mathbf{z} \in B_R^M \setminus D$.** Let us consider a given point $\mathbf{z} \in B_R^M \setminus D$ and a sequence $\{f_{\mathbf{z}}^n\}_{n\in\mathbb{N}} \subset L^2(\Sigma^{SL})$ such that

$$(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f_{\mathbf{z}}^n \to \mathrm{R}\Phi_{\mathbf{z}} \quad \text{in } L^2(\Sigma^S)\,.$$

Assume that $\{||f_{\mathbf{z}}^n||_{L^2(\Sigma^{SL})}\}_{n\in\mathbb{N}}$ is bounded. Then, by the compactness of the operator $(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}})$, there is some $f_{\mathbf{z}} \in L^2(\Sigma^{SL})$ such that for a subsequence of the original sequence

$$(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f_{\mathbf{z}}^n \to (\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f_{\mathbf{z}} \quad \text{in } L^2(\Sigma^S)\,,$$

so that $(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f_{\mathbf{z}} = \mathrm{R}\Phi_{\mathbf{z}}$. As we saw in Lemma 4.6, this is equivalent to

$$\int_{\Sigma^{SL}} u_{\mathbf{x}_0}^s\, f_{\mathbf{z}}\, dS = \Phi_{\mathbf{z}}(\mathbf{x}_0) \quad \text{for a.e. } \mathbf{x}_0 \in \Sigma^S\,.$$

Let us define

$$F_{\mathbf{z}}(\mathbf{x}) := \int_{\Sigma^{SL}} u_{\mathbf{x}}^s\, f_{\mathbf{z}}\, dS - \Phi_{\mathbf{z}}(\mathbf{x}) \quad \text{for a.e. } \mathbf{x} \in B_R \setminus (\{\mathbf{z}\} \cup \overline{D})\,.$$

Notice that, by the reciprocity relation (Lemma 4.5), $\triangle F_{\mathbf{z}} + k^2\, F_{\mathbf{z}} = 0$ in $B_R \setminus (\{\mathbf{z}\} \cup \overline{D})$. In consequence, $F_{\mathbf{z}} \in \mathbb{H}_{-R}((-R, a_S) \times \Sigma)$ with $F_{\mathbf{z}}|_{\Sigma^S} = 0$, so that [3, Lemma 1] guarantees that $F_{\mathbf{z}} = 0$ in $(-R, a_S) \times \Sigma$. Then, by the unique continuation principle, $F_{\mathbf{z}} = 0$ in $B_R \setminus (\{\mathbf{z}\} \cup \overline{D})$. In other words,

$$\int_{\Sigma^{SL}} u_{\mathbf{x}}^s\, f_{\mathbf{z}}\, dS = \Phi_{\mathbf{z}}(\mathbf{x}) \quad \text{for a.e. } \mathbf{x} \in B_R \setminus (\{\mathbf{z}\} \cup \overline{D})\,,$$

and we arrive to a contradiction by taking the limit when we approach $\mathbf{z}$ by points $\mathbf{x} \in B_R \setminus (\{\mathbf{z}\} \cup \overline{D})$.

Summing up, if a sequence $\{f_{\mathbf{z}}^n\}_{n\in\mathbb{N}} \subset L^2(\Sigma^{SL})$ satisfies $(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f_{\mathbf{z}}^n \to \mathrm{R}\Phi_{\mathbf{z}}$ in $L^2(\Sigma^S)$, then $\{||f_{\mathbf{z}}^n||_{L^2(\Sigma^{SL})}\}_{n\in\mathbb{N}}$ cannot be bounded.

We have shown the following result that justifies the usage of the RGM, as well as its theoretically equivalent scheme, the near field LSM.

Theorem 4.7. *Assume that k is not a transmission eigenvalue and not a mixed eigenvalue for D (see Proposition 9). Then given $\mathbf{z} \in B_R^M$, consider the function $r_{\mathbf{z}} \in L^2(\Sigma^S)$ defined by*

$$r_{\mathbf{z}}(\mathbf{x}_0) := \mathrm{R}\Phi_{\mathbf{z}}(\mathbf{x}_0) \quad \text{for a.e. } \mathbf{x}_0 \in \Sigma^S\,.$$

(i) *For any $\mathbf{z} \in D$ and a given $\varepsilon > 0$, there exists $f_{\mathbf{z}}^{\varepsilon} \in L^2(\Sigma^{SL})$ such that*

$$||(\mathrm{R} \circ \mathcal{S}_{\Sigma^{SL}}) f_{\mathbf{z}}^{\varepsilon} - r_{\mathbf{z}}||_{L^2(\Sigma^S)} < \varepsilon\,, \tag{20}$$

and the corresponding sequence of functions $\{\mathcal{S}_{\Sigma^{SL}} f_{\mathbf{z}}^{\varepsilon};\ \varepsilon > 0\} \subset L^2(\Sigma^{SL})$ converges in $L^2(\Sigma^{SL})$ as $\varepsilon \to 0$.

(ii) *For any $\mathbf{z} \in \partial D$ and a fixed $\varepsilon > 0$, we have that*

$$||f_{\mathbf{z}_n}^{\varepsilon}||_{L^2(\Sigma^{SL})} \to \infty \quad \text{when } \mathbf{z}_n \in D,\ \mathbf{z}_n \to \mathbf{z}\,,$$

for any sequence of functions $\{f_{\mathbf{z}}^{\varepsilon};\ \varepsilon > 0\} \subset L^2(\Sigma^{SL})$ satisfying (20).

(iii) *For each $\mathbf{z} \in B_R^M \setminus \overline{D}$ and any sequence of functions $\{f_{\mathbf{z}}^{\varepsilon};\ \varepsilon > 0\} \subset L^2(\Sigma^{SL})$ that satisfy (20), it holds that*

$$||f_{\mathbf{z}}^{\varepsilon}||_{L^2(\Sigma^{SL})} \to \infty \quad \text{for } \varepsilon \to 0\,.$$

Proof. We have already proved (i) and (iii) in Paragraphs 4.5.1 and 4.5.2, whereas (ii) is just a particular case of (iii). □

5. **Numerical results.** The Helmholtz equation that we use for the numerics is a particular case of the theory that we have presented. In particular, we suppose that u satisfies

$$\mathrm{div}\left(\frac{\rho_{air}}{\rho(\mathbf{x})}\nabla u\right) + \frac{\omega^2}{c_{air}^2}\frac{\rho_{air}\, c_{air}^2}{\rho(\mathbf{x})\, c(\mathbf{x})^2} u = 0$$

where $\omega = f/2\pi$, and f is the frequency of the sound field. In addition ρ_{air} is the density of air and c_{air} is the speed of sound in air. Thus in our previous formulation

$$A(\mathbf{x}) = \frac{\rho_{air}}{\rho(\mathbf{x})} I \quad \text{and} \quad \mathrm{n}(\mathbf{x}) = \frac{\rho_{air} c_{air}^2}{\rho(\mathbf{x}) c(\mathbf{x})^2},$$

where I is the 3×3 identity matrix.

The wave number is given by $k = \omega/c_{air}$. We take $c(\mathbf{x})$ and $\rho(\mathbf{x})$ to be piecewise constant (constant in the air and in the obstruction). This forward problem is solved using the UWVF with Dirichlet-to-Neumann boundary conditions on either end of a short section of the pipe.

In this study the pipe is a cylinder with axis of rotation along the x_1-axis and radius 0.075 m (so the cross section is $\Sigma = \{\hat{\mathbf{x}} = (x_2, x_3);\ x_2^2 + x_3^2 < 0.075^2\}$). Three wave-numbers are investigated corresponding to $f = 10KHz$, $6KHz$ and $2KHz$. We take the sound speed in the pipe to be $c_{air} = 343$ m/s and the density of air to be $\rho_{air} = 1.2$ kg/m^3.

The forward problem is surprisingly hard to solve using our implementation of the UWVF. To obtain good geometry approximation we needed a fine grid near the curved surface of the pipe (because our implementation of UWVF uses standard tetrahedra) and this resulted in many small elements. Perhaps as a result the bi-conjugate gradient solver often failed to converge. The implementation of the UWVF will be described in detail in a separate publication.

In all cases the field is measured on Σ^M at $x_1 = a_M := -4$m while the sources are at $x_1 = a_S := -4.1$m. We use point sources positioned at M_q Gauss-Jacobi

Wave number k	Number of propagating modes	Total number of modes	Wavelength
3.6637	3	38	1.71
10.9910	21	38	0.57
18.1530	53	77	0.2

TABLE 1. Characteristics of the forward problem. Three wave numbers were used: as the wavenumber increases the number of propagating modes increases. The wavelength is $2\pi/k$.

Wave number k	M_q	N_q
3.6637	4	12
10.9910	4	12
18.1530	8	30
18.1530	6	20

TABLE 2. Number of incident source points.

points radially (computed using the "Sandia_Rules" quadrature package[1]) and N_q azimuthal points. This results in $N_q M_q$ source points. The number of points is chosen depending on the wave number k (see Table 2).

To provide a simple test of the inverse solver we assumed that the Fourier coefficients of the scattered field could be measured on Σ^M for each point source on Σ^S (as was done in [3]). Thus we have at our disposal coefficients $\{u_n(\mathbf{x}_{s,j})\}$ for $n = 1, \cdots, N$, where $N > N_p$ for each source point $\mathbf{x}_{s,j}$ with $j = 1, \cdots, N_q M_q$. Notice that $u_n \approx 0$ if $n > N_p$ since such modes are evanescent, and the measurement surface is far from the scatterer even at the lowest frequency. The the scattered field u^s which depends on the source point $\mathbf{x}_{s,j}$ and measurement point $\mathbf{x} = (x_1, \hat{\mathbf{x}})$ with for $x_1 < a_M$ is given by

$$u^s_{\mathbf{x}_{s,j}}(\mathbf{x}) = u^s(\mathbf{x}; \mathbf{x}_{s,j}) \approx \sum_{n=1}^{N} u_n(\mathbf{x}_{s,j}) \theta_n(\hat{\mathbf{x}}) \exp(i\beta_n(a_M - x_1))$$

Using the reciprocity relation for the scattered field (Lemma 4.5) we thus have the near field equation where we seek $g \in L^2(\Sigma^M)$ such that

$$\int_{\Sigma^M} u^s(\mathbf{x}; \mathbf{x}_{s,j}) g(\mathbf{x})\, dS_{\mathbf{x}} = \Phi_{\mathbf{z}}(\mathbf{x}_{s,j})$$

for $1 \leq j \leq N_q M_q$. This can be solved approximately using the usual LSM Morozov/Tikhonov approach as given in [10]. Although there is no relationship between our forward and inverse solvers, to further avoid numerical crimes, we perturb each Fourier coefficient by a random number uniformly distribution in $\xi_{n,j} \in (-\epsilon, \epsilon)$ with $\epsilon = 0.01$ and the perturbed data is then given by

$$u^\epsilon_n(\mathbf{x}_{s,j}) = u_n(\mathbf{x}_{s,j})(1 + \xi_{n,j}).$$

This gives almost 1% error in the spectral norm for the resulting matrix, and this error is used for the Morozov/Tikhonov algorithm.

[1] Available from `http://people.sc.fsu.edu/~jburkardt/f_src/sandia_rules/sandia_rules.html`

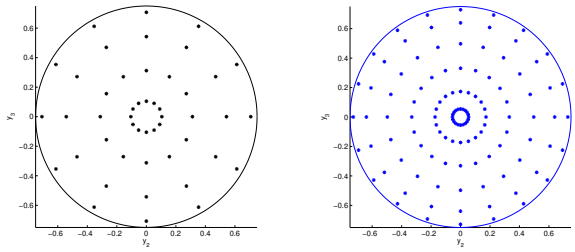


FIGURE 3. Distribution of source points in the cross section Σ of the pipe for the numerical experiments reported here. Each star marks the position of a point source (see Table 2 for parameters). Left: Points used for 2KHz and 6KHz. Right: Points used for 10KHz.

5.1. **The examples.**

5.1.1. *Pipe with ball.* In [3], several examples are shown of reconstructions of impenetrable balls in a waveguide. Our first example is a penetrable ball suspended in the waveguide having wave speed and density (chosen arbitrarily to give some penetration of the field into the ball) of $c = 600$ m/s and density $\rho = 20$ kg/m^3. Thus

$$A = \frac{1.2}{20} < 1 \quad \text{and} \quad \mathrm{n} = \left(\frac{1.2}{20}\right)\left(\frac{343}{600}\right)^2 < 1.$$

So the conditons of Lemma 3.1 are met.

The example is intended to check that our results for penetrable media are similar to published data for sound hard objects. The scatterer is a ball of radius 0.03m centered at the origin and completely contained inside the pipe. The configuration is shown in Figure 4.

5.1.2. *Pipe with hard obstruction.* The next example is a sound hard obstruction formed by cutting out a portion of the pipe. The computational domain is the cylinder of radius 0.075m with axis along the x_1-axis having removed the intersection of this pipe with a cylinder of the same radius with axis parallel to the x_2-axis translated down parallel to the x_3 axis by 0.125m. This results in a partial cutout as shown in Figure 8. This example is not covered by our theory, but is covered by an extension of the theory in [3]. Since hard obstructions occur in practice we study this case here.

5.1.3. *Pipe with penetrable obstruction.* Our third example, covered by our theory but not by that from [3], is a penetrable obstruction occupying a bounded region in the pipe formed by cutting out a portion of the pipe. The computational domain is the entire cylinder of radius 0.075m with axis along the x_1-axis. The penetrable obstruction is the region formed by intersecting this pipe with another cylinder of the same radius and with axis parallel to the x_2-axis translated down parallel to the x_3 axis by 0.125m. This results in a partial blockage as shown in Figure 12. The boundary of this obstacle is the same as the hard boundary of the previous example, but now the wave can penetrate the obstruction. We choose the same parameters for the obstruction as for the ball in Section 5.1.1.

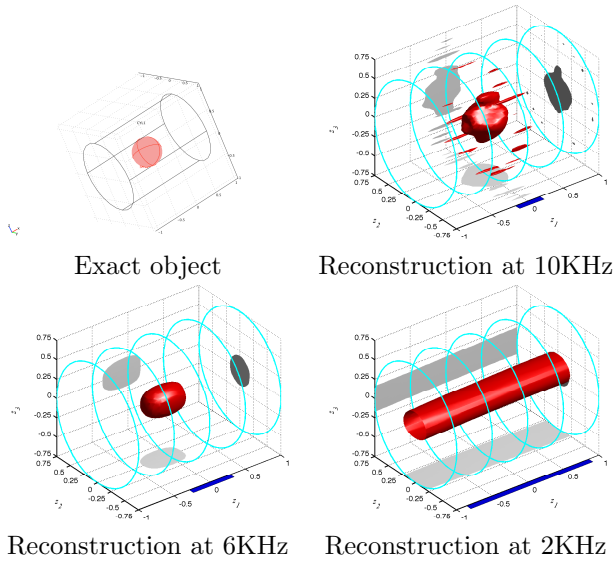


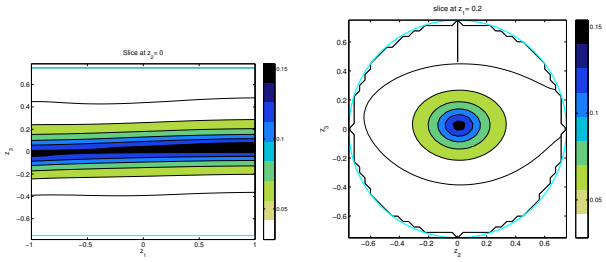
FIGURE 4. Ball scatterer: exact object and reconstruction iso-surfaces of $1/\|g\|$ as a function of the source point $\mathbf{z}$ using the iso-value defined in the text. At 2KHz the scatterer is not localized along the axis of the pipe.

5.1.4. *Blocked pipe.* For our last example, the pipe is the semi-infinite domain $(-\infty, 0) \times \Sigma$. The sound hard boundary condition $\partial u/\partial x_1 = 0$ is applied on the end Σ_0. This models a fully blocked pipe. In this case the solution is easily written using a modal expansion together with the modal expansion of the fundamental solution Φ given in [3]. Neither the theoretical results of [3] or our paper cover this example.

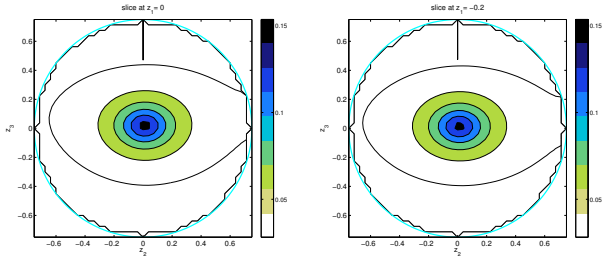
5.2. **Results.**

5.2.1. *Pipe with ball.* Results are shown in Figures 4 - 7. At 2KHz results are shown in Figures 4 (bottom right panel) and Figure 5. At this frequency there are just 3 propagating modes, and we see that there is insufficient data to reconstruct the ball. At 6KHz (see Figures 4 (bottom left panel) and Figure 6) there are 21 propagating modes and the reconstruction is elongated but the single scatterer is clearly visible. At 10KHz (see Figures 4 (top right panel) and Figure 7) the size of the ball is improved, but the result is very noisy, perhaps due to poor performance of the forward solver (i.e. error in the computed data).

5.2.2. *Pipe with hard obstruction.* Results are shown in Figures 8 - 11. Again at 2KHz the scatterer is not reconstructed (see the bottom right panel of Figure 8 and Figure 9). At 6KHz and 10KHz (see Figure 8 and Figures 10 - 11) only the surface of the obstruction closest to the measurement surface (i.e. on the left) is well reconstructed. This is not surprising since sources and measurements are all located to the left of the obstruction. Using sources and measurements from either side of the obstacle would improve the reconstruction.

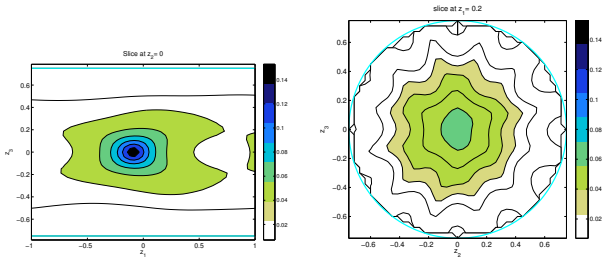


z_1, z_3 plane at $z_2 = 0$ z_2, z_3 plane at $z_1 = 0.2$

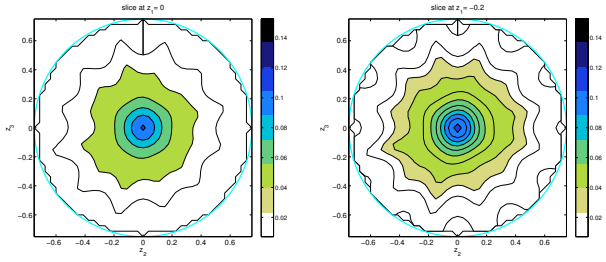


z_2, z_3 plane at $z_1 = 0$ z_2, z_3 plane at $z_1 = -0.2$

FIGURE 5. Ball scatterer, 2KHz: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region. At this frequency the method does not localize the scatterer along the axis of the pipe.

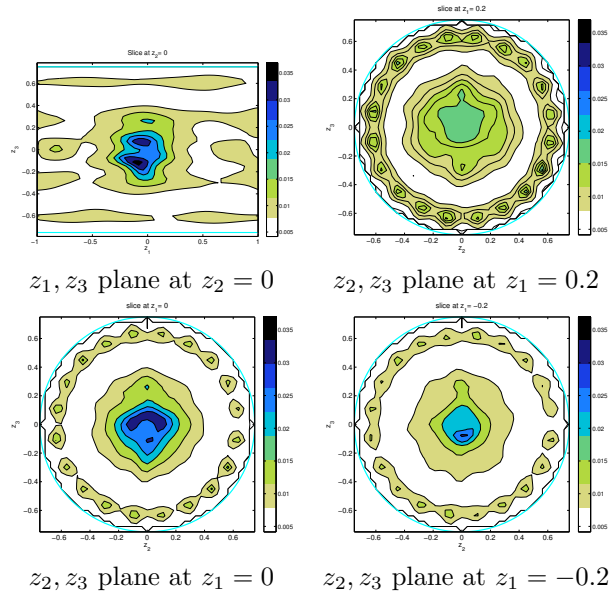


z_1, z_3 plane at $z_2 = 0$ z_2, z_3 plane at $z_1 = 0.2$



z_2, z_3 plane at $z_1 = 0$ z_2, z_3 plane at $z_1 = -0.2$

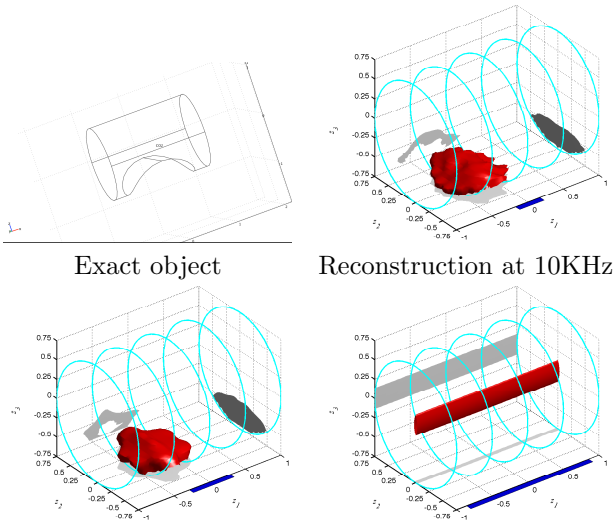
FIGURE 6. Ball scatterer, 6KHz: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region. At this frequency the method now localizes the scatterer along the axis of the pipe (compare to Figure 5).



z_1, z_3 plane at $z_2 = 0$ z_2, z_3 plane at $z_1 = 0.2$

z_2, z_3 plane at $z_1 = 0$ z_2, z_3 plane at $z_1 = -0.2$

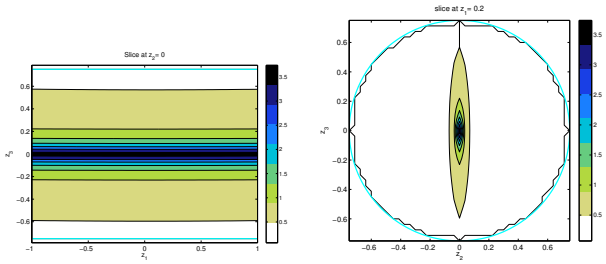
FIGURE 7. Ball scatterer, 10KHz: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region. At this frequency the method now suggests the size of the scatterer along the axis of the pipe but is rather noisy (compare to Figure 6).



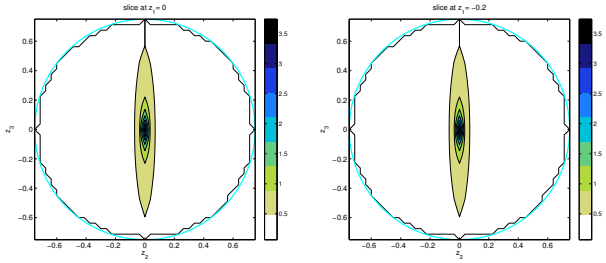
Exact object Reconstruction at 10KHz

Reconstruction at 6KHz Reconstruction at 2KHz

FIGURE 8. Hard obstruction: exact boundary and reconstruction iso-surfaces of $1/\|g\|$ as a function of the source point $\mathbf{z}$ using the iso-value defined in the text. At 2KHz the scatterer is not well localized.

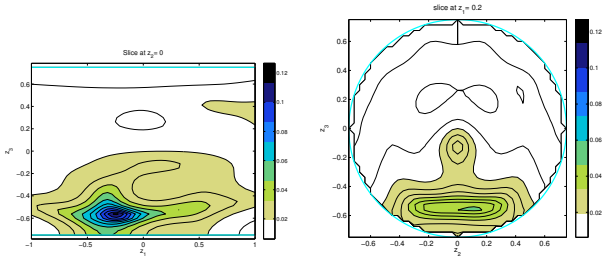


z_1, z_3 plane at $z_2 = 0$ z_2, z_3 plane at $z_1 = 0.2$

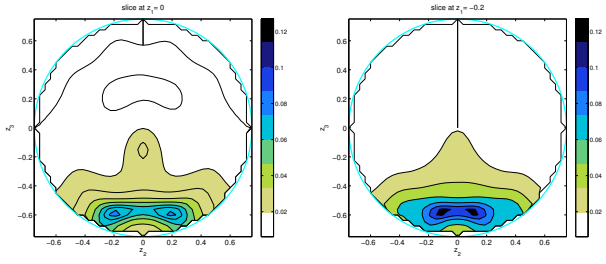


z_2, z_3 plane at $z_1 = 0$ z_2, z_3 plane at $z_1 = -0.2$

FIGURE 9. Hard obstruction, 2KHz: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region.

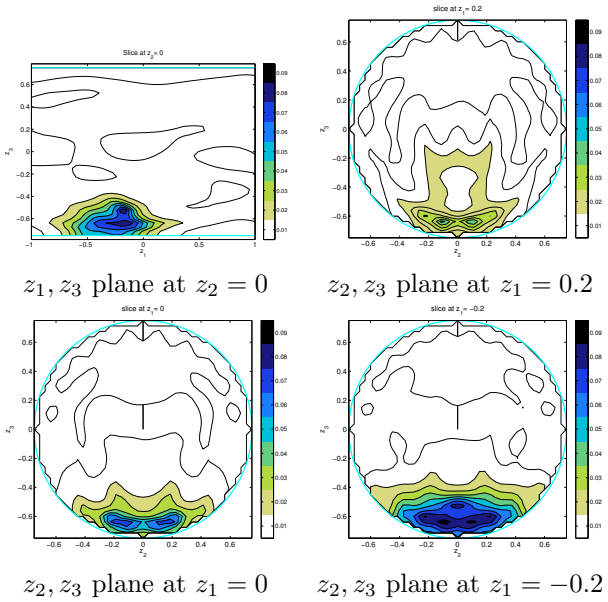


z_1, z_3 plane at $z_2 = 0$ z_2, z_3 plane at $z_1 = 0.2$



z_2, z_3 plane at $z_1 = 0$ z_2, z_3 plane at $z_1 = -0.2$

FIGURE 10. Hard obstruction, 6KHz: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region.



z_1, z_3 plane at $z_2 = 0$ z_2, z_3 plane at $z_1 = 0.2$

z_2, z_3 plane at $z_1 = 0$ z_2, z_3 plane at $z_1 = -0.2$

FIGURE 11. Hard obstruction, 10KHz: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region.

5.2.3. *Pipe with penetrable obstruction.* For the penetrable partial obstruction, results are shown in Figures 12 - 15. Results are broadly comparable to those for the sound hard obstruction in Section 5.2.2.

5.2.4. *Blocked pipe.* Results for this problem are the most surprising in this paper. In Figure 16 (top row) we show the exact scatterer and an attempt to reconstruct the object using standard data at 6KHz. The cylindrical wall of the pipe is clearly visible, but the wall at $x_1 = 0$ is not visible as shown in the isovalue plots in Figure 17. Other tests at different frequencies and using a different number of sources (not shown) reveal the same problem. As far as we are aware, this is the first example where the LSM or RGM methods fail completely to detect a scatterer.

We have no rigorous explanation of why the method fails in this case. One possible reason for the failure is to note that, for each point source, the scattered field is the field due to an image point at $x_1 = 4.1$m, and that the reconstruction of the pipe is consistent with this data. Another possible reason for the failure is that we make measurements at $x_1 = -4$m which is far from the end of the pipe. Evanescent modes excited near $x_1 = 0$ have decayed strongly by $x_1 = -4$m and perhaps the absence of these modes prevents the reconstruction of the end wall. To test this we placed the sources at $x_1 = -0.05$m and receivers at the same place. Then using this data the LSM reconstructed the end wall as shown in the lower panel of Figure 16 and the cross-sections in Figure 18.

6. **Conclusions.** We have proved that the Linear Sampling Method and Reciprocity Gap Method have the standard properties of blowup of appropriate kernels when applied to the detection of penetrable partial blockages in a cylindrical waveguide or pipe. In this application the RGM is particularly attractive because it can

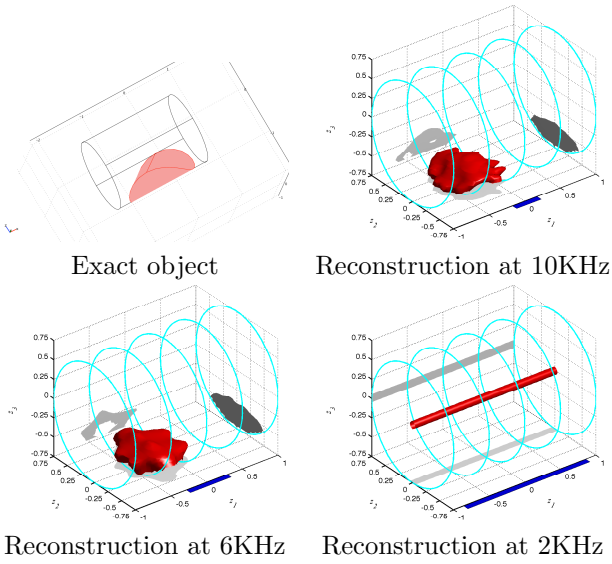


FIGURE 12. Penetrable obstruction: exact domain and reconstruction iso-surfaces of $1/\|g\|$ as a function of the source point $\mathbf{z}$ using the iso-value defined in the text. Results are similar to those for the hard object in Figure 8.

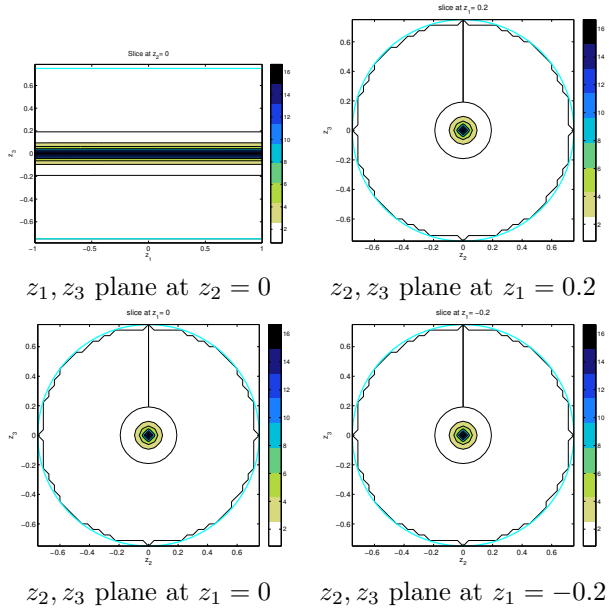
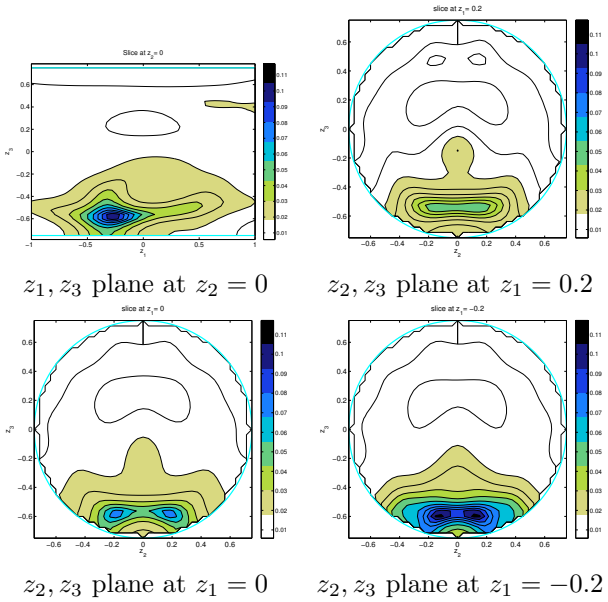


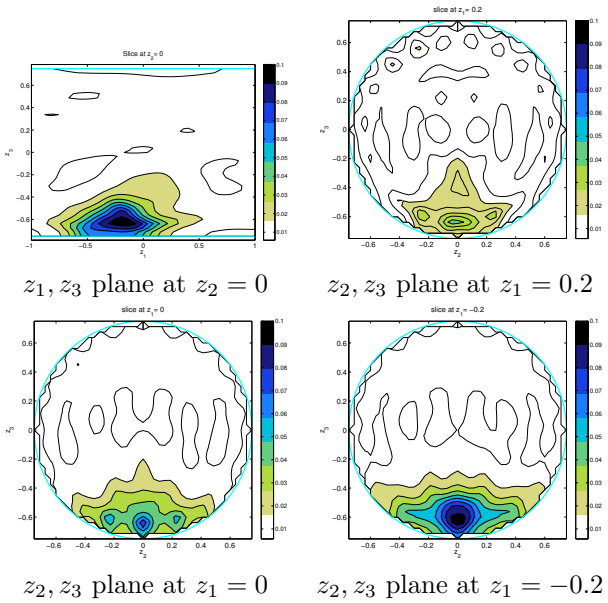
FIGURE 13. Penetrable obstruction, 2KHz: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region.



z_1, z_3 plane at $z_2 = 0$ z_2, z_3 plane at $z_1 = 0.2$

z_2, z_3 plane at $z_1 = 0$ z_2, z_3 plane at $z_1 = -0.2$

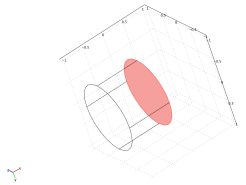
FIGURE 14. Penetrable obstruction, 6KHz: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region.



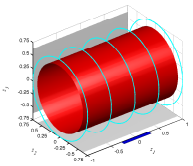
z_1, z_3 plane at $z_2 = 0$ z_2, z_3 plane at $z_1 = 0.2$

z_2, z_3 plane at $z_1 = 0$ z_2, z_3 plane at $z_1 = -0.2$

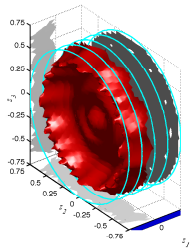
FIGURE 15. Penetrable obstruction, 10KHz: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region.



Exact object

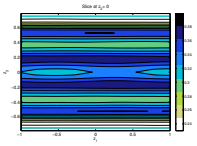


Reconstruction at 6KHz

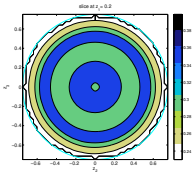


Reconstruction at 6KHz (near data)

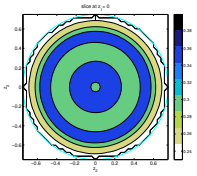
FIGURE 16. Blocked pipe: exact domain, reconstruction isosurfaces of $1/\|g\|$ at 6KHz as a function of the source point $\mathbf{z}$ using the iso-value defined in the text, and near field reconstruction isosurfaces of $1/\|g\|$ at 6KHz (using the isovalue 0.075).



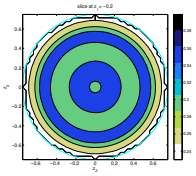
z_1, z_3 plane at $z_2 = 0$



z_2, z_3 plane at $z_1 = 0.2$



z_2, z_3 plane at $z_1 = 0$



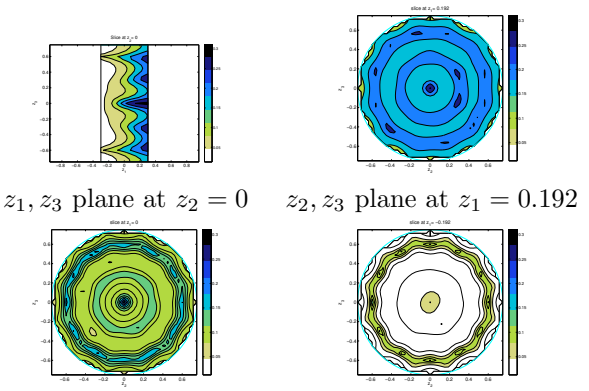
z_2, z_3 plane at $z_1 = -0.2$

FIGURE 17. Blocked pipe, 6KHz: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region.

be used to avoid the need to find the background fundamental solution for the pipe with access holes.

Numerical results show that, provided there are sufficiently many propagating modes, these methods can reconstruct hard or penetrable blockages in the pipe using distant measurements. These results suggest that methods based on multistatic data could have an application to surveying sewer pipes.

We also encountered one unusual case: the method did not detect a completely blocked pipe (semi-infinite sound hard cylinder).



z_1, z_3 plane at $z_2 = 0$ z_2, z_3 plane at $z_1 = 0.192$

z_2, z_3 plane at $z_1 = 0$ z_2, z_3 plane at $z_1 = -0.192$

FIGURE 18. Blocked pipe, 6KHz, near field data: contour plots of $1/\|g\|$ as a function of $\mathbf{z}$ at selected planes intersecting the search region.

These observations point to the need for further study of the method in this application. We need to characterize the scatterers that cannot be detected by this method, and determine ways to detect these anomalous cases (possibly combining RGM with other inversion techniques). We also need to study the quality of the reconstructions as a function of frequency and number and position of the sources in order to determine the simplest setup that can provide sufficiently realistic reconstructions. The application of multifrequency data to this problem also needs to be investigated.

Appendix A. **Proof of existence for the solution of the forward problem.** In this appendix we outline the proof Proposition 2.

To start with, we recall that in [3, pp.17–18] it is shown that

$$(\sum_{n\in\mathbb{N}} k_n \, |(\cdot,\theta_n)_{\Sigma_{\pm R}}|^2)^{1/2} \tag{21}$$

defines a norm in $H^{1/2}(\Sigma_{\pm R})$ that is equivalent to $||\cdot||_{H^{1/2}(\Sigma_{\pm R})}$. Then, by duality, we also know that

$$(\sum_{n\in\mathbb{N}} \frac{1}{k_n} \, |\langle\cdot,\theta_n\rangle_{\Sigma_{\pm R}}|^2)^{1/2} \tag{22}$$

is a norm in $\tilde{H}^{-1/2}(\Sigma_{\pm R})$ equivalent to $||\cdot||_{\tilde{H}^{-1/2}(\Sigma_{\pm R})}$.

Using the above equivalence of norms (21) and the expression of the Dirichlet-to-Neumann operator (1), we can reason as in [2, Lemma 2.1] to deduce that there exists $C > 0$ (a constant independent of k) such that

$$||T^{\pm R} g||_{\tilde{H}^{-1/2}(\Sigma_{\pm R})} \leq C\,(|k|^2+1)\,||g||^2_{H^{1/2}(\Sigma_{\pm R})} \quad \forall g \in H^{1/2}(\Sigma_{\pm R}). \tag{23}$$

Thus, $T^{\pm R} : H^{1/2}(\Sigma_{\pm R}) \to \tilde{H}^{-1/2}(\Sigma_{\pm R})$ is linear continuous for each $k \in \mathbb{C}$ with $\mathrm{Re}(k) > 0$ and $\mathrm{Im}(k) \geq 0$.

Let us choose $k_0 \in \mathbb{R}$ such that $k_1 > k_0 > 0$, and consider $k \in \mathbb{R}$ with $k_0 \geq k > 0$. Then, for every $n \in \mathbb{N}$, $k_n > k_0$, so that $\beta_n = \imath|\beta_n|$. Besides,

$$|\beta_n|^2 = k_n^2 - k^2 \geq k^2(\frac{k_n^2}{k_0^2} - 1) \geq C^2 k^2 \,.$$

Thus, we can use the expression of the Dirichlet-to-Neumann operator (1) to deduce the following counterpart of [2, Lemma 2.2]:

$$\mp \langle T^{\pm R} g, g\rangle_{\Sigma_{\pm R}} \geq C\, k \sum_{n\in\mathbb{N}} |(g,\theta_n)_{\Sigma_{\pm R}}|^2 \geq C' k\, ||g||^2_{L^2(\Sigma_{\pm R})} \quad \forall g \in H^{1/2}(\Sigma_{\pm R}). \tag{24}$$

Furthermore, if $k \in \mathbb{R}$ with $k > 0$, we can take $N \in \mathbb{N}$ such that $k_N < k < k_{N+1}$. In this case,

$$\beta_n \in \mathbb{R} \quad \text{with } \beta_n > 0 \qquad \text{if } n \leq N,$$

whereas

$$-\imath\beta_n \in \mathbb{R} \quad \text{with } -\imath\beta_n > 0 \qquad \text{if } n > N.$$

Therefore, we can adapt [2, Lemma 2.3] to show that

$$\mathrm{Re}(\mp\langle T^{\pm R} g, g\rangle_{\Sigma_{\pm R}}) \geq 0, \tag{25}$$

and there exists a constant $C > 0$ such that

$$\mathrm{Im}(\mp\langle T^{\pm R} g, g\rangle_{\Sigma_{\pm R}}) \geq -\,k \sum_{n=1}^{N} |(g,\theta_n)_{\Sigma_{\pm R}}|^2 \geq -Ck\, ||g||^2_{L^2(\Sigma_{\pm R})}, \tag{26}$$

for all $g \in H^{1/2}(\Sigma_{\pm R})$.

We now have all the ingredients needed to show that Lemma 2.4 of [2] holds in our case. Indeed, the proof is just a simplification of that given in the aforementioned reference.

Lemma A.1. *Suppose $k^* \neq k_n$ for all $n \in \mathbb{N}$. Then, there is a neighborhood U of k^* such that for all $k \in U$ the Dirichlet-to-Neumann operator $T^{\pm R} : H^{1/2}(\Sigma_{\pm R}) \to \tilde{H}^{-1/2}(\Sigma_{\pm R})$ defined by (1) is analytic with respect to k.*

Taking advantage of (25), we can show that the following counterpart of [2, Lemma 3.3] holds true:

$$-\,\mathrm{Re}(a(w,w)) \geq \alpha |w|^2_{H^1(B_R)} - |k|^2\, ||\mathrm{n}||_{L^\infty(D)} ||w||^2_{L^2(B_R)} \quad \forall w \in H^1(B_R), \tag{27}$$

where $\alpha > 0$ is independent of k, and $k \in \mathbb{R}$, $k > 0$. Therefore, we can apply Fredholm alternative as in [2, Corollary 3.4] to deduce that the forward problem (2) is well-posed if and only if it admits at most one solution.

We finally conclude the statement of Proposition 2 reasoning as in [2, Theorem 3.7].

REFERENCES

[1] T. Arens, D. Gintides, and A. Lechleiter, *Direct and inverse medium scattering in a 3D homogeneous planar waveguide*, SIAM J. Appl. Math., 71 (2011), pp. 753–72.

[2] T. Arens, D. Gintides, and A. Lechleiter, *Variational formulations for scattering in a three-dimensional acoustic waveguide*, Math. Meth. Appl. Sci., 31 (2007), pp. 821-847.

[3] L. Bourgeois and E. Lunéville, *The linear sampling method in a waveguide: a modal formulation*, Inv. Prob., 24 (2008), pp. 15–18.

[4] F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer-Verlag, New-York, 2006.

[5] F. Cakoni, M. Fares, and H. Haddar, *Analysis of two linear sampling methods applied to electromagnetic imaging of buried objects*, Inv. Prob., 22 (2006), pp. 845–67.
[6] F. Cakoni and H. Haddar, *Interior transmission problem for anisotropic media*, in Mathematical and Numerical Aspects of Wave Propagation -WAVES 2003 (Cohen et al., eds.), Springer-Verlag, 2003, pp. 613-618.
[7] D. Colton and H. Haddar, *An application of the reciprocity gap functional to inverse scattering theory*, Inv. Prob., 21 (2005), pp. 383–98.
[8] D. Colton and A. Kirsch, *A simple method for solving inverse scattering problems in the resonance region*, Inv. Prob., 12 (1996), pp. 383–93.
[9] D. Colton, L. Paivarinta and J. Sylvester, *The interior transmission problem*, Inverse Problems and Imaging, 1(1) (2007), pp. 13–28.
[10] D. Colton, M. Piana, and R. Potthast, *A simple method using Morozov's discrepancy principle for solving inverse scattering problems*, Inv. Prob., 13 (1997), pp. 1477–93.
[11] K. Horoshenkov, R. Ashley, and J. Blanksby, *Determination of sewer roughness and sediment properties using acoustic techniques*, Water Science and Technology, 47 (2003), p. 87–93.
[12] P. Monk and V. Selgas, *Near field sampling type methods for the inverse fluid–solid interaction problem*, Inverse Problems and Imaging, 5 (2011), pp. 465–83.
[13] B. Pincon and K. Ramdani, *Selective focusing on small scatterers in acoustic waveguides using time reversal mirrors*, Inv. Prob., 23 (2007), pp. 1–25.
[14] F. Podd, M. Ali, K. Horoshenkov, A. Wood, S. Tait, J. Boot, R. Long, and A. Saul, *Rapid sonic characterisation of sewer change and obstructions*, Water Science and Technology, 56 (2007), pp. 131–9.
[15] P. Roux and M. Fink, *Time reversal in a waveguide: Study of the temporal and spatial focusing*, J. Acoust. Soc. Am., 107 (2000), pp. 2418–29.
[16] P. Roux, B. Roman, and M. Fink, *Time-reversal in an ultrasonic waveguide*, Appl. Phys. Lett., 70 (1997), pp. 1811–3.
[17] A. Tolstoy, K. Horoshenkov, and M. Bin Ali, *Detecting pipe changes via acoustic matched field processing*, Applied Acoustics, 70 (2009), pp. 695–702.
[18] Y. Xu, C. Matawa, and W. Lin, *Generalized dual space indicator method for underwater imaging*, Inv. Prob., 16 (2000), pp. 1761–76.

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