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Embedded Systems and Industrial Controller EE5563 (2)

Today's discussions

- Modelling and Simulation of Dynamic Systems
  - I/O models
  - State-space models
  - Linear state equations
  - Time invariant systems
  - Open Loop closed Loop systems
- Laplace Transforms, Transfer Functions, Block Diagrams

1. Define system boundaries and specifications
2. Specify system Inputs/Outputs (i.e. *system variables*)
3. Understand and estimate the components of the system using models
4. Draw the systems components diagram
5. Write systems equations:
   - physical laws (constitutive),
   - continuity equation for *through variables, e.g. equilibrium of forces at joints, current balance at nodes, …)*
   - *Cross variables (equations for velocity, voltage, pressure drop, …)*
6. Express system boundary conditions and response initial conditions using system variables.

Basic Steps Modelling Dynamic Systems

- State variables are the minimal set of variables that describe the dynamic state of a system.

- The state space represents the dynamics of an *n*th order system is defined *n* first-order differential equations – normally coupled.
- Example of first order linear differential equation:

$$\acute{x}_1 = x_1 + 2x_2$$
$$\acute{x}_2 = 3x_1 + 2x_2$$

We call it coupled because you need the knowledge of one variable to find the other.

State-Space Models

- A state vector ***x*** is a column vector of $(x_1, x_2, \dots x_n)$ variables that describe the state of a dynamic system.
- n is the number of state variables and represents the *order* of the system.

- ***Property 1:*** The transformation $\boldsymbol{g}$ of $x(t_0)$ to $x(t_1)$ with respect to input $\boldsymbol{u}[t_0, t_1]$ in the interval $[t_0, t_1]$:

$$x(t_1) = \boldsymbol{g}(t_0, t_1, x(t_0), \boldsymbol{u}[t_0, t_1] \qquad (1)$$

Each excitation creates a trajectory – *n* trajectory will create **state-space**

Properties of State-Space

Properties of State-Space

» ***Property 2:*** System output $\boldsymbol{y}(t_1)$is determined by the state $x(t_1)$ and input (excitation) $u(t_1)$ at any given time t_1. Expressed as:

$$y(t_1) = \boldsymbol{h}(t_1, x(t_1), \boldsymbol{u}(t_1)) \qquad (2)$$

In other words system output at time t_1 depends on the following factors:

1. Time
2. State vector
3. The input

Transformation h has no memory

State model

- A state model thus consists of *n state equations* (first-order ordinary differential equations (time domain), which are coupled.
- In the vector for expressed as:

$$\dot{x} = f(x, u, t) \qquad (3)$$
$$y = h(x, u, t) \qquad (4)$$

First order Differential Equation

Algebraic Output equation

- Example of first order linear differential equation:

$$\acute{x}_1 = x_1 + 2x_2$$
$$\acute{x}_2 = 3x_1 + 2x_2$$

We call it coupled because you need the knowledge of one variable to find the other.

An *n*th order linear state model is given by the differential state equations as:

$$
\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \ldots + b_{1r}u_r \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \ldots + b_{2r}u_r \\
&. \\
&. \\
\dot{x}_n &= a_{n1}x_1 + a_{12}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \ldots + b_{nr}u_r
\end{aligned} \qquad (5)
$$

Where, $\dot{x}_1 = \dfrac{dx}{dt}$; $x_1, x_2, \ldots x_n$ are the state variables and $u_{1\ldots r}$

Are the input variables.

Linear State Equation

The Output Equation

The algebraic output equations are thus:

$$y_1 = c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \ldots + d_{1r}u_r$$
$$y_2 = c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \ldots + d_{2r}u_r$$
$$.$$
$$.$$
$$.$$
$$y_m = c_{m1}x_1 + c_{m2}x_2 + \ldots + c_{mn}x_n + d_{m1}u_1 + d_{m2}u_2 + \ldots + d_{mr}u_r \qquad (6)$$

Where $y_{1 \ldots m}$ are the output variables of the system

» The vector-matrix form of equations (5 and 6):

$$\dot{x} = \mathbf{Ax} + \mathbf{Bu} \qquad (7)$$

$$y = \mathbf{Cx} + \mathbf{Du} \qquad (8)$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}; \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ . \\ \dot{x}_n \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & & \\ . & . & & \\ a_{n1} & a_{n1} & \dots & a_{2n} \end{bmatrix};$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1r} \\ b_{21} & b_{22} & \dots & b_{2r} \\ . & & & . \\ . & \dots & \dots & . \\ . & & & . \\ b_{n1} & b_{n2} & \dots & b_{nr} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ . & & & . \\ . & \dots & \dots & . \\ . & & & . \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1r} \\ d_{21} & d_{22} & \dots & d_{2r} \\ . & & & . \\ . & \dots & \dots & . \\ . & & & . \\ d_{m1} & d_{m2} & \dots & d_{mr} \end{bmatrix}$$

The Vector-Matrix of State Model

- State, output, order of a system, and system initial state
- According to Newton's second Law, the rectilinear motion of a mass (*m*) with respect to an *input force u(t),* the position *x* can be expressed as:

$$m\frac{d^2x}{dt^2}=u(t) \quad \text{or } m\ddot{x}-u=0 \qquad (9)$$

- Developing the I/O and state models:

The I/O models:

1. *When x to be the output : y=x* → $m\frac{d^2y}{dt^2}=u(t)$

2. *When Output* $y=v=\dot{x}$ → $m\frac{dy}{dt}=u(t)$

3. the two outputs $y_1=x$ and $y_2=v=\dot{x}$ → $m\frac{d^2y_1}{dt^2}=u(t)$

$$m\frac{dy_2}{dt}=u(t)$$

Example 1

The state space model:

1. Define the two state variables when $x_1 = x \; ; \; x_2 = \frac{dx}{dt}$
   then:

$$\dot{x}_1 = x_2 \quad \text{the State equation:} \quad \dot{x}_2 = \frac{1}{m} u(t)$$

$$\text{output equation:} \quad y = x_1$$

$$\dot{x}_1 = x_2 \quad \text{the State equation:} \quad \dot{x}_2 = \frac{1}{m} u(t)$$

$$\text{output equation:} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Vector Matrix Illustration

Example 1 Continued

The state space model:

2. Define the two state variable when $x_1 = -6x$ and $x_2 = -\frac{1}{2}(\dot{x}_1)$

State equations: $\dot{x}_1 = -2x_2$

$$\dot{x}_2 = \frac{3}{m}u(t)$$

output equation: $y = -\frac{1}{6}x_1$

$$\dot{x}_1 = -2x_2$$

$$\dot{x}_2 = \frac{3}{m}u(t)$$

then one state equation is given by one of the definitions itself, and the other state equation is obtained by substituting the two definitions in into equation (9).

$$\text{output equation}: \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1/6 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Vector Matrix Illustration

$-2 \times (-\frac{1}{6})$

Example 1 Continued

Conclusions

- State vector is the minimum set of variables that determines the dynamic state of a system.
- State vectors are not unique and may choices are possible for a given system
- Output variables can be determined from any choice of state variable.
- State variables may or may not have physical representation.

If $\dot{x} = f(x, u, t)$ and $y = h(x, u, t)$ are not explicitly time dependent pointing to the fact that equations (7 & 8):

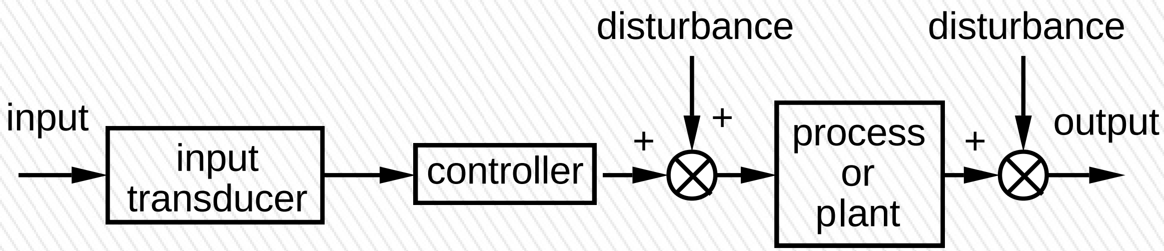
$$\dot{x} = \mathbf{Ax} + \mathbf{Bu}$$

$$y = \mathbf{Cx} + \mathbf{Du}$$

are **constant.**

Brief about stationary dynamic systems

Open Loop Systems

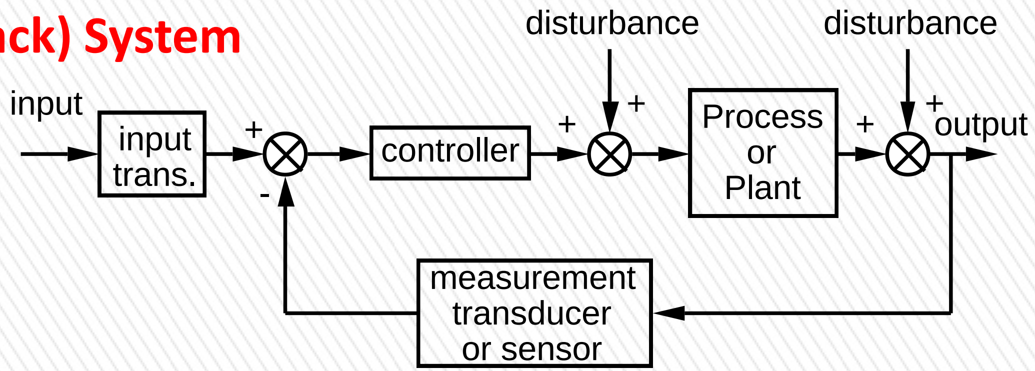


- A good model of a process/plant allows us to work out the input values to achieve the desired output

- The controller <u>cannot</u> compensate for any disturbances.

Control and Modelling of Open Loop Systems

Closed Loop (Feedback) System



- » all the examples of early control systems involved some form of measurement of the output.
- » The output then is then fedback to be compared to the reference value.
- » **Note:** the controller now acts on the difference between the input and output values (scaled by the transducer).
- » if the transducers have unity gain i.e. amplify the signal by 1
  - > the controller acts on the error signal.

Control and Modelling of Open Loop Systems

» Transfer-function models (Laplace transfer-functions) are based on the Laplace transform.

» Means to represent dynamic models in Laplace domains

» Frequency Domain models (frequency transfer functions) are special case of Laplace domain models, and are based on Fourier transform

» They are interchangeable

» Systems with single I/O can be represented uniquely by one transfer function

» System with 2 or more I/O (input and output vectors) need several transfer functions

» Only minimum knowledge of the theory of Laplace and Fourier transform is needed to use transfer functions for modelling

Transfer-Function & Frequency Domain

Laplace & Fourier transform

- The Laplace transform converts :
  - differentiation into a multiplication by the Laplace variable "*s*". *And*
  - Integration into division by "*s*".

- Fourier transform – special case of the Laplace transform

By setting the $s = j\omega$

Complex (non-real) variable

Where ω is the frequency variable.

- The preference of which domain to use depends on the nature of the problem, input, duration, and the measures.
- Most functions used are in the for of t^n or e^t or $sin\omega t$ so $f(t) = y$

Laplace Transform Methods

- The Laplace transform, transforms from time domain to Laplace domain (complex frequency domain)
- The Laplace transform is an integral transform defined as:

$$Y(s) = \int_0^\infty y(t)\exp(-st)\,dt \quad or\; Y(s) = Ly(t) \qquad (10)$$

$s = j\omega$ Laplace variable and j is the complex number

L: Laplace operator

- The inverse of Laplace transform: $y(t) = L^{-1}Y(s)$ (11)

The Laplace transform is linear so as:

$$if\ y(t) = k.y(t);\quad then\ Y(s) = k.Y(s)\ \ where\ k\ is\ contant$$

And $Y_1(s)$ $is\ the\ Laplace\ Transform\ of\ y_1(t)$

In other words $\mathbf{L}\ \{y_1(t)\} = Y_1(s)$

If $y(t) = \ y_1(t) + y_2(t);\ \ then\ Y(s) = Y_1(s) + Y_2(s)$ (12)

Laplace Transform Methods Cont.

Laplace Transform of a Derivative

Based on equation (10):

Initial condition

$$L\dot{y} = \int_0^\infty e^{-st}\frac{dy}{dt}dt = \mathrm{sY(s) - y(0)} \qquad (13)$$

By continuously applying equation (11) we can achieve Laplace transform of the higher derivatives:

$$L\ddot{y} = sL[\dot{y}t] - \dot{y}(0) = s[sY(s) - y(0)] - \dot{y}(0)$$

$$\therefore L\ddot{y} = s^2L[y(t)] - sy(0) - \dot{y}(0) \qquad (14)$$

And

$$L\dddot{y} = s^3Y(s) - s^2y(0) - s\dot{y}(0) - \ddot{y}(0) \qquad (15)$$

$$L\frac{d^ny}{dt^n} = s^nY(s) - s^{n-1}y(0) - s^{n-2}\dot{y}(0) - \cdots - \frac{d^{n-1}y}{dt^{n-1}}(0) \qquad (16)$$

- The Laplace transform time integral $\int_0^t y(\tau)d\tau$ is reached by the direction application of equation (10):

ie $Y(s) = \int_0^{\infty} y(t)\exp(-st)\,dt \quad or\ Y(s) = Ly(t)$ (10)

$$\longrightarrow \quad L\int_0^t y(\tau)d\tau = \int_0^{\infty} e^{-st}\int_0^t y(\tau)d\,\tau dt = \int_0^{\infty}\left(-\frac{1}{s}\right)\frac{d}{dt}(e^{-st})\int_0^t y(\tau)d\tau dt$$

Integrating by parts: $\int udv = uv - \int vdu$ results into:

$$L\int_0^t y(\tau)d\tau = \left(-\frac{1}{s}\right)e^{-st}\int_0^t y(\tau)d\tau\Big|_0^{\infty} - \int_0^{\infty}\left(-\frac{1}{s}\right)e^{-st}y(t)dt$$

$= 0 - 0 + \int_0^{\infty}\left(\frac{1}{s}\right)e^{-st}y(t)dt \rightarrow L\int_0^t y(\tau)d\tau = \frac{1}{s}Y(s)$ (17)

Laplace Transform of an Integral

» The Fourier transform is the process of converting from time domain to frequency domain:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\exp(-j\omega t)\,dt \;\; or\; Y(j\omega = Fy(t) \qquad (18)$$

Where f = cycle frequency and

$\omega = 2\pi$ is the angular frequency variable

Fourier inverse:

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)\exp(j\omega t) \;\; or\; y(t) = F^{-1}Y(j\omega) \qquad (19)$$

Fourier Transform

1^{st} order system I/O model:

output $y(t)$: L $\{y(t) = Y(s)\}$

input $x(t)$: L$\{x(t)\} = X(s)$

$$\frac{dy}{dt} + 2y(t) = 6u(t)$$

Taking the Laplace Transform: L $\dot{y} = sY(s) - y(0)$

$$sY(s) - y(0) + 2(y) = 6U(s)$$

$$\rightarrow (s+2)Y(s) = y(0) + 6U(s)$$

$$\rightarrow Y(s) = \frac{y(0)}{s+2} + \frac{6U(s)}{s+2}$$ Forced response

Free response

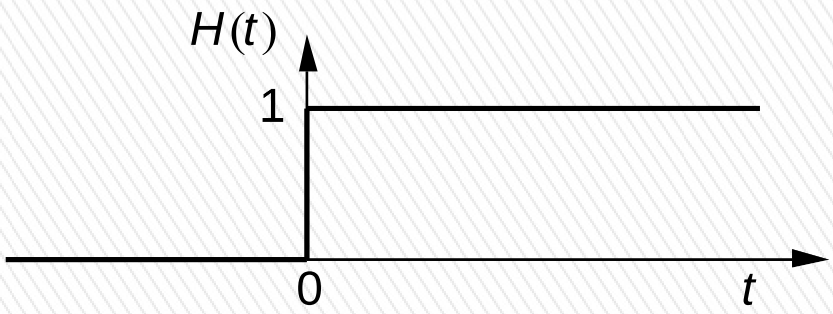
for a linear system, the total response is the sum of the responses to the initial conditions (the free response) and the input (the forced response) applied separately (superimposition).

Example, the principle of superimposition

Laplace Transforms for basic signals

- can determine the Laplace transform of signals from its definition - rather use table of standard transforms.
- Two basic functions are:
  - Unit impulse $\delta(t)$
  - Unit step H(t)=$\begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

looks like



Corresponding Laplace transforms: $\delta(t) \leftrightarrow 1$ unit impulse

$H(t) \leftrightarrow \frac{1}{s}$ unit step

» Delayed time function: $y(t-\tau)H(t-\tau) \leftrightarrow e^{-s\tau}Y(s)$

Note that the exponential function is important in solving LDE

Exponential decay: $e^{-at}H(t) \leftrightarrow \frac{1}{s+a}$

For sine and cosine:

$$sin\omega t\ H(t) \leftrightarrow \frac{\omega}{s^2+\omega^2}$$

$$cos\omega t\ H(t) \leftrightarrow \frac{s}{s^2+\omega^2}$$

More on Laplace Transforms

And various functions can be included

$$e^{-at}y(t)H(t) \leftrightarrow Y(s+a)$$

$$e^{-at}sin\omega tH(t) \leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$$

$$e^{-at}cos\omega tH(t) \leftrightarrow \frac{(s+a)}{(s+a)^2+\omega^2}$$

and

$$tH(t) \leftrightarrow \frac{1}{s^2}$$

$$t^nH(t) \leftrightarrow \frac{n!}{s^{n+1}}$$

$$t^ny(t)H(t) \leftrightarrow (-1)^n\frac{d^n}{ds^n}[Y(s)]$$

$$te^{-at}H(t) \leftrightarrow \frac{1}{(s+a)^2}$$

Laplace Transforms cont.

» we are interested in the value at which a signal “settles down” i.e. *the steady state or final value*

$$\lim_{t \to \infty} f(t)$$

Given by:

$$\lim_{t \to \infty} f(t) = \lim_{s \to \infty} F(s)$$

The theorem stands if only the limit exist (i.e. signalling reaching a steady state)

Final Value Theorem

» Remember the 1st order LDE I/O example:

If $u(t) = 0 \quad \forall t \quad \therefore \quad U(s) = 0$

So

$Y(s) = \frac{y(0)}{s+2} = y(0)\frac{1}{s+2}$ y(0) is a constant

From the linearity problem and standard transform of exponential function:

$$\therefore y(t) = y(0)e^{-2t}H(t)$$

Now consider $y(0) = 0 \;\; and \; u(t) = H(t) \; (the \; unit \; step \; function)$

$$\therefore U(s) = \frac{1}{s}$$ (see slide 25)

And $Y(s) = \frac{6}{s(s+2)} = \frac{3}{s} - \frac{3}{s+2}$ by partial fraction expansion:

$$\therefore y(t) = 3H(t) - 3e^{-2t}H(t) = 3H(t)(1 - e^{-2t})$$

Standard transform table

- This response is called the (**unit**) **step response.**
- The steady state or final value can be calculated:

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} Y(s)$$
$$= \lim_{s \to 0} s \times \frac{6}{s(s+2)} = 3$$

Next week more on Transfer functions and dynamical systems.

Example continued