AN AUGMENTED LAGRANGIAN CONTACT ALGORITHM EMPLOYING A VERTEX-BASED FINITE VOLUME METHOD.

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Introduction

Recently a considerable number of researchers have investigated the suitability of applying Finite Volume (FV) methods to problems in Computational Solid Mechanics (CSM) [8]. Generally, these investigations were motivated by the acknowledged success of applying FV methods, to problems in Computational Fluid Dynamics (CFD) [7].
Presently, there are two main approaches when applying FV methods in CSM, cell-centred [1, 3, 10] or vertex-based [2, 6, 8, 9]. In both approaches the name refers to the discretised location of the solved variable, which in CSM is generally displacement. The first approach is based on traditional FV methods [7] as applied to problems in CFD and suffers from the same difficulties when applied to complex geometries involving arbitrarily structured meshes [1, 3]. The second approach is based on traditional Finite Element (FE) methods [13] and employs shape functions to describe the variation of a solved variable over an element, and is therefore well suited to complex geometries [2, 6]. Both approaches apply strict conservation over a control volume and have demonstrated superiority over traditional FE methods with regard to accuracy [10, 8]. Some researchers have attributed this superiority to the local conservation of a variable as enforced by the control volumes employed [2] and others have attributed it to the enforced continuity of the derivatives of variables across cell or control volume boundaries [10].
In this paper a vertex-based FV method is presented for the computational solution of quasi-static solid mechanics problems involving linear elastic materials undergoing deformable-deformable contact and infinitesimal strains. With regard to the contact algorithm, an augmented Lagrangian method is employed and the problems are analysed numerically with non-coincident meshes [5]. A comparison between the vertex-based FV and the standard Galerkin Finite Element (FE) methods is provided with regard to discretisation and solution accuracy.

Equilibrium Equations and Boundary Conditions

In matrix form the quasi-static equilibrium equation is

$$[L]^T\{\sigma\} + \{b\} = \{0\} \qquad \text{in } \Omega^{(i)}, \tag{1}$$

where $[L]$ is the differential operator, $\{\sigma\}$ is the Cauchy stress, $\{b\}$ is the body force and $\Omega^{(i)}$ is the domain relating to body (i). The boundary conditions on the surface

$\Gamma^{(i)} = \Gamma_t^{(i)} \cup \Gamma_u^{(i)}$ of the domain $\Omega^{(i)}$ can be defined as [13, 6]

$$[R]^T\{\sigma\} = \{t_p\} \qquad \text{on } \Gamma_t^{(i)} \text{ and} \tag{2}$$

$$\{u\} = \{u_p\} \qquad \text{on } \Gamma_u^{(i)}, \tag{3}$$

where $\{t_p\}$ are the prescribed tractions on the boundary $\Gamma_t^{(i)}$, $\{u_p\}$ are the prescribed displacements on the boundary $\Gamma_u^{(i)}$ and $[R]$ is the outward normal operator [8]. The stress is related to the infinitesimal elastic strain as follows; $\{\sigma\} = [D]\{\epsilon_e\} = [D][L]\{u\}$, where $[D]$ is the elasticity matrix and $\{u\}$ is the displacement.
Considering the possible contact surfaces $\Gamma_c^{(i)}$ between the bodies, an auxiliary equilibrium equation can be assumed [5]

$$\mu[G]\{u\} - \{\lambda\} = \{0\} \qquad \text{on } \Gamma_c^{(i)}, \tag{4}$$

where the first term represents a fictitious internal energy and the second term is related to the external tractions. In terms of an augmented Lagrangian method, μ represents the penalty parameter, $\{\lambda\}$ represents the Lagrange multipliers and $[G]\{u\}$ represents the constraints on the displacements of the contacting bodies [5].

Vertex-based Discretisation

Employing the method of weighted residuals to equations (1), (2), (3) and (4), it is possible to obtain the following weak form of the equilibrium equation;

$$\begin{aligned} &- \int_{\Omega^{(i)}} [LW]^T[D][L]\{u\}\, d\Omega + \int_{\Omega^{(i)}} [W]^T\{b\}\, d\Omega + \int_{\Gamma_u^{(i)}} [RW]^T[D][L]\{u\}\, d\Gamma \\ &+ \int_{\Gamma_t^{(i)}} [W]^T\{t_p\}\, d\Gamma + \int_{\Gamma_c^{(i)}} [W_c]^T\mu[G]\{u\}\, d\Gamma - \int_{\Gamma_c^{(i)}} [W_c]^T\{\lambda\}\, d\Gamma = \{0\}, \end{aligned} \tag{5}$$

where $[W]$ is a diagonal matrix of arbitrary weighting functions [6, 9]. It is now possible to approximate the displacements using shape functions as follows; $u \approx \hat{u} = \sum_{j=1}^{nv}[N]_j\bar{u}_j$ where $\bar{u}_j$ are the unknown displacements and $[N]_j$ are the shape functions, both relating to the nv vertices. Dispensing with the contact body indicial notation in order to simplify the inclusion of the approximation in equation (5) and assuming the displacement approximation satisfies the contact constraints, it is possible to obtain the following contributions to the overall system of equations;

$$[K]_{ij} = \int_{\Omega_i} [LW]_i^T[D][LN]_j\, \mathrm{d}\Omega - \int_{\Gamma_{u_i}} [RW]_i^T[D][LN]_j\, \mathrm{d}\Gamma + \int_{\Gamma_{c_i}} [W]_i^T\mu[N]_j\, \mathrm{d}\Gamma, \tag{6}$$

$$\{f\}_i = \int_{\Omega_i} [W]_i^T\{b\}\, \mathrm{d}\Omega + \int_{\Gamma_{t_i}} [W]_i^T\{t_p\}\, \mathrm{d}\Gamma + \int_{\Gamma_{c_i}} [W]_i^T\{\lambda\}\, \mathrm{d}\Gamma, \tag{7}$$

where Ω_i is the control volume associated with the vertex i and $\Gamma_i = \Gamma_{u_i} \cup \Gamma_{t_i} \cup \Gamma_{c_i}$ is the boundary of the control volume. Additionally, $[W_c] = -[W]$ and the arbitrary weighting functions $[W]$ are replaced by a finite set of functions $\sum_{i=1}^{nv}[W]_i$ [13, 6].

Standard Galerkin FE method

It is now possible to obtain the standard FE formulation by specifying $[W]_i = [N]_i$ in equations (6) and (7), such that

$$[K]_{ij} = \int_{\Omega_i} [B]_i^T[D][B]_j\, \mathrm{d}\Omega + \int_{\Gamma_{c_i}} [N]_i^T\mu[N]_j\, \mathrm{d}\Gamma \quad \text{and}$$

$$\{f\}_i = \int_{\Omega_i} [N]_i^T \{b\}\, d\Omega \; + \int_{\Gamma_{t_i}} [N]_i^T \{t_p\}\, d\Gamma \; + \int_{\Gamma_{c_i}} [N]_i^T \{\lambda\}\, d\Gamma,$$

where $[B]_i = [LN]_i$. In this case the integral term around the control volume boundary in equation (6) disappears as the shape functions are zero on the boundary [6].

Vertex-based FV method

Alternatively, for the vertex-based FV we can specify $[W]_i = [I]$ over the control volume and zero elsewhere, such that

$$\begin{aligned} [K]_{ij} &= -\int_{\Gamma_{u_i}} [R]_i^T [D][B]_j\, d\Gamma \; + \int_{\Gamma_{c_i}} \mu [N]_j\, d\Gamma \quad \text{and} \\ \{f\}_i &= \int_{\Omega_i} \{b\}\, d\Omega \; + \int_{\Gamma_{t_i}} \{t_p\}\, d\Gamma \; + \int_{\Gamma_{c_i}} \{\lambda\}\, d\Gamma. \end{aligned}$$

In this case the integral term over the control volume in equation (6) has disappeared as a consequence of the application of the differential operator [L]. It is important to note that the material and contact stiffness contributions are no longer definitely symmetric as they were in the previous FE case.
Standard augmented Lagrangian contact algorithms [5] can now be employed with regard to both methods. In this research the algorithms were implemented within the programming environment of Mathematica [11].

Results and Conclusions

The Hertzian case of a linear elastic cylinder under the action an applied force and in contact with a rigid foundation is illustrated in Figure 1(a) [4]. The loads I and II are applied at the point F and the same mesh was employed for both the FE and FV numerical analyses. It is important to note that the accuracy of the numerical results is dependent upon the meshing of the contact region via contact face elements. A contact point, either Gauss or integration depending upon the numerical technique employed, must exist at the end of the contact region [12]. Regarding the contact face elements employed for the FE and FV methods, the Gauss and integration points associated with both methods are located at equivalent positions within the contact elements.

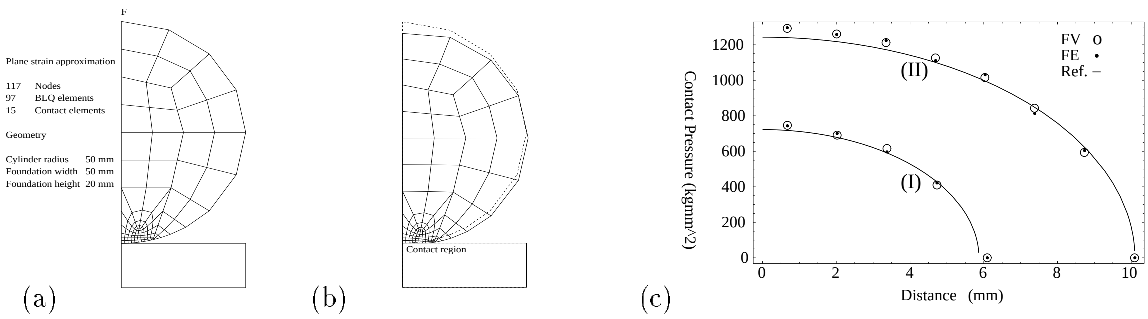

Figure 1: Contact region (a) before and (b) after deformation, and (c) pressure profiles.

The deformation and contact region are illustrated in Figure 1 (b) for the load case (II). The cylinder has deformed against the rigid foundation with no penetration occurring.

The associated contact pressure profile is plotted in Figure 1 (c), with an additional plot for load case (I). Both the FE and FV methods are in good agreement with analytical reference solution. Although the numerical results differ slightly for this case there is no immediate evidence of superiority or inferiority with regard to the accuracy of either numerical method. This agrees with 2D numerical comparisons of the two methods for problems involving material non-linearity [8]. Finally, it should be noted that in the case of material non-linearity further comparison of the two methods involving 3D cases was more informative [8].

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