A VERTEX BASED DISCRETISATION SCHEME APPLIED TO MATERIAL NON-LINEARITY WITHIN A MULTI-PHYSICS FINITE VOLUME FRAMEWORK

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A thesis submitted in partial fulfilment of the
requirements of the University of Greenwich
for the degree of Doctor of Philosophy

November 1996

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Dedications

For my grandmother,
Florence Taylor,
3rd September 1911 – 18th November 1985,

and my grandfather,
George David Taylor,
17th June 1911 – 4th September 1986.

Acknowledgements

Firstly, I acknowledge the people directly associated with this research project. This group includes, primarily, Prof. Mark Cross and Dr. Chris Bailey, who have both provided careful supervision and displayed significant amounts of patience.

This group also includes a considerable number of acedemic, research and technical staff associated with the School of Computing and Mathematical Sciences at the University of Greenwich. This collection of people have all provided invaluable support to varying degrees, but two people deserve a particular acknowledgement, Nick Croft for his academic contribution and Frank for his technical support.

Secondly, I acknowledge both my parents and my brother for their complete support and also my friends for their entertaining distractions and polite interest.

Finally and maybe most importantly, I acknowledge the Engineering and Physical Sciences Research Council without whose financial support this research may not have been possible.

Abstract

The objective of this research is the development of novel three dimensional Finite Volume (FV) algorithms for the solution of small strain, quasi-static, Computational Solid Mechanics (CSM) problems involving non-linear material behavior, specifically materials described by an elasto-visco-plastic constitutive relationship and a von-Mises yield criterion. The motivation is to contribute the non-linear CSM capability to an integrated FV framework for the comprehensive solution of the thermo-mechanical behaviour exhibited by the shape casting of metals.

A study of novel two and three dimensional FV algorithms associated with CSM is presented. The algorithms employ a variety of two and three dimensional elements and are compared with the standard Bubnov-Galerkin Finite Element Method, with regard to algorithmic procedure, linear solvers, accuracy and computational cost. A variety of benchmark solid mechanics problems involving elasto-plastic and elasto-visco-plastic material behaviour are studied. These include the plane stress analysis of a perforated tensile strip of aluminium, the plane strain analysis of a hollow metal cylinder and the three dimensional analysis of a hollow metal sphere.

The control volume-unstructured mesh, vertex based, FV algorithm for CSM problems is integrated within a multi-physics FV framework PHYSICA, which includes cell-centred FV procedures for the solution of problems involving simultaneous heat transfer, solidification and fluid flow. The thermo-mechanical coupling is described in detail. A variety of thermo-mechanical benchmark problems involving thermo-elasto-plastic and thermo-elasto-visco-plastic behaviour are studied, these include the quenching and the solidification of an infinite steel plate. Finally, the completely coupled capability of the FV framework PHYSICA is validated against experimental observations obtained from the gravity die casting of a hollow aluminium cylinder.

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