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Analysis of Boundary-Domain Integral Equations for Variable-Coefficient Mixed BVP in 2D

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Abstract The direct segregated Boundary-Domain Integral Equations (BDIEs) for the mixed boundary value problem for a second order elliptic partial differential equation with variable coefficient in 2D is considered in this paper. An appropriate parametrix (Levi function) is used to reduce this BVP to the BDIEs. Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to insure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BDIEs is analysed.

1 Preliminaries

The direct segregated Boundary-Domain Integral Equations (BDIEs) for the mixed boundary value problem for a second order elliptic partial differential equation with variable coefficient in 2D is considered in this paper. An appropriate parametrix (Levi function) is used to reduce this BVP to the BDIEs. Although the theory of BDIEs in 3D is well developed, cf. [8, 9, 2, 3], the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to insure the

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invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BDIEs is analysed.

Let Ω be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$, and let $n(x)$ be the exterior unit normal vector defined on $\partial\Omega$. The set of all infinitely differentiable functions on Ω with compact support is denoted by $\mathscr{D}(\Omega)$. The function space $\mathscr{D}'(\Omega)$ consists of all continuous linear functionals over $\mathscr{D}(\Omega)$. The space $H^s(\mathbb{R}^2)$, $s\in\mathbb{R}$, denotes the Bessel potential space, and $H^{-s}(\mathbb{R}^2)$ is the dual space to $H^s(\mathbb{R}^2)$. We define $H^s(\Omega)=\{u\in\mathscr{D}'(\Omega): u=U|_\Omega \text{ for some } U\in H^s(\mathbb{R}^2)\}$. The space $\widetilde{H}^s(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^2)$, and for $s\in(-\frac{1}{2},\frac{1}{2})$, the space $H^s(\Omega)$ can be identified with $\widetilde{H}(\Omega)$, see, e.g., [7].

We shall consider the scalar elliptic differential equation

$$Au(x)=\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right]=f(x) \qquad \text{in } \Omega,$$

where u is unknown function and f is a given function in Ω. We assume that

$$a\in C^\infty(\mathbb{R}^2),\quad 0<a_{\min}\le a(x)\le a_{\max}<\infty,\quad \forall x\in\mathbb{R}^2. \tag{1}$$

For $u\in H^2(\Omega)$ and $v\in H^1(\Omega)$ if we put $h(x)=a(x)\frac{\partial u(x)}{\partial x_j}v(x)$ and apply the Gauss-Ostrogradski Theorem, we obtain the following *Green's first identity*:

$$\mathscr{E}(u,v)=-\int_\Omega (Au)(x)v(x)dx+\int_{\partial\Omega} T^{c+}u(x)\gamma^+v(x)ds_x, \tag{2}$$

where $\mathscr{E}(u,v):=\sum_{i=1}^{2}\int_\Omega a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}dx$ is the symmetric bilinear form, γ^+ is the trace operator and

$$T^{c+}u(x):=\sum_{i=1}^{2}n_i(x)\gamma^+\left[a(x)\frac{\partial}{\partial x_i}u(x)\right] \text{ for } x\in\partial\Omega, \tag{3}$$

is the *classical co-normal derivative*.

Remark 1.1 *For $v\in\mathscr{D}(\Omega)$, $\gamma^+v=0$. If $u\in H^1(\Omega)$, then we can define Au as a distribution on Ω by, $(Au,v)=-\mathscr{E}(u,v)$ for $v\in\mathscr{D}(\Omega)$.*

The subspace $H^{1,0}(\Omega;A)$ is defined as in [5] (see also, [10])

$$H^{1,0}(\Omega;A):=\{g\in H^1(\Omega): Ag\in L_2(\Omega)\},$$

with the norm $\|g\|^2_{H^{1,0}(\Omega;A)}:=\|g\|^2_{H^1(\Omega)}+\|Ag\|^2_{L_2(\Omega)}$.

For $u \in H^1(\Omega)$ the classical co-normal derivative (3) is not well defined, but for $u \in H^{1,0}(\Omega;A)$, there exists the following continuous extension of this definition hinted by the first Green identity (2) (see, e.g., [5, 10] and the references therein).

Definition 1.2 *For $u \in H^{1,0}(\Omega;A)$ the (canonical) co-normal derivative $T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ is defined in the following weak form,*

$$\langle T^+u, w\rangle_{\partial\Omega} := \mathscr{E}(u, \gamma^+_{-1}w) + \int_\Omega (Au)\gamma^+_{-1}w dx \quad \text{for all } w \in H^{\frac{1}{2}}(\partial\Omega) \tag{4}$$

where $\gamma^+_{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is a continuous right inverse of the trace operator γ^+, which maps $H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, while $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denote the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, which extend the usual $L_2(\partial\Omega)$ inner product.

Remark 1.3 *The first Green identity* (2) *also holds for $u \in H^{1,0}(\Omega;A)$ and $v \in H^1(\Omega)$ if we replace there T^{c+} by T^+, cf. [5, 10].*

By interchanging the roles of u and v in the first Green identity and subtracting the result, we obtain *the second Green identity* for $u, v \in H^{1,0}(\Omega;A)$,

$$\int_\Omega (vAu - uAv)dx = \langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T^+v, \gamma^+u\rangle_{\partial\Omega}. \tag{5}$$

Let $\partial\Omega = \overline{\partial\Omega}_D \cup \overline{\partial\Omega}_N$ where $\partial\Omega_D$ and $\partial\Omega_N$ are nonempty and nonintersecting parts of $\partial\Omega$. We shall derive and investigate BDIEs for the following mixed BVP: *Find a function $u \in H^1(\Omega)$ satisfying conditions*

$$Au = f \quad \text{in } \Omega, \tag{6}$$
$$\gamma^+u = \varphi_0 \quad \text{on} \quad \partial\Omega_D, \tag{7}$$
$$T^+u = \psi_0 \quad \text{on} \quad \partial\Omega_N, \tag{8}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ and $f \in L_2(\Omega)$ are given functions. Equation (6) is understood in distributional sense as in Remark 1.1, equation (7) is understood in trace sense and equation (8) is understood in functional sense (4).

Theorem 1.4 *The homogeneous version of BVP* (6) – (8)*, i.e., with $f = 0$, $\varphi_0 = 0$, $\psi_0 = 0$ has only the trivial solution. Hence the nonhomogeneous problem* (6) – (8) *may posses at most one solution.*

Proof. The proof follows from Green's formula (2) with $v = u$ as a solution of the homogeneous mixed BVP (cf. [2, Theorem 2.1]). □

2 Parametrix-Based Potential Operators

Definition 2.1 *A function $P(x,y)$ is a parametrix (Levi function) for the operator A if*

$$A_x P(x,y) = \delta(x-y) + R(x,y)$$

where δ is the Dirac-delta distribution, while $R(x,y)$ is a remainder possessing at most a weak singularity at $x = y$.

For 2D, the parametrix and hence the corresponding remainder can be chosen as in [8],

$$P(x,y) = \frac{\ln|x-y|}{2\pi a(y)}, \quad R(x,y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x-y|^2} \frac{\partial a(x)}{\partial x_i}, \quad x,y \in \mathbb{R}^2.$$

Similar to [8, 2], we define the parametrix-based Newtonian and remaineder potential operators as

$$\mathscr{P}g(y) := \int_{\Omega} P(x,y)g(x)dx, \quad \mathscr{R}g(y) := \int_{\Omega} R(x,y)g(x)dx. \tag{9}$$

The single and double layer potential operators corresponding to the parametrix $P(x,y)$, are defined for $y \notin \partial\Omega$ as

$$Vg(y) := -\int_{\partial\Omega} P(x,y)g(x)ds_x, \qquad Wg(y) := -\int_{\partial\Omega} T_x^+ P(x,y)g(x)ds_x, \tag{10}$$

where g is some scalar density function. The following boundary integral (pseudo-differential) operators are also defined for $y \in \partial\Omega$,

$$\mathscr{V}g(y) := -\int_{\partial\Omega} P(x,y)g(x)ds_x, \qquad \mathscr{W}g(y) := -\int_{\partial\Omega} T_x^+ P(x,y)g(x)ds_x, \tag{11}$$

$$\mathscr{W}'g(y) := -\int_{\partial\Omega} T_y^+ P(x,y)g(x)ds_x. \tag{12}$$

Let $V_\Delta, W_\Delta, \mathscr{V}_\Delta, \mathscr{W}_\Delta$ denote the potentials and the boundary operators corresponding to the Laplace operator $A = \Delta$. Then the relations similar to [1, Eq. (3.9)–(3.12)] hold (cf.[2] for the 3D case),

$$Vg = \frac{1}{a} V_\Delta g, \qquad Wg = \frac{1}{a} W_\Delta (ag) \tag{13}$$

$$\mathscr{V} g = \frac{1}{a} \mathscr{V}_\Delta g, \qquad \mathscr{W} g = \frac{1}{a} \mathscr{W}_\Delta (ag), \tag{14}$$

$$\mathscr{W}' g = \mathscr{W}'_\Delta g + \left[a \frac{\partial}{\partial n} \left(\frac{1}{a} \right) \right] \mathscr{V}_\Delta g, \tag{15}$$

$$T^+ W g = T^+_\Delta W_\Delta (ag) + \left[a \frac{\partial}{\partial n} \left(\frac{1}{a} \right) \right] W^+_\Delta (ag). \tag{16}$$

The mapping and jump properties of the operators (9)-(12) follow from relations (13)-(16) and are described in details in [6, Theorems 1-3]. Particularly, we have the following jump relations.

Theorem 2.2 *Let $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$, $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$ and $y \in \partial\Omega$. Then*

$$\gamma^\pm V g_1(y) = \mathscr{V} g_1(y) \tag{17}$$

$$\gamma^\pm W g_2(y) = \mp \frac{1}{2} g_2(y) + \mathscr{W} g_2(y). \tag{18}$$

$$T^\pm V g_1(y) = \pm \frac{1}{2} g_1(y) + \mathscr{W}' g_1(y), \tag{19}$$

$$T^\pm W g_2(y) = \hat{\mathscr{L}} g_2(y) - \frac{\partial a}{\partial n} \left(\mp \frac{1}{2} I + \mathscr{W} \right) g_2(y), \tag{20}$$

where

$$\hat{\mathscr{L}} g_2 := T^+_\Delta W_\Delta (a g_2) = T^-_\Delta W_\Delta (a g_2) =: \hat{\mathscr{L}}_\Delta (a g_2) \quad \text{on} \quad \partial\Omega. \tag{21}$$

If $u \in H^{1,0}(\Omega; A)$, then substituting $v(x)$ by $P(x,y)$ in the second Green identity (5) for $\Omega \setminus B(y,\varepsilon)$, where $B(y,\varepsilon)$ is a disc of radius ε centered at y, and taking the limit $\varepsilon \to 0$, we arrive at the following parametrix-based third Green identity (cf. e.g. [11, 8, 2]),

$$u + \mathscr{R} u - V T^+ u + W \gamma^+ u = \mathscr{P} A u \quad \text{in } \Omega. \tag{22}$$

Applying the *trace operator* to equation (22) and using the jump relations (17) and (18), we have

$$\frac{1}{2} \gamma^+ u + \gamma^+ \mathscr{R} u - \mathscr{V} T^+ u + \mathscr{W} \gamma^+ u = \gamma^+ \mathscr{P} A u \quad \text{on } \partial\Omega. \tag{23}$$

Similarly, applying *co-normal derivative operator* to equation (22), and using the jump relation (19), we obtain

$$\frac{1}{2} T^+ u + T^+ \mathscr{R} u - \mathscr{W}' T^+ u + T^+ W \gamma^+ u = T^+ \mathscr{P} A u \quad \text{on } \partial\Omega. \tag{24}$$

For some functions f, Ψ and Φ let us consider a more general indirect integral relation associated with equation (22),

$$u + \mathscr{R}u - V\Psi + W\Phi = \mathscr{P}f \quad \text{in } \Omega. \tag{25}$$

Lemma 2.3 *Let $u \in H^1(\Omega), f \in L_2(\Omega), \Psi \in H^{-\frac{1}{2}}(\partial\Omega), \Phi \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy equation (25). Then u belongs to $H^{1,0}(\Omega;A)$ and is a solution of PDE* (6)*, i.e., $Au = f$ in Ω, and $V(\Psi - T^+u)(y) - W(\Phi - \gamma^+u)(y) = 0, \quad y \in \Omega$.*

Proof. The proof is similar to the one in 3D case in [2, Lemma 4.1]. □

For $s \in \mathbb{R}$ and $\Gamma_1 \subset \partial\Omega$, let us define the subspaces (cf., e.g., [12, p. 147])

$$H^s_{**}(\partial\Omega) := \{g \in H^s(\partial\Omega) : \langle g, 1\rangle_{\partial\Omega} = 0\}, \; \widetilde{H}^s_{**}(\Gamma_1) := \{g \in \widetilde{H}^s(\Gamma_1) : \langle g, 1\rangle_{\Gamma_1} = 0\}.$$

The following result is proved in [6, Theorem 4].

Theorem 2.4 *If $\psi \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$ satisfies $\mathscr{V}\psi = 0$ on $\partial\Omega$, then $\psi = 0$.*

Proof. The theorem holds for the operator $\mathscr{V}_\Delta$ (see, e.g., [7, Corollary 8.11(ii)]), which due to (14) implies it for the operator $\mathscr{V}$ as well. □

The following theorem is proved in [7, Theorem 8.16].

Theorem 2.5 *(i) The operator $\mathscr{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, is $H^{-\frac{1}{2}}(\partial\Omega)$- elliptic, i.e., $\langle \mathscr{V}_\Delta\psi, \psi\rangle_{\partial\Omega} \ge c\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)}$ for all $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, if and only if $Cap_{\partial\Omega} <$ 1.*

(ii) The operator $\mathscr{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, has a bounded inverse if and only if $Cap_{\partial\Omega} \ne 1$.

The following result is proved in [6, Theorem 5].

Theorem 2.6 *Let $\Omega \subset \mathbb{R}^2$ have* $\operatorname{diam}(\Omega) < 1$. *Then the single layer potential $\mathscr{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is invertible.*

Proof. Since $\operatorname{Cap}_{\partial\Omega} \le \operatorname{diam}(\Omega)$, (see, [13, p.553, properties 1 and 3]), then $\operatorname{diam}(\Omega) <$ 1 implies $\operatorname{Cap}_{\partial\Omega} < 1$. The result follows from Theorem 2.5(ii) and the first relation in (14). □

Corollary 2.7 *Let Γ_1 be a non-empty part of the boundary curve $\partial\Omega$.*

(i) The operator

$$r_{\Gamma_1}\mathscr{V} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1) \tag{26}$$

is bounded and Fredholm of index zero.

(ii) If $\widetilde{\psi} \in \widetilde{H}^{-\frac{1}{2}}_{**}(\Gamma_1)$ satisfies $r_{\Gamma_1}\mathscr{V}\widetilde{\psi} = 0$ on Γ_1, then $\widetilde{\psi} = 0$.

Proof. (i) The operator $\mathscr{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is bounded, which implies that operator (26) is bounded as well.

The operator $\mathscr{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ admits the decomposition $\mathscr{V}_\Delta = \mathscr{V}_0 + K$, where the operators $\mathscr{V}_0$ is positive and bounded below and K is a compact linear operator from $H^{-\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ (cf. [7, Theorm 7.6], and [5, Theorem 2]). If $\widetilde{\psi} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_1)$, then $\operatorname{supp}\widetilde{\psi} \subset \overline{\Gamma}_1$ and

$$\langle r_{\Gamma_1} \mathscr{V}_0 \widetilde{\psi}, \widetilde{\psi} \rangle_{\Gamma_1} = \langle \mathscr{V}_0 \widetilde{\psi}, \widetilde{\psi} \rangle_{\partial\Omega} \geq c \|\widetilde{\psi}\|_{H^{-\frac{1}{2}}(\partial\Omega)} = c \|\widetilde{\psi}\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_1)},$$

which means, the operator $r_{\Gamma_1} \mathscr{V}_0 : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ is positive and bounded below. Also, the operator $r_{\Gamma_1} K : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ is compact. Since $r_{\Gamma_1} \mathscr{V}_\Delta = r_{\Gamma_1} \mathscr{V}_0 + r_{\Gamma_1} K$, the operator $r_{\Gamma_1} \mathscr{V}\Delta : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ is Fredholm of index zero, (cf. [7, Theorem 2.33]). Since $\mathscr{V} = \frac{1}{a}\mathscr{V}_\Delta$ and the multiplication by $\frac{1}{a}$ is an isomorphism in $H^{\frac{1}{2}}(\Gamma_1)$ under condition (1), we obtain (cf. e.g. [7, Theorem 2.21]) that operator (26) is Fredholm of index zero as well.

To prove item (ii), suppose $\widetilde{\psi} \in \widetilde{H}^{-\frac{1}{2}}_{**}(\Gamma_1)$, i.e., $\langle \widetilde{\psi}, 1 \rangle_{\Gamma_1} = \langle \widetilde{\psi}, 1 \rangle_{\partial\Omega} = 0$, which implies $\widetilde{\psi} \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$. For $\widetilde{\psi} \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$, we have $\langle \mathscr{V}_\Delta \widetilde{\psi}, \widetilde{\psi} \rangle_{\partial\Omega} \geq 0$, moreover, if $\langle \mathscr{V}_\Delta \widetilde{\psi}, \widetilde{\psi} \rangle_{\partial\Omega} = 0$, then $\widetilde{\psi} = 0$ on $\partial\Omega$ (cf. [7, Theorm 8.12]. Hence, if $r_{\Gamma_1} \mathscr{V} \widetilde{\psi} = 0$, then $r_{\Gamma_1} \mathscr{V}_\Delta \widetilde{\psi} = 0$ and $\langle \mathscr{V}_\Delta \widetilde{\psi}, \widetilde{\psi} \rangle_{\partial\Omega} = \langle r_{\Gamma_1} \mathscr{V}_\Delta \widetilde{\psi}, \widetilde{\psi} \rangle_{\Gamma_1} = 0$, which implies $\widetilde{\psi} = 0$. □

The following assertion can be proved similar to [7, Theorem 8.16].

Theorem 2.8 *Let Γ_1 be a non-empty part of the boundary curve $\partial\Omega$.*

(i) The operator $r_{\Gamma_1} \mathscr{V}_\Delta : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$, is $\widetilde{H}^{-\frac{1}{2}}(\Gamma_1)-$elliptic if and only if $Cap_{\Gamma_1} < 1$.

(ii) The operators $r_{\Gamma_1} \mathscr{V}_\Delta : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ and $r_{\Gamma_1} \mathscr{V} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ are continuously invertible if and only if $Cap_{\Gamma_1} \neq 1$.

Corollary 2.9 *Let Γ_1 be a non-empty part of the boundary curve and* $\operatorname{diam}(\Gamma_1) < 1$. *Then the operator $r_{\Gamma_1} \mathscr{V} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ has a bounded inverse.*

Proof. Since $\operatorname{Cap}_{\Gamma_1} \leq \operatorname{diam}(\Gamma_1)$, (see, [13, p.553, properties 1 and 3]), then $\operatorname{diam}(\Gamma_1) < 1$ implies $\operatorname{Cap}_{\Gamma_1} < 1$. The result follows from Theorem 2.8(ii). □

Theorem 2.10 *Let Γ_2 be a non-empty open part of the boundary curve $\partial\Omega$. The operator*

$$r_{\Gamma_2} \hat{\mathscr{L}}_\Delta := r_{\Gamma_2} T^{\pm}_\Delta W_\Delta : \widetilde{H}^{\frac{1}{2}}(\Gamma_2) \to H^{-\frac{1}{2}}(\Gamma_2) \tag{27}$$

is $\widetilde{H}^{\frac{1}{2}}(\Gamma_2)-$elliptic. Operator (27) *and the operator*

$$r_{\Gamma_2} \hat{\mathscr{L}} : \widetilde{H}^{\frac{1}{2}}(\Gamma_2) \to H^{-\frac{1}{2}}(\Gamma_2) \tag{28}$$

are continuously invertible.

Proof. The ellipticity of operator (27) follows from inequality (6.39) in [12]. The continuity of this operator and the Lax-Milgram Lemma then imply its invertibility. Together with relation (21) this implies the invertibility of operator (28).

The following result is proved in [6, Lemma 2].

Lemma 2.11

(i) Let either $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ *and* $\operatorname{diam}(\Omega) < 1$, *or* $\Psi^* \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$. *If* $V\Psi^*(y) = 0$ *in* Ω *, then* $\Psi^* = 0$ *on* $\partial\Omega$.
(ii) Let $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$. *If* $W\Phi^*(y) = 0$ *in* Ω, *then* $\Phi^* = 0$ *on* $\partial\Omega$.

Lemma 2.12 *Let* $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, *where* Γ_1 *and* Γ_2 *are non-empty non-intersecting parts of the boundary curve* $\partial\Omega$. *Let* $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\Gamma_2)$ *and either* $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}_{**}(\Gamma_1)$ *or* $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_1)$ *but* $\operatorname{diam}(\Gamma_1) < 1$. *If*

$$V\Psi^*(y) - W\Phi^*(y) = 0, \qquad y \in \Omega, \tag{29}$$

then $\Psi^* = 0$ *and* $\Phi^* = 0$.

Proof. The proof follows from Theorem 2.8 (i) and Theorem 2.10 similar to [2, Lemma 4.2(iii)]. □

3 BDIEs for Mixed BVP

We shall use the following notations for product spaces.

$$\begin{aligned}
\mathbb{X}^0 &:= H^{1,0}(\Omega;A) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N),\\
\mathbb{Y}^{11,0} &:= H^{1,0}(\Omega;A) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N),\\
\mathbb{Y}^{22,0} &:= H^{1,0}(\Omega;A) \times H^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N),\\
\mathbb{Y}^{12,0} &:= H^{1,0}(\Omega;A) \times H^{\frac{1}{2}}(\partial\Omega),\\
\mathbb{Y}^{21,0} &:= H^{1,0}(\Omega;A) \times H^{-\frac{1}{2}}(\partial\Omega).
\end{aligned}$$

Let further in this Section $u \in H^{1,0}(\Omega;A)$ be a solution of BVP (6)-(8) with $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ and $f \in L_2(\Omega)$.

Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of the given data $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ from $\partial\Omega_D$ to $\partial\Omega$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ from $\partial\Omega_N$ to $\partial\Omega$, respectively. Similar to [2], to reduce BVP (6)-(8) to one or another BDIE system, we shall use equation (22) in Ω, and restrictions of equation (23) or (24) to appropriate parts of the boundary. We shall substitute f for Au, $\Phi_0 + \varphi$ for γ^+u and $\Psi_0 + \psi$ for T^+u, where $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ are considered as known, while ψ belongs to $\widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ and φ to $\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ due to the boundary conditions (7) – (8) and are to be found along with $u \in H^{1,0}(\Omega;A)$. This will lead us to four different *segregated* BDIE systems with respect to the unknown triplet $[u,\psi,\varphi]^\top =: \mathscr{U} \in \mathbb{X}^0 \subset \mathbb{X}$.

BDIE System (M11) is obtained from equation (22) in Ω, the restriction of equation (23) on $\partial\Omega_D$ and the restriction of equation (24) on $\partial\Omega_N$. Then we arrive at the following segregated system of BDIEs:

$$u + \mathscr{R}u - V\psi + W\varphi = F_0 \quad \text{in} \quad \Omega, \tag{30}$$

$$\gamma^+\mathscr{R}u - \mathscr{V}\psi + \mathscr{W}\varphi = \gamma^+F_0 - \varphi_0 \quad \text{on} \quad \partial\Omega_D, \tag{31}$$

$$T^+\mathscr{R}u - \mathscr{W}'\psi + T^+W\varphi = T^+F_0 - \psi_0 \quad \text{on} \quad \partial\Omega_N, \tag{32}$$

where $F_0 := \mathscr{P}f + V\Psi_0 - W\Phi_0$.

System (30) – (32) can be written in the form $\mathscr{M}^{11}\mathscr{U} = \mathscr{F}^{11}$, where

$$\mathscr{M}^{11} := \begin{bmatrix} I+\mathscr{R} & -V & W \\ r_{\partial\Omega_D}\gamma^+\mathscr{R} & -r_{\partial\Omega_D}\mathscr{V} & r_{\partial\Omega_D}\mathscr{W} \\ r_{\partial\Omega_N}T^+\mathscr{R} & -r_{\partial\Omega_N}\mathscr{W}' & r_{\partial\Omega_N}T^+W \end{bmatrix}, \quad \mathscr{F}^{11} := \begin{bmatrix} F_0 \\ r_{\partial\Omega_D}\gamma^+F_0 - \varphi_0 \\ r_{\partial\Omega_N}T^+F_0 - \psi_0 \end{bmatrix}.$$

Due to the mapping properties of participating operators, $\mathscr{F}^{11} \in \mathbb{Y}^{11,0}$ and the operator $\mathscr{M}^{11} : \mathbb{X}^0 \to \mathbb{Y}^{11,0}$ is bounded.

Remark 3.1 $\mathscr{F}^{11} = 0$ *if and only if* $(f, \Phi_0, \Psi_0) = 0$.

Proof. The proof follows in the similar way as in the corresponding proof in 3D case in [2, Remark 5.1]. □

BDIE system (M12), obtained using equation (22) in Ω and equation (23) on the whole boundary $\partial\Omega$, is:

$$u + \mathscr{R}u - V\psi + W\varphi = F_0 \quad \text{in} \quad \Omega, \tag{33}$$

$$\frac{1}{2}\varphi + \gamma^+\mathscr{R}u - \mathscr{V}\psi + \mathscr{W}\varphi = \gamma^+F_0 - \Phi_0 \quad \text{on} \quad \partial\Omega\ . \tag{34}$$

System (33) – (34) can be written in the form $\mathscr{M}^{12}\mathscr{U} = \mathscr{F}^{12}$, where

$$\mathscr{M}^{12} := \begin{bmatrix} I+\mathscr{R} & -V & W \\ \gamma^+\mathscr{R} & -\mathscr{V} & \frac{1}{2}I+\mathscr{W} \end{bmatrix}, \quad \mathscr{F}^{12} := \begin{bmatrix} F_0 \\ \gamma^+F_0 - \Phi_0 \end{bmatrix}.$$

Note that $\mathscr{F}^{12}$ belongs to $\mathbb{Y}^{12,0}$ and due to the mapping properties of operators involved in $\mathscr{M}^{12}$, the operator $\mathscr{M}^{12} : \mathbb{X}^0 \to \mathbb{Y}^{12,0}$ is bounded.

Remark 3.2 *Let* $\Psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ *or* $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ *but* $\operatorname{diam}(\Omega) < 1$. *Then* $\mathscr{F}^{12} = 0$ *if and only if* $(f, \Phi_0, \Psi_0) = 0$.

Proof. Indeed, the latter equality evidently implies the former. Conversely, let $\mathscr{F}^{12} = (F_0, \gamma^+F_0 - \Phi_0) = 0$. This implies $-V\Psi_0 + W\Phi_0 = \mathscr{P}f$ in Ω. Due to Lemma 2.3, $f = 0$ and $V\Psi_0 - W\Phi_0 = 0$ in Ω. The equality $\gamma^+F_0 - \Phi_0 = 0$ implies $\Phi_0 = 0$ on $\partial\Omega$. Thus $V\Psi_0 = 0$, hence by Theorem 2.4 it follows $\Psi_0 = 0$. □

BDIE system (M21) is another system obtained using equation (22) in Ω and equation (24) on $\partial\Omega$, i.e.,

$$u+\mathscr{R}u-V\psi+W\varphi=F_0 \quad \text{in} \quad \Omega, \tag{35}$$

$$\frac{1}{2}\psi+T^+\mathscr{R}u-\mathscr{W}'\psi+T^+W\varphi=T^+F_0-\Psi_0 \quad \text{on} \quad \partial\Omega. \tag{36}$$

System (35)–(36) can be written in the form $\mathscr{M}^{21}\mathscr{U}=\mathscr{F}^{21}$, where

$$\mathscr{M}^{21}:=\begin{bmatrix} I+\mathscr{R} & -V & W \\ T^+\mathscr{R} & \frac{1}{2}I-\mathscr{W}' & T^+W \end{bmatrix}, \quad \mathscr{F}^{21}:=\begin{bmatrix} F_0 \\ T^+F_0-\Psi_0 \end{bmatrix}.$$

Note that $\mathscr{F}^{21}$ belongs to $\mathbb{Y}^{21,0}$ and due to the mapping properties of operators involved in $\mathscr{M}^{21}$, the operator $\mathscr{M}^{21}:\mathbb{X}^0\to\mathbb{Y}^{21,0}$ is bounded.

Remark 3.3 $\mathscr{F}^{21}=0$ *if and only if* $(f,\Phi_0,\Psi_0)=0$.

Proof. The proof follows in the similar way as in Remark 3.2.

BDIE system (M22), a system of almost second kind (up to the spaces) obtained using equation (22) in Ω, the restriction of equation (24) to $\partial\Omega_D$ and the restriction of equation (23) to $\partial\Omega_N$ is:

$$u+\mathscr{R}u-V\psi+W\varphi=F_0 \quad \text{in} \quad \Omega, \tag{37}$$

$$\frac{1}{2}\psi+T^+\mathscr{R}u-\mathscr{W}'\psi+T^+W\varphi=T_a^+F_0-\Psi_0 \quad \text{on} \quad \partial\Omega_D, \tag{38}$$

$$\frac{1}{2}\varphi+\gamma^+\mathscr{R}_bu-\mathscr{V}_a\psi+\mathscr{W}_a\varphi=F_0^+-\Phi_0 \quad \text{on} \quad \partial\Omega_N. \tag{39}$$

System (37) – (39) can be rewritten in the form $\mathscr{M}^{22}\mathscr{U}=\mathscr{F}^{22}$, where

$$\mathscr{M}^{22}:=\begin{bmatrix} I+\mathscr{R} & -V & W \\ r_{\partial\Omega_D}T^+\mathscr{R} & r_{\partial\Omega_D}(\frac{1}{2}I-\mathscr{W}') & r_{\partial\Omega_D}T^+W \\ r_{\partial\Omega_N}\gamma^+\mathscr{R} & -r_{\partial\Omega_N}\mathscr{V} & r_{\partial\Omega_N}(\frac{1}{2}I+\mathscr{W}) \end{bmatrix}, \quad \mathscr{F}^{22}:=\begin{bmatrix} F_0 \\ r_{\partial\Omega_D}\{T^+F_0-\Psi_0\} \\ r_{\partial\Omega_N}\{\gamma^+F_0-\Phi_0\} \end{bmatrix}.$$

Note that $\mathscr{F}^{22}$ belongs to $\mathbb{Y}^{22,0}$ and due to the mapping properties of operators involved in $\mathscr{M}^{22}$, the operator $\mathscr{M}^{22}:\mathbb{X}^0\to\mathbb{Y}^{22,0}$ is bounded.

Remark 3.4 $\mathscr{F}^{22}=0$ *if and only if* $(f,\Phi_0,\Psi_0)=0$.

Proof. The proof follows in the similar way as in the corresponding proof in 3D case in [2, Remark 5.11]. □

4 Equivalence

In what follows, we shall prove the equivalence of the mixed BVP (6) – (8) to BDIE systems (M11), (M12), (M21) and (M22).

Theorem 4.1 *Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$, $f \in L_2(\Omega)$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of φ_0 and ψ_0, respectively.*

(i) If some $u \in H^{1,0}(\Omega;A)$ solves the mixed BVP (6)-(8) in Ω, then the solution is unique and the triplet $(u,\psi,\varphi)^T \in \mathbb{X}^0$, where

$$\psi = T^+u - \Psi_0, \quad \varphi = \gamma^+u - \Phi_0, \quad \text{on } \partial\Omega \tag{40}$$

solves the BDIE systems (M11), (M12), (M21) and (M22).

(ii) If $\operatorname{diam}(\Omega) < 1$ *and a triplet $(u,\psi,\varphi)^T \in \mathbb{X}^0$ solves one of the BDIE systems (M11) or (M21) or (M12) or (M22), then this solution is unique and solves all the BDIE systems, while u solves BVP* (6) - (8) *and relations* (40) *hold.*

Proof. (i) Let $u \in H^{1,0}(\Omega;A)$ be a solution to BVP (6) - (8). Due to Theorem 1.4 it is unique. Set $\psi := T^+u - \Psi_0$ and $\varphi := \gamma^+u - \Phi_0$. Then $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$, $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and recalling how BDIE systems (M11), (M12), (M21) and (M22) were constructed, we obtain that the triplet $(u,\psi,\varphi)^T$ solves systems (M11), (M12), (M21) and (M22).

(ii) Let a triplet $(u,\psi,\varphi)^T \in \mathbb{X}^0$ solve BDIE system (M11) or (M12) or (M21) or (M22). The hypotheses of Lemma 2.3 are satisfied for the first equation in BDIE system, implying that u solves PDE (6) in Ω, while the following equation holds,

$$V\Psi^* - W\Phi^* = 0 \quad \text{in} \quad \Omega, \tag{41}$$

where $\Psi^* = \Psi_0 + \psi - T^+u$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+u$.

Suppose first that the triplet $(u,\psi,\varphi)^T \in \mathbb{X}^0$ solves BDIE system (M11). Taking trace of equation (30) on $\partial\Omega_D$ using the jump relations (17)-(18), and subtracting equation (31) from it, we obtain,

$$\gamma^+u = \varphi_0 \quad \text{on} \quad \partial\Omega_D, \tag{42}$$

i.e., u satisfies the Dirichlet condition (7). Taking the co-normal derivative of equation (30) on $\partial\Omega_N$, using the jump relations (19)-(20) and subtracting equation (32) from it, we obtain

$$T^+u = \psi_0 \quad \text{on} \quad \partial\Omega_N, \tag{43}$$

i.e., u satisfies the Neumann condition (8). Hence u solves the mixed BVP (6)-(8).

Taking into account $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$ and $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$, equation (42) and (43) imply that the first equation in (40) is satisfied on $\partial\Omega_N$ and the second equation in (40) is satisfied on $\partial\Omega_D$. Thus we have $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ in (41). Let $\Gamma_1 = \partial\Omega_D$, $\Gamma_2 = \partial\Omega_N$. Then $\operatorname{diam}(\Gamma_1) \le \operatorname{diam}(\Omega) < 1$ and Lemma 2.12 implies $\Psi^* = \Phi^* = 0$, which completes the proof of conditions in (40). Uniqueness of the solution to BDIE systems (M11) follows from (40) along with Remark 3.1 and Theorem 1.4.

Finally, item (i) implies that triplet $(u,\psi,\varphi)^T \in \mathbb{X}^0$ solves also BDIE systems (M12), (M21) and (M22).

Similar arguments work if we suppose that instead of the BDIE systems (M11), the triplet $(u, \psi, \varphi)^T \in \mathbb{X}^0$ solves BDIE systems (M21) or (M12) or (M22).

□

5 Conclusion

In this paper, we considered the mixed BVP problem for variable coefficient PDE in a two-dimensional bounded domain, where the right hand side function is from $L_2(\Omega)$ and the Dirichlet data from the space $H^{\frac{1}{2}}(\partial\Omega_D)$ and the Neumann data from the space $H^{-\frac{1}{2}}(\partial\Omega_N)$. The BVP was reduced to four systems of Boundary-Domain Integral Equations and their equivalence to the original BVP was shown. The invertibility of the associated operators in the corresponding Sobolev spaces can be also proved. In a similar way one can consider also the 2D versions of the BDIEs for mixed problem in exterior domains, united BDIEs as well as the localised BDIEs, which were analysied for 3D case in [2, 4, 9, 3].

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References

1. Ayele, T. G., Dufera, T. T. and Mikhailov, S. E.: Analysis of boundary-domain integral equations for variable-coefficient Neuman BVP in 2D. In: *Integral Methods in Science and Engineering* vol. 1, Birkhäuser, New York, 2017, pp. 21–32.
2. Chkadua, O., Mikhailov, S. E. and Natroshvili, D.: Analysis of direct boundary-domain integral equations for a mixed bvp with variable coefficent, I: Equivalence and invertibility. *J. Integral Equations and Appl.* **21**, 499–543 (2009).
3. Chkadua, O., Mikhailov, S. E. and Natroshvili, D.: Analysis of some localized boundary-domain integral equations. *J. Integral Equations and Appl.* **21**, 405–445 (2009).
4. Chkadua, O., Mikhailov, S. E. and Natroshvili, D.: Analysis of direct segregated boundary-domain integral equations for variable-coefficient mixed BVPs in exterior domains. *Analysis and Appl.* **11**, 1350006, (1-33) (2013).
5. Costabel, M.: Boundary integral operators on Lipschiz domains: elementary results. *SIAM J. on Mathematical Anal.* **19**, 613–626 (1988).
6. Dufera T.T., Mikhailov S.E.: Analysis of boundary-domain integral equations for variable-coefficient Dirichlet BVP in 2D, In: *Integral Methods in Science and Engineering: Theoretical and Computational Advances.* C. Constanda and A. Kirsh, eds., Springer (Birkhäuser): Boston, ISBN 978-3-319-16727-5, 2015, 163-175, DOI: 10.1007/978-3-319-16727-5_15.
7. McLean, W.: *Strongly Elliptic Systems and Boundary Integral Equations*. Cambrige University Press, Cambrige (2000).

8. Mikhailov, S. E.: Localized boundary-domain integral formulations for problems with variable coefficients. *Int. J. Engineering Analysis with Boundary Elements*, **26**, 681–690, 2002.
9. Mikhailov, S. E.: Analysis of united boundary-domain integral and integro-differential equations for a mixed BVP with variable coeffecients. *Math. Meth. Appl. Sci.*, **29**,715–739 (2006).
10. Mikhailov, S. E.: Traces, extensions and co-normal derivatives for elliptic systems on Lipschiz domains. *Mathematical Analysis and Appl.* **378**, 324-342 (2011).
11. Miranda, C.: *Partial Differential Equations of Elliptic Type*. Springer, Berlin-Heidelberg-New York, (1970).
12. Steinbach, O.: *Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements*, Springer (2007).
13. Yan, Y. and Sloan, L. H.: On integral equations of the first kind with logarthmic kernels. *Journal of Integral Equations and Applications,* 1(4): 549-579 (1988).