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ON AN INTEGRAL EQUATION OF SOME BOUNDARY VALUE PROBLEMS FOR HARMONIC FUNCTIONS IN PLANE MULTIPLY CONNECTED DOMAINS WITH NONREGULAR BOUNDARY

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ABSTRACT. An integral equation of boundary value problems for the Laplace equation is investigated.

Bibliography: 14 titles.

§1. Introduction

Let D be a finite or infinite domain with boundary Γ which is a collection of simple closed contours of bounded rotation without cusps: $\Gamma = \bigcup_0^m \Gamma_i$, where the contour Γ_0 encloses all the remaining contours Γ_i; it may be absent, in which case the domain D is infinite. The two-dimensional Dirichlet and Neumann problems for the Laplace equation in domains with nonregular boundary have been studied rather well [1]–[3]. The Dirichlet problem is unconditionally and uniquely solvable in the class of functions harmonic in D and continuous in $\bar{D}$ if the prescribed boundary values of the function sought are continuous. Up to an arbitrary additive constant the Neumann problem is uniquely solvable in the class of functions harmonic in D which have a prescribed boundary flux on the boundary which is a regular, countable additive set function on Γ, if this flux over the entire contour Γ is equal to zero [2], [3].

On the other hand [4], solutions of elliptic boundary value problems, in particular for the Laplace equation, in domains with corners may have derivatives possessing power singularities at a corner even if the prescribed boundary conditions are sufficiently smooth. These derivatives, or, more correctly, the coefficients of the power singularity, are frequently of fundamental interest for applications, in particular, for elasticity theory (see, for example, [5] and [6]).

An effective method of solving boundary value problems is to reduce them to integral equations on the boundary and then solve the latter numerically . Here, depending on the integral representation of the solution of the original problem, the density of the equation may have the order of the solution itself (i.e., be bounded) or the order of derivatives of

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the solution (i.e., have a power singularity) in a neighborhood of a corner. In the first case it is rather difficult to obtain the coefficient for increasing derivatives with sufficient accuracy. In the second case, to effectively solve the integral equation and obtain the coefficient of the singularity it is necessary to explicitly distinguish the singular character of the desired density at corners of the contour; this is impeded by insufficient study of the equation itself and hence of the smoothness of its solution. Therefore, as far as is known to the author, the second method has not been adequately applied to problems in domains with a nonregular boundary.

In this paper the solution of the Dirichlet problem is sought in the form of a nontraditional integral representation in terms of an angular potential, while a solution of the problem with a skew derivative is sought in the form of a generalized logarithmic potential. This makes it possible to reduce both problems to a single integral equation on the contour Γ with a density of the order of the derivatives of the solution of the original problem; here the kernel of the equation has strong stationary singularities at the corners. This type of equation was considered in [7] and [8]. A formula is then obtained for the Fredholm radius of the equation obtained, and its solvability and uniqueness in the case of a finite or infinite multiply connected domain D are investigated. For the case where the equation lies on the spectrum a perturbing operator is introduced which removes the equation from the spectrum and leads to an equation whose solution coincides with one of the solutions of the original equation. The spectral properties of the perturbed equation are studied, and a perturbing operator of a certain special form is proposed which leads to an equation whose solution can be expanded in a convergent Neumann series for any right-hand side.

§2. Reduction to an integral equation

Let s be the arc length of the contour Γ oriented so that in passing around it in a positive direction the domain D lies to the left. We consider the Dirichlet problem $u|_\Gamma = f_1(s) \in W_r^1(\Gamma)$ and the problem with a skew derivative $\partial u/\partial \vec{l}|_\Gamma = f_2(s) \in L_r(\Gamma)$, $1 < r < \infty$, where at smooth points of Γ the unit vector $\vec{l}$ makes a constant angle, distinct from zero and 2π, with the tangent to the contour. Let $z = x + iy$ be the complex coordinate of a point of the plane, let $t = t(s)$ be the complex coordinate of a point of the contour Γ, let $k(s) = dt/ds \equiv e^{i\varphi(s)}$, where φ is the angle between the tangent to the contour and the axis Ox, and let $k_l(s) = dz/d\vec{l}(s)|_\Gamma \equiv k(s)\beta$, where β is a complex constant with $|\beta| = 1$ and $\operatorname{Im}\beta < 0$. We henceforth do not distinguish functions of the arguments t and s, i.e., $f(t) = f(t(s)) = f(s)$.

We seek a solution of the Dirichlet problem in the form

$$u(x, y) = \operatorname{Re}\left\{\int_{z_0}^{z}\left[\Phi(z_1) + \sum_{i=1}^{m}\frac{A_i}{z_1 + a_i}\right]dz_1\right\} + A_0, \tag{2.1}$$

and a solution of the problem with a skew derivative in the form

$$u(x, y) = \operatorname{Re}\left\{\frac{1}{\beta}\int_{z_0}^{z}\Phi(z_1)\,dz_1\right\} + C. \tag{2.2}$$

Here z_0 is an arbitrary fixed point in D, the a_i, $i = 1,\dots,m$, are arbitrary fixed points within the contours Γ_i, the A_i, $i = 0,\dots,m$, are real constants which will be determined in §4, and C is an arbitrary constant.

We seek the analytic function $\Phi(z)$ in the Smirnov classes E_p, $p > 1$ [3], and we require that $\Phi(\infty) = 0$ if the domain D contains the point $z = \infty$. After substitution into the boundary conditions (and differentiating them with respect to s in the Dirichlet problem) for $\Phi(z)$ we obtain the Riemann-Hilbert problem

$$\operatorname{Re}[k(t)\Phi^+(t)] = f(t), \qquad t \in \Gamma, f(t) \in L_r, 1 < r, \tag{2.3}$$

where

$$f(t) = \frac{df_1(s)}{ds} - \operatorname{Re} \sum_{i=1}^{m} \frac{k(s)A_i}{t(s) - a_i}$$

for the Dirichlet problem and $f(t) = f_2(t)$ for the problem with a skew derivative. Any solution of (2.3) in E_p for $p > 1$ on the basis of (2.1) and (2.2) generates a solution of the original problems in the class $u \in C^2(D) \cap C(\overline{D})$ (with an appropriate choice of the constants A_i in the Dirichlet problem) if the function u thus obtained is single-valued and also bounded at infinity for an infinite domain D. Since solutions of the original Dirichlet problem and the Neumann problem (as a special case of the problem with a skew derivative) are known to exist for the boundary conditions described (see [2] and [3]), while the choice of them in the form (2.1), (2.2) can only restrict the class of functions admissible as a solution, in order to not miss a solution, we consider $\Phi(z) \in E_p$ with the smallest possible $p > 1$.

A solution of (2.3), in turn, is sought in the form

$$\Phi(z) = \frac{1}{\pi i} \int_\Gamma \frac{\mu(s)\, ds}{t(s) - z}. \tag{2.4}$$

Here the real function $\mu \in L_p(\Gamma)$, $1 < p < \infty$; then $\Phi(z) \in E_p(D)$.

If we substitute (2.4) into (2.2), then we obtain a representation of the solution of the problem with a skew derivative which practically coincides with the generalized logarithmic potential introduced by Gabov [9] and with the ordinary simple-layer potential if $\beta = -i$, and the problem with a skew derivative goes over into the Neumann problem. Substitution of (2.4) into (2.1) shows that (2.1) and (2.4) essentially constitute an integral representation of the solution of the Dirichlet problem with density μ and a kernel which is an integral along the contour of a double-layer potential, i.e., simply the angle subtended by the segment of the path of integration from the point (x, y), with the addition of some logarithmic terms (in the case of a multiply connected domain) and an additive constant. This representation can also be exressed in terms of the angular potential used in [9] for simply connected domains with a smooth boundary. In [10] the smoothness of the angular potential in a domain bounded by a curve of bounded variation is investigated.

Substituting (2.4) into (2.3), we arrive at an integral equation for $\mu(s)$ ($\lambda = 1$):

$$(I - \lambda K)\mu = f,$$

$$(K\mu)(s_0) = -\frac{1}{\pi} \int_\Gamma \mu(s) \operatorname{Im}\left[\frac{k(s_0)}{t - t_0}\right] ds, \qquad \mu(s) \in L_p, t = t(s), t_0 = t(s_0). \tag{2.5}$$

The adjoint to (2.5) is the equation

$$(I - \lambda K^*)\nu = f^*,$$

$$(K^*\nu)(s_0) = \frac{1}{\pi} \int_\Gamma \nu(s) \operatorname{Im}\left[\frac{k(s)}{t - t_0}\right] ds, \qquad \nu(s) \in L_q(\Gamma), q = \frac{p}{p-1}. \tag{2.6}$$

For $\lambda = 1$, (2.5) coincides with the equation corresponding to the Neumann problem for the domain D, while for $\lambda = -1$ it coincides with the equation corresponding to the Neumann problem for a domain exterior relative to D if its solution is sought in the form of a simple-layer potential with density $\mu(s)$. Equation (2.6) for $\lambda = -1$ coincides with the equation corresponding to the Dirichlet problem for the domain D, and for $\lambda = 1$ it coincides with the equation corresponding to the problem for a domain exterior relative to D if its solution is sought in the form of a double-layer potential with density $\nu(s)$. We note that in our formulation $\lambda = 1$ both for finite and infinite domains, but the function $k(s)$ in the case of a simply connected domain changes sign on passing to the exterior problem, since the positive direction changes.

The kernels of equations (2.5) and (2.6) are bounded if $s \neq s_0$, while on the diagonal at points of Lyapunov smoothness they have no more than an integrable singularity. When s and s_0 tend to a corner point from different sides the kernel has a strong singularity. Equations of this type were considered by Lopatinskiĭ [7] in the space $L_1(\Gamma)$ with a power weight on a simple, closed, piecewise smooth contour, and by Shelepov [8] in the spaces L_p, $1 < p < \infty$, on a simple contour of bounded rotation without cusps. In those notes formulas are obtained for the index of the equation which, as is easily seen from the arguments of [7], remain valid for a finite collection of simple, closed, nonintersecting contours of bounded rotation. In [8] the Fredholm radius of the operator K^* in the space $L_q(\Gamma)$, $1 < q < \infty$, is also presented, and it is equal to

$$\Omega^*(q) = \min_k \frac{\sin(\pi/q)}{\sin(|\pi - \omega_k|/q)}.$$

The minimum is taken over all corner points; ω_k is the magnitude of the interior angle. In [2] equation (2.5) with $\mu\, ds$ replaced by dM is investigated in a space of functions M which are absolutely additive set functions on Γ, and (2.6) is investigated in a space of continuous functions on a simple contour of bounded rotation without cusps. The Fredholm radius of the operators K and K^*, respectively, in these spaces is equal to $\Omega(\infty) = \min_k(\pi/|\pi - \omega_k|) > \Omega^*(q)$. Since in these spaces for $|\lambda| < \Omega^*(q)$, $1 < q < \infty$, the equations in question are Fredholm, while the spaces $L_p(\Gamma)$ are contained in the first set and contain the second, the spectrum and eigenfunctions of the operator (2.5) in the space L_p, $1 < p < \infty$, and of the operator (2.6) in L_q, $1 < q < \infty$, in the regions $|\lambda| < \Omega(p) \equiv \Omega^*(p/(p-1))$ and $|\lambda| < \Omega^*(q)$, respectively, coincide with those obtained in [2]. Further, as for equations on a smooth contour [11], [6], all remaining properties of equations (2.5) and (2.6) are proved in the usual way, and this carries over directly to multiply connected domains. The results can be formulated in the following manner.

THEOREM 1. *If*

$$|\lambda| < \min_k \frac{\sin[\pi(1 - 1/p)]}{\sin[|\pi - \omega_k|(1 - 1/p)]}$$

then equation (2.5) *in the space* L_p, $1 < p < \infty$, *is Fredholm, the poles of the resolvent of this equation are real and simple, and there are no poles in the interval* $-1 < \lambda < 1$. *If there is a contour* Γ_0 ($\infty \notin D$), *then* $\lambda = 1$ *is a pole of the resolvent in any case, while* $\lambda = -1$ *is a pole in the case where the domain is not simply connected* ($m > 0$). *If there is no such contour* ($\infty \in D$), *then* $\lambda = 1$ *is not a pole of the resolvent, while* $\lambda = -1$ *is a pole.*

From Theorem 1 it is possible, in particular, to extract a result obtained in [8]: equation (2.5) for $\lambda = 1$ is Fredholm in the spaces $L_p(\Gamma)$, $1 < p < p_0 = 1 + 1/\max_k|1 - \omega_k/\pi|$, where the maximum is taken over all corners. Since for contours without cusps $0 < \omega_k < 2\pi$, equation (2.5) for $\lambda = 1$ is Fredholm in $L_2(\Gamma)$.

It is easy to see by direct verification that $\nu_0(s) = 1$ is a solution of the homogeneous equation (2.6) in the case where Γ_0 is present and $\lambda = 1$. We shall also write out a solution of the homogeneous equation (2.5) for this case.

Let $z = \omega(\xi)$ be a conformal mapping of the disk $|\xi| < 1$ onto the exterior of Γ_0 such that $\omega(0) = \infty$. The contour Γ_0 thus goes over into a contour L. A solution of the homogeneous equation (2.5) is then given by

$$\mu_0(t_0) = \begin{cases} e^{\psi(t_0)}, & t_0 \in \Gamma_0, \\ 0, & t_0 \in \Gamma_i, i \neq 0, \end{cases} \tag{2.7}$$

$$\psi(t_0) = \frac{1}{\pi i}\int_L \ln[i\tau k(t)]\left\{\frac{1}{\tau - \tau_0} - \frac{1}{2\tau}\right\} d\tau, \qquad t = \omega(\tau), \tau_0 = \omega^{-1}(t_0).$$

Here we assume that at the corners the modulus of the jump $|\arg k^* - \arg k^-| < \pi$, and for $\ln[i\tau k(t)]$ any branch is taken which under this condition is single-valued on the contour L. This expression is obtained by formal use of the methods of [1] for smooth contours in solving the homogeneous Riemann-Hilbert problem in the exterior of Γ_0. The expression (2.7) is the limit value of the function holomorphic outside Γ_0

$$\Psi(z) = i\omega^{-1}(z)e^{\psi(z)}, \qquad \psi(z) = -\frac{1}{\pi i}\int_L \ln[i\tau k(t)]\left\{\frac{1}{\tau - \omega^{-1}(z)} - \frac{1}{2\tau}\right\} d\tau, \tag{2.8}$$

multiplied by $k(t_0)$. This implies that the piecewise holomorphic function $\Phi(z)$ generated by (2.4) with density $\mu_0(s)$ is equal to zero inside Γ_0 and is hence a solution of the homogeneous problem (2.3), while $\mu_0(s)$ is a solution of the homogeneous equation (2.5) for $\lambda = 1$. Direct verification shows that $\mu_0(s)$ is a real function. It is further necessary that $\mu_0(s) \in L_p(\Gamma_0)$. This result follows from §§19.2–19.8 of [3]. Thus, (2.7) actually represents a solution of the homogeneous equation (2.5) for $\lambda = 1$ in the spaces L_p, $1 < p < p_0$. Application of the results of [2] and [3] enables us to generalize the methods of [11] and show that the homogeneous equations (2.5) and (2.6) for $\lambda = 1$ have no other solutions in the spaces $L_p(\Gamma)$, $1 < p < p_0$, and $L_q(\Gamma)$, $p_0/(p_0 - 1) < q < \infty$, respectively (in [3], for example, this assertion is proved for the case of a simply connected domain D). We have the following result.

THEOREM 2. *If the contour* Γ_0 *is present, then in the spaces* L_p, $1 < p < p_0$, *the homogeneous equation* (2.5) *for* $\lambda = 1$ *has a unique nontrivial solution* (2.7) *which by* (2.4) *generates only the zero function* $\Phi(z)$ *inside* Γ_0, *and the corresponding inhomogeneous equation is solvable if the following condition is satisfied*:

$$\int_\Gamma f(s)\, ds = 0. \tag{2.9}$$

§3. Perturbation of the equation lying on the spectrum

Theorem 1 enables us to solve (2.5) for $\lambda = 1$ by expanding the resolvent in a Neumann series; this is preferable to methods requiring the reduction of the original integral equation to a system of linear algebraic equations and subsequent solution of this system by some general method which does not take account of its connection with the integral equation and the spectral properties of the latter. Suppose that there is no contour Γ_0, i.e., the domain D is infinite; in this case $\lambda = 1$ is not a pole, while $\lambda = -1$ is a simple pole of the resolvent of (2.5). Then for $\lambda = 1$ the Neumann series, generally speaking, will not converge, but the series obtained by expanding the resolvent multiplied by the function $(\lambda + 1)$ in powers of λ will converge [11] (see also [6]); hence

$$\mu = \frac{f}{2} + \sum_{i=0}^{\infty} \mu_i, \quad \mu_0 = (I + K)\frac{f}{2}, \quad \mu_i = K\mu_{i-1}, \qquad i = 1, 2, \ldots. \tag{3.1}$$

Suppose now that the contour Γ_0 is present, i.e.,the domain D is bounded. The resolvent of (2.5) has a pole for $\lambda = 1$, and if D is not simply connected it also has a pole for $\lambda = -1$. If the solvability condition (2.9) for equation (2.5) with $\lambda = 1$ is satisfied, then, as described in [11], a solution of it could be represented in the form of an ordinary Neumann series for a simply connected domain or its modification (3.1) for a domain not simply connected. However, in this case the solution of (2.5) is not stable relative to small changes of the right side which violate condition (2.9); these are always possible in numerical computation.

We consider the linear operator equation

$$Ax = y, \tag{3.2}$$

where x and y are elements of Banach spaces B_1 and B_2 respectively, and A is a bounded linear operator from B_1 to B_2. The adjoint to (3.2) is the equation

$$A^*x = y^*, \tag{3.3}$$

where x^* and y^* are elements of the dual spaces B_2^* and B_1^* respectively, and A^* is the adjoint of A. A generalized Schmidt lemma holds.

LEMMA 1. *Suppose that* (3.2) *is a generalized Fredholm equation and* x_i *and* x_i^*, $i = 1, \ldots, n$, *are all linearly independent solutions of the homogeneous equations* (3.2) *and* (3.3) *respectively. Then the equation*

$$(A - A_1)x = y, \qquad A_1 x = \sum_{i=1}^{n} \psi_i \varphi_i(x), \tag{3.4}$$

is uniquely and unconditionally solvable. Here φ_i *and* ψ_i *are, respectively, arbitrary elements of* B_1^* *and* B_2 *such that*

$$\det\left[x_i^*(\psi_j)\right] \neq 0, \quad \det\left[\varphi_i(x_j)\right] \neq 0, \qquad i, j = 1, \ldots, n. \tag{3.5}$$

If the solvability conditions $x_i^*(y) = 0$ *for* (3.2) *are satisfied, then the solution* x *of* (3.4) *is the one of the solutions of* (3.2) *such that* $\varphi_i(x) = 0$, $i = 1, \ldots, n$.

In somewhat different terms this lemma is proved in [12] (see also [13]). It makes it possible to take equation (3.2) off the spectrum in the presence of rather little available information regarding solutions of the homogeneous equations (3.2) and (3.3) sufficient only to verify conditions (3.5). However, it is not clear how the spectral properties of the equation change here if A is an operator from B to B written in the form $A = I - \lambda A_0$; it is therefore difficult to draw conclusions regarding the solution of the equation obtained by the method of successive approximations. For this case we rewrite (3.2) in the form

$$(I - \lambda A_0)x = y, \tag{3.6}$$

and (3.4) in the form

$$[I - \lambda(A_0 + A_{01})]x = y; \tag{3.7}$$

suppose that again $A_{01}x = \Sigma_1^n \psi_i \varphi_i(x)$. We introduce the notation R_λ and $R_{\lambda 1}$ for the respective resolvents of (3.6) and (3.7), i.e., $R_\lambda(I - \lambda A_0) = I$ and $R_{\lambda 1}[I - \lambda(A_0 + A_{01})] = I$. We shall express $R_{\lambda 1}$ in terms of R_λ. For this we act with the operator R_λ on (3.7) and obtain as a result

$$x - \lambda \sum_{j=1}^{n} (R_\lambda \psi_j)\varphi_j(x) = R_\lambda y. \tag{3.8}$$

We now act on (3.8) with the functionals φ_i; as a result, we have a system of linear algebraic equations for determining the constants $\varphi_i(x)$:

$$\sum_{j=1}^{n} \left[\delta_{ij} - \lambda\varphi_i(R_\lambda\psi_j)\right]\varphi_j(x) = \varphi_i(R_\lambda y), \qquad i = 1,\dots,n, \tag{3.9}$$

where δ_{ij} is the Kronecker symbol. Let $\Delta = \det[\delta_{ij} - \lambda\varphi_i(R_\lambda\psi_j)]$ be the determinant and d_{ij} the algebraic complements of the matrix of this system. Solving (3.9) and substituting the expressions for $\varphi_i(x)$ into (3.8), we find that

$$x = R_{\lambda 1}y = R_\lambda y + \frac{\lambda}{\Delta(\lambda)} \sum_{j=1}^{n} (R_\lambda\psi_j) \sum_{i=1}^{n} d_{ij}(\lambda)\varphi_i(R_\lambda y). \tag{3.10}$$

From this it is evident that the set of poles of the resolvent of the operator $A_0 + A_{01}$ belongs to the union of the poles of the resolvent of A_0 and the zeros of the determinant $\Delta(\lambda)$. Using Lemma 1 or directly investigating the representation (3.10) with consideration of the expansion of the resolvent in a neighborhood of a pole [14], we obtain the following result.

LEMMA 2. *The set of poles of the resolvent of the operator* $A_0 + A_{01}$ *consists of the poles of the resolvent of* A_0 *and the zeros of the function* $\Delta(\lambda) = \det[\delta_{ij} - \lambda\varphi_i(R_\lambda\psi_j)]$. *Here if* $\lambda = \lambda_0$ *is a pole of the resolvent* R_λ *and*

$$\det\left[x_i^*(\psi_j)\right] \neq 0, \quad \det\left[\varphi_i(\psi_j)\right] \neq 0, \qquad i, j = 1,\dots,n,$$

where the x_i *and* x_i^*, $i = 1,\dots,n$, *are all solutions of the homogeneous equation* (3.6) *and its adjoint, respectively, for* $\lambda = \lambda_0$, *then* $\lambda = \lambda_0$ *is a regular point of the resolvent* $R_{\lambda 1}$ *of* (3.7).

We now attempt to choose the elements φ_i and ψ_i so that the resolvent $R_{\lambda 1}$ is regular at the poiont $\lambda = \lambda_0$ where R_λ has a pole and, moreover, so that it does not acquire additional singular points as compared with R_λ in the finite part of the λ plane.

LEMMA 3. *Suppose that* λ_0 *is a simple pole of the resolvent* R_λ *of* (3.6). *If* $\varphi_i = x_i^*$, $x_i^*(\psi_j) = (-1/\lambda_0)\delta_{ij}$ *or* $\psi_j = x_j$, $\varphi_i(x_j) = (-1/\lambda_0)\delta_{ij}$, $i, j = 1,\dots,n$, *where the* x_i *and* x_i^*, $i, j = 1,\dots,n$, *are all solutions of the homogeneous equation* (3.6) *and its adjoint, respectively, for* $\lambda = \lambda_0$, *then the singular points of the resolvent* $R_{\lambda 1}$ *of* (3.7) *coincide with the singular points of the resolvent* R_λ *of* (3.6) *with the exception of the point* $\lambda = \lambda_0$ *where the resolvent* $R_{\lambda 1}$ *is regular.*

PROOF. Suppose, for example, $\varphi_i = x_i^*$; then $\Delta(\lambda) = \det[\delta_{ij} - \lambda x_i^*(R_\lambda \psi_j)]$. For sufficiently small $|\lambda|$ we have

$$x_i^*(R_\lambda, \psi_j) = x_i^*\left(\sum_{k=0}^{\infty} \lambda^k A_0^k \psi_j\right) = \sum_{k=0}^{\infty} \lambda^k x_i^*\left(A_0^k \psi_j\right)$$
$$= \sum_{k=0}^{\infty} \lambda^k \lambda_0^{-k} x_i^*(\psi_j) = \left(I - \frac{\lambda}{\lambda_0}\right)^{-1} x_i^*(\psi_j).$$

Continuing this function analytically to the entire complex λ plane, we find that

$$\Delta(\lambda) = \det\left[\delta_{ij} - \frac{\lambda\lambda_0}{\lambda_0 - \lambda} x_i^*(\psi_j)\right].$$

If $x_i^*(\psi_j) = (-1/\lambda_0)\delta_{ij}$, then

$$\Delta(\lambda) = \left(\frac{\lambda_0}{\lambda_0 - \lambda}\right)^n,$$

and hence $\Delta(\lambda)$ has no zeros in the finite part of the λ-plane.

We claim that $\det[\varphi_i(x_j)] = \det[x_i^*(x_j)] \neq 0$. Indeed,

$$\Delta(\lambda) = \det\left[\delta_{ij} - \lambda x_i^*\left\{\sum_{k=1}^{n} a_{kj} x_k (\lambda - \lambda_0)^{-1} + f(\lambda)\right\}\right],$$

where $f(\lambda)$ is a regular function at the point λ_0. Then

$$\Delta(\lambda) = (-\lambda)^n (\lambda - \lambda_0)^{-n} \det\left[\sum_{k=1}^{n} a_{kj} x_i^*(x_k)\right] + O\left[(\lambda - \lambda_0)^{1-n}\right]$$
$$= (-\lambda)^n (\lambda - \lambda_0)^{-n} \det[a_{kj}] \det[x_i^*(x_k)] + O\left[(\lambda - \lambda_0)^{1-n}\right],$$

and if $\det[x_i^*(x_k)] = 0$, then $\Delta(\lambda) = O[(\lambda - \lambda_0)^{1-n}]$, which is impossible, since $\Delta(\lambda) = \lambda_0^n(\lambda_0 - \lambda)^{-n}$. Invoking Lemma 2, we obtain the desired result. When $\psi_j = x_j$ and $\varphi_i(x_j) = -\lambda_0^{-1}\delta_{ij}$, the lemma is proved similarly.

REMARK 2. To prove the applicability of the method of successive approximations it is frequently sufficient that all the zeros of $\Delta(\lambda)$ be greater than some number in modulus, say, $|\lambda_i| > |\lambda_0|$. Naturally, to satisfy this inequality the conditions of Lemma 3 can be considerably relaxed.

We return to the integral equation (2.5) for $\lambda = 1$ corresponding to a bounded domain D, i.e., in the presence of the contour Γ_0, and we replace this equation by

$$[I - \lambda(K + K_1)]\mu = f, \tag{3.11}$$

$$(K_1\mu)(s_0) = \psi_1(s_0)\int_\Gamma \varphi_1(s)\mu(s)\,ds, \qquad \psi_1 \in L_p(\Gamma),\ \varphi_1 \in L_q(\Gamma).$$

If $\int_\Gamma \psi_1(s)\,ds \neq 0$ and $\int_\Gamma \varphi_1(s)\mu_0(s)\,ds \neq 0$, where $\mu_0(s)$ is given by (2.7), then the conditions of Lemma 1 are satisfied, and (3.11) is unconditionally and uniquely solvable for $\lambda = 1$; its solution coincides with one of the solutions of (2.5). For example, it is possible to take $\psi_1(s) = \mathrm{E}$ and $\varphi_1(s) = 1$, $s \in \Gamma$, i.e., to represent K_1 in the form

$$(K_1\mu)(s_0) = \mathrm{E}\int_\Gamma \mu(s)\,ds, \tag{3.12}$$

where E is a nonzero constant. Integrating (2.7) with consideration of (2.8), we find that the condition $\int_\Gamma \varphi_1(s)\mu_0(s)\,ds \neq 0$ is satisfied with the function $\varphi_1(s)$ so chosen. If, moreover, in (3.12) we take $E = -(\int_\Gamma ds)^{-1}$, then the conditions of Lemma 3 are satisfied, and the characteristic numbers of equation (3.11) are the same as those of (2.5) with the exception of the point $\lambda = 1$. We have thus proved the following result.

THEOREM 3. *Suppose that the contour* Γ_0 *is present. Then equation* (3.11), *where the operator K is represented in* (2.5) *and*

$$(K_1\mu)(s_0) = -\int_\Gamma \mu(s)\,ds \Big/ \int_\Gamma ds,$$

is unconditionally and uniquely solvable in L_p, $1 < p < p_0$, *for* $\lambda = 1$, *and its solution coincides with the solution of* (2.5) *for* $\lambda = 1$ *such that* $\int_\Gamma \mu(s)\,ds = 0$. *If the domain D is simply connected, then all the singular points of the resolvent of* (3.11) *lie in the region* $|\lambda| > 1$; *if D is not simply connected, then the point* $\lambda = -1$ *is a simple pole of the resolvent of* (3.11), *and the remaining singular points lie in the region* $|\lambda| > 1$.

From this we immediately find that the solution of (3.11) with the operator K_1 for $\lambda = 1$ indicated in Theorem 3 can be represented as an ordinary Neumann series for a simply connected domain and as the modified series (3.1) with K replaced by $K + K_1$ for a domain that is not simply connected, and these series converge stably for any right side of (2.5). We further note that for the solution of the boundary value problems in question it is immaterial which of the solutions of (2.5) ($\lambda = 1$) is obtained, since by Theorem 2 they all yield the same function $\Phi(z)$.

§4. Solutions of the original problems

LEMMA 4. *Suppose* $\mu(s)$ *is a solution of* (2.5) *for* $\lambda = 1$ *on the contour* $\Gamma = \bigcup_0^m \Gamma_i$ (Γ_0 *may be absent*). *Then*

$$\int_{\Gamma_i} \mu(s)\,ds = \frac{1}{2}\int_{\Gamma_i} f(s)\,ds, \qquad i = 1,\dots,m.$$

This lemma can be proved by integrating (2.5).

LEMMA 5. *Let* $\Phi(z)$ *be the function represented by* (2.4) *where* $\mu(s)$ *is a solution of* (2.5) *for* $\lambda = 1$, *and let* $\tilde{\Gamma}$ *be a simple closed contour in D oriented in a counterclockwise direction. Then*

$$\int_{\tilde{\Gamma}} \Phi(z)\,dz = -\sum_i \int_{\Gamma_i} f(s)\,ds,$$

where the summation is over all contours enclosed by Γ.

Indeed, direct integration gives

$$\int_{\tilde{\Gamma}} \Phi(z)\,dz = -2\sum_i \int_{\Gamma_i} \mu(s)\,ds,$$

and use of Lemma 4 leads to the required result.

THEOREM 4. *Let D be a connected domain with boundary* $\Gamma = \bigcup_0^m \Gamma_i$ *which is a collection of closed curves of bounded rotation. Then the problem with a skew derivative for the Laplace equation* $du/d\vec{l} = f_2(s) \in L_r(\Gamma)$, $1 < r$, *where* $(\vec{l}, \vec{n}) = \text{const} \neq 0$, *is solvable in the class of functions belonging to* $C^2(D) \cap C(\overline{D})$ *if* $\int_{\Gamma_i} f_2(s)\,ds = 0$ *on all the* Γ_i, *and its solution is given by* (2.2) *and* (2.4), *where* $\mu(s)$ *is a solution of* (2.5) *for* $\lambda = 1$.

Indeed, any solution of (2.5) generates by (2.2) and (2.4) a solution of the problem with a skew derivative if the function u obtained on the basis of it is single-valued and bounded at infinity for an infinite domain D. To satisfy these conditions it is necessary and sufficient that

$$\operatorname{Re}\left[\bar{\beta}\int_{\tilde{\Gamma}}\Phi(z)\,dz\right]=0 \tag{4.1}$$

for any contour $\tilde{\Gamma}$ within D. Considering Lemma 5, we find that to satisfy this condition it is necessary and sufficient that $\int_{\Gamma_i} f_2(s)\,ds = 0$, $i = 1,\ldots,m$. If Γ_0 is absent, then by Theorem 1 equation (2.5) is solvable, and the theorem is proved. If Γ_0 is present, then for the solvability of (2.5) by Theorem 2 it is further necessary that condition (2.9) be satisfied, and for this it is required that $\int_{\Gamma_0} f_2(s)\,ds = 0$. The theorem is proved.

REMARK 3. If $\vec{l} = \vec{n}$, i.e., $\beta = -i$, then condition (4.1) is satisfied for any function $f_2(s)$, since

$$\operatorname{Re}\left[i\int_{\Gamma_j}\Phi(z)\,dz\right] = -\operatorname{Im}\int_{\Gamma_j} f_2(s)\,ds = 0,$$

and for the solvability of (2.5) in the case of a bounded domain it is only necessary that condition (2.9) be satisfied. Therefore, condition (2.9) is sufficient for the solvability of the Neumann problem and for its representation in the form (2.2), (2.4) in a bounded domain, while in an unbounded domain it is unconditionally solvable.

We proceed to the Dirichlet problem. For fixed constants A_i the representation (2.1) is single-valued if $f_1(s) \in W_r^1(\Gamma)$ for any A_i and a_i, since in this case

$$\operatorname{Re}\left\{\int_{\tilde{\Gamma}}\left[\Phi(z)+\sum_{i=0}^{m}\frac{A_i}{z-a_i}\right]dz\right\} = \sum_i\int_{\Gamma_i} f(s)\,ds = \sum_i\int_{\Gamma_i}\frac{df_1}{ds}\,ds = 0.$$

However, if the constants A_i are fixed in some arbitrary manner, then the representation (2.1), (2.4) gives a solution only of the altered Dirichlet problem [1], i.e., of the Dirichlet problem with boundary conditions coinciding with the given conditions up to constants which may be different for each contour. To determine the A_i providing the solution of the original Dirichlet problem we carry out a procedure close to that used in [1] in solving the Dirichlet problem by means of a double-layer potential. We impose the following conditions on the function $u(x, y)$ given by (2.1) and (2.4) where $\mu(s)$ is a solution of (2.5) for $\lambda = 1$:

$$u(x(s_i), y(s_i)) = f_1(s_i), \qquad i = 0,\ldots,m. \tag{4.2}$$

Here the s_i are arbitrary points of the respective contours Γ_i. If there is no contour Γ_0, then in place of condition (4.2) for $i = 0$ we add the condition

$$\sum_{i=1}^{m} A_i = 0, \tag{4.3}$$

which ensures that u will be bounded at infinity. If it is possible to satisfy (4.2) and (4.3) for some collection of A_i, then this collection gives the solution of the original Dirichlet problem.

Conditions (4.2) form a system of linear algebraic equations for determining A_i:

$$\begin{aligned}\sum_{j=1}^{m} A_j\left(\ln\left|\frac{t_i - a_j}{z_0 - a_j}\right| - \operatorname{Re}\int_{z_0}^{t_i}\Phi_j(z)\,dz\right) + A_0 \\ = f_1(t_i) - \operatorname{Re}\int_{z_0}^{t_i}\Phi_0(z)\,dz, \qquad i = 0,\dots,m.\end{aligned} \tag{4.4}$$

If there is no contour Γ_0, then in place of (4.4) for $i = 0$ we add the condition (4.3). Here the $\Phi_j(z)$, $j = 1,\dots,m$, are the functions obtained by means of (2.4) if for μ we there insert the solution of (2.5) with the right side $\operatorname{Re} k(s)/(t(s) - a_j)$, while $\Phi_0(z)$ is generated by the solution of (2.5) with right side $df_1(s)/ds$. The determinant of the system of equations thus obtained is nonzero, since otherwise there would exist nonzero constants A_i satisfying (4.2) and (4.3) with a zero right side of the system (4.4), (4.3), in particular, when the Dirichlet problem with zero boundary conditions can be solved. Indeed, then $f_1(t_i) = 0$, $df_1/ds = 0$, and if there is no contour Γ_0, then (2.5) for $\lambda = 1$ with a zero right side has only the trivial solution $\mu_0 = 0$, and hence $\Phi_0(z) = 0$; if Γ_0 is present, then (2.5) has the nontrivial solution μ_0 given by (2.7), which by Theorem 2 generates only the zero function $\Phi_0(z)$. Thus, if the determinant of the system (4.4), (4.3) is equal to zero, then it is possible to construct by means of (2.1) a nonzero solution of the Dirichlet problem, since by what has been proved above the integral of $\Phi(z)$ in (2.1) for the Dirichlet problem with continuous boundary conditions is single-valued and not equal to a nonzero constant, while $\sum_1^m A_i \ln(z - a_i)/(z_0 - a_i)$ is not a single-valued function. However, the Dirichlet problem with zero boundary conditions cannot have nonzero solutions in the domains we are considering. Thus, the constants A_i are uniquely determined.

In actual numerical solution of the Dirichlet problem in a multiply connected domain by our method it is first necessary to solve (2.5) (or (3.11) for a bounded domain) with the $m + 1$ right sides $df_1(s)/ds$ and $\operatorname{Re} k(s)/(t(s) - a_j)$, $j = 1,\dots,m$, to obtain the functions $\Phi_j(z)$, $j = 0,\dots,m$ by (2.4), to find the constants A_j from the system (4.4), (4.3), to construct the function $\Phi(z) = \Phi_0(z) - \sum_1^m A_j\Phi_j(z)$, and to find the final solution by substituting this expression into (2.1).

REMARK 4. If the domain D is simply connected, i.e., there is only the single contour Γ_0 or Γ_1, then it is not necessary to determine the constants A_j. Indeed, if D is bounded, then the single constant A_0 present can be taken in the form $A_0 = f_1(t_0)$ by letting the point z_0 in (2.1) tend simultaneously to t_0 along a path lying in D. If D is unbounded, then for the boundedness of the solution at infinity it is necessary that $A_1 = 0$, while it is possible to proceed with the constant A_0 as in the case of a bounded domain. Thus, it is necessary to solve (2.5) or (3.11) only for a single right side.

Thus, it has been possible to reduce the original boundary value problems to integral equations solvable by the method of successive approximations.

On the basis of our results it is further possible to investigate the smoothness of the solution of the integral equations obtained and to show that it can be represented as the product of a known weight function having singularity at the corners and an unknown smooth function whose values at the corners give the desired values of the coefficients of the power singularity in the derivatives of the solution of the original problems. A detailed exposition of these propositions goes beyond the framework of the present paper and will form the subject of a separate work.

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