Pergamon

Engineering Fracture Mechanics Vol. 52, No. 4, pp. 745–754, 1995

0013-7944/95 \$9.50 + 0.00
0013–7944(95)00029–1

A FUNCTIONAL APPROACH TO NON-LOCAL STRENGTH CONDITIONS AND FRACTURE CRITERIA—II. DISCRETE FRACTURE

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Abstract—Some new notions of discrete fracture mechanics are introduced: fracture quantum, homogeneous and isotropic fracture quanta sets, and fracture quanta sets that are weakly sensitive to the boundary. Non-local functional strength conditions for a fracture quantum and for a body are considered. Definitions are given for strength homogeneity, isotropy, finite non-locality, and weak sensitivity to the boundary for a discretely fracturable body. Quantum-point dualism in a fracture description is analyzed, which allows one to go from the continuous point of view to the discrete one and vice versa. Presented approaches are applied to generalizations of some known non-local strength conditions on near-boundary points.

INTRODUCTION

A non-local strength concept for a body and for a point has been presented in [1]. We considered their non-locality with respect to stresses only. It was assumed that a point fracture (or strength instability) occurs at some point y^* of a body when the stress field $\sigma(x)$ achieves some critical value $\sigma^*(x)$. However, staying in the frame of the point non-local strength concept, it is difficult in some cases to describe clearly non-local fracture of near-boundary points. For this reason, let us now drop this assumption and assume that fracture zone can be non-point.

Some notions and designations from [1] will be used below without additional comment.

1. FRACTURE QUANTA

Definition 1. A body (material) D will be called discretely fracturable if fracture in it can only occur at once on a point set $F \in D$ called fracture quantum. The set of all fracture quanta of a body D will be designated as $\mathbb{F}(D)$.

In a given body D, each fracturing stress field $\sigma^*(x)$ can have, generally speaking, its own fracture quantum. Fracture quanta can differ in position, orientation, form, and dimension. One and the same body point can belong to different fracture quanta. Microcracks of a characteristic dimension, micro-pores of characteristic radius and so on can figure as fracture quanta. If isolated body points are fracture quanta then discrete fracture degenerates into point fracture.

Let us associate with each fracture quantum F a local right rectilinear coordinate system with origin at a point $y(F)$—the center of F—and with coordinate vectors $\eta(F)$, $\zeta(F)$, $\xi(F)$ directed along the axes of F. For example, the geometrical mass center of F can be chosen as $y(F)$, and vectors $\eta(F)$, $\zeta(F)$, $\xi(F)$ can be directed along the principal inertia axes of F.

Let us introduce the notions of homogeneity, isotropy, boundary layers and weak sensitivity to the boundary for fracture quanta sets.

Definition 2. A fracture quanta set $\mathbb{F}^\chi$ for an unbounded body will be called homogeneous if the point set F', obtained by an arbitrary shift of any fracture quantum $F \in \mathbb{F}^\chi$, is also a fracture quantum: $F' \in \mathbb{F}^\chi$.

Thus, it is possible in a homogeneous set $\mathbb{F}^\chi$ of fracture quanta to extract the subset $\mathbb{F}^0$ of generating quanta, from which it is possible to obtain any other quantum of the set $\mathbb{F}^\chi$ by a proper shift. Meanwhile, each quantum $F^0 \in \mathbb{F}^0$ generates its own subset $\mathbb{F}^\chi(F^0) \supset \mathbb{F}^0$; $\mathbb{F}^\chi = \cup \mathbb{F}^\chi(F^0)$; and

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no quantum F^0 from $\mathbb{F}^0$ can be obtained by a shift of some other quantum $F^{0\prime}$ from $\mathbb{F}^0$ (i.e. the quanta $F^0 \in \mathbb{F}^0$ are irreducible by shifts). Any quantum $F \in \mathbb{F}^\infty$ can be represented in the form:

$$F = F(F^0; y), \quad F^0 \in \mathbb{F}^0, \tag{1}$$

where $y = y(F)$ is center of the quantum F.

Definition 3. We will say that the material of a bounded body D possesses a homogeneous set of fracture quanta, if the body D can be completed to an unbounded body possessing a homogeneous set $\mathbb{F}^\infty$ of fracture quanta. □

Definition 4. A homogeneous fracture quanta set $\mathbb{F}^\infty$ for an unbounded body will be called isotropic if the point set F'', obtained by an arbitrary rotation of any fracture quantum $F \in \mathbb{F}^\infty$ with respect to any point, is also a fracture quantum: $F'' \in \mathbb{F}^\infty$.

Thus, it is possible in a homogeneous isotropic set $\mathbb{F}^\infty$ of fracture quanta to extract the subset $\mathbb{F}^{0c}$ of generating quanta, from which it is possible to obtain any other quantum of the set $\mathbb{F}^\infty$ by a proper shift and rotation. Meanwhile, each quantum $F^{0c} \in \mathbb{F}^{0c}$ generates its own subset $\mathbb{F}^\infty(F^{0c}) \supset \mathbb{F}^{0c}$; $\mathbb{F}^\infty = \cup \mathbb{F}^\infty(F^{0c})$; and no quantum F^{0c} from $\mathbb{F}^{0c}$ can be obtained by shifting and rotating some other quantum F^{0c}, from $\mathbb{F}^{0c}$ (i.e. quanta $F^{0c} \in \mathbb{F}^{0c}$ are irreducible by shifts and rotations). Any quantum $F \in \mathbb{F}^\infty$ can be represented in the form:

$$F = F(F^{0c}; y, \eta, \zeta), \quad F^{0c} \in \mathbb{F}^{0c}, \tag{2}$$

where $y = y(F)$ is center of the quantum F, and $\eta = \eta(F)$, $\zeta = \zeta(F)$ are its axes.

Definition 5. We will say that the material of a bounded body D possesses a homogeneous isotropic set of fracture quanta if the body D can be completed to an unbounded body possessing a homogeneous isotropic set $\mathbb{F}^\infty$ of fracture quanta.

We had to introduce somewhat artificial Definitions 3 and 5 of homogeneity and isotropy of fracture quanta sets for bounded body material because bounded body on lintels, for example, can have fracture quanta differing from ones in the middle of body. Meanwhile, the arbitrary shift and turn demanded in Definitions 2 and 4 are not available for bounded body.

Definition 6. For a body D and a quanta set $\hat{\mathbb{F}}$, a quantum $F \in \hat{\mathbb{F}}$ is internal if $F \subset D$; the subset of all internal quanta from $\hat{\mathbb{F}}$ is denoted by $\mathbb{F}^i(D; \hat{\mathbb{F}}) \subset \hat{\mathbb{F}}$; and their union is the internal subdomain $D^i(D; \hat{\mathbb{F}}) := \cup F$, $F \in \mathbb{F}^i(D; \hat{\mathbb{F}})$; the double boundary layer is the subdomain $B^2(D; \hat{F}) := D \backslash D^i(D; \hat{\mathbb{F}})$.

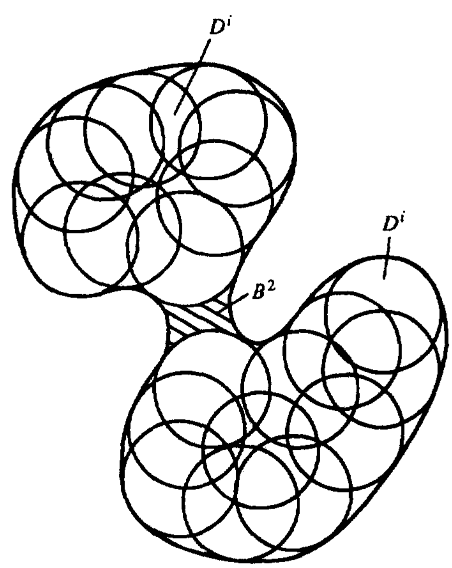

Fig. 1. Internal quanta $\mathbb{F}^i(D; \hat{\mathbb{F}})$, internal domain $D^i(D; \hat{\mathbb{F}})$, and boundary layer $B^2(D; \hat{\mathbb{F}})$ for fracture quanta set $\hat{\mathbb{F}}$, consisting of circles of a fixed radius.

For example, the internal quanta set, consisting of all kinds of circles of a fixed radius, is shown in Fig. 1. The double boundary layer B^2 for the body shown is part of the lintel where the circles cannot be inscribed.

Definition 7. Let the material of a bounded body D possess a homogeneous set of fracture quanta $\mathbb{F}^\infty = \cup \mathbb{F}^\infty(F^0)$, $F^0 \in \mathbb{F}^0$, where the generating quanta F^0 are irreducible by shifts. The fracture quanta set $\mathbb{F}(D)$ of the body D will be called weakly sensitive to the boundary if it can be represented as a union of some sets $\mathbb{F}(D; F^0)$ in the form $\mathbb{F}(D) = \cup \mathbb{F}(D; F^0)$, $F^0 \in \mathbb{F}^0$, such that for each $F^0 \in \mathbb{F}^0$ the subset of all fracture quanta from $\mathbb{F}(D; F^0)$, belonging to $D^i(D; \mathbb{F}^\infty(F^0))$, coincides with the subset of internal fracture quanta $\mathbb{F}^i(D; \mathbb{F}^\infty(F^0))$.

Note that each quantum $F^0 \in \mathbb{F}^0$ generates a set $\mathbb{F}^\infty(F^0)$ which possesses its own internal domain $D^i(D; \mathbb{F}^\infty(F^0))$ and boundary layer $B^2(D; \mathbb{F}^\infty(F^0))$. It is a consequence of this definition that, if a body D possesses a fracture quanta set $\mathbb{F}(D)$ which is weakly sensitive to the boundary then the set can have fracture quanta not belonging to the set $\mathbb{F}^\infty$, but these quanta must include points from $B^2(D; \mathbb{F}^\infty(F^0))$ for some F^0.

2. NON-LOCAL STRENGTH CONDITIONS AT DISCRETE FRACTURE

Definition 8. Let a stress field $\sigma(x)$ be given in a discretely fracturable body D possessing a fracture quanta set $\mathbb{F}(D)$. Then for each virtual quantum $F \in \mathbb{F}(D)$ there are parameters $\lambda'(\sigma; F) \geqslant 0$ such that the stress field $\sigma'(x; F) = \lambda'(\sigma; F)\sigma(x)$ causes no fracture of F. The supremum of $\lambda'(\sigma; F)$ for the given field $\sigma(x)$ and for the quantum F will be called the strength functional $\underline{\lambda}(\sigma; F)$. If $\underline{\lambda}(\sigma; F) > 0$, then the stress field $\sigma(x)$ will be called admissible for the quantum F; if $\underline{\lambda}(\sigma; F) = 0$, then inadmissible. We will call the set consisting of all admissible stress fields for a quantum F the admissible stress set $\mathbb{S}(D; F)$.

For each $\sigma(x) \in \mathbb{S}(D; F)$, the critical stress field $\sigma^*(x; F) := \underline{\lambda}(\sigma; F)\sigma(x)$ causes either fracture or strength instability of the quantum F. The strength instability of F under the stress field $\sigma^*(x; F)$ means that $\sigma^*(x; F)$ causes no fracture of F but for each $\lambda' > 1$ the stress field $\sigma'(x) = \lambda'\sigma^*(x; F)$ exceeds the fracturing stress field for F.

Repeating all reasoning from Part 2 of [1] with obvious variations, we come to the notions of strength functionals $\underline{\Lambda}(\sigma; F) := 1/\underline{\lambda}(\sigma; F)$, $\underline{\mu}(\sigma/\|\sigma\|; F) := \underline{\Lambda}(\sigma; F)/\|\sigma\|$ for a quantum $F \in \mathbb{F}(D)$. The conditions of non-fracture of a quantum F (conditions of non-activation of the fracture quantum F) will have the form:

$$\lambda(\sigma; F) > 1$$

$$\underline{\Lambda}(\sigma; F) < 1$$

$$\|\sigma\| < \underline{\lambda}(\sigma/\|\sigma\|; F). \tag{3}$$

These conditions coincide with conditions (11)–(13) of [1] after replacing a presumed fracture point y by fracture quantum F. When inequalities (3) go into equalities, we obtain the criteria of body fracture through the quantum F or of strength instability of the quantum F.

Hence, the body admissible set is $\mathbb{S}(D) = \cap_F \mathbb{S}(D; F)$, and the body strength functionals are $\underline{\lambda}(\sigma) = \inf_F \underline{\lambda}(\sigma; F)$, $\underline{\Lambda}(\sigma) = \sup_F \underline{\Lambda}(\sigma; F)$, $F \in \mathbb{F}(D)$. The criterion of fracture (i.e. of an activation at least of one fracture quantum in the body) or strength instability for a body D, can be written in one of the following equivalent forms:

$$\inf_F \underline{\lambda}(\sigma; F) = 1, \tag{4}$$

$$\sup_F \underline{\Lambda}(\sigma; F) = 1, \tag{5}$$

$$\|\sigma\| = \inf_F \underline{\lambda}(\sigma/\|\sigma\|; F), \quad \sigma \in \mathbb{S}(D). \tag{6}$$

The infimum in (4) and (6) and the supremum in (5) are taken among all fracture quanta $F \in \mathbb{F}(D)$. Any critical stress field $\sigma^*(x)$ for the body is a solution of these equations.

The case, when the supremum and the infimum in these equations are achieved at a critical fracture quantum (or at some critical quanta) $F^* \in \mathbb{F}(D)$ will be called the isolated quantum fracture. Then the supremum can be replaced by maximum, and the infimum by minimum. For the isolated fracture we obtain three equivalent equations for determination of the quantum F^* of fracture or strength instability:

$$\underline{\lambda}(\sigma^*; F^*) = 1$$

$$\underline{\Lambda}(\sigma^*; F^*) = 1,$$

$$\|\sigma^*\| = \underline{\lambda}(\sigma^*/\|\sigma^*\|; F^*).$$

The other case, when the infimum and the supremum can be achieved at no quantum $F \in \mathbb{F}(D)$, will be called the infinitesimal (quantum) fracture. In this case $\underline{\lambda}(\sigma^*; F) > 1$ for each $F \in \mathbb{F}(D)$, i.e. each quantum F is strength stable for $\sigma(x) = \sigma^*(x)$. On the other hand if $\lambda' = 1 + \mathrm{d}\lambda$ for any $\mathrm{d}\lambda > 0$, then the inequalities (3) will be violated for $\sigma'(x) = \lambda'\sigma^*(x) = (1 + \mathrm{d}\lambda)\sigma^*(x)$ in the critical set $\mathrm{d}\mathbb{F}^*$ consisting of an infinite number of such quanta F^* that

$$1 < \underline{\lambda}(\sigma^*; F^*) \leqslant 1 + \mathrm{d}\lambda,$$

$$(1 + \mathrm{d}\lambda)^{-1} \leqslant \underline{\Lambda}(\sigma^*; F^*) < 1,$$

$$\|\sigma^*\| < \underline{\lambda}(\sigma^*/\|\sigma^*\|; F^*) \leqslant \|\sigma^*\|(1 + \mathrm{d}\lambda).$$

The critical set $\mathrm{d}\mathbb{F}^*$ monotonically decreases when $\mathrm{d}\lambda$ decreases. Moreover $\mathrm{d}\mathbb{F}^* \to \emptyset$ when $\mathrm{d}\lambda \to 0$ because of closeness of the set of parameters λ' for which $\sigma'(x) = \lambda'\sigma(x)$ causes fracture trough or strength instability of a quantum F (see Part 2 of [1]). Consequently if $\mathrm{d}\lambda$ is infinitesimal then $\mathrm{d}\mathbb{F}^*$ is infinitesimal too.

By analogy with the corresponding non-local strength conditions for point, it is possible also in the case of discrete fracture to introduce the notions of strength homogeneity, isotropy, finite non-locality, and also weak sensitivity to boundary with respect to strength.

The statements below in this paper are formulated for the strength functional $\underline{\Lambda}$, but analogous relations also hold for $\underline{\lambda}$ and $\underline{\mu}$.

It is possible for each fracture quantum F to go from the stress field $\sigma(x)$ in the global coordinate system to stress field $\sigma^{(F)}(w) = \sigma^{(y,\eta,\zeta)}(w)$ in the related with F local coordinate system originating at $y(F)$ and having the coordinate vectors $\eta(F)$, $\zeta(F)$, $\xi(F)$; here $w = x - y(F)$. Then we obtain the relative admissible stress set $\tilde{\tilde{\mathbb{S}}}(D; F)$ consisting of relative functions $\sigma^{(F)}(w)$, and the relative strength functional $\tilde{\tilde{\underline{\Lambda}}}$ for a set F: $\tilde{\tilde{\underline{\Lambda}}}(\sigma^{(F)}; F) := \underline{\Lambda}(\sigma; F)$.

In view of (1), the admissible stress set and the strength functionals for a quantum F at an unbounded body with a homogeneous set of fracture quanta $\mathbb{F}^\infty \supset \mathbb{F}^0$ can be represented in terms of a generating quantum $F^0 \in \mathbb{F}^0$ and the center $y(F)$ of the quantum F: $\tilde{\tilde{\mathbb{S}}}^0(D^\infty; F^0; y) := \tilde{\tilde{\mathbb{S}}}(D^\infty; F(F^0; y))$, $\tilde{\tilde{\underline{\Lambda}}}^{\infty 0}(\sigma^{(y,\eta,\zeta)}; F^0; y) := \tilde{\tilde{\underline{\Lambda}}}^\infty(\sigma^{(F)}; F(F^0; y))$. Note that $\eta(F) = \eta(F^0)$, $\zeta(F) = \zeta(F^0)$, $\xi(F^0) = \xi(F^0)$.

Definition 9. A discretely fracturable unbounded body will be called strength homogeneous, if it possesses a homogeneous fracture quanta set $\mathbb{F}^\infty$, and if the relative admissible stress set as well as the relative strength functionals do not depend on the position of fracture quantum center, i.e. $\tilde{\tilde{\mathbb{S}}}^0(D^\infty; F^0; y) = \tilde{\tilde{\mathbb{S}}}^0(D^\infty; F^0)$, $\tilde{\tilde{\underline{\Lambda}}}^{\infty 0}(\sigma^{(y,\eta,\zeta)}; F^0; y) = \tilde{\tilde{\underline{\Lambda}}}^{\infty 0}(\sigma^{(y,\eta,\zeta)}; F^0)$.

The material of a discretely fracturable bounded body will be called strength homogeneous if the body can be completed to a strength homogeneous unbounded body.

In view of (2), the relative admissible stress set and the strength functionals for a quantum F at an unbounded body with an isotropic homogeneous set of fracture quanta $\mathbb{F}^\infty$ can be represented in terms of a generating quantum F^{0c}, the center y and the axes η, ζ of the quantum F:

$$\tilde{\tilde{\mathbb{S}}}^{0c}(D^\infty; F^{0c}; y, \eta, \zeta) := \tilde{\tilde{\mathbb{S}}}(D^\infty; F(F^{0c}; y, \eta, \zeta)),$$

$$\tilde{\tilde{\underline{\Lambda}}}^{\infty 0c}(\sigma^{(y,\eta,\zeta)}; F^{0c}; y, \eta, \zeta) := \tilde{\tilde{\underline{\Lambda}}}^\infty(\sigma^{(F)}; F(F^{0c}; y, \eta, \zeta)).$$

Definition 10. A discretely fracturable unbounded strength homogeneous body will be called strength isotropic, if it possesses a homogeneous isotropic fracture quanta set $\mathbb{F}^{\infty}$, and if the relative admissible stress set as well as the relative strength functionals do not depend on the position and orientation of the fracture quantum i.e.

$$\tilde{\tilde{\mathbb{S}}}^{0c}(D^{\infty}; F^{0c}; y, \eta, \zeta) = \tilde{\tilde{\mathbb{S}}}^{0c}(D^{\infty}; F^{0c}), \; \tilde{\tilde{\underline{\Lambda}}}^{\infty 0c}(\sigma^{(y,\eta,\zeta)}; F^{0c}; y, \eta, \zeta) = \tilde{\tilde{\underline{\Lambda}}}^{\infty 0c}(\sigma^{y,\eta,\zeta}; F^{0c}).$$

The material of a discretely fracturable bounded body will be called strength homogeneous and isotropic if the body can be completed to a strength homogeneous and isotropic unbounded body.

Definition 11. A discretely fracturable bounded body D, consisting of a strength homogeneous material, will be called weakly sensitive to the boundary with respect to strength, if its fracture quanta set $\mathbb{F}(D)$ is weakly sensitive to the boundary, and if for each $F \in \mathbb{F}(D) \cap \mathbb{F}^{\infty}$ there is an extension of each admissible stress field $\sigma(x) \in \mathbb{S}(D; F)$ to a field $\sigma^{\infty}(x) \in \mathbb{S}(D^{\infty}; F)$ defined in unbounded space D^{∞}, such that $\sigma(x) = \chi(D; x)\sigma^{\infty}(x)$ and the values of the strength functionals for the initial body D and for the corresponding unbounded body D^{∞}, consisting of the same material, coincide for all corresponding stress fields: $\underline{\Lambda}(\sigma; F) = \underline{\Lambda}^{\infty}(\sigma^{\infty}; F)$.

Definition 12. For a discretely fracturable body D, we will call the strength functionals finitely non-local for a fracture quantum $F \in \mathbb{F}(D)$ if a finite domain $D'(F)$ exists such that $\mathrm{mes}[D'(F)] < \mathrm{mes}[D]$, and $\underline{\Lambda}(\sigma; F) = \underline{\Lambda}(\chi[D'(F)]\sigma; F)$ for all $\sigma \in \mathbb{S}(D; F)$. We will call the intersection of such domains for a given quantum F the domain of non-locality $\Omega(F)$.

Note that when the domain of non-locality degenerates into a point $y \in D$, and the fracture quantum does not, then we come to the local concept of discrete fracture. On the other hand, when the fracture quantum degenerates into a point $y \in D$, and the domain of non-locality does not, then we come to the non-local concept of point fracture.

Definition 13. We will say that a fracture quantum F possesses an internal domain of non-locality if its strength functionals are finitely non-local and the domain of non-locality $\Omega(F) \subset F$.

Thus, activation of a fracture quantum with an internal domain of non-locality is defined by the stress field on this quantum only. Hence, for a body D, consisting of a material for which all fracture quanta from $\mathbb{F}^{\infty}$ possess internal domains of non-locality, it is not necessary for weak sensitivity to the boundary to extend the stress field from D to the whole space, i.e. in this case we come to the following simplified formulation of weak sensitivity to the boundary.

Statement. Let the material of a discretely fracturable bounded body D be strength homogeneous and let all fracture quanta from $\mathbb{F}^{\infty}$ possess internal domains of non-locality. Then the body D is weakly sensitive to the boundary with respect to strength, if its fracture quanta set $\mathbb{F}(D)$ is weakly sensitive to the boundary and if for each $F \in \mathbb{F}(D) \cap \mathbb{F}^{\infty}$ the strength functionals for the body D and for corresponding unbounded body, consisting of the same material, coincide for all stress fields $\sigma \in \mathbb{S}(D; F)$: $\underline{\Lambda}(\sigma; F) = \underline{\Lambda}^{\infty}(\sigma; F)$.

Note that notions of internal non-locality and weak sensitivity to the boundary, given by the last two definitions, are the main reasons for the introduction and description of discrete fracture, because, unlike the point fracture concept, these notions allow sufficiently natural and simple modeling of non-local strength conditions for boundary and near-boundary points (maybe excluding points from boundary layers of the second kind).

3. QUANTUM-POINT DUALISM IN STRENGTH DESCRIPTION

The author believes that discrete modeling of fracture is sufficiently close to the true fracture nature (see also [2–5]) and in many cases there is a reason to stay successively in the discrete point of view. However, experimental data in some cases do not present reliable information about the real configuration of the fracture zone. For this reason, it would be preferable to have a possibility to stay, depending on convenience, either in the point or in the discrete point of view, and, if necessary, to go without difficulties from one point of view to another and vice versa. Some ways of description of such dualism are proposed below.

3.1. *Point strength for discretely fracturable body*

The non-local strength conditions for a discretely fracturable body can be represented in point form (11)–(13) or (15) of [1]. Let us suppose for this, that fracture or strength instability at a point y' occurs when it occurs at a fracture quantum including the point y'. Then for each point y' we obtain the admissible stress set, the strength functional, and the strength condition:

$$\mathbb{S}(D; y') = \bigcap_{F \ni y'} \mathbb{S}(D; F); \quad \underline{\Lambda}(\sigma; y') = \sup_{F \ni y'} \underline{\Lambda}(\sigma; F), \; F \in \mathbb{F}(D), \quad \sigma \in \mathbb{S}(D; y'); \tag{7}$$

$$\Lambda(\sigma; y') < 1. \tag{8}$$

If the functional $\underline{\Lambda}(\sigma; F)$ is finitely non-local, then the functional $\underline{\Lambda}(\sigma; y')$ is finitely non-local too, and the domain of non-locality is

$$\Omega(y') = \sum_{F \ni y} \Omega(F).$$

Let us now consider the axis $\eta(F)$ of the minimal inertia moment of the quantum F as the fracture direction, and the axis $\zeta(F)$ of the maximal inertia moment of the quantum F as the normal to the fracture plane. Then for each point y', direction η' and plane ζ' we come to the admissible stress set, the strength functional, and the strength conditions:

$$\mathbb{S}(D; y', \eta', \zeta') = \bigcap_{F} \mathbb{S}(D; F); \underline{\Lambda}(\sigma; y', \eta', \zeta') = \sup_{F} \underline{\Lambda}(\sigma; F), \quad \sigma \in \mathbb{S}(D; y', \eta', \zeta'); \tag{9}$$

$$\underline{\Lambda}(\sigma; y', \eta', \zeta') < 1. \tag{10}$$

If the functional $\Lambda(\sigma; F)$ is finitely non-local, then the functional $\Lambda(\sigma; y', \eta', \zeta')$ is finitely non-local too, and the domain of non-locality is $\Omega(y', \eta', \zeta') = \cup_F \Omega(F)$. Here and in (9) the quanta F meet the conditions $F \in \mathbb{F}(D)$, $F \ni y'$, $\eta(F) = \eta'$, $\zeta(F) = \zeta'$, and the scalar product $(\eta', y(F) - y') \geqslant 0$.

According to the strength conditions (8) and (10), fracture passes through y' at the transition of the inequalities to equalities. We can also demand that the point y' is, in addition, the starting point of fracture, i.e. it belongs to the boundary of the corresponding fracture quantum. Then, in the representation (7) of the admissible stress set and of the strength functionals as well as in the representation of the domain of non-locality for the point y', it is necessary to take the intersection and union of sets, infimum and supremum over not all fracture quanta including y' but only over the quanta F whose boundaries include y', i.e., F: $\{\partial F \ni y'\}$.

Analogously, if we demand that fracture starts from a point y' and goes in a direction η' in a plane ζ', then in addition to the point y' belonging to the boundary of some fracture quantum F, whose axis $\eta(F)$ is directed along η' and axis $\zeta(F)$ along ζ', it is necessary to demand that the point y' is the most distant from the quantum center $y(F)$ in a sense. For example, it is possible to demand that $y' \in \partial F$ and the point y' is situated on the axis $\eta(F)$, i.e. the scalar product $(\eta', y(F) - y') = |y(F) - y'|$, and to take the intersection and union of sets, infimum and supremum in (9) only over the quanta meeting these conditions.

If fracture quanta have special points which can be fracture starting points, e.g. crack ends when cracks are fracture quanta, then only quanta having y' as such a special point can be used in the intersection and union of sets, infimum and supremum in (7) and (9) for the representation of the admissible stress set and the strength functionals for the fracture starting point y'.

3.2. *Fracture quanta and point strength conditions*

In using point strength conditions (11)–(13) or (15) [1], which coincide with conditions (8) and (10) of this paper, we get the point fracture or strength instability occuring at a point y when these conditions are violated, i.e. go into equalities, in that point. However, in contrast to that, it is possible also to suppose that upon violation of the point strength conditions at a point y, fracture or strength instability occurs at a fracture quantum assigned to this point. Then the point non-local strength conditions (8) and (10) can be represented in the discrete form (3). Such a point of view is realized below in this section.

Let us assign in some way a fracture quantum $F(\sigma, y, \eta, \zeta)$ to each point y and directions η, ζ and to each stress field $\sigma(x)$ (more precisely to each function $\sigma(x)/\|\sigma\|$). In modeling a real situation usually $F(\sigma, y, \eta, \zeta)$ or $\partial F(\sigma, y, \eta, \zeta)$ includes the point y, and $F(\sigma, y, \eta, \zeta)$ is directed in a sense along η in plane ζ.

We will suppose that when a stress field $\sigma(x)$ achieves its critical value $\sigma^*(x) = \underset{\frown}{\lambda}(\sigma; y, \eta, \zeta)\sigma(x)$ for a point y and directions η, ζ then fracture or strength instability occurs through the quantum $F(\sigma, y, \eta, \zeta)$. Let $\hat{\mathbb{S}}(D; F'; y, \eta, \zeta) \subset \mathbb{S}(D; y, \eta, \zeta)$ be the subset of all admissible stress fields $\sigma(x)$ for which one and the same fracture quantum $F' = F(\sigma, y, \eta, \zeta)$ is assigned to the point y and directions η, ζ. Then we introduce the quantum strength functional $\underset{\frown}{\lambda}(\sigma; F'; y, \eta, \zeta)$ assigned to the parameters y, η, ζ which equals $\underset{\frown}{\lambda}(\sigma; y, \eta, \zeta)$ for $\sigma(x) \in \hat{\mathbb{S}}(D; F'; y, \eta, \zeta)$, and equals infinity for $\sigma(x) \in \mathbb{S}(D; y, \eta, \zeta) \backslash \hat{\mathbb{S}}(D; F'; y, \eta, \zeta)$. As usual, $\underset{\frown}{\Lambda}(\sigma; F'; y, \eta, \zeta) = 1/\underset{\frown}{\lambda}(\sigma; F'; y, \eta, \zeta) = \|\sigma\| \underset{\frown}{\mu}(\sigma/\|\sigma\|; F'; y, \eta, \zeta)$. Thus, the use of the fracture quantum F' is extended from $\sigma(x) \in \hat{\mathbb{S}}(D; F'; y, \eta, \zeta)$ to all $\sigma(x) \in \mathbb{S}(D; y, \eta, \zeta)$. For this reason we will below drop the parameter σ from fracture quantum arguments. The set of admissible stress fields $\hat{\mathbb{S}}(D; F'; y, \eta, \zeta)$ assigned to the parameters y, η, ζ for a quantum $F' = F(y, \eta, \zeta)$ coincides thus with $\mathbb{S}(D; y, \eta, \zeta)$.

Analogously, it is possible to introduce the notions of finite non-locality for discrete strength functionals $\underset{\frown}{\Lambda}(\sigma; F'; y, \eta, \zeta)$ and of the domain of non-locality $\Omega(F'; y, \eta, \zeta)$ assigned to the parameters y, η, ζ. The notions coincide with the corresponding ones given in Definition 12 after replacing there the set $\mathbb{S}(D; F')$ by the set $\hat{\mathbb{S}}(S; F'; y, \eta, \zeta)$.

Let us denote by $\mathbb{F}(D)$ the set of all fracture quanta $F(y, \eta, \zeta)$ obtained for all choices of parameters $(y, \eta, \zeta) \in D$. Taking into account that one and the same fracture quantum $F' \in \mathbb{F}(D)$ can be assigned to different parameters y, η, ζ, we obtain the following expressions for the admissible stress field set and for the strength functionals of fracture quantum $F' \in \mathbb{F}(D)$:

$$\mathbb{S}(D; F') = \bigcap_{y,\eta,\zeta} \mathbb{S}(D; F'; y, \eta, \zeta), \quad F(y, \eta, \zeta) = F', \quad (y, \eta, \zeta) \in D;$$

$$\underset{\frown}{\Lambda}(\sigma; F') = \sup_{y,\eta,\zeta} \underset{\frown}{\Lambda}(\sigma; F'; y, \eta, \zeta), \quad F(y, \eta, \zeta) = F', \quad (y, \eta, \zeta) \in D, \quad \sigma \in \mathbb{S}(D; F').$$

Thus, the strength of a quantum F' can be described by the strength conditions (3) with the obtained functionals $\Lambda(\sigma; F')$, $\underset{\frown}{\lambda}(\sigma; F')$, $\mu(\sigma/\|\sigma\|; F')$.

If the functional $\underset{\frown}{\Lambda}(\sigma; y, \eta, \zeta)$ is finitely non-local then the functional $\underset{\frown}{\Lambda}(\sigma; F')$ is finitely non-local too, and its domain of non-locality is $\Omega(F') = \cup_{y,\eta,\zeta} \Omega(F'; y, \eta, \zeta)$, $F(y, \eta, \zeta) = F'$, $(y, \eta, \zeta) \in D$.

The choice of assigned fracture quanta can be made in many different ways if it is not related to a real picture of fracture. In such cases it seems natural to use assigned fracture quanta with internal domains of non-locality (i.e. fracture quanta including their own domains of non-locality) if the point strength functional $\underset{\frown}{\Lambda}(\sigma; y, \eta, \zeta)$ and, consequently, the quantum strength functional $\Lambda(\sigma; F')$, are finitely non-local.

Note that in view of the previous reasoning of Part 3, it is possible in principal to return completely from the discrete fracture concept given above, to the point fracture concept of [1], and to consider the fracture quanta introduced above, only as some internal parameters of the model, which are useful means for describing strength functionals in near-boundary points but have no immediate relation to a real fracture picture.

This dualism allows a simple transition from the point concept to the discrete one and vice versa. It will be used in Part 4 for generalizations of the known strength conditions quoted in [1].

4. GENERALIZATIONS OF SOME NON-LOCAL STRENGTH CONDITIONS FOR NEAR-BOUNDARY POINTS

Let us consider the following non-local strength conditions, quoted in Part 1 of [1] and represented in point non-local strength functional form in Part 4 of [1]: (1) the condition, based on average stress over a characteristic length d_1; (2) the condition based on a minimum stress over a characteristic length d_2; (3) the condition based on the model of a fictitious crack with a characteristic length d_3.

Let $k = 1/3$ be the order number of the corresponding strength condition. The functional representations of Sections 4.1–4.3 of [1] are quite applicable for body internal points, i.e. points distant more than d_k from the body boundary ∂D (y_1-type points in Fig. 2). For strength analysis of near-boundary points of y_2- or y_3-type in Fig. 2, the strength conditions presented there must be generalized.

The simplest way to do this is a transition to the discrete fracture concept according to Section 3.2. We assign to each internal point y and direction $\eta(\theta)$ the fracture quantum $F_k(y, \eta)$ coinciding with the domain of non-locality $\Omega_k(y, \eta)$ of the functional $\Delta_k(y, \eta)$. Thus, the fracture quanta set $\mathbb{F}_k$ includes all kinds of segments (i.e. cracks) with lengths d_k belonging to the body D. One and the same fracture quantum is assigned to two points which are segment ends, and vectors η at these points are opposite and directed to the quantum center.

If the body D is unbounded then for each $k = 1/3$ the fracture quanta set $\mathbb{F}_k^\infty$ is homogeneous and isotropic. Any element $F \in \mathbb{F}_k^\infty$ can be represented in the form $F(y, \eta) = F(F_k^{0c}(\eta); y) = F(F_k^{0c}; y, \eta)$. Here the generating quantum $F_k^0(\eta)$, from which each other quantum $F \in \mathbb{F}_k^\infty$ can be obtained by a proper shift, can be taken in the form $F_k^0(\eta) = \Omega_k(0, \eta(\theta))$; the generating quantum F_k^{0c}, from which each other quantum $F \in \mathbb{F}_k^\infty$ can be obtained by a proper shift and rotation, can be taken into the form $F_k^{0c} = \Omega_k(0, \eta(0))$.

Then for a fracture quantum $F' \in \mathbb{F}_k^\infty$, having a center y' and direction $\eta(\theta)$, we obtain from the $\Delta_k(\sigma; y, \eta(\theta))$ given by (17)–(19) of [1] their quantum analogs $\Delta_k(\sigma; F') = \Delta_k(\sigma; y' - \eta d_k/2, \eta)$. So, the discretely fracturable material obeying one of the quoted strength conditions is strength homogeneous and isotropic and possesses internal domains of non-locality $\Omega_k(F'(y', \eta)) = \Omega_k(y' - \eta d_k/2, \eta)$.

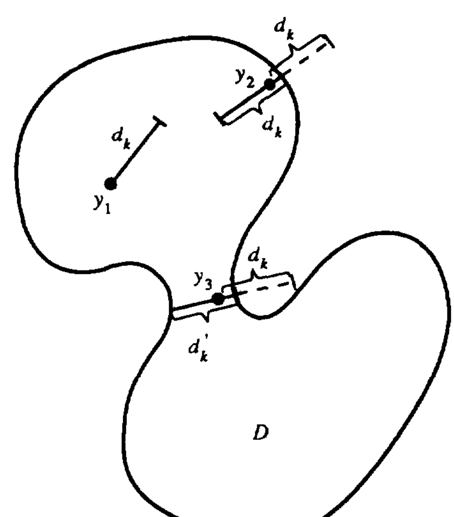

Fig. 2. Fracture quanta corresponding to generalized strength conditions (2)–(4) of ref. [1].

For the description of fracture of a bounded body trough near-boundary quanta, including points of y_2- and y_3-type in Fig. 2, it is natural to suppose that fracture quanta sets $\mathbb{F}_k(D)$ for all three strength conditions are weakly sensitive to the boundary, i.e. they include all kinds of segments with length d_k which do not fall outside the domain D. For a boundary layer of the second kind, including points of y_3-type, it is natural also to suppose that fracture quantum sets $\mathbb{F}_k(D)$ include also all segments with length $d'_k < d_k$ which belong to D, not to ∂D, and join some boundary points.

Strength functionals $\Lambda_k(\sigma; F_k)$ for the internal fracture quanta F_k (which belong to the internal subdomains $D'(D, \mathbb{F}^\infty(F_k^0(\eta)))$ for the corresponding η) can be defined in the same way as for the unbounded body. For the functionals $\Lambda_1(\alpha; F_1)$, $\Lambda_2(\sigma; F_2)$ it is equivalent to the supposition of weak sensitivity to the boundary with respect to strength. For the strength condition with a fictitious crack ($k = 3$) the sensitivity to the boundary remains non-weak and moreover, when one end of the fracture quantum F_3 falls on the boundary ∂D, i.e. an edge fictitious crack is considered, we should use the stress intensity factor only on the internal end of the crack in the condition.

The strength functionals for fracture quanta intersecting $B^2(D, \mathbb{F}^\infty(F_k^0(\eta)))$ i.e. having length $d'_k < d_k$, can be defined analogously by use of the same formulas (17)–(19) of [1] if we replace their d_k by d'_k. However, calculation of the strength functional $\Lambda_3(\sigma; F')$ in this case is reduced to calculation of limits of stress intensity factors on the ends of a shortened crack of length d''_3, going along a line which joins two points of ∂D and has length d'_3, when $d''_3 \to d'_3$ (see lintel near point y_3 in Fig. 2). It is not difficult to see that these values of the functional $\Lambda_3(\sigma; F')$ may be equal either to zero or to infinity, i.e. each stress field, opening such crack at both its ends, becomes inadmissible.

We can see that for such a generalization the functionals $\Lambda_1(\sigma; F')$ and $\Lambda_2(\sigma; F')$ are weakly sensitive to the boundary, in contrast to the functional $\Lambda_3(\sigma; F')$.

Analogous generalizations can be also given to the modified forms of the functionals given in Section 4 of [1].

CONCLUSIONS

A concept of non-local strength for a discretely fracturable body is presented in this work, in addition to the concept of non-local strength for a point fracturable body given in [1]. Notions of fracture quanta sets are introduced. Non-local strength functionals are considered which are associated with the supremum of positive factors by which a given stress field may be multiplied to obtain a stress field which does not activate the considered fracture quantum. Definitions are given for strength homogeneity, isotropy, finite non-locality, and weak sensitivity to the boundary for a discretely fracturable body. Quantum-point dualism in the fracture description is analyzed which allows one to go from the quantum point of view to the point one and vice versa. The approaches presented are applied to generalizations of some known non-local strength conditions on near-boundary body points.

The concept of non-local strength for a discretely fracturable body is sufficiently useful for strength modeling of heterogeneous materials such as concrete, composites, particleboard, and so on. Corresponding methods of approximation of fracture quanta sets and non-local strength functionals by use of experimental data would be the subject of another investigation.

The non-local strength of a discretely fracturable body is elaborated in this work by starting from general abstract reasoning. For the construction of non-local strength conditions, another approach is also possible which is based on explicit introduction of some structural material parameters. Such an approach was used by Cherepanov [2–5] for modeling an atomic-dislocation mechanism of fracturing using the additional structure parameter τ_0—the critical share stress for sliding in a crystal.

Acknowledgement—This work was completed under support of the author by an Alexander von Humboldt Foundation fellowship as a guest professor at the University of Stuttgart.

REFERENCES

[1] S. E. Mikhailov, A functional approach to non-local strength conditions and fracture criteria—I. Body and point fracture. *Engng Fracture Mech.* **52,** 731–743 (1995).
[2] G. P. Cherepanov, Initiation of micro-cracks and dislocations. *Appl. Mech.* **23** (12), 67–81 (1987).
[3] G. P. Cherepanov, Growth of micro-cracks under monotonic loading. *Appl. Mech.* **24** (4), 86–88 (1988).
[4] G. P. Cherepanov, Closure of micro-cracks under unloading and occurring of reverse dislocations. *Appl. Mech.* **24** (7), 3–18 (1988).
[5] G. P. Cherepanov, Quantum fracture mechanics. *Probl. Strength* **2**(2), 3–9 (1990).

(*Received* 9 *March* 1994)