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A FUNCTIONAL APPROACH TO NON-LOCAL STRENGTH CONDITIONS AND FRACTURE CRITERIA—I. BODY AND POINT FRACTURE

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Abstract—A general form of non-local strength condition based on a nonlinear space strength functional is proposed, and its relation with some known non-local strength conditions is discussed. The strength functional is associated with the supremum of a positive factor by which a given stress field may be multiplied to obtain a non-fracturing stress field. Mathematical constraints on the functional form caused by the demand of functional boundedness on admissible stress fields are explored. The notions of strength homogeneity, strength isotropy and finite non-locality for non-local strength conditions are introduced.

INTRODUCTION

IN THE traditional approach, strength of a solid in a given point is characterized by the value of some function of stress and/or strain tensor components in the same point without consideration of the stress state in neighboring points. This is the essence of so-called local strength conditions (LSC) and corresponding local fracture criteria (LFC). They give a good description of experimental data when macro-stress variations are small enough on dimensions of the order of the material structure scale.

When a stress field evaluated on the assumption of a solid homogeneity varies abruptly enough on such dimensions, for example near concentrators, so-called scale effects are observed. They consist of the fact that the body survives larger stresses without fracture under these conditions than under the corresponding uniform stress state. The scale effects can be apparently accounted for by means of a LFC considering the micro-structure stress state and exploring the strength of smaller structure elements. But this way is very tedious and demands knowledge of strength and deformability characteristics of the structure elements, their mutual dispositions, and statistics of these parameters.

It is preferable to construct macroscopic non-local strength conditions (NLSC) and corresponding non-local fracture criteria (NLFC), which use stresses or strains not only at the investigated point but also in some neighborhood of it. The criteria use stresses or strains obtained on the assumption of a solid micro-homogeneity. All micro-heterogeneity is implicitly taken into account by corresponding non-locality of the criterion form. NLFC parameters should be obtained from macro-tests, although they could in principle be obtained also by proper elaboration of adequate microstructural models where LFC operate.

On the other hand, the fracture criteria parameters in linear fracture mechanics such as crack intensity factors K_{1c}, K_{2c}, K_{3c} or other parameters expressed in terms of them are in no way connected with LFC parameters for a solid without crack-like global concentrators. NLFC give the possibility to connect these two groups of strength parameters, that is, to evaluate solid strength with, as well as without, cracks from the same position. NLFC also allow one to analyze the strength of solids with singular concentrators which gives rise to stress singularities different from the square root, such as corner points, bonded solids, and so on.

There is a number of NLSC used now for strength evaluation, for example, the Neuber–Novozhilov condition based on the average stress over a fixed interval, a condition based on

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the minimum stress over a fixed interval, and a condition based on a fictitious crack of a fixed length.

However, to the author's knowledge, there is no sufficiently general theory now which would allow one to choose an adequate form of NLSC for a given material together with a rational series of experimental tests for the determination of NLSC parameters.

In this work, a general form of NLSC based on a nonlinear space strength functional is proposed, and its relation with the known NLSC is discussed. The strength functional is associated with the supremum of a positive factor, by which a given stress field may be multiplied to obtain a non-fracturing stress field. Mathematical constraints on the form of the functional, caused by the demand of functional boundedness on admissible stress fields, are explored. The notions of strength homogeneity, strength isotropy and finite non-locality for non-local strength conditions are introduced. Some of these ideas were briefly presented in the author's papers [1, 2].

1. SOME NON-LOCAL STRENGTH CONDITIONS

The classical local strength conditions for a point y can be represented in the form:

$$f(\sigma_{ij}(y)) < \sigma_c, \tag{1}$$

where f is a material function and σ_c is a material constant.

Some known non-local strength conditions and fracture criteria for two-dimensional cases are quoted below in this section. Let (ρ, θ) be a local polar coordinate system with respect to an analyzed point y of a body D with boundary ∂D; let $\eta(\theta)$ be a unit vector making an angle θ with an axis; and let $\sigma_{\rho\rho}, \sigma_{\rho\theta}, \sigma_{\theta\theta}$ be the stress components in this coordinate system.

1.1. *Strength condition based on average stress over a characteristic length* d_1

This approach was used by Neuber [3], Novozhilov [4], Morozov [5], and other authors. Somewhat generalized, its form is the following:

$$\frac{1}{d_1} \max_{-\pi \leqslant \theta < \pi} \int_0^{d_1} \sigma_{\theta\theta}(y + \rho\eta(\theta))\, d\rho < \sigma_c. \tag{2}$$

Here σ_c and d_1 are material constants, σ_c is the strength of a body without concentrators under uniform traction, and the left hand side of the inequality is maximum of the hoop stress average over an interval, originating at the analyzed point y, with length d_1.

1.2. *Strength condition based on a minimum stress over a characteristic length* d_2

This is a generalization of criteria used by Cruse [6], Whitney and Nusmer [7], and other authors. It may be represented in the form:

$$\max_{-\pi \leqslant \theta < \pi} \left[\min_{0 \leqslant \rho \leqslant d_2} \sigma_{\theta\theta}(y + \rho\eta(\theta)) \right] < \sigma_c. \tag{3}$$

Here σ_c and d_2 are material constants.

1.3. *Strength condition based on a model of a fictitious crack with characteristic length* d_3

This was used by Waddoups *et al.* [8], Caprino *et al.* [9], and other authors. In the simplest case it may be represented in the form:

$$\max_{\pi \leqslant \theta < \pi} \min_i K_{1i}(y, \theta, d_3) = \max_{-\pi \leqslant \theta < \pi} \min_i \int_0^{d_3} [N_{1i\rho}(\rho, y, \theta)\sigma_{\theta\rho}(y + \rho\eta(\theta)) + N_{1i\theta}(\rho, y, \theta)\sigma_{\theta\theta}(y + \rho\eta(\theta))]\, \mathrm{d}\rho < K_{1c}. \tag{4}$$

Here K_{1c} and d_3 are material constants, $K_{11} := K_1(y)$, $K_{12} := K_1(y + d_3\eta(\theta))$ are the stress intensity factors at the ends of a fictitious crack originating from y and directed along $\eta(\theta)$; $N_{1i\rho}(\rho, y, \theta)$ and $N_{1i\theta}(\rho, y, \theta)$ $(i = 1/2)$ are Green-type functions for the body with the crack, which give the values of the stress intensity factors K_{1i} induced at the crack ends by two unit equal and oppositely directed normal or tangent concentrated forces applied to opposite crack shores points.

The strength conditions go into fracture criteria when inequalities are replaced with equalities.

Conditions (2)–(4) can be directly applied to any internal point y_1 which is distant more than d_k from the body boundary (see Fig. 1, point y_1). Otherwise, the integration intervals in (2), (4) and the minimization interval in (3) do not belong to the body for several angles θ (see Fig. 1, points y_2 and y_3), and the criteria demand further generalization. Some possible generalizations are proposed in [10], Part 4.

Each of the quoted strength conditions includes two parameters which are material constants. The parameters can be determined, in principle, from two macro-experiments, for example, in tension of a smooth specimen and of the specimen with a crack. If the stress field σ_{ij} is smooth and if the characteristic lengths d_k are small enough, then all the non-local strength conditions quoted above go into traditional local conditions (1).

Conditions (2) and (3) can be further generalized by replacing the stress component $\sigma_{\theta\theta}$ by some function of the stress tensor components σ_{ij}, and conditions (4) can be generalized by taking into account the influence of the shear stress intensity factor K_2.

There are some other NLSC in which the parameters d_k are considered as dependent on the global body geometry. In the paper of Pipes *et al.* [11] the parameters depend on the macro-crack dimension if the analyzed point is its tip. In papers by Mileiko *et al.* [12], and by Polilov [13, 14] some models of changes of concentrator geometry before fracture are used.

All these strength conditions can be represented in the form of inequalities for some functionals, acting on a stress field. The functionals depend, in general, on the analyzed point and presumed fracture direction and on the boundary of the body.

An attempt is made below to obtain a general form of, and some restrictions on, such functionals.

2. A GENERAL FORM OF BODY NON-LOCAL STRENGTH CONDITIONS

Let us consider a body D having a stress field $\sigma(x)$ induced by some mass and/or boundary loading. We will consider only materials whose fracture is determined by stress field values alone and is independent of the stress field history. In particular, if a stress field $\sigma(s)$ is a fracturing one, then the stress field $\lambda'\sigma(x)$ is also fracturing for all constants $\lambda' \geqslant 1$.

Definition 1. Let a stress field $\sigma(x)$ be given in a body D. Then there are parameters $\lambda'(\sigma) \geqslant 0$ such that the stress field $\sigma'(x) = \lambda'\sigma(x)$ causes no fracture for every point of the body D. The supremum of $\lambda'(\sigma)$ for the given field $\sigma(x)$ will be called the strength functional $\underline{\lambda}(\sigma)$. If $\underline{\lambda}(\sigma) > 0$, then the stress field $\sigma(x)$ will be called admissible; if $\underline{\lambda}(\sigma) = 0$, then $\sigma(x)$ is inadmissible. We will call the set consisting of all admissible stress fields the admissible stress set $\mathbb{S}(D)$.

It is obvious, that if $\sigma(x) \in \mathbb{S}(D)$ then $\sigma'(x) = \lambda'\sigma(x) \in \mathbb{S}(D)$ for each $\lambda' \in [0, \infty)$.

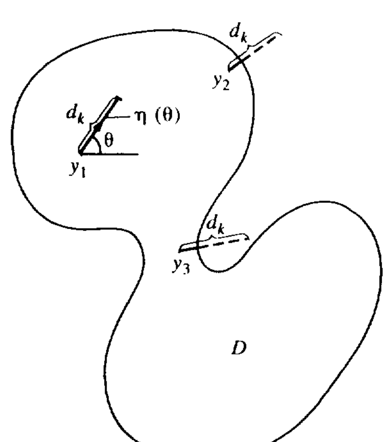

Fig. 1. Characteristic intervals for strength conditions (2)–(4).

The choice of some function set as an admissible one outlines the class of stresses, each of which the body can carry without fracture (may be after proportional diminishing of stresses but not to zero).

For example, it seems natural to take the space of stresses with bounded energy as an admissible one, that is $\mathbb{S}(D) = L_2(D)$ with the norm

$$\|\sigma\|_{L_2} = \left[\int_D \sigma_{ij}^2(x)\,\mathrm{d}D\right]^{1/2} < \infty. \tag{5}$$

Then, for a two-dimensional body, a function $\sigma_{ij}(x)$ having power singularities of the order of $O(|x|^{-\alpha})(\alpha < 1)$ for $x \to 0$ will cause fracture only if the norm of this function is large enough, but for $\alpha \geqslant 1$ such $\sigma_{ij}(x)$ is not admissible. Particularly a concentrated force at $x = 0$ generates the elastic stress field $\sigma_{ij}(x) = O(|x|^{-1})$ which consequently is not admissible, i.e. fracture must occur at any non-zero concentrated force for the choice $\mathbb{S}(D) = L_2(D)$.

So, let some admissible stress set $\mathbb{S}$ be chosen. The goal is then to extract the subset of stress fields from $\mathbb{S}$ which cause no fracture, and also to describe the subset boundary. Let $\sigma(x) \in \mathbb{S}(D)$. The stress field $\sigma^*(x) = \underline{\lambda}(\sigma)\sigma(x)$ will be called the critical one for the stress field $\sigma(x)$. The set of all such critical fields gives the critical hyper-surface in $\mathbb{S}(D)$ being the mentioned boundary between the subsets of fracturing and non-fracturing stresses.

This hyper-surface is schematically sketched in Fig. 2, where the stress field $\sigma(x)$ is shown as a two-dimensional vector. Such a two-dimensional representation would be thoroughly exact for the body D consisting only of two points x_1, x_2 with one stress value in each point. The other finite-dimensional analog of this non-local approach is the description of a fracture (or yield) surface in the six-dimensional stress-space in case of a three-axis stress state at a point in the traditional (local) approach.

If $\sigma^*(x)$ causes fracture then this boundary point is included into the fracturing subset of $\mathbb{S}(D)$. If $\sigma^*(x)$ causes no fracture then it gives the strength unstable state because for each $\lambda' > 1$ the stress field $\sigma'(x) = \lambda'\sigma^*(x)$ exceeds the fracturing stress field. Thus for each $\sigma(x) \in \mathbb{S}(D)$ the set of parameters λ', for which $\sigma'(x) = \lambda'\sigma(x)$ causes fracture or strength instability, is closed, i.e. includes its boundary point $\lambda' = \underline{\lambda}(\sigma)$. Hence for $\sigma(x) \in \mathbb{S}(D)$, $\underline{\lambda}(\sigma)$ may be also defined as the minimum of positive factors $\lambda'(\sigma)$ for which the stress field $\sigma'(x) = \lambda'\sigma(x)$ causes fracture or strength unstable state.

If $\underline{\lambda}(\sigma) > 1$, then the field $\sigma(x)$ is not fracturing; if $\underline{\lambda}(\sigma) = 1$, then $\sigma(x) = \sigma^*(x)$ is fracturing or gives the strength unstable state; if $\underline{\lambda}(\sigma) < 1$, then $\sigma(x)$ exceeds the fracturing stress. If for all $\lambda' > 0$ the stress field $\lambda'\sigma(x)$ does not cause fracture, then $\underline{\lambda}(\sigma) = \infty$.

Thus, the strength condition

$$\underline{\lambda}(\sigma) > 1 \tag{6}$$

is obtained where $\underline{\lambda}(\sigma)$ is a positively-defined and, in general, nonlinear unbounded functional on $\mathbb{S}$.

If we put $\underline{\lambda}(\sigma) = \sigma_c/f(\sigma)$, where f is a function and σ_c is a constant, then (6) goes into the usual local strength condition (1).

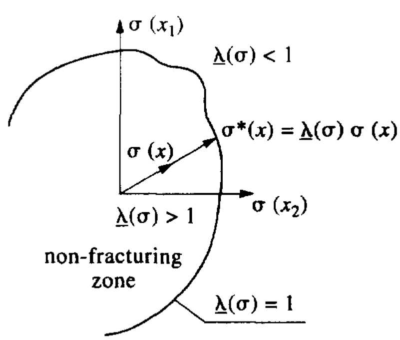

Fig. 2. Fracture hyper-surface in $\mathbb{S}$.

The strength condition (6) and the analogous expressions below mean that if the strict inequality is fulfilled then there is no fracture of the body, and that fracture or strength instability occurs if equality is achieved. That is the fracture (or instability) criterion is $\underline{\lambda}(\sigma) = 1$, and the solutions $\sigma(x) = \sigma^*(x)$ of this equation specify the mentioned fracture (critical) hypersurface in $\mathbb{S}$.

Let $\underline{\Lambda}(\sigma) := 1/\underline{\lambda}(\sigma)$, then we have the stress conditions (6) in more usual form

$$\underline{\Lambda}(\sigma) < 1. \tag{7}$$

Because $\lambda(\sigma)$ is the factor by which the $\sigma(x)$ must be multiplied to obtain the stress of fracture or instability, we have for any positive constant k:

$$\underline{\lambda}(k\sigma) = (1/k)\underline{\lambda}(\sigma), \tag{8}$$

i.e. $\underline{\lambda}$ is a positively-uniform functional of order -1.

Let the admissible set $\mathbb{S}$ be a subset of a normed space $\mathbb{L}$, and let $\|\sigma\|$ denote $\|\sigma\|_{\mathbb{L}}$. Then by virtue of (8) we obtain that for $\sigma(x)$ with norm $\|\sigma\| \neq 0$ the strength functional $\underline{\lambda}(\sigma) = (1/\|\sigma\|)\underline{\lambda}(\sigma/\|\sigma\|)$, and the critical stress $\sigma^*(x)$ of the same form is

$$\sigma^*(x) = \underline{\lambda}(\sigma)\sigma(x) = (\sigma(x)/\|\sigma\|)\underline{\lambda}(\sigma/\|\sigma\|).$$

In particular, taking norms of both sides of this equality, we obtain $\|\sigma^*\| = \underline{\lambda}(\sigma/\|\sigma\|)$. Hence, the strength condition (6) can be rewritten also in the form

$$\|\sigma\| < \underline{\lambda}(\sigma/\|\sigma\|). \tag{9}$$

On the other hand $\Lambda(\sigma) = 1/\underline{\lambda}(\sigma) = \|\sigma\|/\underline{\lambda}(\sigma/\|\sigma\|) = \|\sigma\|\underline{\mu}(\sigma/\|\sigma\|)$, so $\underline{\Lambda}$ is a positively-uniform functional of order 1, and the functional $\underline{\mu}(\sigma_0) := 1/\underline{\lambda}(\sigma_0)$ is defined on the intersection of the set $\mathbb{S}$ and the unit sphere $\mathbb{S}_0(\mathbb{L})$: $\{\|\sigma_0\|_{\mathbb{L}} = 1\}$.

Thus, we succeed in splitting the functional $\Lambda(\sigma)$ into a product of two functionals, one of which is the stress field norm $\|\sigma\|$, and the other is the functional $\underline{\mu}(\sigma/\|\sigma\|)$, independent of the stress field norm $\|\sigma\|$.

In addition to $\lambda(\sigma)$, let us call the functionals $\underline{\Lambda}$ and $\underline{\mu}$ strength functionals too.

Let us consider the functionals for which there is uniform separateness of $\underline{\lambda}(\sigma_0)$ from zero for all $\sigma_0 \in \mathbb{S}_0 \cap \mathbb{S}$, i.e. not only $\underline{\lambda}(\sigma_0) \neq 0$, but stronger condition holds:

Definition 2. The functional $\underline{\lambda}(\sigma)$ will be called non-vanishing in $\mathbb{L}$ if for all $\sigma_0 \in \mathbb{S}_0(\mathbb{L}) \cap \mathbb{S}$ there exists a constant C_0 such that

$$\underline{\lambda}(\sigma_0) > C_0 > 0. \tag{10}$$

In other words, the norm of any critical stress $\|\sigma^*\| = \underline{\lambda}(\sigma/\|\sigma\|)$ is not less than some positive number C_0 which is independent of $\sigma(x)$ for a non-vanishing strength functional $\underline{\lambda}(\sigma)$. If we consider the finite-dimensional analogy for a fracture surface, shown in Fig. 2, there is a finite gap between the fracture surface and the coordinate origin in the stress space for the case of a non-vanishing functional.

Non-vanishing of the functional $\underline{\lambda}(\sigma)$ is equivalent to boundedness of the functional $\underline{\mu}(\sigma_0)$ on $\mathbb{S}_0 \cap \mathbb{S}$, because $\|\underline{\mu}(\sigma_0)\| < 1/C_0$ for all $\sigma_0 \in \mathbb{S}_0 \cap \mathbb{S}$, and to boundedness of the functional $\underline{\Lambda}(\sigma)$ on $\mathbb{S}$ because $\|\underline{\Lambda}(\sigma)\| < \|\sigma\|/C_0$ for all $\sigma \in \mathbb{S}$.

3. A GENERAL FORM OF POINT NON-LOCAL STRENGTH CONDITIONS

3.1. *Point non-local strength conditions*

In Part 2 we discussed body fracture without pointing out the point of fracture. Let us consider now the condition that fracture occurs at some point y.

Repeating all reasoning of Part 2, we come to the same notions of admissible stress set $\mathbb{S}(D; y)$ and of strength functionals $\underline{\lambda}(\sigma; y)$, $\underline{\Lambda}(\sigma; y)$, $\underline{\mu}(\sigma; y)$ which depend in this case not only on the domain D and the stress field $\sigma(x)$, but on the analyzed point y too. The strength conditions for a point y and $\sigma(x) \in \mathbb{S}(D; y)$ will have the form:

$$\underline{\lambda}(\sigma; y) > 1, \tag{11}$$

$$\underline{\Lambda}(\sigma; y) < 1, \tag{12}$$

$$\|\sigma\| < \underline{\lambda}(\sigma/\|\sigma\|; y). \tag{13}$$

We will believe that the fracture occurs in a body, if fracture occurs at least at one point y^* of the body; and strength instability of the body occurs at a stress field $\sigma^*(x)$ if $\sigma^*(x)$ causes no fracture, and fracture occurs at $\sigma'(x) = \lambda'\sigma^*(x)$ for each $\lambda' > 1$. Then according to the Definition 1, the body admissible set is $\mathbb{S}(D) = \bigcap_{y \in D} \mathbb{S}(D; y)$ and the body strength functionals are $\underline{\lambda}(\sigma) = \inf_y \underline{\lambda}(\sigma; y)$, $\underline{\Lambda}(\sigma) = \sup_y \underline{\Lambda}(\sigma; y)$, $y \in D$. Consequently, the criterion of fracture (or strength instability) for the body for $\sigma(x) \in \mathbb{S}(D)$ may be written in one of the equivalent forms:

$$\inf_y \underline{\lambda}(\sigma; y) = 1,$$

$$\sup_y \underline{\Lambda}(\sigma; y) = 1,$$

$$\|\sigma\| = \inf_y \underline{\lambda}(\sigma/\|\sigma\|; y).$$

Any critical stress field $\sigma^*(x)$ for the body is a solution of these equations.

The case, when the supremum and the infimum in these equations are achieved at a critical (fracture) point (or at some points) $y^* \in D$, will be called the isolated fracture. Then the supremum can be replaced by maximum, and the infimum by minimum. For the isolated fracture we obtain the three equivalent equations for determination of the points y^* of fracture or strength instability:

$$\underline{\lambda}(\sigma^*; y^*) = 1,$$

$$\underline{\Lambda}(\sigma^*; y^*) = 1,$$

$$\|\sigma^*\| = \underline{\lambda}(\sigma^*/\|\sigma^*\|; y^*).$$

The other case, when the infimum and the supremum can be achieved at no point of D, will be called the infinitesimal fracture. [It occurs for example for the strength functionals $\underline{\lambda}(\sigma; y)$ discontinuous with respect to y. It takes place particularly for bonded bodies when the interface has higher strength than the points of the body parts.] In this case $\underline{\lambda}(\sigma^*; y) > 1$ for each $y \in D$, i.e. each point y is strength stable for $\sigma(x) = \sigma^*(x)$. On the other hand if $\lambda' = 1 + d\lambda$ for any $d\lambda > 0$, then the inequalities (11)–(13) will be violated for $\sigma'(x) = \lambda'\sigma^*(x) = (1 + d\lambda)\sigma^*(x)$ in the critical zone dY^* consisting of infinite number of such points y^* that

$$1 < \underline{\lambda}(\sigma^*; y^*) \leqslant 1 + d\lambda,$$

$$(1 + d\lambda)^{-1} \leqslant \underline{\Lambda}(\sigma^*; y^*) < 1,$$

$$\|\sigma^*\| < \underline{\lambda}(\sigma^*/\|\sigma^*\|; y^*) \leqslant \|\sigma^*\|(1 + d\lambda).$$

The critical zone dY^* monotonically decreases when $d\lambda$ decreases. Moreover $dY^* \to \phi$ when $d\lambda \to 0$ because of closeness of the set of parameters λ' for which $\sigma'(x) = \lambda'\sigma(x)$ causes fracture or strength instability at a point y (see Part 2). Consequently if $d\lambda$ is infinitesimal then dY^* is infinitesimal too.

The strength functional dependence on an analyzed body point y reflects the influence of several factors:

(1) Nonuniformity of stresses in a body.
(2) Nonhomogeneity of strength in a body.
(3) The influence of point vicinity to a body boundary.

We will try below to separate the influence of these factors on the strength functionals. The effect of presumed fracture direction at a point will be considered too.

3.2. *Strength homogeneity*

Let us introduce the function

$$\sigma^{(y)}(w) := \sigma(y + w), \quad y + w \in D \tag{14}$$

for any given stress field $\sigma(x) \in D$ and any analyzed point y. Then it is possible to go from the admissible set $\mathbb{S}(D; y)$, consisting of functions $\sigma(x)$, into the relative admissible set $\tilde{\mathbb{S}}(D; y)$ consisting of relative functions $\sigma^{(y)}(w)$, and to obtain for all strength functionals their relative analogs, acting on $\sigma^{(y)}(w) \in \tilde{\mathbb{S}}(D; y)$ instead of $\sigma(x) \in \mathbb{S}(D; y)$. For example the functional $\underline{\Lambda}(\sigma; y)$ goes into the relative functional

$$\tilde{\underline{\Lambda}}(\sigma^{(y)}; y) := \underline{\Lambda}(\sigma; y),$$

where $\sigma^{(y)}$ and σ are connected by relation (14).

Definition 3. We will call an unbounded body D^{∞} a strength homogeneous one, if explicit dependence of the relative admissible stress set and of the relative strength functionals on the second argument y is absent, i.e. $\tilde{\mathbb{S}}(D^{\infty}; y) = \tilde{\mathbb{S}}(D^{\infty})$,

$$\tilde{\underline{\Lambda}}^{\infty}(\sigma^{(y)}; y) = \tilde{\underline{\Lambda}}^{\infty}(\sigma^{(y)}), \tilde{\underline{\lambda}}^{\infty}(\sigma^{(y)}; y) = \tilde{\underline{\lambda}}^{\infty}(\sigma^{(y)}), \tilde{\underline{\mu}}^{\infty}(\sigma^{(y)}; y) = \tilde{\underline{\mu}}^{\infty}(\sigma^{(y)}).$$

The strength functionals corresponding to unbounded space are marked here by the superscript ∞.

Thus, if there is some stress field and we move the whole stress field pattern by a constant shift Δ, then the strength functional values in any point $y + \Delta$ will for a homogeneous body be the same as for the initial stress field in a point y. In other words, strength at each point y of a strength homogeneous body is characterized only by the stress distribution relative to the point y, i.e. $\underline{\Lambda}(\sigma; y) = \tilde{\underline{\Lambda}}^{\infty}(\sigma^{(y)})$ and so on.

The definition above is not applicable directly to a bounded body D because it is, in general, impossible to move a given stress field pattern while remaining within the body, whereas the strength functionals act only on functions defined in D.

Definition 4. We will call the material of a bounded body a strength homogeneous material if the body may be completed to an unbounded strength homogeneous body.

It is worth noting that just the strength functionals $\tilde{\underline{\Lambda}}^{\infty}$, $\tilde{\underline{\lambda}}^{\infty}$, $\tilde{\underline{\mu}}^{\infty}$ of an unbounded strength homogeneous body are characteristics of material in non-local strength theory, whereas strength functionals of a bounded body are characteristics of the analyzed body as a whole, because they are determined by body material as well as by body form, i.e. body boundary, and by boundary condition type too.

We note also that because of boundary influence, strength functional values for a bounded body consisting of a homogeneous material can differ from functional values for a corresponding unbounded body even for stresses being non-zero only inside the bounded body. The NLSC with fictitious crack (4) can be an example of such a situation.

3.3. *Account of presumed fracture direction*

If fracture in a body is realized by nucleation and propagation of a crack, it is possible to consider the condition that micro-fracture goes from a point y in a direction η. In the three-dimensional case it is possible to mark by a normal ζ also a plane of micro-fracture. Here η and ζ are mutually orthogonal unit vectors, and in the two-dimensional case ζ lies in the problem plane.

Thus, for each point $y \in D$ and directions η, ζ we come to the notions of admissible stress set $\mathbb{S}(D; y, \eta, \zeta)$ and of strength functionals $\underline{\lambda}(\sigma; y, \eta, \zeta)$, $\underline{\Lambda}(\sigma; y, \eta, \zeta)$, $\underline{\mu}(\sigma; y, \eta, \zeta)$ which depend in this case on directions η, ζ too. The strength conditions analogous to (11)–(13) for a body point y and directions η, ζ have the form:

$$\underline{\lambda}(\sigma;y,\eta,\zeta) > 1,$$

$$\Lambda(\sigma;y,\eta,\zeta) < 1,$$

$$\|\sigma\| < \underline{\lambda}(\sigma/\|\sigma\|;y,\eta,\zeta). \tag{15}$$

The point y^* and directions η^*, ζ^* where fracture occurs are characterized by the fact that inequality will go into equality at this point and in these directions earlier than other ones. Then the body admissible stress set is $\mathbb{S}(D) = \bigcap_{y,\eta,\zeta} \mathbb{S}(D;y,\eta,\zeta)$ and the body strength functionals are $\lambda(\sigma) = \inf_{y,\eta,\zeta} \underline{\lambda}(\sigma;y,\eta,\zeta)$, $\underline{\Lambda}(\sigma) = \sup_{y,\eta,\zeta} \Lambda(\sigma;y,\eta,\zeta)$, $y,\eta,\zeta \in D$. Consequently, the fracture (or instability) criterion for the body for $\sigma(x) \in \mathbb{S}(D)$ may be written in one of the equivalent forms:

$$\inf_{y,\eta,\zeta} \underline{\lambda}(\sigma;y,\eta,\zeta) = 1,$$

$$\sup_{y,\eta,\zeta} \underline{\Lambda}(\sigma;y,\eta,\zeta) = 1,$$

$$\|\sigma\| = \inf_{y,\eta,\zeta} \underline{\lambda}(\sigma/\|\sigma\|;y,\eta,\zeta). \tag{16}$$

Any critical stress field $\sigma^*(x)$ for the body is a solution of these equations.

Analogous to the Section 3.1, we have the isolated fracture if the supremum and the infimum in (16) are achieved at a critical (fracture) point and directions $y^*, \eta^*, \zeta^* \in D$. Then the supremum can be replaced by maximum, and the infimum by minimum. For the isolated fracture we obtain the three equivalent equations for determination of the point y^* and directions η^*, ζ^* of fracture or strength instability:

$$\underline{\lambda}(\sigma^*;y^*,\eta^*,\zeta^*) = 1,$$

$$\underline{\Lambda}(\sigma^*;y^*,\eta^*,\zeta^*) = 1,$$

$$\|\sigma^*\| = \lambda(\sigma^*/\|\sigma^*\|;y^*,\eta^*,\zeta^*).$$

If the infimum and the supremum can be achieved at no point and no directions in D, we obtain the infinitesimal fracture. If $\lambda' = 1 + \mathrm{d}\lambda$, $\mathrm{d}\lambda > 0$, then the inequalities (15) will be violated for $\sigma'(x) = \lambda'\sigma^*(x) = (1+\mathrm{d}\lambda)\sigma^*(x)$ in the critical parametric zone $\mathrm{d}Y^*$ consisting of infinite number of such point y^* and directions η^*, ζ^* that

$$1 < \underline{\lambda}(\sigma^*;y^*,\eta^*,\zeta^*) \leqslant 1 + \mathrm{d}\lambda,$$

$$(1+\mathrm{d}\lambda)^{-1} \leqslant \underline{\Lambda}(\sigma^*;y^*,\eta^*,\zeta^*) < 1,$$

$$\|\sigma^*\| < \underline{\lambda}(\sigma^*/\|\sigma^*\|;y^*,\eta^*,\zeta^*) \leqslant \|\sigma^*\|(1+\mathrm{d}\lambda).$$

Because each of the strength functionals for any fixed tensor function $\sigma(x) \in \mathbb{S}(D)$ generates a function of y, η, ζ, it is possible to introduce the strength operators $\hat{\underline{\lambda}}(\sigma)$, $\hat{\underline{\mu}}(\sigma_0)$, $\hat{\underline{\Lambda}}(\sigma)$, which put functions $l(y,\eta,\zeta)$, $m(y,\eta,\zeta)$, $L(y,\eta,\zeta)$ in correspondence with a function $\sigma(x)$. The operators will have the form:

$$\hat{\underline{\lambda}}(s)\colon \mathbb{S}(D) \ni \sigma(x) \to l(y,\eta,\zeta) = [\hat{\underline{\lambda}}(\sigma)](y,\eta,\zeta) := \underline{\lambda}(\sigma;y,\eta,\zeta),$$

$$\hat{\underline{\mu}}(\sigma_0)\colon \mathbb{S}_0 \cap \mathbb{S}(D) \ni \sigma_0(x) \to m(y,\eta,\zeta) = [\hat{\underline{\mu}}(\sigma_0)](y,\eta,\zeta) := \underline{\mu}(\sigma_0;y,\eta,\zeta),$$

$$\hat{\underline{\Lambda}}(\sigma)\colon \mathbb{S}(D) \ni \sigma(x) \to L(y,\eta,\zeta) = [\hat{\underline{\Lambda}}(\sigma)](y,\eta,\zeta) := \underline{\Lambda}(\sigma;y,\eta,\zeta).$$

If the strength functionals $\underline{\mu}(\sigma_0;y,\eta,\zeta)$ $(\sigma_0 \in \mathbb{S}_0 \cap \mathbb{S}(D))$ and $\underline{\Lambda}(\sigma;y,\eta,\zeta)$ $(\sigma \in \mathbb{S}(D))$ are bounded for all $(y,\eta,\zeta) \in D$, then we obtain that the operator $\hat{\underline{\mu}}(\sigma_0)$ acts boundedly from $\mathbb{S}_0 \cap \mathbb{S}$ to $\tilde{C}$, and that the operator $\hat{\underline{\Lambda}}(\sigma)$ acts boundedly from $\mathbb{S}$ to $\tilde{C}$, where $\tilde{C}$ is the set of bounded functions of the parameters y, η, ζ. The operator $\hat{\underline{\lambda}}(\sigma)$ is then defined, strictly speaking, not on all $\mathbb{S}(D)$, because the functions $l(y,\eta,\zeta) = \underline{\lambda}(\sigma;y,\eta,\zeta)$ can become infinite for some $\sigma(x) \in \mathbb{S}(D)$ and $(y,\eta,\zeta) \in D$, which means that the stress field $\lambda'\sigma(x)$ does not induce fracture for any $\lambda' < \infty$.

3.4. *Strength isotropy*

For the multi-dimensional case, in addition to strength homogeneity with respect to a point y it is possible to consider also strength homogeneity with respect to directions η and ζ, i.e. strength isotropy. Let us relate a local right handed coordinate system with an analyzed point y and direct the first two coordinate vectors along the presumed fracture vectors η and ζ. Aggregate of components of tensor $\sigma^{(y)}(w)$ in this coordinate system will be denoted as $\sigma^{(y,\eta,\zeta)}(w)$, where $w = x - y$. Then it is possible to go from the admissible set $\mathbb{S}(D; y, \eta, \zeta)$ consisting of functions $\sigma(x)$ into the relative admissible set $\tilde{\tilde{\mathbb{S}}}(D; y, \eta, \zeta)$ consisting of relative functions $\sigma^{(y,\eta,\zeta)}(w)$ and to obtain for all strength functionals their relative analogs, acting on components of the tensor $\sigma^{(y,\eta,\zeta)}(w) \in \tilde{\tilde{\mathbb{S}}}(D; y, \eta, \zeta)$ in the local coordinate system, instead of components of the tensor $\sigma(x) \in \mathbb{S}(D; y, \eta, \zeta)$ in a fixed global coordinate system. For example, the relative functional $\tilde{\tilde{\Lambda}}(\sigma^{(y,\eta,\zeta)}; y, \eta, \zeta) := \Lambda(\sigma; y, \eta, \zeta)$.

Definition 5. We will call an unbounded body as strength isotropic in a point y if explicit dependence of the relative admissible stress set and of the relative strength functionals on their last two arguments η and ζ is absent:

$$\tilde{\tilde{\mathbb{S}}}(D^\infty; y, \eta, \zeta) = \tilde{\tilde{\mathbb{S}}}(D^\infty; y), \quad \tilde{\tilde{\Lambda}}^\infty(\sigma^{(y,\eta,\zeta)}; y, \eta, \zeta) = \tilde{\tilde{\Lambda}}^\infty(\sigma^{(y,\eta,\zeta)}; y),$$
$$\tilde{\tilde{\lambda}}^\infty(\sigma^{(y,\eta,\zeta)}; y, \eta, \zeta) = \tilde{\tilde{\lambda}}^\infty(\sigma^{(y,\eta,\zeta)}; y), \tilde{\tilde{\mu}}^\infty(\sigma^{(y,\eta,\zeta)}; y, \eta, \zeta) = \tilde{\tilde{\mu}}^\infty(\sigma^{(y,\eta,\zeta)}; y).$$

We will call a material of a bounded body strength isotropic in a point y, if the body can be completed to an unbounded strength isotropic body.

Thus, if there is some stress field in an unbounded strength isotropic body and we rotate the stress field about a point y, then the values of all the relative strength functionals in the point y for the rotated stress field (for all vectors η', ζ' obtained by the same rotation from corresponding vectors η, ζ) will have the same values as for initial stress field (for vectors η, ζ).

In other words, for an unbounded strength isotropic body the strength in a point y with respect to fracture in any direction η in any plane ζ is determined only by the stress distribution relative to this direction and plane.

3.5. *Finite non-locality*

Let mes$[D]$ be length for a one-dimensional body, area for a two-dimensional and volume for a three-dimensional one. Denote the characteristic function of a set D' by

$$\chi[D'; x] := \begin{cases} 1, & x \in D' \\ 0, & x \notin D' \end{cases}.$$

Definition 6. For a body D, we will call the strength functional $\Lambda(\sigma; y, \eta, \zeta)$ finitely non-local for a point y and directions η and ζ if a finite domain $D'(y, \eta, \zeta)$ exists such that mes$[D'(y, \eta, \zeta)] <$ mes$[D]$ and $\Lambda(\sigma; y, \eta, \zeta) = \Lambda(\sigma\chi[D'(y, \eta, \zeta)]; y, \eta, \zeta)$ for all $\sigma \in \mathbb{S}(D; y, \eta, \zeta)$. We will call the intersection of such domains for a given point y and given directions η, ζ as the domain of non-locality $\Omega(y, \eta, \zeta)$.

In the local coordinate system relative to the point y and directions η, ζ, the domain of non-locality $\Omega(y, \eta, \zeta)$ goes into the relative domain of non-locality $\tilde{\tilde{\Omega}}(y, \eta, \zeta)$. For a strength homogeneous unbounded body, naturally, the relative domain of non-locality does not depend on the point y: $\tilde{\tilde{\Omega}}(y, \eta, \zeta) = \tilde{\tilde{\Omega}}(\eta, \zeta)$; for a strength isotropic body it does not depend on the directions η and ζ: $\tilde{\tilde{\Omega}}(y, \eta, \zeta) = \tilde{\tilde{\Omega}}(y)$.

3.6. *Body boundary influence*

Let a body D have a boundary ∂D. In the general case, there is a dependence of strength functionals on the body boundary ∂D and on the vicinity of an analyzed point y to the boundary, in addition to a dependence on material properties for the point y and directions η, ζ. The dependence can be expressed in the form:

$$\Lambda(\sigma; y, \eta, \zeta) = \tilde{\tilde{\Lambda}}(\sigma^{(y,\eta,\zeta)}; y, \eta, \zeta) = \tilde{\tilde{\Lambda}}^\partial(\sigma^{(y,\eta,\zeta)}; y, \eta, \zeta; \partial^{(y,\eta,\zeta)}D),$$

and in analogous forms for the rest of the strength functionals. Here, $\partial^{(y,\eta,\zeta)}D$ designates equations of the boundary ∂D in the local coordinate system relative to the point y and directions η, ζ. Unlike

$\tilde{\tilde{\Lambda}}$, the dependence of the strength functional $\tilde{\tilde{\Lambda}}^c$ on the position of the point y and on directions η, ζ relative to other body points (arguments y, η, ζ), which characterize strength heterogeneity and anisotropy of material, is separated from the dependence on the boundary (the last argument). If the body boundary is varied under the condition that the stress field remains unchanged, then the arguments y, η, ζ in $\tilde{\tilde{\Lambda}}^c$ do not vary. If the body boundary is shifted together with an analyzed geometrical point or turned around it remaining material points fixed, then the last argument in $\tilde{\tilde{\Lambda}}^c$ does not vary. If the body boundary is moved away from an analyzed point (i.e. if material is added to the body) then the strength functionals must go into the corresponding ones for unbounded body, i.e. $\tilde{\tilde{\Lambda}}^c(\sigma^{(y,\eta,\zeta)};y,\eta,\zeta;\partial^{(y,\eta,\zeta)}D)\to\tilde{\tilde{\Lambda}}^\infty(\sigma^{(y,\eta,\zeta)};y,\eta,\zeta)$, $(\partial^{(y)}D\to\infty)$.

For a homogeneous isotropic material

$$\Lambda(\sigma;y,\eta,\zeta)=\tilde{\tilde{\Lambda}}^c(\sigma^{(y,\eta,\zeta)};y,\eta,\zeta;\partial^{(y,\eta,\zeta)}D)\to\tilde{\tilde{\Lambda}}^\infty(\sigma^{(y,\eta,\zeta)}),\quad(\partial^{(y)}D\to\infty).$$

Not only the body boundary but also the types of boundary conditions are essential for several strength functionals. For example, in the two-dimensional case the boundary conditions enter into the strength condition (4), based on the model of a fictitious crack. The stress intensity factor K_1 in a crack tip depends not only on the compensated tractions at the crack shores and on the distance from the crack to the boundary, but also on the boundary geometry and the types of boundary conditions, which are implicitly included in Green type functions $N_{1i\rho}(\rho,\theta)$ and $N_{1i\theta}(\rho,\theta)$. These functions may be represented in terms of a classical Green function as a solution of the corresponding integral equation on the crack contour, where the classical Green function is the kernel of the equation.

In the general case, for an elastic body the corresponding Green function G includes the information about the boundary conditions. Consequently, the dependence of the strength functional on the boundary can be replaced by dependence on the relative Green function $G^{(y,\eta,\zeta)}(w)$ for an elastic body, i.e. the strength functionals may be presented in the form:

$$\Lambda(\sigma;y,\eta,\zeta)=\tilde{\tilde{\Lambda}}^G(\sigma^{(y,\eta,\zeta)};y,\eta,\zeta;G^{(y,\eta,\zeta)}).$$

In traditional (local) fracture criteria, the possibility of fracture at an analyzed point y depends on the stress at this point only, and consequently does not depend on the body boundary and on the proximity of the point y to the boundary. That is, the strength at the point y in bounded body is the same as for an unbounded body with the same stress at y.

It is possible in non-local strength conditions for bounded bodies also to try to use the strength functionals found for corresponding unbounded bodies. It is necessary in this case to extend in some way the bounded body stress field beyond the body boundary.

Definition 7. We will call a bounded body D, consisting of strength homogeneous material, non-sensitive to the boundary with respect to strength if, for each point $y\in D$, there is an extension of each admissible stress field $\sigma(x)\in\mathbb{S}(D;y)$ to some field $\sigma^\infty(x)\in\mathbb{S}(D^\infty;y)$ defined in the unbounded space D^∞, such that $\sigma(x)=\chi(D;x)\sigma^\infty(x)$ and the values of the strength functional Λ for the initial body D coincide with the values of the strength functional Λ^∞ for the corresponding unbounded body D^∞, consisting of the same material, for all corresponding stress fields:

$$\Lambda(\sigma;y,\eta,\zeta)=\Lambda^\infty(\sigma^\infty;y,\eta,\zeta),\quad(y,\eta,\zeta)\in D,$$

and analogously for the rest of the strength functionals λ and μ.

4. FUNCTIONAL FORM AND SEVERAL MODIFICATIONS OF SOME NON-LOCAL STRENGTH CONDITIONS

Let us consider the non-local strength conditions, quoted in Part 1 for the plane case, represent them in the form (15), and classify them in accordance with the classification given above. In the plane case the direction of presumed fracture is a function of the angle θ: $\eta=\eta(\theta)$, the fracture plane normal ζ lies in the plane of the problem normal to $\eta(\theta)$, i.e. $\zeta=\zeta(\theta)$, therefore dependence on η and ζ is reduced to dependence on θ.

4.1. *Neuber-Novozhilov condition*

Let us divide inequality (2) by σ_c, then for an internal point y which is distant more than d_1 from the boundary we come to the strength condition in the form (15):

$$\Lambda_1(\sigma; y, \eta(\theta)) = \tilde{\tilde{\Lambda}}_1^{\infty}(\sigma^{(y,\theta)}) = \max[\Lambda_{1+}(\sigma^{(y,\theta)}), 0] < 1,$$

$$\Lambda_{1+}(\sigma^{(y,\theta)}) := \frac{1}{d_1\sigma_c}\int_0^{d_1} \sigma_{\theta\theta}^{(y)}(y + \rho\eta(\theta))\,\mathrm{d}\rho. \tag{17}$$

According to the strength condition (2), fracture can not occur in the case of a negative value of the functional Λ_+, for this reason the strength functional Λ_1 is set to zero for such cases. The strength functional Λ_1 corresponds to a strength homogeneous isotropic material and is finitely non-local, its domain of non-locality is the segment $\Omega_1(y, \eta) = [y, y + \eta d_1]$. The functional Λ_{1+} is linear with respect to σ.

The admissible stress set $\mathbb{S}_1(D; y, \eta)$ for a point y and a direction η consists of the stress fields $\sigma(x)$ for which the integral in (17) is defined. Thus, the set $L_1[y, y + d_1\eta]$ of stress fields integrable on the segment, or some subset of $L_1[y, y + d_1\eta]$, can be taken as the admissible stress set $\mathbb{S}_1(D; y, \eta)$. Note that the more convenient space $L_2(D)$ with norm (5) (the space of stresses with finite energy) is not admissible for the functional Λ_1 because the functional is not defined for all $\sigma(x)$ from $L_2(D)$. For example, the functional Λ_1 is not defined on stresses σ_{ij} of the form $\sigma_{ij}(x_1, x_2) = C_{ij}|x_1|^{-1/4}$, belonging to $L_2(D)$. [Here (x_1, x_2) are cartesian coordinates of point x.] If the subset of all stress fields from $L_2(D)$, where the functional Λ_1 is defined, is chosen as an admissible set then the functional is not bounded on the set.

To make the space $L_2(D)$ admissible it is necessary to modify the strength functional $\Lambda_1(\sigma)$ to a functional $\Lambda_{1m}(\sigma)$ which is close to $\Lambda_1(\sigma)$ on sufficiently smooth functions and bounded on $L_2(D)$. For example, we can put

$$\Lambda_{1m}(\sigma; y, \eta(\theta)) = \tilde{\tilde{\Lambda}}_{1m}^{\infty}(\sigma^{(y,\theta)}) = \max[\Lambda_{1m+}(\sigma^{(y,\theta)}), 0],$$

$$\Lambda_{1m+}(\sigma^{(y,\theta)}) := \frac{1}{|\Omega_{1m}|\sigma_c}\iint_{\Omega_{1m}} \sigma_{\theta\theta}^{(y)}\,\mathrm{d}\Omega,$$

where the domain of non-locality $\Omega_{1m}(y, \eta(\theta))$ has a small width and lies along the segment $[y, y + \eta(\theta)d_1]$, and where $|\Omega_{1m}|$ is the area of Ω_{1m}. If the domain Ω_{1m} is narrow enough and if the stresses are continuous with respect to θ, then the functional Λ_{1m} is reduced to the functional Λ_1. The admissible stress sets are $\mathbb{S}_{1m}(D; y, \eta) = L_2(\Omega_{1m}(y, \eta))$, $\mathbb{S}_{1m}(D) = L_2(D)$.

4.2 *Minimum stress condition*

Let us divide the inequality (3) by σ_c, then for an internal point y which is distant more than d_2 from the boundary we come to the strength condition in the form (15):

$$\Lambda_2(\sigma; y, \eta(\theta)) = \tilde{\tilde{\Lambda}}_2^{\infty}(\sigma^{(y,\theta)}) = \max[\Lambda_{2+}(\sigma^{(y,\theta)}), 0] < 1,$$

$$\Lambda_{2+}(\sigma^{(y,\theta)}) := \frac{1}{\sigma_c}\min_{0 \leqslant \rho \leqslant d_2}[\sigma_{\theta\theta}^{(y)}(y + \rho\eta(\theta))]. \tag{18}$$

The functional Λ_{2+} is not linear. The functional Λ_2 corresponds to a strength homogeneous isotropic material and is finitely non-local, having as its domain of non-locality the segment $\Omega_2(y, \eta) = [y, y + \eta d_2]$. The functional Λ_2 is defined on the set of functions which do not equal $+\infty$ at least at one point at each one-dimensional segment. But the example of Section 4.1 shows that the functions from $L_2(D)$ exist, which do not satisfy this condition.

The functional Λ_2 can be modified in different ways. Some modification of the functional for functions from $L_2(D)$ can be presented in the form:

$$\Lambda_{2m}(\sigma; y, \eta(\theta)) = \tilde{\tilde{\Lambda}}_{2m}^{\infty}(\sigma^{(y,\theta)}) = \max[\Lambda_{2m+}(\sigma^{(y,\theta)}), 0],$$

$$\underline{\Lambda}_{2m+}(\sigma^{(y,\theta)}) := \min_{0 \leqslant \rho \leqslant d_2} \left[\frac{1}{|\Omega'_{2m}|\sigma_c} \iint_{\Omega'_{2m}} \sigma_{\theta\theta}^{(y)} \, d\Omega \right],$$

where the region $\Omega'_{2m}(y, \theta, \rho)$ is a neighborhood of the point $y + \rho\eta(\theta)$ and has a small diameter. The modified domain of non-locality is $\Omega_{2m}(y, \theta) = \cup_{0 \leqslant \rho \leqslant d_2} \Omega'_{2m}(y, \theta, \rho)$. If the region Ω'_{2m} is small enough and if the stresses are continuous in $\Omega_{2m}(y, \theta)$, then the functional $\underline{\Lambda}_{2m}$ is reduced to the functional Λ_2. The admissible stress sets are $\mathbb{S}_{2m}(D; y, \eta) = L_2(\Omega_{2m}(y, \eta))$, $\mathbb{S}_{2m}(D) = L_2(D)$.

4.3 *Condition with fictitious crack*

Let us divide the inequality (4) by K_c, then for an internal point y, which is distant more than d_3 from boundary we come to the strength condition in the form (15);

$$\underline{\Lambda}_3(\sigma; y, \eta(\theta)) = \tilde{\tilde{\underline{\lambda}}}^G 3(\sigma^{(y,\eta)}; G^{(y,\eta)}) = \max[\underline{\Lambda}_{3+}(\sigma^{(y,\theta)}; G^{(y,\eta)}), 0] < 1,$$

$$\underline{\Lambda}_{3+}(\sigma^{(y,\eta)}; G^{(y,\eta)}) := \frac{1}{K_{1c}} \min_i \int_0^{d_3} [N_{1i\rho}(\rho; G^{(y,\eta)}) \sigma_{\theta\rho}^{(y)}(y + \rho\eta(\theta))$$

$$+ N_{1i\theta}(\rho; G^{(y,\eta)}) \sigma_{\theta\theta}^{(y)}(y + \rho\eta(\theta))] \, d\rho. \tag{19}$$

It is supposed here that the Green-type functions $N_{1i\rho}$, $N_{1i\theta}$ can be expressed in terms of classical Green functions $G_{kl}(x, y)$ for a body without crack as a result of solving a corresponding integral equation on the crack contour. The kernel of the integral equation includes $G_{kl}(x, y)$.

The functional $\underline{\Lambda}_{3+}$ is linear and finitely non-local, and its domain of non-locality is the segment $\Omega_3(y, \eta) = [y, y + \eta d_3]$. For an elastically homogeneous and isotropic material the functional corresponds to a strength homogeneous and isotropic material because $N_{1ij}(\rho; G^{(y,\eta)}) \to N_{1ij}^{\infty}(\rho; G^{\infty})$ when $\partial D \to \infty$. For a bounded body, however, unlike previous strength conditions (17), (18), this functional depends on the position of the analyzed point y relative to the boundary ∂D and also on the orientation $\eta(\theta)$ relative to the body, via the dependence on the Green function $G^{(y,\eta)}$. That is, the strength functional $\underline{\Lambda}_3$ is sufficiently sensitive to boundary.

It is known that the Green-type functions $N_{1i\rho}(\rho)$, $N_{1i\theta}(\rho)$ for a stress intensity factor have a singularity of the order $1/\sqrt{\rho}$ when $\rho \to 0$. Thus, the functional $\underline{\Lambda}_3$ is defined on the admissible stress set consisting of functions $\sigma(x)$ integrable with the weight $1/\sqrt{\rho}$ on the segment $[y, y + \eta d_3]$. As above, the functional $\underline{\Lambda}_{3+}$ is not defined on all functions $\sigma \in L_2(D)$, therefore there is a reason to modify the function $\underline{\Lambda}_3$ in order to make the space $L_2(D)$ admissible.

The functional representations of Part 4 are quite applicable for body internal points, i.e. points distant more than d_k from the body boundary ∂D (y_1-type points in Fig. 1). For strength analysis of near-boundary points y_2- or y_3-type at Fig. 1, the presented strength conditions must be generalized. The simplest way to do this is to use the discrete fracture concept presented in [10].

CONCLUSIONS

The main results presented in this work are:

(1) The notion of an admissible stress set is introduced.

(2) General nonlinear strength functionals are proposed, which have a specific mechanical meaning and are positively-defined and positively-uniform.

(3) The strength functional $\underline{\Lambda}(\sigma)$ is represented in the form of the product of a stress field function norm and of a nonlinear functional, dependent only on the form of the stress field and independent of its norm.

(4) The notions of strength homogeneity, isotropy and finite non-locality are introduced.

(5) Possible revisions of some known non-local strength conditions are discussed.

The extension of non-local strength concept to discrete fracture of fracture quanta is given in [10]. The non-local strength concept presented here can be applied not only to classification of available strength conditions but also to the description of strength properties of specific materials. Strength functional approximation methods and their applications for the description of strength of specific materials by use of experimental data will be the subject of another paper.

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