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On Some Weighted Hardy Type Classes of One-Parametric Holomorphic Functions. II. Partial Volterra Operators in Parameter

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Some classes of partial Volterra operators acting (with respect to a real variable) on a function of one complex and one real variable are explored. The case when the operator kernels depend additionally on a complex parameter is also considered. It is proved that Volterra operators from these classes and the resolvent operators act in weighted Hardy type classes of one parametric holomorphic functions defined on a wedge or on a strip of the complex plane. Moreover, the Volterra operators commutate with the Mellin operators in these classes. The half line Paley–Wiener theorem for the resolvent of the Volterra operator is extended to operators with parameter. Volterra operators tending to operators of the convolution type as well as Volterra operators with fading memory are studied too.

INTRODUCTION

Several weighted Hardy type classes of one-parametric holomorphic functions defined on a wedge or on a strip of the complex plane are introduced in [5]. Properties of functions from these classes and of their Mellin images are also investigated there. Here we consider properties of the Volterra operators (in a real variable t) acting on such functions. These considerations are necessary for solving boundary value problems for a hereditarily-elastic (visco-elastic) body by use of the complex

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Kolosov−Muskhelishvili potentials (see, e.g., [2−4]). The potentials are functions holomorphic in a complex variable (space coordinate) and depending also on a real variable (time). The hereditary properties of materials are described by Volterra operators of the second kind with respect to time.

Section 1 of the paper includes a definition of a class of Volterra operators without parameter and a proof that these operators act in the weighted Hardy type classes and commute with the Mellin operator acting with respect to another variable. In Section 2 a class of Volterra operators with a complex parameter on a strip is considered and it is proved that their resolvents on a finite segment belong to the same operator class too; such Volterra operators act in the corresponding Hardy type classes of one-parametric functions in the strip. In Section 3 the half line Paley−Wiener theorem for a resolvent of a Volterra operator of the convolution type is extended to the considered class of operators with parameter when the operators are, or tend to, operators of the convolution type as $t \to \infty$. The Volterra operators with fading memory are also considered. It is proved that they act in the Hardy type classes of functions that tend to harmonically oscillating functions in t as $t \to \infty$. It is shown that a resolvent of an operator with fading memory (without or with a complex parameter) is also an operator with fading memory. Asymptotic relations for $t \to \infty$ are given too.

The results presented here were applied in [3, 4] to the stress singularity analysis of visco-elastic bonded wedges. It was particularly shown there that the presence of Volterra operators in the material model changes the stress singular behavior at finite times drastically in comparison with the classical elastic predictions.

The results of this paper are based on [7, 1], where Volterra operators, acting on functions of one variable, are considered.

We shall use here the notions and the notations of Part I of this paper given in [5]. The references on the numbers of formulas and statements of [5] will be preceded by the symbol I.

We shall consider the matrix Volterra operators (i.e., integral operators with $(n \times n)$ matrix kernels $K(\gamma, t, \tau))$ acting on n-dimensional vector functions $g(\rho, t)$, $\Phi(\gamma, t)$, etc.; the symbol $|\cdot|$ will be used for a corresponding matrix or vector norm. When $n = 1$, we come to the scalar case.

1. VOLTERRA OPERATORS INDEPENDENT OF COMPLEX VARIABLES

Let us give some definitions and describe several known properties of the Volterra operators.

1.1. DEFINITION. (1) An operator $\underline{K}$ belongs to the class $VB(J)$ if it is an integral Volterra operator

$$(\underline{K}g)(t) := \int_J K(t,\tau) g(\tau)\, d\tau \qquad (t \in J), K(t,\tau) = 0 \ \forall \tau > t;$$

and the kernel norm is finite:

$$|||\, K; J \,||| := \sup_{t \in J} \int_J |K(t,\tau)|\, d\tau < \infty. \tag{1.1}$$

(2) An operator $\underline{K}$ belongs to the class $VC(J)$ if $\underline{K}$ belongs to $VB(J)$ and $\varepsilon(K; t, \Delta t) := \int_J |K(t+\Delta t, \tau) - K(t,\tau)|\, d\tau \to 0$ as $\Delta t \to 0$ for any $t \in J$.

1.2. THEOREM. (1) *If $\underline{K}$ belongs to $VB(J)$ ($\underline{K}$ belongs to $VC(J)$), then $\underline{K}$ acts from the spaces $L_\infty(J)$, $B(J)$, $C(J)$ into the spaces $B(J)$ ($C(J)$) and is bounded; furthermore, $\|\underline{K}\| \le |||\, K \,|||$, where $\|\underline{K}\|$ signifies the operator norm in the corresponding spaces.*

(2) *If, moreover $J = [t_1, t_2]$ and $t_2 < \infty$, then $(\underline{I} + \underline{K})^{-1}$ acts in $L_\infty(J)$, $B(J)$ (and in $C(J)$) and is bounded; furthermore, $(\underline{I} + \underline{K})^{-1} = \underline{I} - \underline{R}$, where the resolvent operator $\underline{R}$ belongs to $VB(J)$ ($\underline{R}$ belongs to $VC(J)$).*

Proofs of these claims are given, e.g., in [1, Chap. 9.5].

Let us describe some properties of a Volterra operator as a partial integral operator acting with respect to the second variable on a function of two variables.

1.3. THEOREM. *Let $\underline{K}$ belong to $VC(J)$.*

(1) *If g belongs to $L_\infty \hat{L}_p(\delta_0, \delta_\infty; J)$, then $\underline{K}g$ belongs to $C\hat{L}_p(\delta_0, \delta_\infty; J)$ and for $p = 2$, $\underline{\mathscr{M}}\underline{K}g = \underline{K}\underline{\mathscr{M}}g \in CH_2^0(S(\delta_0, \delta_\infty); J)$.*

(2) *If $\Phi(\gamma, t)$ belongs to $L_\infty H_p^0(S; J)$, $S := S(\delta_0, \delta_\infty)$, then $\underline{K}\Phi$ belongs to $CH_p^0(S; J)$, $(\partial/\partial\gamma)(\underline{K}\Phi) = \underline{K}(\partial\Phi/\partial\gamma) \in CH_p^0(S; J)$, and for $p = 2$, $\underline{\mathscr{M}}^{-1}(S)\underline{K}\Phi = \underline{K}\underline{\mathscr{M}}^{-1}(S)\Phi \in C\hat{L}_2(\delta_0, \delta_\infty; J)$.*

(3) *If $h(z,t)$ belongs to $L_\infty H_p(\delta_0, \delta_\infty; W; J)$, $W := W(\theta_-, \theta_+)$, then $\underline{K}h$ belongs to $CH_p(\delta_0, \delta_\infty; W; J)$, $(\partial/\partial z)(\underline{K}h) = (\underline{K}(\partial h/\partial z)) \in CH_p(\delta_0, \delta_\infty; W'; J)$ $\forall \overline{W'} \subset W$, and for $p = 2$, $\underline{\mathfrak{M}}(W)\underline{K}h = \underline{K}\underline{\mathfrak{M}}(W)h \in CH_2^\vee(\theta_-, \theta_+; S(\delta_0, \delta_\infty); J)$.*

(4) *If $\Phi(\gamma, t)$ belongs to $L_\infty H_p^\vee(\theta_-, \theta_+; S; J)$, $S := S(\delta_0, \delta_\infty)$, then $\underline{K}\Phi$ belongs to $CH_p^\vee(\theta_-, \theta_+; S; J)$, $(\partial/\partial\gamma)(\underline{K}\Phi) = (K(\partial\Phi/\partial\gamma)) \in CH_p^\vee(\theta_-, \theta_+; S; J)$, and for $p = 2$, $\underline{\mathfrak{M}}^{-1}(S)\underline{K}\Phi = \underline{K}\underline{\mathfrak{M}}^{-1}(S)\Phi \in CH_2(\delta_0, \delta_\infty; W(\theta_-, \theta_+); J)$.*

Proof. Let g belong to $L_\infty \hat{L}_p(\delta_0, \delta_\infty; J)$. Then according to the generalized Minkowski inequality, we get

$$\begin{aligned}
\|\underline{K}g; \delta; J\|_{pB} &= \sup_{t\in J} \left[\int_0^\infty \left| \int_J K(t,\tau) g(\rho,\tau)\, d\tau \right|^p \rho^{p\delta-1}\, d\rho \right]^{1/p} \\
&\le \sup_{t\in J} \int_J \left[\int_0^\infty \left| K(t,\tau) g(\rho,\tau) \rho^\delta \right|^p \rho^{-1}\, d\rho \right]^{1/p} d\tau \\
&= \sup_{t\in J} \int_J |K(t,\tau)|\, \|g(\cdot,\tau); \delta\|_p\, d\tau \\
&\le |||\, K; J\, |||\; \|g; \delta; J\|_{p\infty} < \infty
\end{aligned}$$

for any $\delta \in (\delta_0, \delta_\infty)$. Consequently $\underline{K}g$ belongs to $B\hat{L}_p(\delta_0, \delta_\infty; J)$. Further,

$$\begin{aligned}
&\|[\underline{K}g](\cdot,t) - [\underline{K}g](\cdot,t+\Delta t); \delta\|_p \\
&\quad = \left[\int_0^\infty \left| \int_J [K(t+\Delta t,\tau) - K(t,\tau)] g(\rho,\tau)\, d\tau \right| \rho^{p\delta-1}\, d\rho \right]^{1/p} \\
&\quad \le \int_J \left[\int_0^\infty \left| [K(t+\Delta t,\tau) - K(t,\tau)] g(\rho,\tau) \rho^\delta \right|^p \rho^{-1}\, d\rho \right]^{1/p} d\tau \\
&\quad = \int_J |K(t+\Delta t,\tau) - K(t,\tau)|\, \|g(\cdot,\tau); \delta\|_p\, d\tau \\
&\quad \le \varepsilon(K; t, \Delta t) \|g; \delta; J\|_{p\infty} \to 0, \qquad \Delta t \to 0.
\end{aligned}$$

The first claim of point (1) has been proved. To prove the last claim, let us write an estimate

$$\int_0^\infty \left[\int_J \left| K(t,\tau) g(\rho,\tau) \rho^{\gamma-1} \right| d\tau \right] d\rho \le |||\, K; J\, |||\; \|g; \operatorname{Re}\gamma; J\|_{1\infty} < \infty, \qquad \gamma \in S(\delta_0, \delta_\infty). \tag{1.2}$$

The last inequality follows from Remark I.2.2. Then we have

$$\begin{aligned}
(\underline{\mathscr{M}}\underline{K}g)(\gamma,t) &= \int_0^\infty \left[\int_J K(t,\tau) g(\rho,\tau)\, d\tau \right] \rho^{\gamma-1}\, d\rho \\
&= \int_J K(t,\tau) \left[\int_0^\infty g(\rho,\tau) \rho^{\gamma-1}\, d\rho \right] d\tau = (\underline{K}\underline{\mathscr{M}}g)(\gamma,t).
\end{aligned}$$

The change of the integration order is possible here due to estimate (1.2) and to a corollary from the Fubini theorem (see, e.g., [6, XII, Sect. 4]). The membership $\underline{K}\underline{\mathscr{M}}g \in L_\infty H_2^0(S(\delta_0, \delta_\infty); J)$ follows from the membership $\underline{\mathscr{M}}g \in L_\infty H_2^0(S(\delta_0, \delta_\infty); J)$ (see Theorem I.2.5) and from the first claim of point (2). The proof of point (1) is complete.

Prove now point (2). Let Φ belong to $L_\infty H_p^0(S; J)$. Owing to (1.1) and (I.2.2), we have

$$|[\underline{K}\Phi](\zeta,\tau)| = \left|\int_J K(\tau,\tau_1)\Phi(\zeta,\tau_1)\,d\tau_1\right| \le \int_J |K(\tau,\tau_1)\Phi(\zeta,\tau_1)|\,d\tau_1 \le |||K;J|||\tilde{\tilde{M}}_\infty^0(\Phi;S';J) < \infty \tag{1.3}$$

for any $\tau \in J$. The estimate is uniform with respect to ζ on any $\bar{S}' \in S$.

Let us prove that $[\underline{K}\Phi](\gamma, \tau)$ is holomorphic with respect to $\gamma \in S$ at any $t \in J$, Really $\Phi(\gamma, \tau)$ is holomorphic with respect to $\gamma \in S$ at a.e. $\tau \in J$, and therefore it can be represented (at such τ) as the Cauchy integral:

$$\Phi(\gamma,\tau) = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi(\zeta,\tau)}{\zeta-\gamma}\,d\zeta.$$

Here γ is any point of S and Γ is the boundary of an open disk lying inside of S and containing γ. Then

$$\begin{aligned}[\underline{K}\Phi](\gamma,t) &= \frac{1}{2\pi i}\int_J K(t,\tau)\left[\int_\Gamma \frac{\Phi(\zeta,\tau)}{\zeta-\gamma}\,d\zeta\right]d\tau \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta-\gamma}\left[\int_J K(t,\tau)\Phi(\zeta,\tau)\,d\tau\right] \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{[\underline{K}\Phi](\zeta,t)}{\zeta-\gamma}\,d\zeta,\end{aligned} \tag{1.4}$$

where the order of the integration can be changed owing to estimate (1.3) and the corollary from the Fubini theorem. Hence $[\underline{K}\Phi](\gamma, t)$ is holomorphic at any point $\gamma \in S$ at any $t \in J$ as a Cauchy integral with a bounded density (see (1.3)).

The inequality $M_{pB}^0(\underline{K}\Phi; \delta; J) < \infty$ and the tendency $M_p^0[(\underline{K}\Phi)(\cdot, t) - (\underline{K}\Phi)(\cdot, t + \Delta t); \delta] \to 0$ as $\Delta t \to 0$ for any $t \in J$ uniformly with respect to δ on any $[\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$ are proved as for point (1) above. The first claim of point (2) has been proved.

Prove that the operators $\underline{K}$ and $\partial/\partial\gamma$ commute. Differentiating (1.4), we obtain

$$\begin{aligned}\frac{\partial}{\partial\gamma}[\underline{K}\Phi](\gamma,t) &= \frac{1}{2\pi i}\int_\Gamma \frac{[\underline{K}\Phi](\zeta,t)}{(\zeta-\gamma)^2}\,d\zeta \\ &= \frac{1}{2\pi i}\int_\Gamma d\zeta\int_J K(t,\tau)\frac{\Phi(\zeta,t)}{(\zeta-\gamma)^2}\,d\tau \\ &= \int_J K(t,\tau)\left[\frac{1}{2\pi i}\int_\Gamma \frac{\Phi(\zeta,t)}{(\zeta-\gamma)^2}\,d\zeta\right]d\tau \\ &= \left(\underline{K}\frac{\partial\Phi}{\partial\gamma}\right)(\gamma,t)\in CH_p^0(S;J),\end{aligned}$$

where the change of the integration order is possible owing to estimate (1.3). The last inclusion is implied by $(\partial/\partial\gamma)\Phi$ belonging to $L_\infty H_p^0(S;J)$ and by the first claim of point 2.

The last claim of point (2) is proved as for point (1). Point (2) has been proved. Points (3) and (4) are proved analogously. ∎

1.4. THEOREM. *Suppose an operator* $\underline{K}$ *belongs to* $VC(J)$, *where* $J = [t_1, t_2]$, $-\infty < t_1 < t_1 < \infty$; *then the operator* $(\underline{I}+\underline{K})^{-1}$ *acts in the function classes given in Definitions* I.2.1, I.2.3, I.2.6, I.2.9.

Proof. Let g belongs to $L_\infty\hat{L}_p(\delta_0, \delta_\infty; J)$. Then according to Theorem 1.2, $[(\underline{I}+\underline{K})^{-1}g(\rho,\cdot)](t) = g(\rho,t) - [\underline{R}g(\rho,\cdot)](t)$ for a.e. $\rho\in[0,\infty]$, where $\underline{R}$ belongs to $VC(J)$. According to point (1) of Theorem 1.3, $\underline{R}g$ belongs to $C\hat{L}_p(\delta_0,\delta_\infty;J)\subset L_\infty\hat{L}_p(\delta_0,\delta_\infty;J)$ and consequently $(\underline{I}+\underline{K})^{-1}g$ belongs to $L_\infty\hat{L}_p(\delta_0,\delta_\infty;J)$. The theorem claim for all the other classes is proved analogously. ∎

2. VOLTERRA OPERATORS WITH A COMPLEX PARAMETER ON A STRIP

2.1. DEFINITION. An operator $\underline{K}$ depending on a parameter γ belongs to the class $VCH(S;J)$ if

(i) it is an integral Volterra operator

$$[\underline{K}(\gamma)g](t) := \int_J K(\gamma,t,\tau)g(\tau)\,d\tau \qquad (t\in J),\ K(\gamma,t,\tau)=0 \textit{ for } \tau>t,$$

for any $\gamma\in S$;

(ii) the kernel norm $|||K; \bar{S}'; J||| := \sup_{t\in J}\int_J \sup_{\gamma\in \bar{S}'}|K(\gamma,t,\tau)|\,d\tau$ is finite for any $\bar{S}' \subset S$;

(iii) $\varepsilon(K; \bar{S}; t, \Delta t) := \int_J \sup_{\gamma\in \bar{S}'}|K(\gamma, t+\Delta t, \tau) - K(\gamma,t,\tau)|\,d\tau \to 0$ as $\Delta t \to 0$ for any $\bar{S}' \subset S$; *and*

(iv) the kernel $K(\gamma,t,\tau)$ is holomorphic with respect to $\gamma \in S$ at any t and at almost any τ.

2.2. *Remark.* (1) It follows from the definition that, if $\underline{K}$ belongs to $VCH(S; J)$, then $|||\,K(\gamma); J\,||| \le |||\,K; \bar{S}'; J\,||| < \infty$ for γ on any $\bar{S}' \subset S$ and consequently $\underline{K}(\gamma)$ belongs to $VC(J)$ for any $\gamma \in S$.

(2) A kernel $K(\gamma,t,\tau)$ satisfying condition (ii) of Definition 2.1 is uniformly bounded with respect to γ on any $\bar{S}' \subset S$ for any t and almost any τ.

2.3. LEMMA. *If $\underline{K}$ belongs to $VCH(S; J)$ and γ belongs to S, then*:

(i) *$\underline{K}(\gamma)$ boundedly acts from the spaces $L_\infty(J)$, $B(J)$, $C(J)$ into $C(J)$;*

(ii) *$\partial/\partial\gamma[\underline{K}(\gamma)g](t) = [\underline{K}'(\gamma)g](t)$ for any $g \in L_\infty(J)$ [or $g \in B(J)$, or $g \in C(J)$] and for any $t \in J$, where*

$$[\underline{K}'(\gamma)g](t) := \int_J K'(\gamma,t,\tau)g(\tau)\,d\tau, \qquad K'(\gamma,t,\tau) := \partial K(\gamma,t,\tau)/\partial\gamma, \tag{2.1}$$

and $\underline{K}'$ belongs to $VCH(S; J)$.

Proof. Point (i) is a consequence of point (1) of Theorem 1.2.

To prove (ii), we shall prove at first that the operator $\underline{K}'$ generated by (2.1) belongs to $VCH(S; J)$. Let us choose any $\tilde{S}' := \tilde{S}(\delta_0', \delta_\infty') \subset S = S(\delta_0, \delta_\infty)$ and denote $r := \frac{1}{2}\min(\delta_0' - \delta_0, \delta_\infty - \delta_\infty') > 0$, $\delta_0'' := \delta_0' - r$, $\delta_\infty'' := \delta_\infty' + r$. Let γ be any point of $\bar{S}'$. Then the circle Γ with the center γ and the radius r belongs to $\tilde{S}'' := \bar{S}(\delta_0'', \delta_\infty'')$. Using the Cauchy theorem and point (2) of Remark 2.2, we obtain

$$K(\gamma,t,\tau) = \frac{1}{2\pi i}\int_\Gamma \frac{K(\zeta,t,\tau)}{\zeta-\gamma}\,d\zeta,$$

$$K'(\gamma,t,\tau) = \frac{1}{2\pi i}\int_\Gamma \frac{K(\zeta,t,\tau)}{(\zeta-\gamma)^2}\,d\zeta = \frac{1}{2\pi r}\int_0^{2\pi} K(\gamma+re^{i\theta},t,\tau)e^{-i\theta}\,d\theta,$$

$$|K'(\gamma,t,\tau)| \le \frac{1}{2\pi r}\int_0^{2\pi} \sup_{\zeta\in\bar{S}''}|K(\eta,t,\tau)|\,d\theta = \frac{1}{r}\sup_{\eta\in\bar{S}''}|K(\eta,t,\tau)| < \infty$$

for any t and almost any τ. Consequently,

$$||| K'; \bar{S}'; J ||| \leq \frac{1}{r} ||| K; \bar{S}''; J ||| < \infty. \tag{2.2}$$

Further,

$$\varepsilon(K'; \bar{S}'; t, \Delta t) := \int_J \frac{1}{2\pi r} \sup_{\gamma \in \bar{S}'} \Big| \int_0^{2\pi} \big[K(\gamma + re^{i\theta}, t + \Delta t, \tau) - K(\gamma + re^{i\theta}, t, \tau) \big] e^{-i\theta} \, d\theta \Big| \, d\tau$$

$$\leq \frac{1}{2\pi r} \int_0^{2\pi} \varepsilon[K; \bar{S}''; t, \Delta t] \, d\theta \to 0, \qquad \Delta t \to 0.$$

The membership $\underline{K}' \in VCH(S; J)$ has been proved. Finally, we can change the order of the differentiation in γ and the integration in τ because of the derivatives definition and the Lebesgue dominated convergence theorem together with estimate (2.2). ∎

2.4. *Remark.* If $\underline{K}$ belongs to $VCH(S; J)$ and $g(t)$ belongs to $L_\infty(J)$, $B(J)$, or $C(J)$, then $[\underline{K}(\gamma) g](t)$ is a holomorphic function with respect to $\gamma \in S$ for any $t \in J$. This follows, e.g., from the representation of the derivative $(\partial/\partial\gamma)[\underline{K}(\gamma) g](t)$, according to point (ii) of Lemma 2.3, and from its boundedness owing to the membership $\underline{K}'(\gamma) \in VC(J)$ and to point (1) of Theorem 1.2.

We discuss now properties of the resolvent for an operator from $VCH(S; J)$.

Let us consider the convolutions of functions $a(\gamma, t, \tau)$, $b(\gamma, t, \tau)$:

$$[a * b](\gamma, t, \tau) := \int_J a(\gamma, t, \tau_1) b(\gamma, \tau_1, \tau) \, d\tau_1,$$

$$a^{*1}(\gamma, t, \tau) := a(\gamma, t, \tau), \qquad a^{*n}(\gamma, t, \tau) := [a * a^{*n-1}](\gamma, t, \tau), n > 1.$$

Then from the resolvent definition we have the property

$$\begin{aligned} R(\gamma, t, \tau) &= K(\gamma, t, \tau) - [K * R](\gamma, t, \tau) \\ &= K(\gamma, t, \tau) - [R * K](\gamma, t, \tau). \end{aligned} \tag{2.3}$$

2.5. *Remark.* For operators $\underline{K}_1$, $\underline{K}_2$ satisfying conditions (i) and (ii) of Definition 2.1, we have

$$||| K_1 + K_2; \bar{S}'; J ||| \leq ||| K_1; \bar{S}'; J ||| + ||| K_2; \bar{S}'; J |||,$$

$$||| K_1 * K_2; \bar{S}'; J ||| \leq ||| K_1; \bar{S}'; J ||| \; ||| K_2; \bar{S}'; J |||$$

for any $\bar{S}' \subset S$. Hence the operators $\underline{K}_1 + \underline{K}_2$ and $\underline{K}_1\underline{K}_2$ satisfy conditions (i) and (ii) of Definition 2.1 as well.

2.6. LEMMA. *Suppose $\underline{K}$ belongs to $VCH(S; J)$ and has a resolvent $\underline{R}(\gamma)$ that satisfies conditions* (i) *and* (ii) *of Definition* 2.1 *for any $\gamma \in S$. Then $\underline{R}$ belongs to $VCH(S; J)$.*

Proof. Let us prove that condition (iii) of Definition 2.1 is also fulfilled for $\underline{R}$ under the lemma assumptions. By (2.3), we get

$$\begin{aligned}\varepsilon(R; \bar{S}'; t, \Delta t) &:= \int_J \sup_{\gamma\in\bar{S}'} |R(\gamma, t+\Delta t, \tau) - R(\gamma, t, \tau)|\, d\tau \\ &= \int_J \sup_{\gamma\in\bar{S}'} \Big| K(\gamma, t+\Delta t, \tau) - K(\gamma, t, \tau) \\ &\qquad - \int_J [K(\gamma, t+\Delta t, \tau_1) - K(\gamma, t, \tau_1)] R(\gamma, \tau_1, \tau)\, d\tau_1 \Big| \, d\tau \\ &\le \varepsilon(K; \bar{S}'; t, \Delta t) + |||\, R; \bar{S}'; t_1, t_2 \,|||\, \varepsilon(K; \bar{S}'; t, \Delta t) \to 0 \\ &\qquad \text{as } \Delta t \to 0\end{aligned}$$

for any $\bar{S}' \subset S$. Condition (iii) of Definition 2.1 has been proved.

Let us prove the holomorphy of $R(\gamma, t, \tau)$ at any t and at almost any τ, required by point (iv) of Definition 2.1. Using (2.3), one can see that

$$\begin{aligned}&R(\gamma+\Delta\gamma) - R(\gamma) \\ &\quad = [K(\gamma+\Delta\gamma) - K(\gamma)] - R(\gamma) * [K(\gamma+\Delta\gamma) - K(\gamma)] \\ &\qquad - [K(\gamma+\Delta\gamma) - K(\gamma)] * R(\gamma+\Delta\gamma) \\ &\qquad + R(\gamma) * [K(\gamma+\Delta\gamma) - K(\gamma)] * R(\gamma+\Delta\gamma) \\ &\quad = \Delta\gamma [K'(\gamma_1) - R(\gamma) * K'(\gamma_2) - K'(\gamma_2) * R(\gamma+\Delta\gamma) \\ &\qquad\qquad + R(\gamma) * K'(\gamma_4) * R(\gamma+\Delta\gamma)], \\ &\qquad\qquad\qquad \gamma_i \in [\gamma, \gamma+\Delta\gamma]. \qquad (2.4)\end{aligned}$$

The fixed arguments t, τ are dropped here. Let us fix some $\bar{S}' \subset S$. We take into account that $\underline{K}'$ belongs to $VCH[S(\delta_0, \delta_\infty); J]$ by point (ii) of Lemma 2.3 and that $\underline{R}$ satisfies condition (ii) of Definition 2.1. Then we obtain from Remark 2.5 and point (2) of Remark 2.2 that the functions $K'_m(t, \tau) := \sup_{\gamma\in\bar{S}'} |K'(\gamma, t, \tau)|$, $R_m(t, \tau) := \sup_{\gamma\in\bar{S}'} |R(\gamma, t, \tau)|$, $[R_m * K'_m](t, \tau)$, $[K'_m * R_m](t, \tau)$, and $[R_m * K'_m * R_m](t, \tau)$ are finite at any

$t \in J$ and at almost any $\tau \in J$. Then

$$|R(\gamma + \Delta\gamma, t, \tau) - R(\gamma, t, \tau)| \\ \leq |\Delta\gamma| [K'_m + R_m * K'_m + K'_m * R_m + R_m * K'_m * R_m](t, \tau) \to 0, \\ \Delta\gamma \to 0,$$

at any $t \in J$ and at almost any $\tau \in J$. Hence $R(\gamma, t, \tau)$ is uniformly continuous with respect to γ on any $\overline{S}' \subset S$ at any $t \in J$ and at almost any $\tau \in J$.

Using then the same t, τ, dividing (2.4) by $\Delta\gamma$, tending it to zero, and taking into account the continuity of $R(\gamma, t, \tau)$, we obtain

$$\frac{\partial}{\partial\gamma} R(\gamma, t, \tau) = K'(\gamma, t, \tau) - [R(\gamma) * K'(\gamma)](t, \tau) \\ - [K'(\gamma) * R(\gamma)](t, \tau) \\ + [R(\gamma) * K'(\gamma) * R(\gamma)](t, \tau) \tag{2.5}$$

at any t and at almost any τ. Here the Lebesgue dominated convergence theorem was used with the dominant functions $R_m(t, \tau_1) K'_m(\tau_1, \tau)$, $K'_m(t, \tau_1) R_m(\tau_1, \tau)$, $R_m(t, \tau_1) K'_m(\tau_1, \tau_2) R_m(\tau_2, \tau)$. The right hand side of (2.5) is bounded and unique at any t and at almost any τ. This completes the proof of condition (iv) of Definition 2.1. ∎

2.7. LEMMA. *Let $\underline{K}$ belong to $VCH[S; J(t_1, t_2)]$, $t_2 < \infty$. Then the resolvent operator $\underline{R}$ belongs to $VCH[S; J(t_1, t_2)]$ too.*

Proof. Note that, according to point (1) of Remark 2.2 and to point (2) of Theorem 1.2, $\underline{R}(\gamma)$ belongs to $VC(J)$ for any $\gamma \in S$. Hence point (i) of Definition 2.1 is fulfilled for $\underline{R}(\gamma)$. We shall prove now that the kernel $R(\gamma, t, \tau)$ satisfies condition (ii) of Definition 2.1. Let us fix some $\overline{S}' \subset S$ and consider the function $K_m(\overline{S}'; t, \tau) := \sup_{\gamma \in \overline{S}'} |K(\gamma, t, \tau)|$ as a scalar operator kernel. By conditions (i) and (ii) of Definition 2.1, the operators $\underline{K}_m(\overline{S}')$ and $-\underline{K}_m(\overline{S}')$ belong to $VB(J)$. Consequently the resolvent $\underline{R}_{m-}(\overline{S}')$ of the operator $-\underline{K}_m(\overline{S}')$ also belongs to $VB(J)$ for $t_2 < \infty$ (see part (2) of Theorem 1.2), and $|||\, R_{m-}(\overline{S}'); J \,||| < \infty$.

For the resolvent of a Volterra operator $\underline{K}(\gamma) \in VC(J)$, the Neumann series converges with respect to kernel norm (1.1) for any $\gamma \in S$ (see, e.g., [1, Chap. 9]): $R(\gamma, t, \tau) = \sum_{j=1}^{\infty}(-1)^{j-1}(K)^{*j}(\gamma, t, \tau)$. Then

$$
\begin{aligned}
|R(\gamma, t, \tau)| &\le \sum_{j=1}^{\infty} |K|^{*j}(\gamma, t, \tau) \le \sum_{j=1}^{\infty} (K_m)^{*j}(\bar{S}'; t, \tau) \\
&= -\sum_{j=1}^{\infty} (-1)^{j-1}(-K_m)^{*j}(\bar{S}'; t, \tau) \\
&= -R_{m-}(\bar{S}'; t, \tau), \qquad \forall \gamma \in \bar{S}' \subset S.
\end{aligned}
$$

Thus, $|||\, R; \bar{S}'; J\, ||| \le |||\, R_{m-}(\bar{S}'); J\, ||| < \infty$. Condition (ii) of Definition 2.1 has been proved. Conditions (iii) and (iv) follow from Lemma 2.6. ∎

2.8. THEOREM. *Let $\underline{K}$ belong to $VCH(S; J)$.*

(1) *If Φ belongs to $L_\infty H_p^0(S; J)$, then $\underline{K}\Phi$ belongs to $CH_p^0(S; J)$.*

(2) *If Φ belongs to $L_\infty H_p^\vee(\theta_-, \theta_+; S; J)$, then $\underline{K}\Phi$ belongs to $CH_p^\vee(\theta_-, \theta_+; S; J)$.*

(3) *If Φ belongs to one of the classes from points* (1), (2), *then*

$$
\frac{\partial}{\partial \gamma}[\underline{K}(\gamma)\Phi(\gamma, \cdot)](t) = \left[\underline{K}(\gamma)\frac{\partial}{\partial \gamma}\Phi(\gamma, \cdot)\right](t) + [\underline{K}'(\gamma)\Phi(\gamma, \cdot)](t).
$$

Proof. (1) Let Φ belong to $L_\infty H_p^0(S; J)$, $S = S(\delta_0, \delta_\infty)$. Using the generalized Minkowski inequality, we have for any $\delta \in [\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$

$$
\begin{aligned}
&M_{pB}^0(\underline{K}\Phi; \delta; J) \\
&\quad = \sup_{t \in J}\left[\int_{-\infty}^{\infty}\left|\int_J K(\delta + i\xi, t, \tau)\Phi(\delta + i\xi, \tau)\, d\tau\right|^p d\xi\right]^{1/p} \\
&\quad \le \sup_{t \in J}\int_J\left[\int_{-\infty}^{\infty}|K(\delta + i\xi, t, \tau)|^p|\Phi(\delta + i\xi, \tau)|^p\, d\xi\right]^{1/p} d\tau \\
&\quad \le \sup_{t \in J}\int_J \sup_{\gamma \in \bar{S}(\delta_0', \delta_\infty')}|K(\gamma, t, \tau)|M_p^0[\Phi(\cdot, \tau); \delta]\, d\tau \\
&\quad \le |||\, K; \bar{S}(\delta_0', \delta_\infty'); J\, |||\, M_{p\infty}^0(\Phi; \delta; J) < \infty. \qquad (2.6)
\end{aligned}
$$

Consequently $M_{pB}^0(\underline{K}\Phi; \delta; J)$ is uniformly bounded with respect to δ on any $[\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$.

Further, writing the estimates analogous to (2.6), we have

$$M_p^0\{[\underline{K}\Phi](t+\Delta t)-[\underline{K}\Phi](t);\delta\} \\ \leq \varepsilon\big(K;\overline{S}(\delta_0',\delta_\infty');t,\Delta t\big)M_{p\infty}^0(\Phi;\delta;J)\to 0,\qquad \Delta t\to 0,$$

and the tendency is uniform with respect to δ on any $[\delta_0',\delta_\infty']\subset(\delta_0,\delta_\infty)$.

Let us prove the holomorphy of $[\underline{K}(\gamma)\Phi(\gamma,\cdot)](t)$ with respect to $\gamma\in S$. Reasoning as in the proof of point (2) of Theorem 1.3, we obtain

$$[\underline{K}(\gamma)\Phi(\gamma,\cdot)](t)=\frac{1}{2\pi i}\int_\Gamma\frac{[\underline{K}(\gamma)\Phi(\zeta,\cdot)](t)}{\zeta-\gamma}\,d\zeta. \tag{2.7}$$

It is sufficient to prove the existence of the derivative with respect to γ.

Differentiating (2.7), one may change the order of the differentiation and the integration. It is possible owing to the Lebesgue dominated convergence theorem, since $[\underline{K}'(\gamma^0)\Phi(\zeta,\cdot)](t)$ is uniformly bounded with respect to γ^0 in a vicinity of γ by the force of Remark 2.4 and estimate (I.2.2) holds. Then we have

$$\begin{aligned}\frac{\partial}{\partial\gamma}[\underline{K}(\gamma)\Phi(\gamma,\cdot)](t) &= \frac{1}{2\pi i}\int_\Gamma\frac{[\underline{K}(\gamma)(\zeta,\cdot)](t)}{(\zeta-\gamma)^2}\,d\zeta \\ &\quad+\frac{1}{2\pi i}\int_\Gamma\frac{[\underline{K}'(\gamma)\Phi(\zeta,\cdot)](t)}{(\zeta-\gamma)}\,d\zeta \\ &=\int_J K(\gamma,t,\tau)\left[\frac{1}{2\pi i}\int_\Gamma\frac{\Phi(\zeta,t)}{(\zeta-\gamma)^2}\,d\zeta\right]d\tau \\ &\quad+\int_J\left[\frac{\partial}{\partial\gamma}K(\gamma,t,\tau)\right]\left[\frac{1}{2\pi i}\int_\Gamma\frac{\Phi(\zeta,t)}{\zeta-\gamma}\,d\zeta\right]d\tau \\ &=\left[\underline{K}(\gamma)\frac{\partial}{\partial\gamma}\Phi(\gamma,\cdot)\right](t)+[\underline{K}'(\gamma)\Phi(\gamma,\cdot)](t),\end{aligned} \tag{2.8}$$

where all integrals exist by estimate (I.2.2), point (ii) of Definition 2.1, and point (2) of Lemma 2.3.

The proof of point (1) is complete. Point (2) is proved analogously. Point (3) follows from (2.8). ∎

2.9. THEOREM. *Let $J = [t_1, t_2]$, $t_2 < \infty$. If $\underline{K}$ belongs to $VCH(S; J)$, then $\underline{R}$ acts*:

(i) *from $L_\infty H_p^0(S; J)$ to $CH_p^0(S; J)$;*
(ii) *from $L_\infty H_p^\vee(\theta_-, \theta_+; S; J)$ to $CH_p^\vee(\theta_-, \theta_+; S; J)$.*

Proof. According to Lemma 2.7, $R(\gamma)$ belongs to $VCH(S; J)$. An application of Theorem 2.8 completes the proof. ∎

3. VOLTERRA OPERATOR RESOLVENT PROPERTIES ON THE HALF-AXIS

Up to now, we have considered the resolvent properties only for $t_2 < \infty$. Let us now consider the case $t_2 \le \infty$. Let $J_\infty = J(t_1, \infty) =: [t_1, \infty)$.

3.1. DEFINITION. (1) A Volterra operator $\underline{K}^-$ is of the convolution type if its kernel satisfies the relation $K^-(t, \tau) = K^-(t - \tau)$.

(2) An operator $\underline{K} \in VC(J_\infty)$ tends to an operator of the convolution type $\underline{K}^- \in VC(J_\infty)$ if $|||\, K - K^-; [T, \infty)\, ||| \to 0$ as $T \to \infty$.

(3) An operator $\underline{K} \in VCH(S; J_\infty)$ tends to an operator of the convolution type $\underline{K}^- \in VCH(S; J_\infty)$ if $|||\, K - K^-; \bar{S}'; [T, \infty)\, ||| \to 0$ for any $\bar{S}' \subset S$ at $T \to \infty$.

Let us recall here the classical half line Paley–Wiener theorem [7, Theorem 18] (see also [1, Theorem 2.4.1]).

3.2. THEOREM. *Suppose an operator of the convolution type $\underline{K}^-$ belongs to $VC(J_\infty)$ and* $\det[I + \hat{K}^-(\omega)] \ne 0$, $\operatorname{Re}(\omega) \ge 0$, *where* $\hat{K}^-(\omega) := \int_0^\infty K^-(\tau) e^{-\omega\tau}\, d\tau$. *Then the resolvent of $\underline{K}^-$ is an operator $\underline{R}^-$ of the convolution type belonging to $VC(J_\infty)$.*

Let us consider the following generalization of the formulated theorem.

3.3. THEOREM. *Suppose an operator of the convolution type $\underline{K}^-$ belongs to $VCH(S; J_\infty)$ and* $\det[I + \hat{K}^-(\gamma, \omega)] \ne 0$ *for* $\gamma \in S$, $\operatorname{Re}(\omega) \ge 0$, *where* $\hat{K}^-(\gamma, \omega) := \int_0^\infty K^-(\gamma, \tau) e^{-\omega\tau}\, d\tau$. *Then the resolvent of $\underline{K}^-$ is an operator $\underline{R}^-$ of the convolution type belonging to $VCH(S; J_\infty)$.*

Proof (sketch). From the classical half line Paley–Wiener Theorem 3.2, it follows that $\underline{R}^-(\gamma)$ belongs to $VC(J_\infty)$ and $\underline{R}^-(\gamma)$ is of the convolution type for any $\gamma \in S$ under the theorem assumptions. Consequently, condition (i) of Definition 2.1 is fulfilled for $\underline{R}^-(\gamma)$, and $R^-(\gamma, t, \tau) = R^-(\gamma, t - \tau)$.

Condition (ii) of Definition 2.1 is proved for $\underline{R}^-(\gamma)$ by repeating the long proof of the Paley–Wiener theorem from [1, Chap. 2] with the following modifications. We should replace successively the kernel norms $\|K^-; L\|$ in the Lebesgue spaces $L(t_1, \infty)$ and $L(-\infty, \infty)$ by the corresponding norms for the kernel supremum: $\|\sup_{\gamma \in \bar{S}'}|K^-(\gamma, \cdot)|; L\|$. The property $\|K^-(t + \Delta t) - K^-(t); L\| \to 0$, $\Delta t \to 0$ should be replaced by the corresponding property $\|\sup_{\gamma \in \bar{S}'}|K^-(\gamma, t + \Delta t) - K^-(\gamma, t)|; L\| \to 0$, $\Delta t \to 0$ (provided by point (iii) of Definition 2.1), and $\sup_\omega \|[I + \hat{K}^-(\omega)]^{-1}|$, $\omega \in (-\infty, \infty)$, should be replaced by $\sup_{\gamma, \omega}\|[I + \hat{K}^-(\gamma, \omega)]^{-1}|$, $\gamma \in \bar{S}'$, $\omega \in (-\infty, \infty)$.

The conditions (iii) and (iv) of Definition 2.1 follow from Lemma 2.6. ▮

3.4. LEMMA. *Suppose operators $\underline{K}_1, \underline{K}_2$ belong to $VCH(S; J)$, where $\underline{K}_1$ has a resolvent $\underline{R}_1$ from $VCH(S; J)$ and*

$$||| K_2; \bar{S}'; J ||| < 1/\big(1 + ||| \underline{R}_1; \bar{S}'; J |||\big) \qquad \forall \bar{S}' \subset S. \tag{3.1}$$

Then the operator $\underline{K} = \underline{K}_1 + \underline{K}_2$ has a resolvent $\underline{R}$ from $VCH(s; J)$.

Proof. By Theorem 9.3.9 and Corollary 9.3.18 in [1], it follows from the lemma conditions that the resolvent $\underline{R}(\gamma)$ belongs to $VC(J)$ for any $\gamma \in S$, and the resolvent kernel is given by the Neumann series convergent with respect to kernel norm (1.1):

$$R(\gamma, t, \tau) = R_1(\gamma, t, \tau) - \tilde{R}(\gamma, t, \tau) + \big[\tilde{R} * R_1\big](\gamma, t, \tau),$$

$$\tilde{R}(\gamma, t, \tau) := \sum_{j=1}^{\infty} (-1)^j (K_2 - R_1 * K_2)^{*j}(\gamma, t, \tau).$$

Using Remark 2.5 and inequality (3.1), we get

$$||| \underline{\tilde{R}}; \bar{S}'; J ||| \le \sum_{j=1}^{\infty} \Big(||| \underline{K}_2; \bar{S}'; J ||| + ||| \underline{R}_1; \bar{S}'; J ||| \; ||| \underline{K}_2; \bar{S}'; J ||| \Big)^j < \infty,$$

$$\begin{aligned} ||| \underline{R}; \bar{S}'; J ||| \le{} & ||| \underline{R}_1; \bar{S}'; J ||| + ||| \underline{\tilde{R}}; \bar{S}'; J ||| \\ & + ||| \underline{\tilde{R}}; \bar{S}'; J ||| \; ||| \underline{R}_1; \bar{S}'; J ||| < \infty. \end{aligned}$$

Thus condition (ii) of Definition 2.1 is satisfied. Properties (iii) and (iv) follow from Lemma 2.6. ▮

3.5. LEMMA. *Suppose $\underline{K} \in VCH(S; J_\infty)$ tends to an operator of the convolution type $\underline{K}^- \in VCH(S; J_\infty)$, $\underline{K}^-$ has a resolvent $\underline{R}^- \in VCH(S; J_\infty)$ of the convolution type, and $\underline{K}$ has a resolvent $\underline{R} \in VCH(S; J_\infty)$. Then $\underline{R}$ tends to the operator of the convolution type $\underline{R}^-$.*

Proof. Let $\tilde{K}(\gamma, t, \tau) := K(\gamma, t, \tau) - K^-(\gamma, t - \tau)$, $\tilde{R}(\gamma, t, \tau) := R(\gamma, t, \tau) - R^-(\gamma, t - \tau)$. From the resolvent definition, we get

$$R^- * K^- = K^- - R^- = K^- * R^-, \qquad R * K = K - R = K * R, \quad (3.2)$$

where the arguments γ, t, τ are dropped. Then

$$\tilde{R} + \tilde{R} * K = \tilde{K} - R^- * \tilde{K}. \quad (3.3)$$

Let us convolve this equation by R from the right and subtract the result from (3.3). Then, allowing for (3.2), we have $\tilde{R} = \tilde{K} - \tilde{K} * R - R^- * \tilde{K} + R^- * \tilde{K} * R$. Consequently,

$$\||| \tilde{R}; \bar{S}'; T, \infty ||| \le ||| \tilde{K}; \bar{S}'; T, \infty ||| \Big[1 + ||| R; \bar{S}'; T, \infty ||| + ||| R^-; \bar{S}'; T, \infty ||| + ||| R; \bar{S}'; T, \infty ||| \; ||| R^-; \bar{S}'; T, \infty ||| \Big] \to 0, T \to \infty,$$

since $||| \tilde{K}; \bar{S}'; T, \infty ||| \to 0$ as $T \to 0$. ∎

3.6. THEOREM. *Suppose $\underline{K}$ belongs to $VCH(S; J_\infty)$ and tends to an operator of the convolution type $\underline{K}^-$ that satisfies the conditions of Theorem 3.3. Then the resolvent $\underline{R}$ belongs to $VCH(S; J_\infty)$ and tends to the operator of the convolution type $\underline{R}^- \in VCH(S; J_\infty)$ being the resolvent of $\underline{K}^-$.*

Proof. From [1, Theorems 9.3.19 and 9.11.14], it follow that $\underline{R}(\gamma)$ belongs to $VC(J_\infty)$ for any $\gamma \in S$ under the above assumptions, i.e., condition (i) of Definition 2.1 is fulfilled for $\underline{R}(\gamma)$.

Prove condition (ii) of Definition 2.1 for $\underline{R}(\gamma)$. According to Theorem 3.3, $\underline{R}^-$ belongs to $VCH(S; [T, \infty))$ for any $T < \infty$ and, consequently $||| R^-; \bar{S}'; [T, \infty) ||| = ||| R^-; \bar{S}'; [0, \infty) ||| < \infty$ for any $\bar{S}' \subset S$. Let $T = T(S')$ be sufficiently large such that $||| K - K^-; \bar{S}'; [T, \infty) ||| < 1/(1 + ||| R^-; \bar{S}'; [0, \infty) |||)$. Then it follows from Lemma 3.4 that the operator $\underline{K}$ considered on the interval (T, ∞) has a resolvent $\underline{R}_T$ belonging to $VCH(\bar{S}; [T, \infty))$. Let $\underline{R}_1$ be a resolvent of the operator $\underline{K}$ considered on the segment $[t_1, T]$. The membership $\underline{R}_1 \in VCH(S; [t_1, T])$ follows from Lemma 2.7.

Then, involving the same reasoning as in the proof of Theorem 9.3.13 in [1], we have that the resolvent kernel on all the half axis J_∞ has the representation:

$$R(\gamma, t, \tau) = K(\gamma, t, \tau) - (K * R_1)(\gamma, t, \tau) - (R_T * K)(\gamma, t, \tau) + (R_T * K * R_1)(\gamma, t, \tau), \quad (3.4)$$

where the resolvent kernels $R_1(\gamma, t, \tau)$ and $R_T(\gamma, t, \tau)$ are extended by zero, i.e., $R_1(\gamma, t, \tau) = 0$ for $t > T$ or $\tau > T$, and $R_T(\gamma, t, \tau) = 0$ for $t < T$ or $\tau < T$.

From Remark 2.5, we obtain for any $\bar{S}' \subset S$;

$$||| R; \bar{S}'; J_\infty ||| \le ||| K; \bar{S}'; [t_1, \infty) ||| \Big\{1 + ||| R_1; \bar{S}'; [t_1, T] ||| + ||| R_T; \bar{S}'; [T, \infty) ||| + ||| R_T; \bar{S}'; [T, \infty) ||| \; ||| R_1; \bar{S}'; [t_1, T] ||| \Big\} < \infty.$$

This completes the proof of condition (ii) of Definition 2.1. Conditions (iii) and (iv) follow from Lemma 2.6. Consequently, $\underline{R}$ belongs to $VCH(S; J_\infty)$. The tendency of $\underline{R}$ to $\underline{R}^-$ follows from Lemma 3.5. ∎

For the case when the operators are independent of γ, we obtain the obvious corollary of Theorem 3.6:

3.7. THEOREM. *Suppose $\underline{K}$ belongs to $VC(J_\infty)$ and tends to an operator of the convolution type $\underline{K}^-$ that satisfies the conditions of Theorem* 3.3. *Then the resolvent $\underline{R}$ belongs to $VC(J_\infty)$ and tends to an operator of the convolution type $\underline{R}^- \in VC(J_\infty)$ being the resolvent of $\underline{K}^-$.*

Consider now the Volterra operators with fading memory and investigate their asymptotic properties for $t \to \infty$.

3.8. DEFINITION. An operator $\underline{K}$ belongs to $VC_l(J_\infty; \Omega)$ if:

(i) K belongs to $VC(J_\infty)$;

(ii) $\int_{t_1}^{T} |K(t, \tau)|\, d\tau \to 0$, as $t \to \infty$, $\forall T \in J_\infty$; and

(iii) $\int_{t_1}^{t} K(t, \tau) e^{i\Omega(\tau - t)}\, d\tau \to K_{\Omega \neq \infty}$, $t \to \infty$.

3.9. THEOREM. *Let $\underline{K}$ belong to $VC_l(J_\infty; \Omega)$.*

(1) *Suppose g belongs to $L_{\infty l}\hat{L}_p(\delta_0, \delta_\infty; J_\infty; \Omega)$ and $g(\rho, t) \to g_\Omega(\rho) e^{i\Omega t}$ as $t \to \infty$; then $\underline{K}g$ belongs to $C_l\hat{L}_p(\delta_0, \delta_\infty; J_\infty; \Omega)$ and $(\underline{K}g)(\rho, t) \to K_\Omega g_\Omega(\rho) e^{i\Omega t}$ as $t \to \infty$.*

(2) *Suppose Φ belongs to $L_{\infty l}H_p^0(S; J_\infty; \Omega)$ and $\Phi(\gamma, t) \to \Phi_\Omega(\gamma) e^{i\Omega t}$ as $t \to \infty$; then $\underline{K}\Phi$ belongs to $C_l H_p^0(S; J_\infty; \Omega)$ and $(\underline{K}\Phi)(\gamma, t) \to K_\Omega \Phi_\Omega(\gamma) e^{i\Omega t}$ as $t \to \infty$.*

(3) *Suppose h belongs to $L_{\infty l}H_p(\delta_0, \delta_\infty; W; J_\infty; \Omega)$ and $h(z, t) \to h_\Omega(z) e^{i\Omega t}$ as $t \to \infty$; then $\underline{K}h$ belongs to $C_l H_p(\delta_0, \delta_\infty; W; J_\infty; \Omega)$ and $(\underline{K}h)(z, t) \to K_\Omega h_\Omega(z) e^{i\Omega t}$ as $t \to \infty$.*

(4) *Suppose Φ belongs to $L_{\infty l}H_p^\vee(\theta_-, \theta_+; S; J_\infty; \Omega)$ and $\Phi(\gamma, t) \to \Phi_\Omega(\gamma) e^{i\Omega t}$ as $t \to \infty$; then $\underline{K}\Phi$ belongs to $C_l H_p^\vee(\theta_-, \theta_+; S; J_\infty; \Omega)$ and $(\underline{K}\Phi)(\gamma, t) \to K_\Omega \Phi_\Omega(\gamma) e^{i\Omega t}$ as $t \to \infty$.*

The limits here are understood as in Definition I.2.14.

Proof. Points (2) and (4) are particular cases of points (3) and (4) of Theorem 3.12 given a little bit later. The proof of points (1) and (3) is analogous (with obvious modifications) to the proof of point (3) of Theorem 3.12. ∎

3.10. Definition. An operator $\underline{K}$ belongs to $VC_lH(S; J_\infty; \Omega)$ if:

(i) $\underline{K}$ belongs to $VCH(S; J_\infty)$;

(ii) $\int_{t_1}^T \sup_{\gamma\in\overline{S}'} |K(\gamma, t, \tau)|\, d\tau \to 0$ as $t\to\infty$, $\forall T \in J_\infty$, $\forall \overline{S}' \subset S$; and

(iii) there is a function $K_\Omega(\gamma)$ such that

$$A(t) := \sup_{\gamma\in\overline{S}'} \left| \int_{t_1}^t K(\gamma, t, \tau) e^{i\Omega(\tau - t)}\, d\tau - K_\Omega(\gamma) \right| \to 0 \text{ as } t\to\infty,\ \forall\overline{S}' \subset S. \tag{3.5}$$

3.11. Lemma. *The function $K_\Omega(\gamma)$ from Definition 3.10 is bounded on any $\overline{S}' \subset S$ and holomorphic on S.*

Proof. Due to (3.5), we have

$$\begin{aligned}\sup_{\gamma\in S'} |K_\Omega(\gamma)| &\le \sup_{\gamma\in S'} \left| \int_{t_1}^t K(\gamma, t, \tau) e^{i\Omega(\tau - t)}\, d\tau - K_\Omega(\gamma) \right| \\ &\quad + \int_{t_1}^t \sup_{\gamma\in\overline{S}'} |K(\gamma, t, \tau)|\, d\tau \\ &\le A(t) + |||K; \overline{S}'; J_\infty||| \to |||K; \overline{S}'; J_\infty|||, \qquad t\to\infty.\end{aligned}$$

The boundedness has been proved. The function $\int_{t_1}^t K(\gamma, t, \tau) e^{i\Omega(\tau-t)}\, d\tau$ is holomorphic on S for any $t \in J_\infty$ by Remark 2.4. Taking into account (3.5) and the fact that the uniform limit of holomorphic functions is holomorphic, we get the holomorphy of $K_\Omega(\gamma)$. ∎

Let us formulate an analogue of Theorem 3.9 in this paper and of Theorems 9.6.2, 9.6.4 in [1] for $\underline{K}$ from $VC_lH(S; J_\infty; \Omega)$.

3.12. Theorem. (1) *Suppose $\underline{K}_n$ belongs to $VC_lH(S; J_\infty; \Omega)$ $(n = 1 \div \infty)$ and there is $\underline{K}$ from $VCH(S; J_\infty)$ such that*

$$\lim_{n\to\infty} |||K - K_n; \overline{S}'; J_\infty||| \to 0, \qquad \forall\overline{S}' \subset S. \tag{3.6}$$

Then $\underline{K}$ belongs to $VC_lH(S; J_\infty; \Omega)$ and $K_\Omega(\gamma) = \lim_{n\to\infty} K_{n\Omega}(\gamma)$.

(2) *Let $\underline{K}_1$, $\underline{K}_2$ belong to $VC_lH(S; J_\infty; \Omega)$. Then $\underline{\tilde{K}} := \underline{K}_1\underline{K}_2$ belongs to $VC_lH(S; J_\infty; \Omega)$ and $\tilde{K}_\Omega(\gamma) = K_{1\Omega}(\gamma)K_{2\Omega}(\gamma)$.*

(3) *Let $\underline{K}$ belong to $VC_lH(S; J_\infty; \Omega)$. Suppose Φ belongs to $L_{\infty l}H_p^0(S; J_\infty; \Omega)$ and $\Phi(\gamma, t) \to \Phi_\Omega(\gamma)e^{i\Omega t}$ as $t\to\infty$; then $\underline{K}\Phi$ belongs to $C_lH_p^0(S; J_\infty; \Omega)$ and*

$$(\underline{K}\Phi)(\gamma, t) \to K_\Omega(\gamma)\Phi_\Omega(\gamma)e^{i\Omega t}, \qquad t\to\infty. \tag{3.7}$$

(4) *Let* $\underline{K}$ *belong to* $VC_l H(S; J_\infty; \Omega)$. *Suppose* Φ *belongs to* $L_{\infty l} H_p^\vee(\theta_-, \theta_+; S; J_\infty; \Omega)$ *and* $\Phi(\gamma, t) \to \Phi_\Omega(\gamma) e^{i\Omega t}$ *as* $t \to \infty$; *then* $\underline{K}\Phi$ *belongs to* $C_l H_p^\vee(\theta_-, \theta_+; S; J_\infty; \Omega)$ *and* $(\underline{K}\Phi)(\gamma, t) \to K_\Omega(\gamma) e^{i\Omega t}$ *as* $t \to \infty$.

The limits in points (3) *and* (4) *are understood as in Definition* I.2.14.

Proof. This is obtained by the corresponding modification of Theorems 9.6.2 and 9.6.4 in [1] using the norms and the limits with $\sup_{\gamma \in \overline{S}'}$. We demonstrate here the proof of point (3) only. The other points are proven in an analogous way.

The membership of $\underline{K}\Phi$ in the corresponding class without the symbols l, Ω follows from Theorem 2.8. The function $K_\Omega(\gamma)\Phi_\Omega(\gamma)$ belongs to the required class by Lemma 3.11. We have to prove now only the tendency (3.7) in the sense of Definition I.2.14, point (2), by a modification of the proof of Theorem 9.6.5 in [1].

Let us fix any $\overline{S}' = \overline{S}(\delta_0', \delta_\infty') \subset S$. For any given $\tilde{\varepsilon} > 0$, we can find numbers T_1, T_2 such that $T_1 > t_1$, $T_2 \geq T_1$, $M^0_{p\infty}[\Phi(\cdot, t) - \Phi_\Omega(\cdot) e^{i\Omega t}; \Phi; [T_1, \infty)] < \tilde{\varepsilon}$ for any $t \geq T_1$, and

$$\int_{t_1}^{T_1} \sup_{\gamma \in \overline{S}'} |K(\gamma, t, \tau)| \, d\tau \leq \tilde{\varepsilon},$$

$$\sup_{\gamma \in \overline{S}'} \left| \int_{t_1}^{t} K(\gamma, t, \tau) e^{i\Omega(\tau - t)} \, d\tau - K_\Omega(\gamma) \right| \leq \tilde{\varepsilon}$$

for any $t \geq T_2$. Then for any $t \geq T_2$ and any $\gamma \in \overline{S}'$, we get

$$\begin{aligned}
&\left|(\underline{K}\Phi)(\gamma, t) - K_\Omega(\gamma)\Phi_\Omega(\gamma) e^{i\Omega t}\right| \\
&\quad = \left| \int_{t_1}^{t} K(\gamma, t, \tau) \left[\Phi(\gamma, t) - \Phi_\Omega(\gamma) e^{i\Omega t}\right] d\tau \right. \\
&\qquad \left. + \left[\int_{t_1}^{t} K(\gamma, t, \tau) e^{i\Omega(\tau - t)} \, d\tau - K_\Omega(\gamma) \right] \Phi_\Omega(\gamma) e^{i\Omega t} \right| \\
&\quad \leq \int_{t_1}^{t} |K(\gamma, t, \tau)| \left|\Phi(\gamma, t) - \Phi_\Omega(\gamma) e^{i\Omega t}\right| d\tau \\
&\qquad + \left| \int_{t_1}^{t} K(\gamma, t, \tau) e^{i\Omega(\tau - t)} \, d\tau - K_\Omega(\gamma) \right| |\Phi_\Omega(\gamma)| \\
&\quad \leq \int_{t_1}^{T_1} |K(\gamma, t, \tau)| \left|\Phi(\gamma, t) - \Phi_\Omega(\gamma) e^{i\Omega \tau}\right| d\tau \\
&\qquad + \int_{T_1}^{t} |K(\gamma, t, \tau)| \left|\Phi(\gamma, t) - \Phi_\Omega(\gamma) e^{i\Omega \tau}\right| d\tau + \tilde{\varepsilon} |\Phi_\Omega(\gamma)|.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&M_p^0\big[(\underline{K}\Phi)(\cdot,t) - K_\Omega(\cdot)\Phi_\Omega(\cdot)e^{i\Omega t};\delta\big]\\
&\quad = \left[\int_{-\infty}^{\infty}\left|(\underline{K}\Phi)(\delta+i\xi,t) - K_\Omega(\delta+i\xi)\Phi_\Omega(\delta+i\xi)e^{i\Omega t}\right|^p d\xi\right]^{1/p}\\
&\quad \le \int_{t_1}^{T_1}\sup_{\gamma\in\bar S'}|K(\gamma,t,\tau)|\big[M_p^0(\Phi(\cdot,t);\delta) + M_p^0(\Phi_\Omega;\delta)\big]\,d\tau\\
&\qquad + \int_{T_1}^{t}\sup_{\gamma\in\bar S'}|K(\gamma,t,\tau)|M_p^0\big[\Phi(\cdot,t) - \Phi_\Omega(\cdot)e^{i\Omega\tau};\delta\big]\,d\tau\\
&\qquad + \tilde\varepsilon M_p^0(\Phi_\Omega;\delta)\\
&\quad \le \tilde\varepsilon\big[M_{p\infty}^0(\Phi;\delta;[T,\infty)) + M_p^0(\Phi_\Omega;\delta)\big] + |||K;\bar S';J_\infty|||\,\tilde\varepsilon\\
&\qquad + \tilde\varepsilon M_p^0(\Phi_\Omega;\delta).
\end{aligned}
$$

for $\delta\in[\delta_0',\delta_\infty']$. Hence,

$$
\begin{aligned}
&M_{p\infty}^0\big[(\underline{K}\Phi)(\gamma,t) - K_\Omega(\gamma)\Phi_\Omega(\gamma)e^{i\Omega t};\delta;[T,\infty)\big]\\
&\qquad \le \tilde\varepsilon\big[M_{p\infty}^0(\Phi;\delta;[T,\infty)) + 2M_p^0(\Phi_\Omega;\delta) + |||K;\bar S';J_\infty|||\big] \to 0
\end{aligned}
$$

as $T_1, T_2\to\infty$. The tendency is uniform with respect to $\delta\in[\delta_0',\delta_\infty']$ by the uniform boundedness of $M_{p\infty}^0(\Phi;\delta;[T,\infty))$ and $M_p^0(\Phi_\Omega;\delta)$. ▮

3.13. LEMMA. *Let $\underline{K}_1$, $\underline{K}_2$ belong to $VC_lH(S;J_\infty;\Omega)$, where $\underline{K}_1(\gamma)$ has a resolvent $\underline{R}_1(\gamma)$ from $VC_lH(S;J_\infty;\Omega)$, and $|||K_2;\bar S';J_\infty||| < 1/(1 + |||\underline{R}_1;\bar S';J_\infty|||)$ for any $\bar S'\subset S$. Then the operator $\underline{K}(\gamma) := \underline{K}_1(\gamma) + \underline{K}_2(\gamma)$ has a resolvent $\underline{R}(\gamma)$ from $VC_lH(S;J_\infty;\Omega)$, and $R_\Omega(\gamma) = [I + K_\Omega(\gamma)]^{-1}K_\Omega(\gamma)$.*

Proof. By Lemma 3.4, we have that $\underline{R}$ belongs to $VCH(S;J_\infty)$,

$$\underline{R}(\gamma) = \underline{R}_1(\gamma) - \underline{\tilde R}(\gamma) + \underline{\tilde R}(\gamma)\underline{R}_1(\gamma),$$

$$\underline{\tilde R}(\gamma) = \lim_{n\to\infty}\underline{\tilde R}_n(\gamma),\qquad \underline{\tilde R}_n(\gamma) = \sum_{j=1}^{n}(-1)^j\big[\underline{K}_2(\gamma) - \underline{R}_1(\gamma)\underline{K}_2(\gamma)\big]^j,$$

and the limit exists in the sense of kernel norm (3.6). The lemma claims follow then from points (1) and (2) of Theorem 3.12. ▮

3.14. LEMMA. *If $\underline{K}^-$ belongs to $VCH(S;J_\infty)$ and is an operator of the convolution type, then $\underline{K}^-$ belongs to $VC_lH(S;J_\infty;\Omega)$ for any real Ω and $K_\Omega^-(\gamma) = \hat K^-(\gamma,i\Omega)$ (see Theorem 3.3).*

Proof. Due to condition (ii) of Definition 2.1 for $\underline{K}^{-}(\gamma)$, we have

$$\begin{aligned} |||\, K^{-}; \overline{S}'; J_{\infty} ||| &:= \int_0^{\infty} \sup_{\gamma\in \overline{S}'} |K^{-}(\gamma,\tau)|\, d\tau \\ &= \left\| \sup_{\gamma\in \overline{S}'} |K^{-}(\gamma,\cdot)|; L(0,\infty) \right\| < \infty, \qquad \forall \overline{S}' \subset S. \quad (3.8) \end{aligned}$$

Then

$$\begin{aligned} \int_{t_1}^{T} \sup_{\gamma\in \overline{S}'} |K^{-}(\gamma, t-\tau)|\, d\tau &= \int_{t-T}^{t-t_1} \sup_{\gamma\in \overline{S}'} |K^{-}(\gamma,\tau)|\, d\tau \\ &= \left\| \sup_{\gamma\in \overline{S}'} |K^{-}(\gamma,\cdot)|; L(t-T, t-t_1) \right\| \to 0, \\ & \qquad t\to\infty \end{aligned}$$

for any $T \in J_{\infty}$ and any $\overline{S}' \subset S$. Further,

$$\begin{aligned} K_{\Omega}^{-}(\gamma) &:= \lim_{t\to\infty} \int_{t_1}^{t} K^{-}(\gamma, t-\tau) e^{i\Omega(\tau - t)}\, d\tau \\ &= \lim_{t\to\infty} \int_0^{t-t_1} K^{-}(\gamma,\tau) e^{-i\Omega\tau}\, d\tau \to \int_0^{\infty} K^{-}(\gamma,\tau) e^{-i\Omega\tau}\, d\tau \\ &= \hat{K}^{-}(\gamma, i\Omega). \end{aligned}$$

Here the last integral converges uniformly with respect to γ on any $\overline{S}' \subset S$ for any real Ω by force of (3.8). ∎

3.15. Lemma. *Let an operator* $\underline{K}$ *satisfy conditions* (i) *and* (ii) *of Definition* 3.10 *and tend to an operator of the convolution type* $\underline{K}^{-} \in VCH(S; J_{\infty})$. *Then* $\underline{K}$ *belongs to* $VC_l H(S; J_{\infty}; \Omega)$ *for any real* Ω, *and* $K_{\Omega}(\gamma) = K_{\Omega}^{-}(\gamma) = \hat{K}^{-}(\gamma, i\Omega)$.

Proof. We have to prove only condition (iii) of Definition 3.10. Really, for any $T \in (t_1, t)$,

$$\begin{aligned} A &= \sup_{\gamma\in \overline{S}'} \left| \int_{t_1}^{t} K(\gamma, t, \tau) e^{i\Omega(\tau - t)}\, d\tau - K_{\Omega}^{-}(\gamma) \right| \\ &= \sup_{\gamma\in \overline{S}'} \left| \int_{t_1}^{t} [K(\gamma,t,\tau) - K^{-}(\gamma, t-\tau)] e^{i\Omega(\tau - t)}\, d\tau \right. \\ &\qquad \left. + \int_{t-t_1}^{\infty} K(\gamma,\tau) e^{-i\Omega\tau}\, d\tau \right| \end{aligned}$$

$$\leq \int_{t_1}^{T} \sup_{\gamma\in\bar{S}'} |K(\gamma, t, \tau)| \, d\tau + \int_{t_1}^{T} \sup_{\gamma\in\bar{S}'} |K^{-}(\gamma, t-\tau)| \, d\tau$$

$$+ \int_{T}^{t} \sup_{\gamma\in\bar{S}'} |K(\gamma, t, \tau) - K^{-}(\gamma, t-\tau)| \, d\tau + \int_{t-t_1}^{\infty} \sup_{\gamma\in\bar{S}'} |K(\gamma, \tau)| \, d\tau$$

$$\leq \int_{t_1}^{T} \sup_{\gamma\in\bar{S}'} |K(\gamma, t, \tau)| \, d\tau + \left\| \sup_{\gamma\in\bar{S}'} |K^{-}(\gamma, \cdot)|; L(t-T, t-t_1) \right\|$$

$$+ \, ||| K - K^{-}; \bar{S}'; [T, \infty) ||| + \left\| \sup_{\gamma\in\bar{S}'} |K^{-}(\gamma, \cdot)|; L(t-t_1, \infty) \right\|.$$

Suppose $\varepsilon > 0$. We choose a sufficiently large T such that the third term will be less than $\varepsilon/4$. Choose then sufficiently large t such that the first term is less than $\varepsilon/4$ (according to condition (ii) of Definition 3.10) and the second and the fourth terms are less than $\varepsilon/4$ (according to inequality (3.8)). Since ε is arbitrary, we get $A \to 0$ as $t \to \infty$.

3.16. THEOREM. *Let an operator* $\underline{K}$ *satisfy conditions* (i) *and* (ii) *of Definition* 3.10 *and tend to an operator of the convolution type* $\underline{K}^{-} \in VCH(S; J_\infty)$ *that satisfies the conditions of Theorem* 3.3. *Then both the operator* $\underline{K}$ *and its resolvent* $\underline{R}$ *belong to* $VC_lH(s; J_\infty; \Omega)$ *for any real* Ω, $K_\Omega(\gamma) = K^{-}_\Omega(\gamma) = \hat{K}^{-}(\gamma, i\Omega)$, *and* $R_\Omega(\gamma) = I - [I + K_\Omega(\gamma)]^{-1}$.

Proof. The claims for $\underline{K}$ are proved in Lemma 3.15. By Theorem 3.3, $\underline{R}$ belongs to $VCH(S; J_\infty)$. Repeating the proof of Theorem 3.6, we obtain the same representation (3.4) for the resolvent. Let us check condition (ii) of Definition 3.10 for each term of representation (3.4). The kernel K satisfies it by the theorem assumption. Let $t > T$. Then

$$\int_{t_1}^{T_0} \sup_{\gamma\in\bar{S}'} |(K * R_1)(\gamma, t, \tau)| \, d\tau$$

$$\leq \int_{t_1}^{T_0} \left[\int_{t_1}^{T} \sup_{\gamma\in\bar{S}'} |K(\gamma, t, \xi) R_1(\gamma, \xi, \tau)| \, d\xi \right] d\tau$$

$$\leq \int_{t_1}^{T} \left[\sup_{\gamma\in\bar{S}'} |K(\gamma, t, \xi)| \int_{t_1}^{T_0} \sup_{\gamma\in\bar{S}'} |R_1(\gamma, \xi, \tau)| \, d\tau \right] d\xi$$

$$\leq \int_{t_1}^{T} \sup_{\gamma\in\bar{S}'} |K(\gamma, t, \xi)| \, d\xi \, ||| R_1; \bar{S}'; [t_1, T_0) ||| \to 0, \qquad t \to \infty \quad (3.9)$$

for any $T_0 \geq t_1$ since K meets condition (ii) of Definition 3.10.

By Theorem 3.3, $\underline{R}^-$ belongs to $VCH(S; J_\infty)$. Hence, according to Lemma 3.14, $\underline{R}^-$ belongs to $VC_lH(S; J_\infty; \Omega)$, and consequently, according to Lemma 3.13, $\underline{R}_T$ belongs to $VC_lH(S; J_\infty; \Omega)$. It follows from point (2) of Theorem 3.12, that $\underline{\tilde{K}} := \underline{R}_T \underline{K}$ belongs to $VC_lH(S; J_\infty; \Omega)$, and conditions (ii) and (iii) of Definition 3.10 (where $\tilde{K}_\Omega(\gamma) = R_{T\Omega}(\gamma) K_\Omega(\gamma)$) are fulfilled for $\underline{\tilde{K}}$.

For the last term $R_T * K * R_1 = \tilde{K} * R_1$, we rewrite inequalities (3.9) replacing K by $\tilde{K}$. Thus point (ii) of Definition 3.10 has been proved for $\underline{R}(\gamma)$.

Let us prove point (iii). For $\underline{\tilde{K}}$ it has already been proved. For R_1 in representation (3.4), we have $R_1(\gamma, t, \tau) = 0$ when $t > T$ or $t > T$. Then as in (3.9), we have for any $t \geq T$

$$\sup_{\gamma \in \bar{S}'} \left| \int_{t_1}^{t} (K * R_1)(\gamma, t, \tau) e^{i\Omega(\tau - t)}\, d\tau \right|$$
$$\leq \int_{t_1}^{t} \sup_{\gamma \in \bar{S}'} \left| (K * R_1)(\gamma, t, \tau) \right| d\tau$$
$$\leq \int_{t_1}^{T} \sup_{\gamma \in \bar{S}'} \left| K(\gamma, t, \xi) \right| d\xi \, \||R_1; \bar{S}'; J_\infty\|| \to 0, \qquad t \to \infty.$$

Consequently, $\underline{K}\underline{R}_1$ satisfies Definition 3.10 and $(K * R_1)_\Omega(\gamma) = 0$. Similarly, $\underline{\tilde{K}}\underline{R}_1$ satisfies Definition 3.10 and $(\tilde{K} * R_1)_\Omega(\gamma) = 0$. Therefore $\underline{K}\underline{R}_1$ and $\underline{R}_T \underline{K}\underline{R}_1$ belong to $VC_lH(S; J_\infty; \Omega)$, and $\underline{R}$ belongs to $VC_lH(S; J_\infty; \Omega)$ due to (3.4).

From expression (2.3) and point (2) of Theorem 3.12, we have $K_\Omega(\gamma) R_\Omega(\gamma) = K_\Omega(\gamma) - R_\Omega(\gamma)$. Since the conditions of Theorem 3.3. hold, we get that $\det[I + K_\Omega(\gamma)] = \det[I + \hat{K}^-(\gamma, i\Omega)] \neq 0$ for any real Ω. Hence, the matrix $[I + K_\Omega(\gamma)]^{-1}$ exists, and $R_\Omega(\gamma) = [I + K_\Omega(\gamma)]^{-1} K_\Omega(\gamma) = I - [I + K_\Omega(\gamma)]^{-1}$. ∎

We note in conclusion that similar statements can be also given for a Volterra operator $\underline{K}(z)$ depending on the parameter z on a wedge (in place of $\underline{K}(\gamma)$ depending on the parameter γ on a strip).

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