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On Some Weighted Hardy Type Classes of One-Parametric Holomorphic Functions. I. Function Properties and Mellin Transforms

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Several classes of functions of two variables (one complex and one real) are considered. These functions belong to some weighted Hardy type classes with respect to the complex variable in a wedge or in a strip and are either essentially bounded, or bounded, or continuous in a sense with respect to the real parameter. Properties of such functions and of their Mellin images are investigated. Classes of functions of two real variables are also studied in order to describe boundary properties of the one-parametric holomorphic functions. The analysis is based on properties of the corresponding classes of functions of one variable, which are studied here too. © 1998 Academic Press

INTRODUCTION

For some two-dimensional boundary value problems, particularly, in the continuum mechanics (see, e.g., [14, 5]), solving methods, based on a representation of a general solution in terms of holomorphic functions (complex potentials) are rather popular. Such representations together with the Mellin transform are especially used for the stress singularity analysis near the corner point of an elastic wedge (see, e.g., [1, 2, 6–8]). To provide a rigorous analysis for these problems, it seems natural to consider a solution (as complex potentials) in some weighted Hardy type classes of

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functions of one complex variable in a wedge. Functions from these classes may possess weak singularities in the wedge apex, they satisfy some singularity estimates, and their Mellin images can be expressed in terms of the Mellin images of their boundary values. Such classes and the corresponding weighted Lebesgue type classes of functions of one real variable (for the boundary values) as well as weighted and usual Hardy type classes of functions in a strip (for the Mellin images) are constructed in Section 1 of this paper.

A pair of function classes is introduced first: $\hat{L}_p(\delta_0, \delta_\infty)$, being an intersection of Lebesgue's spaces on the half-axis with different power weights, and the Hardy type class $H_p^0(S)$ on a strip $S(\delta_0, \delta_\infty)$ of the complex plane. It is proved that the Mellin transform (with respect to the real variable) and its inversion map $\hat{L}_2(\delta_0, \delta_\infty)$ and $H_2^0(S(\delta_0, \delta_\infty))$ onto each other. Another pair of classes is also constructed: $H_p(\delta_0, \delta_\infty; W)$, being an intersection of Hardy type classes with power weights in a wedge $W(\theta_-, \theta_+)$ of the complex plane, and $H_p^\vee(\theta_-, \theta_+; S)$ being an intersection of the Hardy type classes with exponential weights on a strip $S(\delta_0, \delta_\infty)$. It occurs that the Mellin transform (with respect to a complex variable) and its inversion map $H_2(\delta_0, \delta_\infty; W(\theta_-, \theta_+))$ and $H_2^\vee(\theta_-, \theta_+; S(\delta_0, \delta_\infty))$ onto each other. Properties of functions from these classes are investigated. Particularly, some pointwise estimates are obtained for functions from the Hardy type classes in a strip or a wedge in terms of Hardy classes integral quantities. The boundary values of the complex Mellin transform images of functions belonging to $H_2(\delta_0, \delta_\infty; W)$ are expressed in terms of the real Mellin transform images of their boundary values. The results presented were applied in [13] to some problems of elasticity.

There are some more complicated problems in applications, when a general solution may be represented in terms of holomorphic functions depending additionally on a real parameter (time). For example, the two-dimensional problems of visco-elasticity (hereditary elasticity) are described by the operator Lamé equations being partial differential equations in space coordinates and integral equations in time. To investigate the stress singularities in such problems (see [9–11]), all results obtained in Section 1 of this paper must be extended to one-parametric holomorphic functions. Since solutions of the visco-elastic problems may be essentially bounded, bounded, or even continuous in time (depending on the boundary loadings smoothness), it is necessary to consider different function classes possessing these properties. In Section 2, classes are constructed for functions of one real or complex variable and of one additional real parameter, such that the functions belong to the corresponding classes given in Section 1 with respect to the first variable and are either essentially bounded, or bounded, or continuous in a sense with respect to the additional parameter. For example, the classes $L_\infty \hat{L}_p(\delta_0, \delta_\infty; J)$, $B\hat{L}_p(\delta_0, \delta_\infty; J)$, $C\hat{L}_p(\delta_0, \delta_\infty; J)$ of functions $g(\rho, t)$ are the corresponding counterparts of the class $\hat{L}_p(\delta_0, \delta_\infty)$ of functions $g(\rho)$, where J is a finite segment or a half-infinite interval.

It is important in visco-elasticity to consider the solution behavior not only at finite time instants but also at time t tending to infinity. In the last case the solution may be in a sense either bounded, or tend to a function harmonically oscillating in time, or tend to a definite limit if the applied loadings possess such properties. To consider these particular cases, the corresponding subclasses of functions having harmonically oscillating limits are constructed in Section 2. For example, the class $L_\infty \hat{L}_p(\delta_0, \delta_\infty; J_\infty)$ includes the subclass $L_{\infty l} \hat{L}_p(\delta_0, \delta_\infty; J_\infty; \Omega)$ of functions tending to ones harmonically oscillating in t with a frequency Ω. All results of Section 1 are extended to these classes.

Some properties of partial Volterra operators in a real variable are investigated in [12] on a finite segment as well as on the half-axis for functions from the classes introduced in Section 2.

The analysis of this paper is essentially based on the results in [18, 3]. Particularly, the complex Mellin transform was considered in [18] for a set of functions satisfying some power estimates at the wedge apex and at infinity. Note, that such sets are too wide and do not possess the needed boundary properties, in contrast to their subsets $H_2(\delta_0, \delta_\infty; W)$ introduced in this paper. Some proprties of the Mellin image of functions from Hardy type classes without weight on a wedge were considered in [3]; however, the functions from these classes cannot have sufficiently arbitrary singularities at the wedge apex.

1. BASIC NOTATIONS, SOME CLASSES OF FUNCTIONS OF ONE VARIABLE, AND MELLIN TRANSFORMS

Let us introduce an intersection of the weighted Lebesgue spaces on the half-axis, where the weight power runs through an interval, and describe some properties of this class.

1.1. DEFINITION. (1) The space $\tilde{L}_p(\delta; 0, \infty)$ consists of functions $g(p)$ defined on the half-axis $0 < \rho < \infty$ with the finite norm $\|g; \delta\|_p := [\int_0^\infty |g(\rho)\rho^\delta|^p \rho^{-1}\, d\rho]^{1/p} < \infty$.

(2) The class $\hat{L}_p(\delta_0, \delta_\infty)$ $(\delta_0 < \delta_\infty)$ consists of the functions $g(\rho)$ belonging to $\tilde{L}_p(\delta; 0, \infty)$ for all $\delta \in (\delta_0, \delta_\infty)$, i.e., $\hat{L}_p(\delta_0, \delta_\infty) := \bigcap_{\delta_0 < \delta < \delta_\infty} \tilde{L}_p(\delta; 0, \infty)$.

Particularly, $\hat{L}_p(\delta_0, \delta_\infty)$ includes all functions $g(\rho)$ locally integrable on $(0, \infty)$ with the power p and such that the estimates $|g(\rho)| < C_0\,\rho^{-\delta_0-\varepsilon}\cdot(\rho \to 0)$, $|g(\rho)| < C_\infty\,\rho^{-\delta_\infty+\varepsilon}(\rho \to \infty)$ hold for any $\varepsilon > 0$.

We shall below suppose $1 \le p \le \infty$ unless otherwise stated.

1.2. LEMMA. *Let $g(\rho) \in \hat{L}_p(\delta_0, \delta_\infty)$, $\delta_0 < \delta_\infty$. Then $\|g;\delta\|_p^p \le \|g;\delta_0'\|_p^p + \|g;\delta_\infty'\|_p^p$ for any $\delta \in [\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$. In particular, the norm $\|g;\delta\|_p$ is uniformly bounded with respect to δ on any internal segment of $(\delta_0, \delta_\infty)$.*

Proof. $\|g;\,\delta\|_p^p = \int_0^1 |g(\rho)\rho^\delta|^p \rho^{-1}\,d\rho + \int_1^\infty |g(\rho)\rho^\delta|^p \rho^{-1}\,d\rho \le \int_0^1 |g(\rho)\rho^{\delta_0'}|^p \rho^{-1}\,d\rho + \int_1^\infty |g(\rho)\rho^{\delta_\infty'}|^p \rho^{-1}\,d\rho \le \|g;\delta_0'\|_p^p + \|g;\delta_\infty'\|_p^p$. ∎

Although the space $\tilde{L}_p(\delta; 0, \infty)$ is not embedded into $\tilde{L}_q(\delta; 0, \infty)$ for $q < p$, it is the case for the classes $\hat{L}_p(\delta_0, \delta_\infty)$ and $\hat{L}_q(\delta_0, \delta_\infty)$. Thus, we have

1.3. LEMMA. *If $1 \le q < p \le \infty$, then $\hat{L}_p(\delta_0, \delta_\infty) \subset \hat{L}_q(\delta_0, \delta_\infty)$.*

Proof. For any $\delta \in (\delta_0, \delta_\infty)$, there is $\varepsilon > 0$ such that $\delta \pm \varepsilon \in (\delta_0, \delta_\infty)$. Then

$$\|g;\delta\|_q^q = \int_0^1 |g(\rho)\rho^{\delta-\varepsilon}|^q \rho^{\varepsilon q}\rho^{-1}\,d\rho + \int_1^\infty |g(\rho)\rho^{\delta+\varepsilon}|^q \rho^{-\varepsilon q}\rho^{-1}\,d\rho$$

$$\le \left[\int_0^1 |g(\rho)\rho^{\delta-\varepsilon}|^p \rho^{-1}\,d\rho\right]^{q/p} \left[\int_0^1 \rho^{\varepsilon pq/(p-q)}\rho^{-1}\,d\rho\right]^{(p-q)/p}$$

$$+ \left[\int_1^\infty |g(\rho)\rho^{\delta+\varepsilon}|^p \rho^{-1}\,d\rho\right]^{q/p} \left[\int_1^\infty \rho^{-\varepsilon pq/(p-q)}\rho^{-1}\,d\rho\right]^{(p-q)/p}$$

$$\le \left[\frac{p-q}{\varepsilon pq}\right]^{(p-q)/p} \left(\|g;\delta-\varepsilon\|_p^q + \|g;\delta+\varepsilon\|_p^q\right)$$

$$\le 2\left[\frac{p-q}{\varepsilon pq}\right]^{(p-q)/p} \left(\|g;\delta-\varepsilon\|_p + \|g;\delta+\varepsilon\|_p\right)^q.$$

Here the Hölder inequality is used with the measure $\rho^{-1}\,d\rho = d(\ln\rho)$. ∎

Let us introduce now some Hardy type classes of holomorphic functions in a strip or in a wedge and consider their properties. We shall use below the complex variables $z = \rho e^{i\theta}$, $\gamma = \delta + i\xi$, $\zeta = x + iy$, where $\rho, \theta, \delta, \xi, x, y$ are real. We denote an open wedge $(0 < \rho < \infty, \theta_- < \theta < \theta_+)$ on the complex z-plane as $W(\theta_-, \theta_+)$ and an open strip $(-\infty < \operatorname{Im}\gamma < \infty, \delta_0 < \operatorname{Re}\gamma < \delta_\infty)$ on the complex γ-plane as $S(\delta_0, \delta_\infty)$; $\overline{W}(\theta_-, \theta_+)$ is the closed wedge $(0 < \rho < \infty, \theta_- \le \theta \le \theta_+)$; $\overline{S}(\delta_0, \delta_\infty)$ is the closed strip $(-\infty < \operatorname{Im}\gamma < \infty, \delta_0 \le \operatorname{Re}\gamma \le \delta_\infty)$.

First we describe some properties of the classical Hardy classes of holomorphic functions in a strip.

1.4. DEFINITION. (1) The class $H_p^0(S)$ consists of the functions $\Phi(\zeta)$ holomorphic on $S = S(a,b)$ such that the quantity

$$M_p^0(\Phi; x) := \left[\int_{-\infty}^{\infty} |\Phi(x+iy)|^p \, dy\right]^{1/p}$$

is uniformly bounded with respect to x on any segment $[a', b'] \subset (a,b)$.

(2) The class $H_p^*(S)$ consists of the functions $\Phi \in H_p^0(S)$ such that $M_p^*(\Phi; S) := \sup_{a<x<b} M_p^0(\Phi; X) < \infty$.

1.5. *Remark*. Obviously, $H_p^*(S) \subset H_p^0(S)$. If $\Phi \in H_p^*(S(a',b'))$ for any segment $[a',b'] \subset (a,b)$, then $\Phi \in H_p^0(S(a,b))$, and vice versa if $\Phi \in H_p^0(S(a,b))$, then $\Phi \in H_p^*(S(a',b'))$ for any segment $[a',b'] \subset (a,b)$.

Let us prove the boundedness of a function from $H_p^*(S)$ (and by Remark 1.5 even from $H_p^0(S)$) on any strictly internal strip and the membership of all derivatives of a function from $H_p^0(S)$ to the same class.

1.6. LEMMA. *Let $\Phi \in H_p^*(S)$, $S = S(a,b)$, $\bar{S}' := \bar{S}(a',b') \subset S$, and $r = \min(a-a', b-b')$. Then*:

$$\text{(i)}\quad |\Phi(\zeta)| \le \left(\frac{2}{\pi r}\right)^{1/p} M_p^*(\Phi; S), \qquad \zeta \in S'; \tag{1.1}$$

$$\text{(ii)}\quad \frac{d}{d\zeta}\Phi(\zeta) \in H_p^0(S) \qquad \textit{and}$$

$$M_p^0\left(\frac{d}{d\zeta}\Phi; x\right) \le M_p^*\left(\frac{d}{d\zeta}\phi; S'\right) \le \frac{1}{r} M_p^*(\Phi; S),\ x \in [a', b'].$$

Proof. (i) Let $\zeta = x + iy \in S'$, then

$$\Phi(\zeta) = \frac{1}{2\pi i}\int_{|\eta-\zeta|=\rho} \frac{\Phi(\eta)}{\eta-\zeta}\, d\eta = \frac{1}{2\pi}\int_0^{2\pi} \Phi(\zeta + \rho e^{i\theta})\, d\theta \tag{1.2}$$

by the Cauchy theorem. Integrating (1.2) with respect to $\rho\, d\rho$, we obtain

$$r^2\Phi(\zeta) = \frac{1}{\pi}\int_0^r\int_0^{2\pi} \Phi(\zeta + \rho e^{i\theta})\, d\theta\, \rho\, d\rho = \frac{1}{\pi}\iint_D \Phi(\eta)\, dD,$$

where D is the disc $|\eta - \zeta| \le r$. Let $S'' := S(x-r, x+r)$, then $\bar{S}'' \subset S$. Using the Hölder inequality, we get

$$|\Phi(\zeta)| \le \frac{1}{\pi r^2}\left[\iint_D |\Phi(\eta)|^p\, dD\right]^{1/p} (\pi r^2)^{1-1/p} \le \left[\frac{1}{\pi r^2}\iint_{S''} |\Phi(\eta)|^p dD\right]^{1/p}$$
$$\le \left[\frac{1}{\pi r^2}\int_{x-r}^{x+r} M_p^0(\Phi; x)\, dx\right]^{1/p} \le \left(\frac{2}{r\pi}\right)^{1/p} M_p^*(\Phi; S).$$

Point (i) has been proved. Now we prove point (ii). Let $\zeta \in S'$. Then by the Cauchy theorem and the generalized Minkowski inequality (see, e.g., [15, point 1.3.2]), we receive

$$\frac{d}{d\zeta}\Phi(\zeta) = \frac{1}{2\pi i}\int_{|\eta-\zeta|=r} \frac{\Phi(\eta)}{(\eta-\zeta)^2}\, d\eta = \frac{1}{2\pi r}\int_0^{2\pi} \Phi(\zeta + re^{i\theta})e^{-i\theta}\, d\theta,$$

$$M_p^0\left(\frac{d}{d\zeta}\Phi; x\right) = \frac{1}{2\pi r}\left[\int_{-\infty}^{\infty}\left|\int_0^{2\pi}\Phi(x+iy+re^{i\theta})e^{-i\theta}\, d\theta\right|^p dy\right]^{1/p}$$
$$\le \frac{1}{2\pi r}\int_0^{2\pi}\left[\int_{-\infty}^{\infty}|\Phi(x+iy+re^{i\theta})|^p\, dy\right]^{1/p} d\theta$$
$$= \frac{1}{2\pi r}\int_0^{2\pi} M_p^0(\Phi; x + r\cos\theta)\, d\theta \le \frac{1}{r}M_p^*(\Phi; S).$$

The estimate of point (ii) has been obtained. Since it is true for any ζ on each $\bar{S}' \subset S$ and is uniform with respect to $x \in [a', b']$, we have $(d/d\zeta)\Phi(\zeta) \in H_p^0(S)$. ∎

Note that the estimate similar to (1.1) for $p = 2$ is contained in [4, Lemma 1.6] (see also [17, point 1.1.4, Lemma 3]), where the result of [16, Theorem III] is refined.

It follows from [18, points 1.29, 3.17] and from the imbedding $\hat{L}_2(\delta_0, \delta_\infty) \subset \hat{L}_1(\delta_0, \delta_\infty)$ that the classes $\hat{L}_2(\delta_0, \delta_\infty)$ and $H_2^0(S(\delta_0, \delta_\infty))$ are mapped onto each other by the direct and inverse Mellin transform, that is, we have the following

1.7. THEOREM. *Let* $\delta_0 < \delta_\infty$, $S = S(\delta_0, \delta_\infty)$.

(I) *For any function* $g(\rho)$ *from* $\hat{L}_2(\delta_0, \delta_\infty)$, *the Mellin transform*

$$(\mathscr{M}g)(\gamma) := \int_0^\infty g(\rho)\rho^{\gamma-1}\, d\rho$$

generates an image

$$\langle g\rangle(\gamma) = (\mathscr{M}g)(\gamma) \tag{1.3}$$

belonging to $H_2^0(S)$, *and*

$$M_2^0(\langle g\rangle;\delta) = (2\pi)^{1/2}\|g;\delta\|_2. \tag{1.4}$$

The inverse Mellin transform has the form

$$[\mathscr{M}^{-1}(S)\langle g\rangle](\rho) := \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\langle g\rangle(\gamma)\rho^{-\gamma}\,d\gamma \qquad (\delta_0<\delta<\delta_\infty),$$

and

$$g(\rho) = [\mathscr{M}^{-1}(S)\langle g\rangle](\rho) \tag{1.5}$$

at the points ρ, *where the function* $g(\rho)$ *is continuous and has a finite variation*.

(II) *For any function* $\langle g\rangle(\gamma)$ *from* $H_2^0(S)$, *the function* $g(\rho)$ *given by* (1.5) *belongs to* $\hat{L}_2(\delta_0, \delta_\infty)$ *and equalities* (1.4), (1.3) *hold*.

We consider now some weighted Hardy type classes in a wedge.

1.8. Definition. Let $W = W(\theta_-, \theta_+)$. The class $H_p(\delta_0, \delta_\infty; W)$ consists of the functions $h(z)$ holomorphic on W such that the quantity

$$M_p(h;\delta;W) := \sup_{\theta_-<\theta<\theta_+}\left[\int_0^\infty |h(\rho e^{i\theta})\rho^\delta|^p\rho^{-1}\,d\rho\right]^{1/p}$$

is bounded for any $\delta\in(\delta_0,\delta_\infty)$.

1.9. *Remark*. If $h(z)\in H_p(\delta_0,\delta_\infty;W)$, then $h(\rho e^{i\theta})\in\hat{L}_p(\delta_0,\delta_\infty)$ with respect to ρ for any $\theta\in(\theta_-,\theta_+)$. Hence by Lemma 1.2, the quantity $M_p(h;\delta;W)$ is uniformly bounded with respect to δ on any segment $[\delta_0', \delta_\infty']\subset(\delta_0,\delta_\infty)$. In addition, it follows from Lemma 1.3 that $H_p(\delta_0,\delta_\infty;W)\subset H_q(\delta_0,\delta_\infty;W)$ if $1\le q<p\le\infty$.

Let us prove an analogue of Lemma 1.6 for the class $H_p(\delta_0,\delta_\infty;W)$.

1.10. Lemma. *Let* $h(z)\in H_p(\delta_0,\delta_\infty;W)$, $W=W(\theta_-,\theta_+)$, $\overline{W}' := \overline{W}(\theta_-',\theta_+')\subset W$, *and* $r=\min(\theta_-'-\theta_-, \theta_+-\theta_+')$. *Then for any segment* $[\delta_0',\delta_\infty']\subset(\delta_0,\delta_\infty)$,

(i) $$|h(z)|\le\tilde{\tilde{M}}(h;\delta_0',\delta_\infty';W')|z|^{-\delta}, \qquad \{z,\delta\}\in W'\times[\delta_0',\delta_\infty'], \tag{1.6}$$

$$\tilde{\tilde{M}}(h;\delta_0',\delta_\infty';W') := \left(\frac{2}{\pi r}\right)^{1/p}\sup_{\delta_0'\le\delta_*\le\delta_\infty'} M_p(h;\delta_*;W);$$

(ii) $$z\frac{d}{dz}h(z)\in H_p(\delta_0,\delta_\infty;W') \qquad \textit{and}$$

$$M_p\left(z\frac{d}{dz}h;\delta;W'\right)\le\left(|\delta|+\frac{1}{r}\right)M_p(h;\delta;W),\ \delta\in[\delta_0',\delta_\infty'].$$

Proof. By the conformal mapping $z = e^{i\zeta}$, the wedge W transforms onto the strip $S := S(\theta_-, \theta_+)$ while W' transforms onto $S' := S(\theta'_-, \theta'_-)$. Since $h(z) \in H_p(\delta_0, \delta_\infty; W)$, we have $\Phi_\delta(\zeta) := h(z(\zeta))e^{i\zeta\delta} \in H_p^*(S)$ and, moreover, $M_p^*(\Phi_\delta; S) = M_p(h; \delta; W)$ for any $\delta \in (\delta_0, \delta_\infty)$. Using estimate (1.1) from lemma 1.6, taking into account that, according to Remark 1.9, the constant $M_p(h; \Delta; W)$ is uniformly bounded with respect to δ on any segment $[\delta'_0, \delta'_\infty] \subset (\delta_0, \delta_\infty)$, and applying the inverse mapping, we obtain estimate (1.6).

To prove point (ii), we take into account $z[(d/dz)h(z)]z^\delta = -\delta\Phi_\delta(\zeta) - i(d/d\zeta)\Phi_\delta(\zeta)$. By point (ii) of Lemma 1.6, we get $(d/d\zeta)\Phi_\delta(\zeta) \in H_p^0(S)$ and $M_p^*((d/d\zeta)\Phi_\delta; S') \leq (1/r)M_p^*(\Phi_\delta; S) = (1/r)M_p(h; \delta; W)$. Hence $M_p(z(d/dz)h; \delta; W') = M_p^*[\delta\Phi_\delta(\zeta) - i(d/d\zeta)\Phi_\delta; S'] \leq (|\delta| + (1/r))M_p(h; \delta; W) < \infty$ for $\delta \in [\delta'_0, \delta'_\infty]$. ∎

Consider now some weighted Hardy type classes in a strip.

1.11. Definition. The class $H_p^\vee(\theta_-, \theta_+; S)$ consists of functions $\Phi(\gamma)$ holomorphic on $S = S(\delta_0, \delta_\infty)$ such that the quantity

$$M_p^\vee(\Phi; \theta_-, \theta_+; \delta) := \sup_{\theta_- < \theta < \theta_+} \left[\int_{-\infty}^{\infty} |\Phi(\delta + i\xi)e^{\xi\theta}|^p \, d\xi\right]^{1/p}$$

is uniformly bounded with respect to δ on any $[\delta'_0, \delta'_\infty] \subset (\delta_0, \delta_\infty)$.

1.12. *Remark.* Obviously, $H_p^\vee(-\theta', \theta'; S) \subset H_p^0(S)$ for any $\theta' > 0$.

Give once more an analogue of Lemma 1.6 for the class $H_p^\vee(\theta_-, \theta_+; S)$.

1.13. Lemma. *Let* $\Phi \in H_p^\vee(\theta_-, \theta_+; S)$, $S = S(\delta_0, \delta_\infty)$, $\bar{S}' := \bar{S}(\delta'_0, \delta'_\infty) \subset S$, *and* $r = \min(\delta_0 - \delta'_0, \delta_\infty - \delta'_\infty)/2$, $\delta''_0 := (\delta_0 + \delta'_0)/2$, $\delta''_\infty := (\delta_\infty + \delta'_\infty)/2$. *Then*:

$$\text{(i)} \quad |\Phi(\gamma)| \leq \tilde{\tilde{M}}^\vee(\Phi; \theta_-, \theta_+; S')|e^{i\gamma\theta}|, \qquad \{\gamma, \theta\} \in S' \times (\theta_-, \zeta_+), \tag{1.7}$$

$$\tilde{\tilde{M}}^\vee(\Phi; \theta_-, \theta_+; S') := \left(\frac{2}{\pi r}\right)^{1/p} \sup_{\delta''_0 < \delta < \delta''_\infty} M_p^\vee(\Phi; \theta_-, \theta_+; \delta);$$

$$\text{(ii)} \quad \frac{d}{d\gamma}(\gamma) \in H_p^\vee(\theta_-, \theta_+; S) \qquad \textit{and}$$

$$M_p^\vee\left(\frac{d}{d\gamma}\Phi; \theta_-, \theta_+; \delta\right) \leq \left(|\theta_-| + |\theta_+| + \frac{1}{r}\right)$$

$$\times \sup_{\delta_0'' < \delta_* < \delta_\infty''} M_p^{\vee}(\Phi; \theta_-, \theta_+; \delta_*), \ \delta \in [\delta_0', \delta_\infty'].$$

Proof. Since $\Phi \in H_p^{\vee}(\theta_-, \theta_+; S)$, we have $\Phi_\theta(\gamma) := \Phi(\gamma)e^{-i\theta\gamma} \in H_p^0(S)$ for any $\theta \in (\theta_-, \theta_+)$. Let $S'' := S(\delta_0'', \delta_\infty'')$. According to Remark 1.5, we have $\Phi_\theta(\gamma) \in H_p^*(S'')$ and $M_p^*(\Phi_\theta; S'') \le \sup_{\delta_0'' < \delta < \delta_\infty''} M_p^{\vee}(\Phi; \theta_-, \theta_+; \delta) < \infty$. Using estimate (1.1), we obtain estimate (1.7).

To prove point (ii), we take into account that $(d/d\gamma)\Phi(\gamma) = [i\theta\Phi_\theta(\gamma) + (d/d\gamma)\Phi_\theta(\gamma)]e^{i\theta\gamma}$. By point (ii) of Lemma 1.6, we have $(d/d\gamma)\Phi_\theta(\gamma) \in H_p^0(S'')$ and $M_p^0((d/d\gamma)\Phi_\theta;\ \delta) \le (1/r)M_p^*(\Phi_\theta;\ S'') \le (1/r) \sup_{\delta_0'' < \delta < \delta_\infty''} M_p^{\vee}(\Phi; \theta_-, \theta_+; \delta)$ for $\delta \in [\delta_0', \delta_\infty']$. Therefore

$$M_p^{\vee}\left(\frac{d}{d\gamma}\Phi; \theta_-, \theta_+; \delta\right) = \sup_{\theta_- < \theta < \theta_+} M_p^0\left[i\theta\Phi_\theta(\gamma) + \frac{d}{d\gamma}\Phi_\theta; \delta\right]$$

$$\le \left(|\theta_-| + |\theta_+| + \frac{1}{r}\right) \sup_{\delta_0'' < \delta_* < \delta_\infty''} M_p^{\vee}(\Phi; \theta_-, \theta_+; \delta_*) < \infty.$$

Since the estimate is valid for any $\overline{S}' \subset S$ and is uniform with respect to $\delta \in [\delta_0', \delta_\infty']$, we have $(d/d\gamma)\Phi(\gamma) \in H_p^{\vee}(\theta_-, \theta_+; S)$. ∎

Let us prove a statement about a product of two holomorphic functions, important for applications in the stress singularity analysis (see, e.g., [10, 11, 13]).

1.14. LEMMA. *Let $\Phi \in H_p^0(S)$. Suppose a function $\Phi_1(\gamma)$ is holomorphic on S and there is a number $\tilde{\tilde{M}}^{\vee}(\Phi_1; S') < \infty$ such that $|\Phi_1(\gamma)| \le \tilde{\tilde{M}}^{\vee}(\Phi_1; S')|e^{i\gamma\theta}|$, $\{\gamma, \theta\} \in S' \times (\theta_-, \theta_+)$ for an interval (θ_-, θ_+) and for every $\overline{S}' \subset S$. Then $\Phi\Phi_1 \in H_p^{\vee}(\theta_-, \theta_+; S)$.*

Proof. Let us fix $\overline{S}' = \overline{S}(\delta_0', \delta_\infty') \subset S = S(\delta_0, \delta_\infty)$ and $\delta \in [\delta_0', \delta_\infty']$. Let $\delta_0'' := (\delta_0 + \delta_0')/2$, $\delta_\infty'' := (\delta_\infty + \delta_\infty')/2$, $S'' := S(\delta_0'', \delta_\infty'')$; then $\overline{S}'' \subset S$, $\overline{S}' \subset S''$. Hence

$$M_p^{\vee}(\Phi\Phi_1; \theta_-, \theta_+; \delta) = \sup_{\theta_- < \theta < \theta_+} \left[\int_{-\infty}^{\infty} |\Phi(\delta + i\xi)\Phi_1(\delta + i\xi)e^{\xi\theta}|^p \, d\xi\right]^{1/p}$$

$$\le \tilde{\tilde{M}}^{\vee}(\Phi_1; S'')\left[\int_{-\infty}^{\infty} |\Phi(\delta + i\xi)|^p \, d\xi\right]^{1/p}$$

$$= \tilde{\tilde{M}}^{\vee}(\Phi_1; S'')M_p^0(\Phi; \delta),$$

where, according to point (1) of Definition 1.4, the quantity $M_p^0(\Phi, \delta)$ is uniformly bounded with respect to δ on any segment $[\delta_0', \delta_\infty'] \in (\delta_0, \delta_\infty)$. ∎

Now we shall prove that $H_2(\delta_0, \delta_\infty; W(\theta_-, \theta_+))$ and $H_2^\vee(\theta_-, \theta_+; S(\delta_0, \delta_\infty))$ are mapped onto each other by the direct and inverse Mellin transform in a complex variable. For $z = \rho e^{i\theta} \in W(\theta_-, \theta_+)$, we shall denote by z^γ the branch $e^{(\ln \rho + i\theta)\gamma}$, where $\theta_- \le \theta \le \theta_+$.

1.15. THEOREM. *Let $\theta_- < \theta_+$, $\delta_0 < \delta_\infty$, $W = W(\theta_-, \theta_+)$, $S = S(\delta_0, \delta_\infty)$.*

(I) *If $h(z) \in H_2(\delta_0, \delta_\infty; W)$, then its Mellin transform with respect to the complex argument z*

$$[\mathfrak{M}(W)h](\gamma) := \int_0^\infty h(z) z^{\gamma-1}\, dz, \qquad z \in W$$

generates an image

$$h^\vee(\gamma) = [\mathfrak{M}(W)h](\gamma), \tag{1.8}$$

which is independent of the integration contour in W and belongs to $H_2^\vee(\theta_-, \theta_+; S)$,

$$M_2^\vee\left(h^\vee; \theta_-, \theta_+; \delta\right) = (2\pi)^{1/2} M_2(h; \delta; W) \qquad (\delta_0 < \delta < \delta_\infty), \tag{1.9}$$

and

$$h(z) = \left[\mathfrak{M}^{-1}(S) h^\vee\right](z) := \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} h^\vee(\gamma) z^{-\gamma}\, d\gamma \qquad (\delta_0 < \delta < \delta_\infty). \tag{1.10}$$

(II) *Vice versa, if a function $h^\vee(\gamma) \in H_2^\vee(\theta_-, \theta_+; S)$, then the function $h(z)$ defined by* (1.10) *belongs to $H_2(\delta_0, \delta_\infty; W)$, is independent of δ, and* (1.9), (1.8) *hold.*

Proof. (I) Let $h(z) \in H_2(\delta_0, \delta_\infty; W)$. Due to point (i) of Lemma 1.10, $h(z)$ satisfies the conditions of [18, Theorem 31] and consequently $h^\vee(\gamma)$ is independent of the integration contour, is holomorphic on S, and (1.10) holds. Then the usual Mellin image $\langle h \rangle(\gamma, \theta)$ of a holomorphic function $h(z)$ with respect to ρ along a ray $z = \rho e^{i\theta}$ with a fixed angle $\theta \in (\theta_-, \theta_+)$ is represented in the form

$$\langle h \rangle(\gamma, \theta) := \int_0^\infty h(\rho e^{i\theta}) \rho^{\gamma-1}\, d\rho = e^{-i\gamma\theta} h^\vee(\gamma).$$

Taking into account the Plancherel theorem (see, e.g., [18, point 3.17]), we get

$$\begin{aligned} M_2^{\vee\,2}\big(h^\vee;\theta_-,\theta_+;\delta\big) &= \sup_{\theta_-<\theta<\theta_+}\int_{-\infty}^{\infty}|h^\vee(\delta+i\xi)e^{\xi\theta}|^2\,d\xi \\ &= \sup_{\theta_-<\theta<\theta_+}\int_{-\infty}^{\infty}|\langle h\rangle(\delta+i\xi,\theta)e^{i(\delta+i\xi)\theta}e^{\xi\theta}|^2\,d\xi \\ &= \sup_{\theta_-<\theta<\theta_+}\int_{-\infty}^{\infty}|\langle h\rangle(\delta+i\xi,\theta)|^2\,d\xi \\ &= 2\pi\sup_{\theta_-<\theta<\theta_+}\int_0^{\infty}|h(\rho e^{i\theta})\rho^{\delta}|^2\rho^{-1}\,d\rho \\ &= 2\pi M_2^2(h;W;\delta)<\infty. \end{aligned} \tag{1.11}$$

Here $M_2(h;W;\delta)$ is uniformly bounded with respect to δ on any segment $[\delta_0',\delta_\infty']\subset(\delta_0,\delta_\infty)$ (see Remark 1.9).

(II) Let $h^\vee\in H_2^\vee(\theta_-,\theta_+;S)$. By virtue of point (i) of Lemma 1.13, $h^\vee(\gamma)$ satisfies the conditions of the second part of Theorem 31 in [18], and, consequently, $h(z)$ is independent of $\delta\in(\delta_0,\delta_\infty)$, is holomorphic on W, and (1.8) holds. Repeating relationships (1.11) in the reverse order we obtain (1.9). ∎

Let us consider the boundary properties of functions from $H_2(\delta_0,\delta_\infty;W)$ and prove that the Mellin transform in a complex variable acting on such functions may be represented in terms of the Mellin transform in a real variable acting on their boundary values.

1.16. Lemma. *Let $W=W(\theta_-,\theta_+)$. If $h(z)\in H_2(\delta_0,\delta_\infty;W)$, then*:

(i) *the functions* $h^\pm(\rho):=\lim\limits_{\theta\to\theta_\pm}h(\rho e^{i\theta})$ *exist for almost all*

$\rho\in(0,\infty)$ *and belong to* $\hat{L}_2(\delta_0,\delta_\infty)$,

$\lim\limits_{\theta\to\theta_\pm}\|h(\rho e^{i\theta})-h^\pm(\rho);\delta\|_2=0$ *for any* $\delta\in(\delta_0,\delta_\infty)$; *and*

(ii)
$$[\mathfrak{M}(W)h](\gamma)=\exp(i\gamma\theta_\pm)[\mathscr{M}h^\pm](\gamma),\qquad \gamma\in S(\delta_0,\delta_\infty). \tag{1.12}$$

Proof. Point (i) is a direct consequence of [3, Theorem 7.5] if we apply it to the function $h(z)z^\delta$. From [3, Lemma 8.7] the proof of point (ii) follows for almost all points γ from each line $\operatorname{Re}\gamma=\delta\in(\delta_0,\delta_\infty)$. But, according to Theorems 1.7, 1.15, the functions $[\mathfrak{M}(W)h](\gamma)$ and $[\mathscr{M}h^{\pm}](\gamma)$ are holomorphic in $S(\delta_0, \delta_\infty)$ and, consequently, (1.12) is fulfilled everywhere on $S(\delta_0, \delta_\infty)$. ∎

Consider now functions from $H_p(\delta_0, \delta_\infty; W)$ and their derivatives as functions of two real variables and prove their (with a weight) membership in the usual Lebesgue space $L_p(W)$.

1.17. LEMMA. *Let* $h \in H_p(\delta_0, \delta_\infty; W(\theta_-, \theta_+))$. *Then*:

(i) $h(z)\rho^{\delta-2/p} \in L_p(W(\theta_-, \theta_+))$, $\delta \in (\delta_0, \delta_\infty)$;

(ii) $(\theta-\theta_-)(d/dz)h(z)\rho^{\delta+1-2/p} \in L_p(W(\theta_-, \theta'))$,

$$(\theta-\theta_+)(d/dz)h(z)\rho^{\delta+1-2/p} \in L_p(W(\theta', \theta_+))$$

for any $\delta \in (\delta_0, \delta_\infty)$, $\theta' \in (\theta_-, \theta_+)$.

Proof. For (i)

$$\begin{aligned}\left\|h(z)\rho^{\delta-2/p}; L_p(W(\theta_-, \theta'))\right\|^p &= \iint_W |h(z)\rho^{\delta-2/p}|^p|dW| \\ &= \int_{\theta_-}^{\theta_+} d\theta \int_0^\infty |h(\rho e^{i\theta})\rho^\delta|^p \rho^{-1}\,d\rho \\ &\le (\theta_+ - \theta_-) M_p^p(h; \delta; W(\theta_-, \theta_+)) < \infty.\end{aligned}$$

Point (i) has been proved. Let $W' := W(\theta_-, \theta')$. Point (ii) of Lemma 1.10 gives

$$\begin{aligned}&\left\|(\theta-\theta_-)\frac{d}{dz}h(z)\rho^{\delta+1-2/p}; L_p(W')\right\|^p \\ &\quad= \iint_{W'} \left|(\theta-\theta_-)\frac{d}{dz}h(z)\rho^{\delta+1-2/p}\right|^p|dW| \\ &\quad= \int_{\theta-}^{\theta'}|\theta-\theta_-|^p\left[\int_0^\infty\left|\frac{d}{dz}h(z)\rho^{\delta+1}\right|^p\rho^{-1}\,d\rho\right]d\theta \\ &\quad\le \int_{\theta_-}^{\theta'}|\theta-\theta_-|^p M_p^p\left(z\frac{d}{dz}h(z); \delta; W(\theta, \theta')\right)d\theta \\ &\quad\le \int_{\theta_-}^{\theta'}|\theta-\theta_-|^p\left[|\delta| + \frac{1}{r(\theta)}\right]^p M_p^p(h; \delta; W(\theta_-, \theta_+))\,d\theta \\ &\quad\le \left[|\delta|(\theta'-\theta_-) + d\right]^p M_p^p(h; \delta; W(\theta_-, \theta_+))(\theta'-\theta_-) < \infty.\end{aligned}$$

Here $r(\theta) = \min(\theta-\theta_-, \theta_+-\theta')$, $d := \sup_{\theta_-<\theta<\theta'}[(\theta-\theta_-)/r(\theta)] \le \max[1, (\theta'-\theta_-)/(\theta_+-\theta')]$.

The second statement of point (ii) is proved analogously. ∎

1.18. LEMMA. *Let* $h \in H_p(\delta_0, \delta_\infty; W(\theta_-, \theta_+))$. *Let a function* $b(\theta) \in C^1[\theta_-, \theta_+]$, $b(\theta_\mp) \neq 0$, *and the following memberships take place for a function* $\tilde{h}(z)$: $z(d/dz)h(z) + b(\theta_-)\tilde{h}(z) \in H_p(\delta_0, \delta_\infty; W(\theta_-, \theta'))$, $z(d/dz)h(z) + b(\theta_+)\tilde{h}(z) \in H_p(\delta_0, \delta_\infty; W(\theta', \theta_+))$, $\forall\ \theta' \in (\theta_-, \theta_+)$. *Then* $[z(d/dz)h(z) + b(\theta)\tilde{h}(z)]\rho^{\delta-2/p} \in L_p(W(\theta_-, \theta_+))$ *for any* $\delta \in (\delta_0, \delta_\infty)$.

Proof. Let $\theta' \in (\theta_-, \theta_+)$. Then

$$\begin{aligned}
&\iint_{W(\theta_-, \theta')} \left| \left[z\frac{d}{dz}h(z) + b(\theta)\tilde{h}(z) \right] \rho^{\delta-2/p} \right|^p |dW| \\
&\quad = \iint_{W(\theta_-, \theta')} \left| \left\{ \frac{b(\theta)}{b(\theta_-)} \left[z\frac{d}{dz}h(z) + b(\theta_-)\tilde{h}(z) \right] \right. \right. \\
&\qquad\qquad \left. \left. + \left[1 - \frac{b(\theta)}{b(\theta_-)} \right] z\frac{d}{dz}h(z) \right\} \rho^{\delta-2/p} \right|^p |dW| \\
&\quad \le \left[\sup_{\theta_- < \theta < \theta'} \left| \frac{b(\theta)}{b(\theta_-)} \right| \right. \\
&\qquad \times \left\| \left[z\frac{d}{dz}h(z) + b(\theta_-)\tilde{h}(z) \right] \rho^{\delta-2/p}; L_p\big(W(\theta_-, \theta')\big) \right\| \\
&\qquad + \sup_{\theta_- < \theta < \theta'} \left| \frac{b(\theta) - b(\theta_-)}{(\theta - \theta_-) b(\theta_-)} \right| \\
&\qquad \left. \times \left\| (\theta - \theta_-) \frac{d}{dz} h(z) \rho^{\delta+1-2/p}; L_p\big(W(\theta_-, \theta')\big) \right\| \right]^p < \infty. \quad (1.13)
\end{aligned}$$

The norms in the last inequality are bounded by Lemma 1.17. The expression $[b(\theta) - b(\theta_-)]/(\theta - \theta_-)$ is bounded since $b(\theta) \in C^1[\theta_-, \theta_+]$. Writing the analogous estimates for the integral along $W(\theta', \theta_+)$ and adding it to (1.13), we obtain the statement desired. ∎

1.19. *Remark.* For any $R \in (0, \infty)$ and for any infinite wedge W, we can define the truncated finite wedge W_{0R}: $\{z \in W, |z| < R\}$ and the truncated infinite wedge $W_{\infty R}$: $\{z \in W, |z| > R\}$. Then the claims of Lemmas 1.17 and 1.18 hold also on W_{0R} for any $\delta > \delta_0$, and on $W_{\infty R}$ for any $\delta < \delta_\infty$.

Really, let $\delta > \delta_0$, $0 < \varepsilon < \min(\delta - \delta_0, \delta_\infty - \delta_0)$. Then

$$\begin{aligned}\left\| h(z)\rho^{\delta-2/p}; L_p(W_{0R}) \right\|^p &= \iint_{W_{0R}} |h(z)\rho^{\delta-2/p}|^p |dW| \\ &= \iint_{W_{0R}} |h(z)\rho^{\delta_0+\varepsilon-2/p}|^p \rho^{(\delta-\delta_0-\varepsilon)p} |dW| \\ &\le \left\| h(z)\rho^{\delta_0+\varepsilon-2/p}; L_p(W_{0R}) \right\|^p R^{(\delta-\delta_0-\varepsilon)p} \\ &\le R^{(\delta-\delta_0-\varepsilon)p} \| h(z)\rho^{\delta_0+\varepsilon-2/p}; L_p(W) \|^p < \infty.\end{aligned}$$

The other claims are proved analogously.

2. SOME CLASSES OF TWO VARIABLES AND THE MELLIN TRANSFORM

In this part, we shall extend the results of the previous one to functions depending additionally on one real parameter t.

Let $-\infty < t_1 < t_2 \le \infty$ and let $J(t_1, t_2)$ be either a segment $[t_1, t_2]$ if $t_2 < \infty$ or a half-infinite interval $[t_1, \infty)$ if $t_2 = \infty$. Keeping in mind the limitations on t_1, t_2, we shall write often J instead of $J(t_1, t_2)$ and J_∞ instead of $J(t_1, \infty) = [t_1, \infty)$. Let $L_\infty(J)$ be the space of measurable and essentially bounded functions on J with the norm $\|f\|_{L_\infty} := \operatorname{ess\,sup}_{t \in J} |f(t)| < \infty$, $B(J)$ be the space of measurable and bounded functions on J with the norm $\|f\|_B := \sup_{t \in J} |f(t)| < \infty$, and $C(J)$ be the space of the continuous and bounded functions on J with the norm $\|f\|_C := \|f\|_B < \infty$.

Consider first a class of functions of two real variables and a class of one-parametric holomorphic functions in a strip, i.e., some classes which are similar to $\hat{L}_p(\delta_0, \delta_\infty)$ and $H_p^0(S)$.

2.1. DEFINITION. Let a function $g(\rho, t)$ be given almost everywhere on $(0, \infty) \times J$ and

$$\|g(.,t); \delta\|_p := \left[\int_0^\infty |g(\rho, t)\rho^\delta|^p \rho^{-1}\, d\rho \right]^{1/p}.$$

(1) $L_\infty \hat{L}_p(\delta_0, \delta_\infty; J)$ consists of functions $g(\rho, t)$ such that $\|g; \delta; J\|_{p\infty} := \operatorname{ess\,sup}_{t \in J} \|g(, t); \delta\|_p < \infty$ for any $\delta \in (\delta_0, \delta_\infty)$.

(2) $B\hat{L}_p(\delta_0, \delta_\infty; J)$ consists of functions $g(\rho, t)$ such that $\|g; \delta; J\|_{pB} := \sup_{t \in J} \|g(., t); \delta\|_p < \infty$ for any $\delta \in (\delta_0, \delta_\infty)$.

(3) $C\hat{L}_p(\delta_0, \delta_\infty; J)$ consists of functions $g(\rho, t) \in B\hat{L}_p(\delta_0, \delta_\infty; J)$ such that $\|g(., t) - g(., t + \Delta t); \delta\|_p \to 0$ as $\Delta t \to 0$ for any $t \in J$ and any $\delta \in (\delta_0, \delta_\infty)$.

2.2. *Remark*. Obviously, $C\hat{L}_p(\delta_0, \delta_\infty; J) \subset B\hat{L}_p(\delta_0, \delta_\infty; J) \subset L_\infty\hat{L}_p(\delta_0, \delta_\infty; J)$. It follows from Lemma 1.3 that $L_\infty\hat{L}_p(\delta_0, \delta_\infty; J) \subset L_\infty\hat{L}_q(\delta_0, \delta_\infty; J)$, $B\hat{L}_p(\delta_0, \delta_\infty; J) \subset B\hat{L}_q(\delta_0, \delta_\infty; J)$, $C\hat{L}_p(\delta_0, \delta_\infty; J) \subset C\hat{L}_q(\delta_0, \delta_\infty; J)$ provided $1 \le q < p \le \infty$. It follows from Lemma 1.2 that

$$\|g; \delta; J\|_{p\infty}^p \le \|g; \delta_0'; J\|_{p\infty}^p + \|g; \delta_\infty'; J\|_{p\infty}^p,$$

$$\|g; \delta; J\|_{pB}^p \le \|g; \delta_0'; J\|_{pB}^p + \|g; \delta_\infty'; J\|_{pB}^p \tag{2.1}$$

for any $\delta \in [\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$ and for $p \ge 1$. Consequently, the norms $\|g; \delta; J\|_{p\infty}$ and $\|g; \delta; J\|_{pB}$ are uniformly bounded with respect to δ on any internal segment of $(\delta_0, \delta_\infty)$.

2.3. Definition. Lt $S = S(\delta_0, \delta_\infty)$, a function $\Phi(\gamma, t)$ be given on $S \times J$, and

$$M_p^0(\Phi(.,t); \delta) := \left[\int_{-\infty}^{\infty} |\Phi(\delta + i\xi, t)|^p \, d\xi\right]^{1/p}.$$

(1) We write $\Phi \in L_\infty H_p^0(S; J)$ if $\Phi(\gamma, t)$ is holomorphic with respect to $\gamma \in S$ at almost any $t \in J$ and if the quantity $M_{p\infty}^0(\Phi; \delta; J) := \operatorname{ess\,sup}_{t\in J} M_p^0(\Phi(.\,t); \delta)$ is uniformly bounded with respect to δ on any segment $[\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$.

(2) We write $\Phi \in BH_p^0(S; J)$ if $\Phi(\gamma, t)$ is holomorphic with respect to $\gamma \in S$ at any $t \in J$ and if the quantity $M_{pB}^0(\Phi; \delta; J) := \sup_{t\in J} M_p^0(\Phi(., t); \delta)$ is uniformly bounded with respect to δ on any segment $[\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$.

(3) We write $\Phi \in CH_p^0(S; J)$ if $\Phi \in BH_p^0(S; J)$ and if $M_p^0[\Phi(., t) - \Phi(., t + \Delta t); \delta] \to 0$ as $\Delta t \to 0$ for any $t \in J$ uniformly with respect to δ on any segment $[\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$.

2.4. Lemma. (1) *If* $\Phi \in L_\infty H_p^0(S; J)$, *then* $(\partial/\partial\gamma)\Phi(\gamma, t) \in L_\infty H_p^0(S; J)$ *and there is a number* $\tilde{\tilde{M}}_\infty^0(\Phi; S'; J) < \infty$ *such that*

$$\operatorname*{ess\,sup}_{t\in J} |\Phi(\gamma, t)| < \tilde{\tilde{M}}_\infty^0(\Phi; S'; J) < \infty \qquad (\gamma \in S') \tag{2.2}$$

for any $\bar{S}' \subset S$.

(2) *If* $\Phi \in BH_p^0(S; J)$, *then* $(\partial/\partial\gamma)\Phi(\gamma, t) \in BH_p^0(S; J)$ *and there is a number* $\tilde{\tilde{M}}_B^0(\Phi; S'; J)$ *such that* $\sup_{t\in J}|\Phi(\gamma, t)| < \tilde{\tilde{M}}_B^0(\Phi; S'; J) < \infty$ $(\gamma \in S')$ *for any* $\bar{S}' \subset S$.

(3) *If* $\Phi \in CH_p^0(S; J)$, *then* $(\partial/\partial\gamma)\Phi(\gamma, t) \in CH_p^0(S; J)$ *and* $\Phi(\gamma, t)$ *is continuous in t uniformly with respect to* γ *on any* $\bar{S}' \subset S$.

Proof. Let $S = S(\delta_0, \delta_\infty)$. If $\Phi \in L_\infty H_p^0(S; J)$, then $\Phi(.,t) \in H_p^0(S)$ for a.e. $t \in J$. Hence the function $\Phi(.,t) \in H_p^*(S'')$ according to Remark 1.5 for any $[\delta'_0, \delta'_\infty] \in (\delta_0, \delta_\infty)$ and $S'' := S[(\delta_0 + \delta'_0)/2, (\delta_\infty + \delta'_\infty)/2]$. Let $S' := S(\delta'_0, \delta'_\infty)$. Then we obtain from Lemma 1.6 for a.e. $t \in J$

$$|\Phi(\gamma, t)| \leq \left(\frac{2}{\pi r}\right)^{1/p} M_p^*(\Phi(.,t); S''), \qquad \gamma \in S';$$

$$M_p^0\left(\frac{\partial}{\partial \gamma}\Phi(\gamma, t); \delta\right) \leq \frac{1}{r} M_p^*(\Phi(.,t); S''), \qquad \delta \in [\delta'_0, \delta'_\infty], \quad (2.3)$$

where $r = \min(\delta_0 - \delta'_0, \delta_\infty - \delta'_\infty)/2$. Taking the essential supremum with respect to t of both sides of these inequalities we obtain point (1).

Point (2) is proved analogously. Applying inequalities (2.3) to the difference $\Phi(\gamma, t + \Delta t) - \Phi(\gamma, t)$, we prove point (3). ∎

Let us present an analogue of Theorem 1.7 for functions of two variables.

2.5. THEOREM. *Let $\theta_- < \theta_+$, $\delta_0 < \delta_\infty$, $W = W(\theta_-, \theta_+)$, $S = S(\delta_0, \delta_\infty)$.*

(I) (1) *If $g(\rho, t) \in L_\infty \hat{L}_2(\delta_0, \delta_\infty; J)$, then its Mellin image (with respect to ρ) $\langle g\rangle(\gamma, t) = (\mathscr{M} g)(\gamma, t)$ belongs to $L_\infty H_2^0(S; J)$, $g(\rho, t) = [\mathscr{M}^{-1} S\langle g\rangle](\rho, t)$ at almost any $t \in J$ and at almost any $\rho \in (0, \infty)$, and*

$$M_{2\infty}^0(\langle g\rangle; \delta; J) = (2\pi)^{1/2} \|g; \delta; J\|_{2\infty} \qquad (\delta_0 < \delta < \delta_\infty). \quad (2.4)$$

(2) *If $g \in B\hat{L}_2(\delta_0, \delta_\infty; J)$, then $\langle g\rangle = (\mathscr{M} g) \in BH_2^0(S; J)$, $g(\rho, t) = [\mathscr{M}^{-1} S\langle g\rangle](\rho, t)$ at any $t \in J$ and at almost any $\rho \in (0, \infty)$, and*

$$M_{2B}^0(\langle g\rangle; \delta; J) = (2\pi)^{1/2} \|g; \delta; J\|_{2B} \qquad (\delta_0 < \delta < \delta_\infty). \quad (2.5)$$

(3) *Moreover, if $g \in C\hat{L}_2(\delta_0, \delta_\infty; J)$, then $(\mathscr{M} g) \in CH_2^0(S; J)$.*

(II) (1) *If a function $\langle g\rangle(\gamma, t) \in L_\infty H_2^0(S; J)$, then $g(\rho, t) = [\mathscr{M}^{-1}(S)\langle g\rangle](\rho, t) \in L_\infty \hat{L}_2(\delta_0, \delta_\infty; J)$, equality* (2.4) *takes place, and $\langle g\rangle(\gamma, t) = (\mathscr{M} g)(\gamma, t)$ at almost any $t \in J$.*

(2) *If $\langle g\rangle \in BH_2^0(S; J)$, then $g = [\mathscr{M}^{-1}(S)\langle g\rangle] \in B\hat{L}_2(\delta_0, \delta_\infty; J)$, equality* (2.5) *takes place, and $\langle g\rangle(\gamma, t) = (\mathscr{M} g)(\gamma, t)$ for any $t \in J$.*

(3) *Moreover, if $\langle g\rangle \in CH_2^0(S; J)$, then $[\mathscr{M}^{-1}(S)\langle g\rangle] \in C\hat{L}_2(\delta_0, \delta_\infty; J)$.*

Proof. (I) (1) Let $g(\rho, t) \in L_\infty \hat{L}_2(\delta_0, \delta_\infty; J)$. Then $g(.,t) \in \hat{L}_2(\delta_0, \delta_\infty)$ for a.e. $t \in J$ and, according to Theorem 1.7, $\langle g\rangle(.,t) \in H_2^0(S)$ for a.e. $t \in J$. Equality (2.4) follows from the Parseval equality (1.4). The uniform boundedness of $M_{2\infty}^0(\Phi; \delta; J)$ with respect to δ on any segment $[\delta'_0, \delta'_\infty] \subset (\delta_0, \delta_\infty)$ follows from the right hand side of (2.4) and from estimate (2.1).

Point (1) has been proved. Point (2) is proved analogously. Point (3) is the consequence of the Parseval equality,

$$\begin{aligned} M_2^{02}&(\langle g\rangle(.,t) - \langle g\rangle(.,t+\Delta t);\delta) \\ &= \int_{-\infty}^{\infty} |\langle g\rangle(\delta + i\xi, t) - \langle g\rangle(\delta + i\xi, t+\delta t)|^2\, d\xi \\ &= \int_{-\infty}^{\infty} |\langle g(.,t) - g(.,t+\Delta t)\rangle(\delta + i\xi)|^2\, d\xi \\ &= 2\pi\|g(.,t) - g(.,t+\Delta t);\delta\|_2^2 \\ &\le 2\pi\|g(.,t) - g(.,t+\Delta t);\delta_0'\|_2^2 \\ &\quad + 2\pi\|g(.,t) - g(.,t+\Delta t);\delta_\infty'\|_2^2 \to 0 \qquad \text{as } \Delta t \to 0. \end{aligned}$$

Part (II) of the theorem is proved analogously by use of part (II) of Theorem 1.7 and the Parseval equalities. ▮

Consider now weighted Hardy type classes of one-parametric holomorphic functions in a strip or in a wedge, i.e., analogues of the classes $H_p(\delta_0, \delta_\infty; W)$ and $H_p^\vee(\theta_-, \theta_+; S)$.

2.6. Definition. Let $W = W(\theta_-, \theta_+)$, a function $h(z,t)$ be given on $W \times J$, and

$$M_p(h(.,t);\delta;W) := \sup_{\theta_- < \theta < \theta_+} \left[\int_0^\infty |h(\rho e^{i\theta}, t)\rho^\delta|^p \rho^{-1}\, d\rho\right]^{1/p}.$$

(1) We write $h \in L_\infty(\delta_0, \delta_\infty; W; J)$ if $h(z,t)$ is holomorphic with respect to $z \in W$ at almost any $t \in J$ and if

$$M_{p\infty}(h;\delta;W;J) := \operatorname*{ess\,sup}_{t\in J} M_p(h(.,t);\delta;W) < \infty \qquad (\delta_0 < \delta < \delta_\infty).$$

(2) We write $h \in BH_p(\delta_0, \delta_\infty; W; J)$ if $h(z,t)$ is holomorphic with respect to $z \in W$ at any $t \in J$ and if

$$M_{pB}(h;\delta;W;J) := \sup_{t\in J} M_p(h(.,t);\delta;W) < \infty \qquad (\delta_0 < \delta < \delta_\infty).$$

(3) We write $h \in CH_p(\delta_0, \delta_\infty; W; J)$ if $h \in BH_p(\delta_0, \delta_\infty; W; J)$ and if $M_p[(h(.,t) - h(.,t+\Delta t); \delta; W] \to 0$ as $\Delta t \to 0$ for any $t \in J$ and for any $\delta \in [\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$.

2.7. *Remark.* Due to Lemma 1.2, the inequalities

$$M_{p\infty}(h;\delta;W;J) \le M_{p\infty}(h;\delta_0';W;J) + M_{p\infty}(h;\delta_\infty';W;J),$$
$$M_{pB}(h;\delta;W;J) \le M_{pB}(h;\delta_0';W;J) + M_{pB}(h;\delta_\infty';W;J)$$

hold for any $\delta \in [\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$, i.e., the quantities $M_{p\infty}(h;\delta;W;J)$, $M_{pB}(h;\delta;W;J)$ are uniformly bounded with respect to δ on any segment $[\delta_0', \delta_\infty] \subset (\delta_0, \delta_\infty)$ when $h \in L_\infty H_p(\delta_0, \delta_\infty;W;J)$ or $h \in BH_p(\delta_0, \delta_\infty;W;J)$, respectively. If $h \in CH_p(\delta_0, \delta_\infty;W;J)$, then $M_p(h(.,t)-h(.,t+\Delta t);\delta;W) \to 0$ uniformly with respect to δ on any $[\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$ as $\Delta t \to 0$.

2.8. Lemma. (1) *Suppose* $h \in L_\infty H_p(\delta_0, \delta_\infty;W;J)$; *then for any* $\overline{W}' \subset W$ *and any segment* $[\delta_0', \delta_\infty'] \in (\delta_0, \delta_\infty)$, *there is a number* $\tilde{\tilde{M}}_\infty(h;\delta_0', \delta_\infty';W';J) < \infty$ *such that*

$$\operatorname*{ess\,sup}_{t\in J}|h(z,t)| \le \tilde{\tilde{M}}_\infty(h;\delta_0', \delta_\infty';W';J)|z|^{-\delta}, \qquad \{z,\delta\} \in W' \times [\delta_0', \delta_\infty'];$$

and $z(\partial/\partial z)h(z,t) \in L_\infty H_p(\delta_0, \delta_\infty;W';J)$.

(2) *Suppose* $h \in BH_p(\delta_0, \delta_\infty;W;J)$; *then for any* $\overline{W}' \subset W$ *and any* $[\delta_0', \delta_\infty'] \in (\delta_0, \delta_\infty)$, *there is a numer* $\tilde{\tilde{M}}_B(h;\delta_0', \delta_\infty';W';J) < \infty$ *such that*

$$\sup_{t\in J}|h(z,t)| \le \tilde{\tilde{M}}_B(h;\delta_0', \delta_\infty';W';J)|z|^{-\delta}, \qquad \{z,\delta\} \in W' \times [\delta_0', \delta_\infty'];$$

and $z(\partial/\partial z)h(z,t) \in BH_p(\delta_0, \delta_\infty;W';J)$.

(3) *Suppose* $h \in CH_p(\delta_0, \delta_\infty;W';J)$; *then for any* $\overline{W}' \subset W$ *and any* $[\delta_0', \delta_\infty'] \in (\delta_0, \delta_\infty)$, *the function* $h(z,t)|z|^\delta$ *is continuous in* t *uniformly with respect to* $\{z,\delta\} \in W' \times [\delta_0', \delta_\infty']$, *and* $z(\partial/\partial z)h(z,t) \in CH_p(\delta_0, \delta_\infty;W';J)$.

Proof. (1) Using Lemma 1.10, we obtain for almost all $t \in J$

$$|h(z,t)| \le \left(\frac{2}{\pi r}\right)^{1/p} \sup_{\delta_0' \le \delta_* \le \delta_\infty'} M_p(h(.,t);\delta_*;W)|z|^{-\delta}, \qquad \{z,\delta\} \in W' \times [\delta_0', \delta_\infty'],$$

$$M_p\left(z\frac{\partial}{\partial z}h(z,t);\delta;W'\right) \le \left(|\delta| + \frac{1}{r}\right)M_p(h(.,t);\delta;W), \qquad \delta \in [\delta_0', \delta_\infty']. \tag{2.6}$$

Taking the essential supremum with respect to t, we get the proof of point (1). Point (2) is proved analogously.

Point (3) is obtained if we apply inequalities (2.6) to the difference $h(z, t+\Delta t)z^{\delta} - h(z, t)z^{\delta}$. ∎

2.9. DEFINITION. Let $S = S(\delta_0, \delta_\infty)$, a function $\Phi(\gamma, t)$ be given on $S \times J$, and

$$M_p^{\vee}\big(\Phi(\,.\,, t); \theta_-, \theta_+; \delta\big) := \sup_{\theta_- < \theta < \theta_+} \left[\int_{-\infty}^{\infty} |\Phi(\delta + i\xi, t)e^{\xi\theta}|^p \, d\xi\right]^{1/p}.$$

(1) We write $\Phi \in L_\infty H_p^{\vee}(\theta_-, \theta_+; S; J)$ if $\Phi(\gamma, t)$ is holomorphic with respect to $\gamma \in S$ at almost any $t \in J$ and if the quantity

$$M_{p\infty}^{\vee}(\Phi; \theta_-, \theta_+; \delta; J) := \operatorname*{ess\,sup}_{t \in J} M_p^{\vee}\big(\Phi(\,.\,, t); \theta_-, \theta_+; \delta\big)$$

is uniformly bounded with respect to δ on any $[\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$.

(2) We write $\Phi \in BH_p^{\vee}(\theta_-, \theta_+; S; J)$ if $\Phi(\gamma, t)$ is holomorphic with respect to $\gamma \in S$ at any $t \in J$ and if the quantity $M_{pB}^{\vee}(\Phi; \theta_-, \theta_+; \delta; J) := \sup_{t \in J} M_p^{\vee}(\Phi(., t); \theta_-, \theta_+; \delta)$ is uniformly bounded with respect to δ on any $[\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$.

(3) We write $\Phi \in CH_p^{\vee}(\theta_-, \theta_+; S; J)$ if $\Phi \in BH_p^{\vee}(\theta_-, \theta_+; S; J)$ and if $M_p^{\vee}[\Phi(., t) - \Phi(., t+\Delta t); \theta_-, \theta_+; \delta] \to 0$ as $\Delta t \to 0$ for any $t \in J$ uniformly with respect to δ on any $[\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$.

2.10. LEMMA. (1) *Suppose* $\Phi \in L_\infty H_p^{\vee}(\theta_-, \theta_+; S; J)$; *then* $(\partial/\partial\gamma)\Phi \cdot (\gamma, t) \in L_\infty H_p^{\vee}(\theta_-, \theta_+; S; J)$ *and there is a number* $\tilde{\tilde{M}}_\infty^{\vee}(\Phi; \theta_-, \theta_+; S'; J)$ *such that*

$$\operatorname*{ess\,sup}_{t \in J} |\Phi(\gamma, t)| < \tilde{\tilde{M}}_\infty^{\vee}(\Phi; \theta_-, \theta_+; S'; J)|e^{i\gamma\theta}|,$$
$$\{\gamma, \theta\} \in S' \times (\theta_-, \theta_+)$$

for any $\overline{S}' \subset S$.

(2) *Suppose* $\Phi \in BH_p^{\vee}(\theta_-, \theta_+; S; J)$; *then* $(\partial/\partial\gamma)\Phi(\gamma, t) \in BH_p^{\vee}(\theta_-, \theta_+; S; J)$ *and there is a number* $\tilde{\tilde{M}}_B^{\vee}(\Phi; \theta_-, \theta_+; S'; J)$ *such that*

$$\sup_{t \in J} |\Phi(\gamma, t)| < \tilde{\tilde{M}}_B^{\vee}(\Phi; \theta_-, \theta_+; S'; J)|e^{i\gamma\theta}|, \qquad \{\gamma, \theta\} \times (\theta_-, \theta_+)$$

for any $\overline{S}' \subset S$.

(3) *Suppose* $\Phi \in CH_p^{\vee}(\theta_-, \theta_+; S; J)$; *then* $(\partial/\partial\gamma)\Phi(\gamma, t) \in CH_p^{\vee}(\theta_-, \theta_+; S; J)$ *and* $\Phi(\gamma, t)|e^{-i\gamma\theta}|$ *is continuous in* t *uniformly with respect to* $\{\gamma, \theta\} \in S' \times (\theta_-, \theta_+)$ *for any* $\overline{S}' \subset S$.

Proof. (1) Using Lemma 1.13, we obtain for a.e. $t \in J$

$$|\Phi(\gamma, t)| \leq \left(\frac{2}{\pi r}\right)^{1/p} \sup_{\delta_0'' < \tilde{\delta} < \delta_\infty''} M_p^{\vee}\left(\Phi(.,t); \theta_-, \theta_+; \tilde{\delta}\right)|e^{i\gamma\theta}|,$$

$$\{\gamma, \theta\} \in S' \times (\theta_-, \theta_+), \quad (2.7)$$

$$M_p^{\vee}\left(\frac{\partial}{\partial\gamma}\Phi(\gamma, t); \theta_-, \theta_+; \delta\right)$$

$$\leq \left(|\theta_-| + |\theta_+| + \frac{1}{r}\right) \sup_{\delta_0'' < \delta < \delta_\infty''} M_p^{\vee}(\delta(.,t); \theta_-, \theta_+; \delta),$$

$$\delta \in [\delta_0', \delta_\infty'].$$

Taking essential supremum with respect to t, we get the proof of point (1). Point (2) is proved analogously.

Point (3) is obtained if we apply inequalities (2.7) to the difference $\Phi(\gamma, t + \Delta t)e^{-i\gamma\theta} - \Phi(\gamma, t)e^{-i\gamma\theta}$. ∎

2.11. Lemma. (1) *Let $\Phi \in L_\infty H_p^0(S; J)$. Suppose a function $\Phi_1(\gamma, t)$ is holomorphic on S with respect to γ at almost any $t \in J$ and there is a number $\tilde{\tilde{M}}_\infty^{\vee}(\Phi_1; S'; J) < \infty$ such that*

$$\operatorname*{ess\,sup}_{t \in J} |\Phi_1(\gamma, t)| \leq \tilde{\tilde{M}}_\infty^{\vee}(\Phi_1; S'; J)|e^{i\gamma\theta}|, \qquad \{\gamma, \theta\} \in S' \times (\theta_-, \theta_+)$$

for an interval (θ_-, θ_+) and for every $\overline{S'} \subset S$. Then $\Phi\Phi_1 \in L_\infty H_p^{\vee}(\theta_-, \theta_+; S; J)$.

(2) *Let $\Phi \in BH_p^0(S; J)$. Suppose $\Phi_1(\gamma, t)$ is holomorphic on S with respect to γ for any $t \in J$, and there is a number $\tilde{\tilde{M}}_B^{\vee}(\Phi_1; S'; J) < \infty$ such that*

$$\sup_{t \in J} |\Phi_1(\gamma, t)| \leq \tilde{\tilde{M}}_B^{\vee}(\Phi_1; S'; J)|e^{i\gamma\theta}|, \qquad \{\gamma, \theta\} \in S' \times (\theta_-, \theta_+)$$

for an interval (θ_-, θ_+) and for every $\overline{S'} \subset S$. Then $\Phi\Phi_1 \in BH_p^{\vee}(\theta_-, \theta_+; S; J)$.

(3) *Let $\Phi \in CH_p^0(S; J)$. Suppose $\Phi_1(\gamma, t)$ satisfies the conditions of point (2) and additionally $\Phi_1(\gamma, t)e^{-i\gamma\theta}$ is continuous in t uniformly with respect to $\{\gamma, \theta\} \in S' \times (\theta_-, \theta_+)$ for every $\overline{S'} \subset S$. Then $\Phi\Phi_1 \in CH_p^{\vee}(\theta_-, \theta_+; S; J)$.*

Proof. (1) Let $S = S(\delta_0, \delta_\infty)$ and $\delta \in [\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$. Then

$$\begin{aligned}
&M_p^\vee(\Phi(.,t)\Phi_1(.,t); \theta_-, \theta_+; \delta)\\
&\quad = \sup_{\theta_- < \theta < \theta_+} \left[\int_{-\infty}^{\infty} |\Phi(\delta + i\xi, t)\Phi_1(\delta + i\xi, t)e^{\xi\theta}|^p \, d\xi\right]^{1/p}\\
&\quad \le \tilde{\tilde{M}}_\infty^\vee(\Phi_1; S'; J)\left[\int_{-\infty}^{\infty} |\Phi(\delta + i\xi, t)|^p \, d\xi\right]^{1/p}\\
&\quad = \tilde{\tilde{M}}_\infty^\vee(\Phi_1; S'; J)M_p^0(\Phi(.,t); \delta)
\end{aligned}$$

for almost any t. Consequently, $M_{p\infty}^\vee(\Phi\Phi_1; \theta_-, \theta_+; \delta; J) \le \tilde{\tilde{M}}_\infty^\vee(\Phi_1; S'; J)M_{p\infty}^0(\Phi; \delta; J) < \infty$, where $M_{p\infty}^0(\Phi; \delta; J)$ is uniformly bounded with respect to δ on any $[\delta_0', \delta_\infty'] \in (\delta_0, \delta_\infty)$ according to point (1) of Definition 2.3.

The proof of point (1) is complete. Point (2) is proved analogously. To prove point (3),

$$\begin{aligned}
&M_p^\vee[\Phi(.,t)\Phi_1(.,t) - \Phi(.,t+\Delta t)\Phi_1(.,t+\Delta t); \theta_-, \theta_+; \delta]\\
&\quad = \sup_{\theta_- < \theta < \theta_+} \left[\int_{-\infty}^{\infty} |\Phi(\delta + i\xi, t)\right.\\
&\qquad\qquad \times[\Phi_1(\delta + i\xi, t) - \Phi_1(\delta + i\xi, t+\Delta t)]e^{\xi\theta}\\
&\qquad\quad + \Phi_1(\delta + i\xi, t+\Delta t)\\
&\qquad\qquad \left.\times[\Phi(\delta + i\xi, t) - \Phi(\delta + i\xi, t+\Delta t)]e^{\xi\theta}|^p \, d\xi\right]^{1/p}\\
&\quad \le \sup_{\theta_- < \theta < \theta_+} \sup_{-\infty < \xi < \infty} \left|[\Phi_1(\delta + i\xi, t) - \Phi_1(\delta + i\xi, t+\Delta t)]e^{\xi\theta}\right|\\
&\qquad \times M_p^0(\Phi(.,t); \delta)\\
&\qquad + \tilde{\tilde{M}}_B^\vee(\Phi_1; S'; J)M_p^0[\Phi(.,t) - \Phi(.,t+\Delta t); \delta] \to 0
\end{aligned}$$

and this tendency is uniform with respect to $\delta \in [\delta_0', \delta_\infty']$ as $\Delta t \to 0$. ∎

Let us present an analogue of Theorem 1.15 for one-parametric holomorphic functions.

2.12. THEOREM. *Let* $\theta_- < \theta_+$, $\delta_0 < \delta_\infty$, $W = W(\theta_-, \theta_+)$, $S = S(\delta_0, \delta_\infty)$.

(I) (1) *Suppose* $h(z,t) \in L_\infty H_2(\delta_0, \delta_\infty; W; J)$; *then its Mellin image* (*with respect to* z) $h^\vee(\gamma, t) := [\mathfrak{M}(W)h](\gamma, t)$ *belongs to* $L_\infty H_2^\vee(\theta_-, \theta_+; S; J)$, $h(z,t) = [\mathfrak{M}^{-1}(S)h^\vee](z,t)$ *for almost any* $t \in J$, *and*

$$M_{2\infty}^\vee\big(h^\vee; \theta_-, \theta_+; \delta; J\big) = (2\pi)^{1/2} M_{2\infty}(h; \delta; W; J) \qquad (\delta_0 < \delta < \delta_\infty). \tag{2.8}$$

(2) *Suppose* $h \in BH_2(\delta_0, \delta_\infty; W; J)$; *then* $h^\vee \in BH_2^\vee(\theta_-, \theta_+; S; J)$, $h(z,t) = [\mathfrak{M}^{-1}(S)h^\vee](z,t)$ *for any* $t \in J$, *and*

$$M_{2B}^\vee\big(h^\vee; \theta_-, \theta_+; \delta; J\big) = (2\pi)^{1/2} M_{2B}(h; \delta; W; J) \qquad (\delta_0 < \delta < \delta_\infty). \tag{2.9}$$

(3) *Moreover, if* $h \in CH_2(\delta_0, \delta_\infty; W; J)$, *then* $h^\vee \in CH_2^\vee(\theta_-, \theta_+; S; J)$.

(II) (1) *Suppose a function* $h^\vee(\gamma, t)$ *belongs to* $L_\infty H_2^\vee(\theta_-, \theta_+; S; J)$; *then* $h(z,t) := [\mathfrak{M}^{-1}(S)h^\vee](z,t)$ *belongs to* $L_\infty H_2(\delta_0, \delta_\infty; W; J)$, *equality* (2.8) *holds, and* $h^\vee(\gamma,t) = [\mathfrak{M}(W)h](\gamma,t)$ *for almost any* $t \in J$.

(2) *Suppose* $h^\vee \in BH_2^\vee(\theta_-, \theta_+; S; J)$; *then* $h \in BH_2(\delta_0, \delta_\infty; W; J)$, *equality* (2.9) *holds, and* $h^\vee(\gamma,t) = [\mathfrak{M}(W)h](\gamma,t)$ *for any* $t \in J$.

(3) *Moreover, if* $h^\vee \in CH_2^\vee(\theta_-, \theta_+; S; J)$, *then* $h \in CH_2(\delta_0, \delta_\infty; W; J)$.

Points (1) and (2) of parts (I) and (II) follow directly from Theorem 1.15 and Remark 2.7. Points (3) are proved as in the proof of Theorem 2.5.

Let us give an extension of Lemma 1.16 and state boundary properties of functions from $L_\infty H_2(\delta_0, \delta_\infty; W; J)$, $BH_2(\delta_0, \delta_\infty; W; J)$, and $CH_2(\delta_0, \delta_\infty; W; J)$ and their relations with the Mellin transforms.

2.13. LEMMA. *Let* $W = W(\theta_-, \theta_+)$.

(1) *Suppose* $h(z,t) \in L_\infty H_2(\delta_0, \delta_\infty; W; J)$; *then the functions* $h^\pm(\rho, t) := \lim_{\theta \to \theta_\pm} h(\rho e^{i\theta}, t)$ *are defined for almost any* $\rho \in (0, \infty)$ *and for almost any* $t \in J$, $h^\pm(\rho, t) \in L_\infty \hat{L}_2(\delta_0, \delta_\infty; J)$, *and*

$$[\mathfrak{M}(W)h](\gamma, t) = \exp(i\gamma\theta_\pm)[\mathscr{M}h^\pm](\gamma, t) \tag{2.10}$$

for $\gamma \in S(\delta_0, \delta_\infty)$ *and for almost any* $t \in J$.

(2) *Moreover, if* $h \in BH_2(\delta_0, \delta_\infty; W; J)$, *then* $h^\pm \in B\hat{L}_2(\delta_0, \delta_\infty; J)$ *and* (2.10) *is satisfied for any* $t \in J$.

(3) *Moreover, if* $h \in CH_2(\delta_0, \delta_\infty; W; J)$, *then* $h^\pm \in C\hat{L}_2(\delta_0, \delta_\infty; J)$.

Proof. (1) The existence of $h^{\pm}(\rho,t)$ at a.e. ρ and t as well as equality (2.10) is the direct consequence of Lemma 1.16. From Lemma 1.16 we also have that $\lim_{\theta\to\theta}\|h(\rho e^{i\theta},t)-h^{\pm}(\rho,t);\delta\|_2=0$ at a.e. $t\in J$. Consequently $\|h^{\pm}(\rho,t);\delta\|_2=\lim_{\theta\to\theta_{\pm}}\|h(\rho e^{i\theta},t)\|_2$ and

$$\|h^{\pm}(\rho,t);\delta\|_2\le\sup_{\theta_-<\theta<\theta_+}\|h\big(\rho e^{i\theta},t\big)\|_2=M_2\big(h(.,t);\delta;W\big). \quad (2.11)$$

Taking the essential supremum of the last inequality with respect to $t\in J$, we get $\|h^{\pm};\ \delta;\ J\|_{2\infty}\le M_{2\infty}(h;\ \delta;\ W;\ J)<\infty$, i.e., $h^{\pm}(\rho,\ t)\in L_\infty\hat{L}_2(\delta_0,\ \delta_\infty;\ J)$.

Point (1) has been proved. Point (2) is proved analogously. To prove point (3) we apply (2.11) to the difference $h^{\pm}(\rho,t)-h^{\pm}(\rho,t+\Delta t)$ and obtain

$$\begin{aligned}&\|h^{\pm}(\rho,t)-h^{\pm}(\rho,t+\Delta t);\delta\|_{2B}\\&\quad\le M_2\big[h(.,t)-h(.,t+\Delta t);\delta;W\big]\to 0\qquad\text{as }\Delta t\to 0.\end{aligned}$$

∎

All the function classes defined in this section can be used also for the case $J=J_\infty:=[t_1,\infty)$. Functions from these classes are then bounded at $t\to\infty$ for (almost) all ρ, z, or γ, but they may have no limits at $t\to\infty$. Consider also more narrow classes consisting of functions that tend to harmonically oscillating ones, or, particularly, have finite limits as $t\to\infty$. Both these classes are sufficiently important for applications, e.g., in visco-elasticity.

2.14. Definition. Let $\theta_-<\theta_+,\delta_0<\delta_\infty, W=W(\theta_-,\theta_+), S=S(\delta_0,\delta_\infty)$.

(1) We write $g\in L_{\infty l}\hat{L}_p(\delta_0,\delta_\infty;J_\infty;\Omega)$ [or $g\in B_l\hat{L}_p(\delta_0,\delta_\infty;J_\infty;\Omega)$, or $g\in C_l\hat{L}_p(\delta_0,\delta_\infty;J;\Omega)$] if g belongs to the corresponding classes without l and Ω and if there is a function $g_\Omega\in\hat{L}_p(\delta_0,\delta_\infty)$ such that $g(\rho,t)\to g_\Omega(\rho)e^{i\Omega t}$, i.e., $\|g(\rho,\ t)-g_\Omega(\rho)e^{i\Omega t};\ \delta;[T,\infty)\|_{p^\infty}\to 0$ [$\|g(\rho,\ t)-g_\Omega(\rho)e^{i\Omega t};\delta;[T,\infty)\|_{pB}\to 0$] as $T\to\infty$ for any $\delta\in(\delta_0,\delta_\infty)$.

(2) We write $\Phi\in L_{\infty l}H_p^0(S;J_\infty;\Omega)$ [or $\Phi\in B_lH_p^0(S;J_\infty;\Omega)$, or $\Phi\in C_lH_p^0(S;J_\infty;\Omega)$] if Φ belongs to the corresponding classes without l and Ω and if there is a function $\Phi_\Omega\in H_p^0(S)$ such that $\Phi(\gamma,t)\to\Phi_\Omega(\gamma)e^{i\Omega t}$, i.e., $M^0_{p\infty}[\Phi(\gamma,t)-\Phi_\Omega(\gamma)e^{i\Omega t};\ \delta;[T,\infty)]\to 0$ [$M^0_{pB}[\Phi(\gamma,t)-\Phi_\Omega(\gamma)e^{i\Omega t};\ \delta;[T,\infty)]\to 0$] as $T\to\infty$, uniformly with respect to δ on any $[\delta_0',\delta_\infty']\subset(\delta_0,\delta_\infty)$.

(3) We write $h\in L_{\infty l}H_p(\delta_0,\delta_\infty;W;J_\infty;\Omega)$ [or $h\in B_lH_p(\delta_0,\delta_\infty;W;J_\infty;\Omega)$, or $h\in C_lH_p(\delta_0,\delta_\infty;W;J_\infty;\Omega)$] if h belongs to the corresponding classes without l and Ω and if there is a function $h_\Omega\in H_p(\delta_0,\delta_\infty;W)$ such that $h(z,t)\to h_\Omega(z)e^{i\Omega t}$, i.e., $M_{p\infty}[h(z,t)-h_\Omega(z)e^{i\Omega t};\delta;W;[T,\infty)]\to 0$. [$M_{pB}[h(z,t-h_\Omega(z)e^{i\Omega t};\delta;W;[T,\infty)]\to 0$] as $T\to\infty$ for any $\delta\in(\delta_0,\delta_\infty)$.

(4) We write $\Phi \in L_{\infty l}H_p^{\vee}(\theta_-, \theta_+; S; J_\infty; \Omega)$ [or $\Phi \in B_l H_p^{\vee}(\theta_-, \theta_+; S; J_\infty; \Omega)$, or $\Phi \in C_l H_p^{\vee}(\theta_-, \theta_+; S; J_\infty; \Omega)$] if Φ belongs to the corresponding classes without l and Ω and if there is a function $\Phi_\Omega \in H_p^{\vee}(\theta_-, \theta_+; S)$) such that $\Phi(\gamma, t) \to \Phi_\Omega(\gamma)e^{i\Omega t}$, i.e., $M_{p\infty}^{\vee}[\Phi(\gamma, t) - \Phi_\Omega(\gamma)e^{i\Omega t}; \theta_-, \theta_+; \delta; [T, \infty)] \to 0$ $[M_{pB}^{\vee}[\Phi(\gamma, t) - \Phi_\Omega(\gamma)e^{i\Omega t}; \theta_-, \theta_+; \delta; [T, \infty)] \to 0]$ as $T \to \infty$, uniformly with respect to δ on any $[\delta_0', \delta_\infty'] \subset (\delta_0, \delta_\infty)$.

2.15. *Remark.* Obviously, if $\Omega = 0$, then we obtain the corresponding classes consisting of functions that tend with respect to t (in the sense of Definition 2.14) to finite limits.

2.16. THEOREM. *Theorems* 2.5 *and* 2.12 *hold if one provides all the function classes in these theorems by the subscript l and the parameter Ω. Moreover, $(\mathscr{M}g)(\gamma, t) \to (\mathscr{M}g_\Omega)(\gamma)e^{i\Omega t}$, $[\mathfrak{M}(W)h](\gamma, t) \to [\mathfrak{M}(W)h_\Omega](\gamma)e^{i\Omega t}$ in the sense of Definition* 2.14 *as $t \to \infty$.*

The proof is easy to carry out by use of the Parseval equalities (2.4), (2.5) and (2.8), (2.10).

2.17. LEMMA. (1) *Suppose $\Phi \in L_{\infty l}H_p^0(S; J_\infty; \Omega)$, a function $\Phi_1(\gamma, t)$ meets the conditions of point* (1) *of Lemma* 2.11, *and there is a limit function $\Phi_1(\gamma, \infty)$ such that*

$$\operatorname*{ess\,sup}_{t\in[T,\infty)} \sup_{\{\gamma,\theta\}\in\overline{S}'\times(\theta_-,\theta_+)} \left[|e^{i\gamma\theta}|\,|\Phi_1(\gamma, t) - \Phi_1(\gamma, \infty)|\right] \to 0, T \to \infty \quad (2.12)$$

for any $\overline{S}' \subset S$. Then $\Phi\Phi_1 \in L_{\infty l}H_p^{\vee}(\theta_-, \theta_+; S; J_\infty; \Omega)$ and $\Phi(\gamma, t)\Phi_1(\gamma, t) \to \Phi_\Omega(\gamma)\Phi_1(\gamma, \infty)e^{i\Omega t}$ in the sense of point (4) *of Definition* 2.14 *as $t \to \infty$.*

(2) *Suppose $\Phi \in B_l H_p^0(S; J_\infty; \Omega)$ [or $\Phi \in C_l H_p^0(S; J_\infty; \Omega)$], a function $\Phi_1(\gamma, t)$ meets the conditions of point* (2) [*and point* (3)] *of Lemma* 2.11, *and there is a limit function $\Phi_1(\gamma, \infty)$ such that*

$$\sup_{t\in[T,\infty)} \sup_{\{\gamma,\theta\}\in\overline{S}'\times(\theta_-,\theta_+)} \left[|e^{i\gamma\theta}|\,|\Phi_1(\gamma, t) - \Phi_1(\gamma, \infty)|\right] \to 0, T \to \infty$$

for any $\overline{S}' \subset S$. Then $\Phi\Phi_1 \in B_l H_p^{\vee}(\theta_-, \theta_+; S; J_\infty; \Omega)$ $[\Phi\Phi_1 \in C_l H_p^{\vee}(\theta_-, \theta_+; S; J_\infty; \Omega)]$ and $\Phi(\gamma, t)\Phi_1(\gamma, t) \to \Phi_\Omega(\gamma)\Phi_1(\gamma, \infty)e^{i\Omega t}$ in the sense of point (4) *of Definition* 2.14 *as $t \to \infty$.*

Proof. (1) From (2.12) and the properties of $\Phi_1(\gamma, t)$, it is easy to see that $\Phi_1(\gamma, \infty)$ meets the conditions of Lemma 1.14 for $\Phi_1(\gamma)$, and, consequently, $\Phi_\Omega(\gamma)\Phi_1(\gamma, \infty) \in H_p^{\vee}(\theta_-, \theta_+; S)$. Let us fix any $\overline{S}' = \overline{S}(\delta_0', \delta_\infty') \subset S$ and $\delta \in [\delta_0', \delta_\infty']$. Then

$$\begin{aligned}
M_{p\infty}^{\vee}&\left[\Phi(\gamma, t)\Phi_1(\gamma, t) - \Phi_\Omega(\gamma)\Phi_1(\gamma, \infty)e^{i\Omega t}; \theta_-, \theta_+; \delta; [T, \infty)\right] \\
&= M_{p\infty}^{\vee}\Big\{\left[\Phi(\gamma, t) - \Phi_\Omega(\gamma)e^{i\Omega t}\right]\Phi_1(\gamma, t) \\
&\qquad + \Phi_\Omega(\gamma)e^{i\Omega t}\left[\Phi_1(\gamma, t) - \Phi_1(\gamma, \infty)\right]; \theta_-, \theta_+; \delta; [T, \infty)\Big\}
\end{aligned}$$

$$\leq M^0_{p\infty}\big[\Phi(\gamma,t)-\Phi_\Omega(\gamma)e^{i\Omega t};\delta;[T,\infty)\big]\tilde{\tilde{M}}^{\vee}_{\infty}(\Phi_1;\theta_-,\theta_+;S';J_\infty)$$
$$+M^0_p(\Phi_\Omega;\delta)$$
$$\times \operatorname*{ess\,sup}_{t\in[T,\infty)}\ \sup_{\{\gamma,\theta\}\in\bar{S}'\times(\theta_-,\theta_+)}\big[|e^{i\gamma\theta}|\,|\Phi_1(\gamma,t)-\Phi_1(\gamma,\infty)|\big]\to 0$$

as $T\to\infty$ due to the first multiplier in the first summand and to the second multiplier in the second summand. Moreover, this tendency is uniform in $\gamma\in\bar{S}'$. Point (1) has been proved. Point (2) is proved analogously. ▮

It is easy to see that all the definitions and the results given above are valid also for n-dimensional vector functions. Then $|.|$ denotes the vector norm, and, in addition, the function Φ_1 in Lemmas 1.14, 2.11, 2.17 can be considered either as a scalar function or as a matrix funciton with an appropriate norm.

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