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# Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains

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ABSTRACT

For functions from the Sobolev space $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, definitions of non-unique generalized and unique canonical co-normal derivative are considered, which are related to possible extensions of a partial differential operator and its right-hand side from the domain $\Omega$, where they are prescribed, to the domain boundary, where they are not. Revision of the boundary value problem settings, which makes them insensitive to the generalized co-normal derivative inherent non-uniqueness are given. It is shown, that the canonical co-normal derivatives, although defined on a more narrow function class than the generalized ones, are continuous extensions of the classical co-normal derivatives. Some new results about trace operator estimates and Sobolev spaces characterizations, are also presented.



## 1. Introduction

While considering a second order partial differential equation for a function from the Sobolev space $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, with a right-hand side from $H^{s-2}(\Omega)$, the *strong* co-normal derivative of $u$ defined on the boundary in the trace sense, does not generally exist. Instead, a *generalized* co-normal derivative operator can be defined by the first Green identity. However this definition is related to an extension of the PDE operator and its right-hand side from the domain $\Omega$, where they are prescribed, to the domain boundary, where they are not. Since the extensions are non-unique, the generalized co-normal derivative operator appears to be non-unique and non-linear unless a linear relation between the PDE solution and the extension of its right-hand side is enforced. This leads to the need of a revision of the boundary value problem settings, which makes them insensitive to the co-normal derivative inherent non-uniqueness. For functions $u$ from a subspace of $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, which can be mapped by the PDE operator into the space $\tilde{H}^t(\Omega)$, $t \geqslant -\frac{1}{2}$, one can still define a *canonical* co-normal derivative, which is unique, linear in $u$ and coincides with the co-normal derivative in the trace sense if the latter does exist.

These notions were developed, to some extent, in [15,16] for a PDE with an infinitely smooth coefficient on a domain with an infinitely smooth boundary, and a right-hand side from $H^{s-2}(\Omega)$, $1 \leqslant s < \frac{3}{2}$, or extendable to $\tilde{H}^t(\Omega)$, $t \geqslant -1/2$. In [17] the analysis was generalized to the co-normal derivative operators for some scalar PDE with a Hölder coefficient and right-hand side from $H^{s-2}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, on a bounded Lipschitz domain $\Omega$.

In this paper updating [18], we extend the previous results on the co-normal derivatives to strongly elliptic second order PDE systems on bounded or unbounded Lipschitz domains with infinitely smooth coefficients, with complete proofs. We

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also give the week BVP settings invariant to the generalized co-normal derivatives non-uniqueness. To obtain these results, some new facts about trace operator estimates and Sobolev spaces characterizations are also proved in the paper.

The paper is arranged as follows. Section 2 provides a number of auxiliary facts on Sobolev spaces, traces and extensions, some of which might be new for Lipschitz domains. Particularly, we proved Lemma 2.4 on two-side estimates of the trace operator, Lemma 2.6 on boundedness of extension operators from boundary to the domain for a wider range of spaces, Theorem 2.9 on characterization of the Sobolev space $H^s_0(\Omega) = \tilde{H}^s(\Omega)$ on the (larger than usual) interval $\frac{1}{2} < s < \frac{3}{2}$, Theorem 2.10 on characterization of the space $H^t_{\partial\Omega}$, $t > -\frac{3}{2}$, Theorem 2.12 on equivalence of $H^s_0(\Omega)$ and $H^s(\Omega)$ for $s \leqslant \frac{1}{2}$, Theorem 2.13 on non-existence of the trace operator, Lemma 2.15 and Theorem 2.16 on extension of $H^s(\Omega)$ to $\tilde{H}^s(\Omega)$ for all $s < \frac{1}{2}$, $s \neq \frac{1}{2} - k$.

The results of Section 2 are applied in Section 3 to introduce and analyze the generalized and canonical co-normal derivative operators on bounded and unbounded Lipschitz domains, associated with strongly elliptic systems of second order PDEs with infinitely smooth coefficients and right-hand side from $H^{s-2}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$. The weak settings of Dirichlet, Neumann and mixed problems (revised versions for the latter two) are considered and it is shown that they are well posed in spite of the inherent non-uniqueness of the generalized co-normal derivatives. It is proved that the canonical co-normal derivative coincides with the classical (strong) one for the cases when they both do exist.

The results of Section 3 are generalized to Hölder–Lipschitz coefficients in [14], see also [18].

## 2. Sobolev spaces, trace operators and extensions

### 2.1. Notations

Suppose $\Omega = \Omega^+$ is a bounded or unbounded open domain of $\mathbb{R}^n$, which boundary $\partial\Omega$ is a simply connected, closed, Lipschitz $(n-1)$-dimensional set. Let $\overline{\Omega}$ denote the closure of $\Omega$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ its complement. In what follows $\mathcal{D}(\Omega) = C^\infty_{comp}(\Omega)$ denotes the space of Schwartz test functions, and $\mathcal{D}^*(\Omega)$ denotes the space of Schwartz distributions; $H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$, $H^s(\partial\Omega) = H^s_2(\partial\Omega)$ are the Sobolev (Bessel potential) spaces, where $s \in \mathbb{R}$ is an arbitrary real number (see, e.g., [12]).

We denote by $\tilde{H}^s(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^s(\mathbb{R}^n)$, which can be characterized as $\tilde{H}^s(\Omega) = \{g\colon\ g \in H^s(\mathbb{R}^n),\ \operatorname{supp} g \subset \overline{\Omega}\}$, see e.g. [13, Theorem 3.29]. The space $H^s(\Omega)$ consists of restrictions on $\Omega$ of distributions from $H^s(\mathbb{R}^n)$, $H^s(\Omega) := \{g|_\Omega\colon\ g \in H^s(\mathbb{R}^n)\}$, and $H^s_0(\Omega)$ is closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$. We recall that $H^s(\Omega)$ coincide with the Sobolev–Slobodetski spaces $W^s_2(\Omega)$ for any non-negative $s$. We denote $H^s_{loc}(\Omega) := \{g\colon\ \varphi g \in H^s(\Omega)\ \forall\varphi \in \mathcal{D}(\Omega)\}$. For infinite (unbounded) domains $\Omega$ we will use also the notation $H^s_{loc}(\overline{\Omega}) := \{g\colon\ \varphi g \in H^s(\Omega)\ \forall\varphi \in \mathcal{D}(\overline{\Omega})\}$ (for bounded domains $H^s_{loc}(\overline{\Omega}) = H^s(\Omega)$).

Note that distributions from $H^s(\Omega)$ and $H^s_0(\Omega)$ are defined only in $\Omega$, while distributions from $\tilde{H}^s(\Omega)$ are defined in $\mathbb{R}^n$ and particularly on the boundary $\partial\Omega$. For $s \geqslant 0$, we can identify $\tilde{H}^s(\Omega)$ with the subset of functions from $H^s(\Omega)$, whose extensions by zero outside $\Omega$ belong to $H^s(\mathbb{R}^n)$, cf. [13, Theorem 3.33], i.e., identify functions $u \in \tilde{H}^s(\Omega)$ with their restrictions, $u|_\Omega \in H^s(\Omega)$. However generally we will distinguish distributions $u \in \tilde{H}^s(\Omega)$ and $u|_\Omega \in H^s(\Omega)$, especially for $s \leqslant -\frac{1}{2}$.

We denote by $H^s_{\partial\Omega}$ the subspace of $H^s(\mathbb{R}^n)$ (and of $\tilde{H}^s(\Omega)$), which elements are supported on $\partial\Omega$, i.e., $H^s_{\partial\Omega} := \{g\colon\ g \in H^s(\mathbb{R}^n),\ \operatorname{supp} g \subset \partial\Omega\}$. To simplify notations for vector-valued functions, $u : \Omega \to \mathbb{C}^m$, we will often write $u \in H^s(\Omega)$ instead of $u \in H^s(\Omega)^m = H^s(\Omega; \mathbb{C}^m)$, etc.

As usual (see e.g. [12,13]), for two elements from dual complex Sobolev spaces the bilinear dual product $\langle\cdot,\cdot\rangle_\Omega$ associated with the sesquilinear inner product $(\cdot,\cdot)_\Omega := (\cdot,\cdot)_{L_2(\Omega)}$ in $L_2(\Omega)$ is defined as

$$\langle u, v\rangle_{\mathbb{R}^n} := \int_{\mathbb{R}^n} \big[\mathcal{F}^{-1}u\big](\xi)[\mathcal{F}v](\xi)\,d\xi =: (\mathcal{F}\bar{u}, \mathcal{F}v)_{\mathbb{R}^n} =: (\bar{u}, v)_{\mathbb{R}^n}, \quad u \in H^s\big(\mathbb{R}^n\big),\ v \in H^{-s}\big(\mathbb{R}^n\big), \tag{2.1}$$

$$\langle u, v\rangle_\Omega := \langle u, V\rangle_{\mathbb{R}^n} =: (\bar{u}, v)_\Omega \quad \text{if } u \in \tilde{H}^s\big(\mathbb{R}^n\big),\ v \in H^{-s}(\Omega), \qquad v = V|_\Omega \quad \text{with } V \in H^{-s}\big(\mathbb{R}^n\big),$$

$$\langle u, v\rangle_\Omega := \langle U, v\rangle_{\mathbb{R}^n} =: (\bar{u}, v)_\Omega \quad \text{if } u \in H^s\big(\mathbb{R}^n\big),\ v \in \tilde{H}^{-s}(\Omega), \qquad u = U|_\Omega \quad \text{with } U \in H^s\big(\mathbb{R}^n\big) \tag{2.2}$$

for $s \in \mathbb{R}$, where $\bar{g}$ is the complex conjugate of $g$, while $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the distributional Fourier transform operator and its inverse, respectively, that for integrable functions take form

$$\hat{g}(\xi) = [\mathcal{F}g](\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} g(x)\,dx, \qquad g(x) = \big[\mathcal{F}^{-1}\hat{g}\big](x) := \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\hat{g}(\xi)\,d\xi.$$

For vector-valued elements $u \in H^s(\mathbb{R}^n)^m$, $v \in H^{-s}(\mathbb{R}^n)^m$, $s \in \mathbb{R}$, definition (2.1) should be understood as

$$\langle u, v\rangle_{\mathbb{R}^n} := \int_{\mathbb{R}^n} \hat{u}(\xi)\cdot\hat{v}(\xi)\,d\xi = \int_{\mathbb{R}^n} \hat{u}(\xi)^\top\hat{v}(\xi)\,d\xi =: (\bar{\hat{u}}, \hat{v})_{\mathbb{R}^n} =: (\bar{u}, v)_{\mathbb{R}^n},$$

where $\hat{u}\cdot\hat{v} = \hat{u}^\top\hat{v} = \sum_{k=1}^m \hat{u}_k\hat{v}_k$ is the scalar product of two vectors.

Let $\mathcal{J}^s$ be the Bessel potential operator defined as

$$\big[\mathcal{J}^s g\big](x) = \mathcal{F}^{-1}_{\xi\to x}\big\{\big(1+|\xi|^2\big)^{s/2}\hat{g}(\xi)\big\}.$$

The inner product in $H^s(\Omega)$, $s \in \mathbb{R}$, is defined as follows,

$$(u,v)_{H^s(\mathbb{R}^n)} := \big(\mathcal{J}^s u, \mathcal{J}^s v\big)_{\mathbb{R}^n} = \int_{\mathbb{R}^n} \big(1+\xi^2\big)^s \overline{\hat{u}(\xi)}\,\hat{v}(\xi)\,d\xi = \big\langle \bar{u}, \mathcal{J}^{2s} v\big\rangle_{\mathbb{R}^n}, \quad u,v \in H^s\big(\mathbb{R}^n\big),$$

$$(u,v)_{H^s(\Omega)} := \big((I-P)U, (I-P)V\big)_{H^s(\mathbb{R}^n)}, \quad u = U|_\Omega,\ v = V|_\Omega, \quad U,V \in H^s\big(\mathbb{R}^n\big). \tag{2.3}$$

Here $P : H^s(\mathbb{R}^n) \to \tilde{H}^s(\mathbb{R}^n \setminus \overline{\Omega})$ is the orthogonal projector, see e.g. [13, p. 77].

For a general Lipschitz domain $\Omega$, let $\{\omega_j\}_{j=1}^J \subset \mathbb{R}^n$ be a finite open cover of $\partial\Omega$ and $\{\varphi_j(x) \in \mathcal{D}(\omega_j)\}_{j=1}^J$ be a partition of unity subordinate to it, $\sum_{j=1}^J \varphi_j(x) = 1$ for any $x \in \partial\Omega$. For any $j$ there exists a half-space domain $\Omega_j$ such that $\omega_j \cap \Omega_j = \omega_j \cap \Omega$ and $\Omega_j$ can be linearly transformed by a rigid translation $\kappa_j$ to a Lipschitz hypograph $\tilde{\Omega}_j = \{x' \in \mathbb{R}^{n-1}:\ x_n > \zeta_j(x')\}$, where $\zeta_j$ are some uniformly Lipschitz functions. Let also $\varkappa_j : \mathbb{R}^n \to \mathbb{R}^n$ be the Lipschitz-smooth invertible functions (evidently related to $\zeta_j$ and $\kappa_j$) such that $\mathbb{R}^n_+ \ni x \mapsto \varkappa_j(x) \in \Omega_j$, while $D_j(x')$ are the Jacobians of the corresponding boundary mappings $\mathbb{R}^{n-1} \ni x' \mapsto \varkappa_j(x') \in \partial\Omega_j$ and $D_j \in L_\infty(\mathbb{R}^{n-1})$.

Similar to [19, p. 85] we introduce the following definition.

**Definition 2.1.** Let $\Omega_k$, $\Omega$ be Lipschitz domains. We say that $\Omega_k \to \Omega$ as $k \to \infty$ if $\partial\Omega_k$ are represented using the same system of covering charts $\omega_j$ as $\partial\Omega$ for all sufficiently large $k$, and

$$\lim_{k\to\infty} |\zeta_{jk} - \zeta_j|_{C^{0,1}(\bar{\omega}_j)} = 0, \tag{2.4}$$

where $\zeta_{jk}$ and $\zeta_j$ are the corresponding Lipschitz functions for the boundary representation.

### 2.2. Sobolev spaces characterization, traces and extensions

To introduce generalized co-normal derivatives in Section 3, we will need several facts about traces and extensions in Sobolev spaces on Lipschitz domain. First we give the following usual definition of the trace operator.

**Definition 2.2.** An operator $\gamma^+ : H^s(\Omega^+) \to H^\sigma(\partial\Omega)$ is a trace operator if for each $u \in H^s(\Omega)$ and for any sequence $\phi_k \in \mathcal{D}(\overline{\Omega})$ converging to $u$ in $H^s(\Omega)$, the sequence of the boundary values $\phi_k|_{\partial\Omega}$ converges to $\gamma^+ u$ in $H^\sigma(\partial\Omega)$. The trace operator $\gamma^- : H^s(\Omega^-) \to H^\sigma(\partial\Omega)$ is defined similarly. If $\gamma^+ u = \gamma^- u$ we denote them as $\gamma u$.

We have the following well-known trace theorem [4, Lemma 3.6].

**Theorem 2.3.** *If $\frac{1}{2} < s < \frac{3}{2}$, then the trace operators*

$$\gamma : H^s\big(\mathbb{R}^n\big) \to H^{s-\frac{1}{2}}(\partial\Omega) \quad \text{and} \quad \gamma^\pm : H^s\big(\Omega^\pm\big) \to H^{s-\frac{1}{2}}(\partial\Omega), \tag{2.5}$$

*are continuous for any Lipschitz domain $\Omega$.*

Let $\gamma^* : H^{\frac{1}{2}-s}(\partial\Omega) \to H^{-s}(\mathbb{R}^n)$ denote the operator adjoined to the trace operator,

$$\big\langle \gamma^* v, w\big\rangle = \langle v, \gamma w\rangle \quad \forall w \in H^s\big(\mathbb{R}^n\big),\ v \in H^{\frac{1}{2}-s}(\partial\Omega).$$

Now we can prove two-side estimates for the trace operator and its adjoined, which particularly imply a statement about the trace operator unboundedness (cf. [12, Chapter 1, Theorem 9.5] for the unboundedness statements in domains with infinitely smooth boundary).

**Lemma 2.4.** *Let $\Omega$ be a Lipschitz domain and $\frac{1}{2} < s \leqslant 1$. Then*

$$C'\sqrt{C_s}\|v\|_{H^{\frac{1}{2}-s}(\partial\Omega)} \leqslant \big\|\gamma^* v\big\|_{H^{-s}(\mathbb{R}^n)} \leqslant C''\sqrt{C_s}\|v\|_{H^{\frac{1}{2}-s}(\partial\Omega)} \quad \forall v \in H^{\frac{1}{2}-s}(\partial\Omega) \tag{2.6}$$

*and thus*

$$C'\sqrt{C_s} \leqslant \|\gamma\|_{H^s(\mathbb{R}^n)\to H^{s-\frac{1}{2}}(\partial\Omega)} = \big\|\gamma^*\big\|_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})\to H^{-s}(\mathbb{R}^n)} \leqslant C''\sqrt{C_s}, \tag{2.7}$$

*where*

$$C_s := \int_{-\infty}^{\infty} \left(1+\eta^2\right)^{-s} d\eta,$$

*$C'$ and $C''$ are positive constants independent of $s$ and $v$. The norm of the trace operator $\gamma : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\partial\Omega)$ tends to infinity as $s \searrow \frac{1}{2}$ since $C_s \to \infty$, while the operator $\gamma : H^{\frac{1}{2}}(\mathbb{R}^n) \to L_2(\partial\Omega)$, if it does exist, is unbounded.*

**Proof.** Let first consider the lemma for the half-space, $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n\colon\ x_n > 0\}$, where $x = \{x', x_n\}$, $x' \in \mathbb{R}^{n-1}$. For $v \in H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})$, taking into account the uniqueness of the trace operator for $s > \frac{1}{2}$, the distributional Fourier transform gives

$$\mathcal{F}_{x\to\xi}\{\gamma^* v\} = \mathcal{F}_{x'\to\xi'}\{v(x')\} =: \hat{v}(\xi').$$

Then we have,

$$\begin{aligned}\|\gamma^* v\|^2_{H^{-s}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-s} \left|\hat{v}(\xi')\right|^2 d\xi \\ &= \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} \left(1+|\xi'|^2+|\xi_n|^2\right)^{-s} d\xi_n\right] \left|\hat{v}(\xi')\right|^2 d\xi' = C_s \|v\|^2_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})},\end{aligned} \tag{2.8}$$

where the substitution $\xi_n = (1+|\xi'|^2)^{\frac{1}{2}}\eta$ was used, cf. [3, Chapter 2, Proposition 4.6]. Thus

$$\|\gamma\|_{H^s(\mathbb{R}^n)\to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} = \|\gamma^*\|_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})\to H^{-s}(\mathbb{R}^n)} = \sqrt{C_s} \to \infty \quad \text{as } s \searrow \frac{1}{2}.$$

On the other hand, by (2.8) the norm $\|\gamma^* v\|_{H^{-\frac{1}{2}}(\mathbb{R}^n)}$ is not finite for any non-zero $v$. This means the operator $\gamma^* : H^0(\mathbb{R}^{n-1}) \to H^{-\frac{1}{2}}(\mathbb{R}^n)$ and thus the operator $\gamma : H^{\frac{1}{2}}(\mathbb{R}^n) \to H^0(\mathbb{R}^{n-1})$ is not bounded, which completes the lemma for $\Omega = \mathbb{R}^n_+$ with $C' = C'' = 1$.

Let now $\Omega$ be a general Lipschitz domain. For $v \in L_2(\partial\Omega)$, $w \in \mathcal{D}(\mathbb{R}^n)$, using the boundary cover and corresponding partition of unity as in Section 2.1 we have,

$$\begin{aligned}\langle\gamma^* v, w\rangle_{\mathbb{R}^n} = \langle v, \gamma w\rangle_{\partial\Omega} &= \int_{\partial\Omega} v(x)w(x)\,d\sigma(x) = \sum_{j=1}^{J}\int_{\partial\Omega} \varphi_j(x)v(x)w(x)\,d\sigma(x) \\ &= \sum_{j=1}^{J}\int_{\mathbb{R}^{n-1}} \left[(\varphi_j v)\circ\varkappa_j\right](x')[w\circ\varkappa_j](x')D_j(x')\,dx' \\ &= \sum_{j=1}^{J}\langle D_j(\varphi_j v)\circ\varkappa_j, \gamma_0[w\circ\varkappa_j]\rangle_{\mathbb{R}^{n-1}} = \sum_{j=1}^{J}\langle\gamma_0^*\left[D_j(\varphi_j v)\circ\varkappa_j\right], w\circ\varkappa_j\rangle_{\mathbb{R}^n},\end{aligned}$$

where $\gamma_0$, $\gamma_0^*$ are the trace operator on $\mathbb{R}^n_+$ and its adjoined, respectively. Taking into account density of $\mathcal{D}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$ and of $L_2(\partial\Omega)$ in $H^{\frac{1}{2}-s}(\partial\Omega)$, we have,

$$\|\gamma^* v\|_{H^{-s}(\mathbb{R}^n)} = \sup_{w\in H^s(\mathbb{R}^n)} \frac{|\langle\gamma^* v, w\rangle_{\mathbb{R}^n}|}{\|w\|_{H^s(\mathbb{R}^n)}} = \sup_{w\in H^s(\mathbb{R}^n)} \left|\sum_{j=1}^{J}\left\langle\gamma_0^*\left[D_j(\varphi_j v)\circ\varkappa_j\right], \frac{w\circ\varkappa_j}{\|w\|_{H^s(\mathbb{R}^n)}}\right\rangle_{\mathbb{R}^n}\right| \tag{2.9}$$

for any $v \in H^{\frac{1}{2}-s}(\partial\Omega)$.

It is well known (see e.g. [13, Theorem 3.23 and p. 98]) that

$$\|v\|^2_{H^{\frac{1}{2}-s}(\partial\Omega)} = \sum_{j=1}^{J}\left\|D_j(\varphi_j v)\circ\varkappa_j\right\|^2_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})}, \quad \frac{1}{2} < s \leqslant \frac{3}{2}, \tag{2.10}$$

$$\tilde{C}'\|w\|_{H^s(\mathbb{R}^n)} \leqslant \|w\circ\varkappa_j\|_{H^s(\mathbb{R}^n)} \leqslant \tilde{C}''\|w\|_{H^s(\mathbb{R}^n)}, \quad j = 1,\dots,J,\ 0 \leqslant s \leqslant 1, \tag{2.11}$$

where $\tilde{C}', \tilde{C}''$ are some positive constants independent of $s$. By (2.8) and (2.10),

$$\left\|\gamma_0^*\left[D_j(\varphi_j v)\circ\varkappa_j\right]\right\|_{H^{-s}(\mathbb{R}^n)} = \sqrt{C_s}\left\|D_j(\varphi_j v)\circ\varkappa_j\right\|_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})} \leqslant \sqrt{C_s}\|v\|_{H^{\frac{1}{2}-s}(\partial\Omega)}.$$

Then (2.9) and (2.11) imply

$$\|\gamma^* v\|_{H^{-s}(\mathbb{R}^n)} \leqslant \tilde{C}'' J\sqrt{C_s}\|v\|_{H^{\frac{1}{2}-s}(\partial\Omega)} \quad \forall v \in H^{\frac{1}{2}-s}(\partial\Omega),$$

which is the right inequality in (2.6).

On the other hand, we have for $v \in L_2(\partial\Omega)$, $w \in \mathcal{D}(\mathbb{R}^n)$,

$$\begin{aligned}
\langle \varphi_j \gamma^* v, w\rangle_{\mathbb{R}^n} &= \langle v, \gamma(\varphi_j w)\rangle_{\partial\Omega} = \int_{\partial\Omega} v(x)\varphi_j(x)w(x)\,d\sigma(x) \\
&= \int_{\partial\Omega\cap\omega_j} v(x)\varphi_j(x)w(x)\,d\sigma(x) = \int_{\mathbb{R}^{n-1}} \big[(\varphi_j v_j)\circ\varkappa_j\big](x')[w\circ\varkappa_j](x')D_j(x')\,dx' \\
&= \big\langle D_j\big[(\varphi_j v_j)\circ\varkappa_j\big], \gamma_0[w\circ\varkappa_j]\big\rangle_{\mathbb{R}^{n-1}} = \big\langle \gamma_0^*\big\{D_j\big[(\varphi_j v_j)\circ\varkappa_j\big]\big\}, w\circ\varkappa_j\big\rangle_{\mathbb{R}^n}.
\end{aligned}$$

By (2.11) this implies,

$$\begin{aligned}
\|\varphi_j\gamma^* v\|_{H^{-s}(\mathbb{R}^n)} &= \sup_{w\in H^s(\mathbb{R}^n)} \left|\left\langle \gamma_0^*\big\{D_j\big[(\varphi_j v)\circ\varkappa_j\big]\big\}, \frac{w\circ\varkappa_j}{\|w\|_{H^s(\mathbb{R}^n)}}\right\rangle_{\mathbb{R}^n}\right| \\
&= \sup_{w\in H^s(\mathbb{R}^n)} \left|\left\langle \gamma_0^*\big\{D_j\big[(\varphi_j v)\circ\varkappa_j\big]\big\}, \frac{w\circ\varkappa_j}{\|w\circ\varkappa_j\|_{H^s(\mathbb{R}^n)}}\right\rangle_{\mathbb{R}^n} \frac{\|w\circ\varkappa_j\|_{H^s(\mathbb{R}^n)}}{\|w\|_{H^s(\mathbb{R}^n)}}\right| \\
&\geqslant \tilde{C}' \sup_{w\in H^s(\mathbb{R}^n)} \left|\left\langle \gamma_0^*\big\{D_j\big[(\varphi_j v)\circ\varkappa_j\big]\big\}, \frac{w\circ\varkappa_j}{\|w\circ\varkappa_j\|_{H^s(\mathbb{R}^n)}}\right\rangle_{\mathbb{R}^n}\right| \\
&= \tilde{C}'\big\|\gamma_0^*\big\{D_j\big[(\varphi_j v)\circ\varkappa_j\big]\big\}\big\|_{H^{-s}(\mathbb{R}^n)},
\end{aligned} \tag{2.12}$$

that is by (2.8) and (2.10),

$$\begin{aligned}
\sum_{j=1}^J \|\varphi_j\gamma^* v\|^2_{H^{-s}(\mathbb{R}^n)} &\geqslant \tilde{C}'^2 \sum_{j=1}^J \big\|\gamma_0^*\big\{D_j\big[(\varphi_j v)\circ\varkappa_j\big]\big\}\big\|^2_{H^{-s}(\mathbb{R}^n)} \\
&= \tilde{C}'^2 C_s \sum_{j=1}^J \big\|D_j\big[(\varphi_j v)\circ\varkappa_j\big]\big\|^2_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})} = \tilde{C}'^2 C_s \|v\|^2_{H^{\frac{1}{2}-s}(\partial\Omega)}.
\end{aligned} \tag{2.13}$$

Since

$$\tilde{C}_j\|\gamma^* v\|_{H^{-s}(\mathbb{R}^n)} \geqslant \|\varphi_j\gamma^* v\|_{H^{-s}(\mathbb{R}^n)} \tag{2.14}$$

for $\varphi_j \in \mathcal{D}(\mathbb{R}^n)$, (2.13) gives the left inequality in (2.6).

Obviously, (2.6) implies (2.7) for $\gamma^*$ and thus for $\gamma$.

As was shown in the first paragraph of the proof, the functional $\gamma_0^*\{D_j[(\varphi_j v)\circ\varkappa_j]\}$ is not bounded on $H^{\frac{1}{2}}(\mathbb{R}^n)$ for any non-zero $v$, then (2.12), (2.14) imply that the operator $\gamma^* : H^0(\partial\Omega) \to H^{-\frac{1}{2}}(\mathbb{R}^n)$ and thus the operator $\gamma : H^{\frac{1}{2}}(\mathbb{R}^n) \to H^0(\partial\Omega)$ is not bounded. □

For $s > 3/2$ the trace operators (2.5) are not continuous on Lipschitz domains, however the following weaker statement holds, which was mentioned in [5] without a proof but can be indeed proved by appropriate estimates of an integral on p. 598 of [5] for this case.

**Lemma 2.5.** *If $\Omega$ is a Lipschitz domain and $s > 3/2$, then the trace operators*

$$\gamma : H^s(\mathbb{R}^n) \to H^1(\partial\Omega) \quad \textit{and} \quad \gamma^\pm : H^s(\Omega^\pm) \to H^1(\partial\Omega)$$

*are continuous.*

**Lemma 2.6.** *For a Lipschitz domain $\Omega$ there exists a linear bounded extension operator $\gamma_{-1} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\mathbb{R}^n)$, $\frac{1}{2} \leqslant s \leqslant \frac{3}{2}$, which is right inverse to the trace operator $\gamma$, i.e., $\gamma\gamma_{-1}g = g$ for any $g \in H^{s-\frac{1}{2}}(\partial\Omega)$. (For $s = \frac{1}{2}$ the trace operator $\gamma$ is understood not as in Definition 2.2 but in the non-tangential sense; see, e.g. [8].) Moreover, $\|\gamma_{-1}\|_{H^{s-\frac{1}{2}}(\partial\Omega)\to H^s(\mathbb{R}^n)} \leqslant C$, where $C$ is independent of $s$.*

**Proof.** For Lipschitz domains and $\frac{1}{2} < s \leqslant 1$, the boundedness of the extension operator is well known, see e.g. [13, Theorem 3.37].

To prove it for the whole range $\frac{1}{2} \leqslant s \leqslant \frac{3}{2}$, let us consider the Green operator $G_\Delta$ that solves the Dirichlet Problem for the Laplace equation in $\Omega$ and continuously maps $H^{s-\frac{1}{2}}(\partial\Omega)$ to $H^s(\Omega)$ if $\Omega$ is a bounded domain and to $H^s_{loc}(\overline{\Omega})$ if $\Omega$ is an unbounded domain. Particularly one can take $G_\Delta = V_\Delta \mathcal{V}_\Delta^{-1}$, where the single layer potential $V_\Delta\varphi$ with a density $\varphi = \mathcal{V}_\Delta^{-1} g \in H^{s-\frac{3}{2}}(\partial\Omega)$, solves the Laplace equation in $\Omega$ with the Dirichlet boundary data $g$ and $\mathcal{V}_\Delta$ is the direct value of the operator $V_\Delta$ on the boundary. The operators $\mathcal{V}_\Delta^{-1} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)$ and $V_\Delta : H^{s-\frac{3}{2}}(\partial\Omega) \to H^s_{loc}(\mathbb{R}^n)$ are continuous for $\frac{1}{2} \leqslant s \leqslant \frac{3}{2}$ as stated in [9,8,10,21,4]. Thus it suffice to take $\gamma_{-1} = \chi G_\Delta$, where $\chi \in \mathcal{D}(\mathbb{R}^n)$ is a cut-off function such that $\chi = 1$ in a sufficiently large open ball such that it includes the boundary $\partial\Omega$. The estimate $\|\gamma_{-1}\|_{H^{s-\frac{1}{2}}(\partial\Omega)\to H^s(\mathbb{R}^n)} \leqslant C$, where $C$ is independent of $s$, then follows. □

Note that continuity of the operator $\gamma$ was not needed in the proof.

Let us denote by $E_0$ the operator of extension of a function defined in $\Omega$ by zero outside $\Omega$ to a function defined in $\mathbb{R}^n$.

**Theorem 2.7.** *Let $\Omega$ be a Lipschitz domain and $s \geqslant 0$ while $s \neq \frac{1}{2} + k$ for any integer $k \geqslant 0$. Then*

$$\tilde{H}^s(\Omega) = H^s_0(\Omega)$$

*in the sense that $u|_\Omega \in H^s_0(\Omega)$ for any $u \in \tilde{H}^s(\Omega)$, and $E_0 v \in \tilde{H}^s(\Omega)$ for any $v \in H^s_0(\Omega)$. Moreover*

$$\|u|_\Omega\|_{H^s(\Omega)} \leqslant \|u\|_{\tilde{H}^s(\Omega)}, \qquad \|E_0 v\|_{\tilde{H}^s(\Omega)} \leqslant C\|v\|_{H^s(\Omega)}, \tag{2.15}$$

*where $C$ depends only on $s$ and on the maximum of the Lipschitz constants of the representation functions $\zeta_j$ for the boundary $\partial\Omega$, see Section* 2.1.

**Proof.** The first claim is proved in [13, Theorem 3.33]. The first estimate in (2.15) is evident, while the second follows from the proofs of the same Theorem 3.33 and Lemma 3.32 in [13]. □

To characterize the space $H^s_0(\Omega) = \tilde{H}^s(\Omega)$ for $\frac{1}{2} < s < \frac{3}{2}$, we will need the following statement.

**Lemma 2.8.** *If $\Omega$ is a Lipschitz domain and $u \in H^s(\Omega)$, $0 < s < \frac{1}{2}$, then*

$$\int_\Omega \operatorname{dist}(x, \partial\Omega)^{-2s} \big|u(x)\big|^2 \, dx \leqslant C\|u\|^2_{H^s(\Omega)} \tag{2.16}$$

*and for a given boundary cover the constant $C$ depends only on $s$ and on the maximum of the Lipschitz constants of the boundary representation functions $\zeta_j$, see Section* 2.1.

**Proof.** Note first that the lemma claim for $u \in \mathcal{D}(\overline{\Omega})$ follows from the proof of [13, Lemma 3.32]. To prove it for $u \in H^s(\Omega)$, let first the domain $\Omega$ be such that

$$\operatorname{dist}(x, \partial\Omega) < C_0 < \infty \tag{2.17}$$

for all $x \in \Omega$, which holds true particularly for bounded domains. Let $\{\phi_k\} \in \mathcal{D}(\overline{\Omega})$ be a sequence converging to $u$ in $H^s(\Omega)$. If we denote $w(x) = \operatorname{dist}(x, \partial\Omega)^{-2s}$, then $w(x) > C_0^{-2s} > 0$. Since (2.16) holds for functions from $\mathcal{D}(\overline{\Omega})$, the sequence $\{\phi_k\} \in \mathcal{D}(\overline{\Omega})$ is fundamental in the weighted space $L_2(\Omega, w)$, which is complete, implying that $\phi_k \in \mathcal{D}(\overline{\Omega})$ converges in this space to a function $u' \in L_2(\Omega, w)$. Since both $L_2(\Omega, w)$ and $H^s(\Omega)$ are continuously imbedded in the non-weighted space $L_2(\Omega)$, the sequence $\{\phi_k\}$ converges in $L_2(\Omega)$ implying the limiting functions $u$ and $u'$ belong to this space and thus coincide. Then from (2.16) for $\phi_k$ we immediately obtain it for arbitrary $u \in H^s(\Omega)$.

For the unbounded domains for which condition (2.17) is not satisfied, let $\chi(x) \in \mathcal{D}(\mathbb{R}^n)$ be a cut-off function such that $0 \leqslant \chi(x) \leqslant 1$ for all $x$, $\chi(x) = 1$ near $\partial\Omega$, while $w(x) < 1$ for $x \in \operatorname{supp}(1-\chi)$. Then (2.17) is satisfied in $\Omega' = \Omega \cap \operatorname{supp}\chi(x)$ and

$$\begin{aligned}\int_\Omega w(x)\big|u(x)\big|^2 dx &= \int_\Omega \big(1-\chi(x)\big) w(x)\big|u(x)\big|^2 dx + \int_\Omega \chi(x) w(x)\big|u(x)\big|^2 dx \\ &\leqslant \|u\|^2_{L_2(\Omega)} + \int_{\Omega'} w(x)\big|\sqrt{\chi(x)}u(x)\big|^2 dx \leqslant \|u\|^2_{H^s(\Omega)} + C\big\|\sqrt{\chi(x)}u\big\|^2_{H^s(\Omega')} \leqslant C_1\|u\|^2_{H^s(\Omega)}\end{aligned}$$

due to the previous paragraph. □

Lemma 2.8 allows now extending the following statement known for $\frac{1}{2} < s \leqslant 1$, see [13, Theorem 3.40(ii)], to a wider range of $s$.

**Theorem 2.9.** *If $\Omega$ is a Lipschitz domain and $\frac{1}{2} < s < \frac{3}{2}$, then*

$$H_0^s(\Omega) = \{u \in H^s(\Omega):\ \gamma^+ u = 0\}. \tag{2.18}$$

**Proof.** Equality (2.18) for $\frac{1}{2} < s \leqslant 1$ is stated in [13, Theorem 3.40(ii)].

Let $1 < s < \frac{3}{2}$. If $u \in H_0^s(\Omega)$ then evidently $\gamma^+ u = 0$ since $\mathcal{D}$ is dense in $H_0^s(\Omega)$ and the trace operator $\gamma^+$ is bounded in $H^s(\Omega)$. To prove that any $u \in H^s(\Omega)$ with $\gamma^+ u = 0$ belongs to $H_0^s(\Omega)$, it remains, due to Theorem 2.7, to prove that $E_0 u \in H^s(\mathbb{R}^n)$. We remark first of all that $E_0 u \in H^1(\mathbb{R}^n)$ due to the previous paragraph and Theorem 2.7, and then make estimates similar to those in the proof of [13, Theorem 3.33],

$$\begin{aligned}
\|E_0 u\|^2_{H^s(\mathbb{R}^n)} &\sim \|E_0 u\|^2_{W_2^1(\mathbb{R}^n)} + \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\nabla E_0 u(x) - \nabla E_0 u(y)|^2}{|x-y|^{2(s-1)+n}}\, dx\, dy \\
&= \|u\|^2_{W_2^1(\Omega)} + \int\limits_{\Omega}\int\limits_{\Omega} \frac{|\nabla u(x) - \nabla u(y)|^2}{|x-y|^{2(s-1)+n}}\, dx\, dy \\
&\quad + \int\limits_{\mathbb{R}^n\setminus\Omega}\int\limits_{\Omega} \frac{|\nabla u(x)|^2}{|x-y|^{2(s-1)+n}}\, dx\, dy + \int\limits_{\Omega}\int\limits_{\mathbb{R}^n\setminus\Omega} \frac{|\nabla u(y)|^2}{|x-y|^{2(s-1)+n}}\, dx\, dy \\
&= \|u\|^2_{W_2^s(\Omega)} + 2\int\limits_{\Omega} \big|w_{s-1}(x)\nabla u(x)\big|^2\, dx,
\end{aligned}$$

where

$$w_{s-1}(x) := \int\limits_{\mathbb{R}^n\setminus\Omega} \frac{dy}{|x-y|^{2(s-1)+n}}, \quad x \in \Omega,$$

and $W_2^s(\Omega)$ is the Sobolev–Slobodetski space. Introducing spherical coordinates with $x$ as an origin, we obtain, $w_{s-1}(x) \leqslant \frac{\alpha_n}{2(s-1)} \operatorname{dist}(x, \partial\Omega)^{-2(s-1)}$ for $x \in \Omega$, where $\alpha_n$ is the area of the unit sphere in $\mathbb{R}^n$. Then, taking into account that $\nabla u \in H^{s-1}(\Omega)$ and $\|\nabla u\|_{H^{s-1}(\Omega)} \leqslant \|u\|_{H^s(\Omega)}$, we have by Lemma 2.8,

$$\|E_0 u\|^2_{H^s(\mathbb{R}^n)} \leqslant \|u\|^2_{W_2^s(\Omega)} + 2C\|u\|^2_{H^s(\Omega)} \leqslant C_s \|u\|^2_{H^s(\Omega)}.$$

Theorem 2.7 completes the proof. □

Let us now give a characterization of the space $H^t_{\partial\Omega}$.

**Theorem 2.10.** *Let $\Omega$ be a Lipschitz domain in $\mathbb{R}^n$.*

(i) *If $t \geqslant -\frac{1}{2}$, then $H^t_{\partial\Omega} = \{0\}$.*
(ii) *If $-\frac{3}{2} < t < -\frac{1}{2}$, then $g \in H^t_{\partial\Omega}$ if and only if $g = \gamma^* v$, i.e.,*

$$\langle g, W\rangle_{\mathbb{R}^n} = \langle v, \gamma W\rangle_{\partial\Omega} \quad \forall W \in H^{-t}(\mathbb{R}^n), \tag{2.19}$$

*with $v = \gamma^*_{-1} g \in H^{t+\frac{1}{2}}(\partial\Omega)$, i.e.,*

$$\langle v, w\rangle_{\partial\Omega} = \langle g, \gamma_{-1} w\rangle_{\mathbb{R}^n} \quad \forall w \in H^{-t-\frac{1}{2}}(\partial\Omega), \tag{2.20}$$

*where $v$ is independent of the choice of the non-unique operators $\gamma_{-1}$, $\gamma^*_{-1}$, and the estimate $\|v\|_{H^{t+\frac{1}{2}}(\partial\Omega)} \leqslant C\|g\|_{H^t(\mathbb{R}^n)}$ holds with $C$ independent of $t$.*

**Proof.** We will follow an idea in the proof of Lemma 3.39 in [13] (see also [3, Proposition 4.8]), extending it from a half-space to a Lipschitz domain $\Omega$.

Let $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$. For any $\phi \in \mathcal{D}(\mathbb{R}^n)$, let us define

$$\phi^\pm(x) = \begin{cases} \phi(x) & \text{if } x \in \Omega^\pm, \\ 0 & \text{if } x \notin \Omega^\pm. \end{cases}$$

Let $t > -\frac{1}{2}$. Then $\phi^{\pm} \in \tilde{H}^{-t}(\Omega^{\pm})$ (see e.g. [13, Theorem 3.40] and Theorem 2.7 for $-\frac{1}{2} < t \leqslant 0$, for greater $t$ it then follows by embedding), $\|\phi - \phi^+ - \phi^-\|_{H^{-t}(\mathbb{R}^n)} = 0$, and there exist sequences $\{\phi_k^{\pm}\} \in \mathcal{D}(\Omega^{\pm})$ converging to $\phi^{\pm}$ in $\tilde{H}^{-t}(\Omega^{\pm})$ as $k \to \infty$. Hence $\langle g, \phi\rangle_{\mathbb{R}^n} = \lim_{k\to\infty}\langle g, \phi_k^+ + \phi_k^-\rangle_{\mathbb{R}^n} = 0$ for any $\phi \in \mathcal{D}(\mathbb{R}^n)$ proving (i) for $t > -\frac{1}{2}$ since $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^{-t}(\mathbb{R}^n) = [H^t(\mathbb{R}^n)]^*$.

Let us prove (ii). For $g \in H^t_{\partial\Omega}$, $-\frac{3}{2} < t < -\frac{1}{2}$, let $v \in H^{t+\frac{1}{2}}(\partial\Omega)$ be defined by (2.20), where existence and continuity of $\gamma_{-1} : H^{-t-\frac{1}{2}}(\partial\Omega) \to H^{-t}(\Omega)$ is proved in Lemma 2.6. Observe that

$$\left|\langle v, w\rangle_{\partial\Omega}\right| \leqslant \|g\|_{H^t(\mathbb{R}^n)}\|w\|_{H^{-t-\frac{1}{2}}(\partial\Omega)}\|\gamma_{-1}\|_{H^{-t-\frac{1}{2}}(\partial\Omega)\to H^{-t}(\mathbb{R}^n)},$$

so $\|v\|_{H^{t+\frac{1}{2}}(\partial\Omega)} \leqslant \|\gamma_{-1}\|_{H^{-t-\frac{1}{2}}(\partial\Omega)\to H^{-t}(\mathbb{R}^n)}\|g\|_{H^t(\mathbb{R}^n)} \leqslant C\|g\|_{H^t(\mathbb{R}^n)}$, where $C$ is independent of $t$ due to Lemma 2.6 if $\gamma_{-1}$ is chosen as in that lemma. We also have that

$$\langle g, W\rangle_{\mathbb{R}^n} - \langle v, \gamma W\rangle_{\partial\Omega} = \langle g, \rho\rangle_{\mathbb{R}^n} \quad \forall W \in H^{-t}\big(\mathbb{R}^n\big),$$

where

$$\rho = W - \gamma_{-1}\gamma W \in H^{-t}\big(\mathbb{R}^n\big).$$

Then we have $\gamma\rho = 0$, which due to Theorems 2.7, 2.9 implies $\tilde{\rho}^{\pm} \in \tilde{H}^{-t}(\Omega^{\pm})$, where $\tilde{\rho}^{\pm}$ are extensions of $\rho|_{\Omega^{\pm}}$ by zero outside $\Omega^{\pm}$, and $\rho = \tilde{\rho}^+ + \tilde{\rho}^-$. Thus there exist sequences $\{\rho_k^{\pm}\} \in \mathcal{D}(\Omega^{\pm})$ converging to $\tilde{\rho}^{\pm}$ in $\tilde{H}^{-t}(\Omega^{\pm})$, implying $\langle g, \rho\rangle_{\mathbb{R}^n} = 0$ since $g \in H^t_{\partial\Omega}$, and thus ansatz (2.19). To prove that $v$ is uniquely determined by $g$, i.e., independent of $\gamma_{-1}$, let us consider $v'$ and $v''$ corresponding to different operators $\gamma'_{-1}$ and $\gamma''_{-1}$. Then by (2.19),

$$\begin{aligned}\big\langle v' - v'', w\big\rangle_{\partial}\Omega &= \big\langle \gamma^{*\prime}_{-1}g - \gamma^{*\prime\prime}_{-1}g, w\big\rangle_{\partial\Omega} = \big\langle g, \gamma'_{-1}w - \gamma''_{-1}w\big\rangle_{\mathbb{R}^n}\\ &= \big\langle v', \gamma\big(\gamma'_{-1}w - \gamma''_{-1}w\big)\big\rangle_{\partial\Omega} = 0 \quad \forall w \in H^{-t-\frac{1}{2}}(\partial\Omega).\end{aligned}$$

It remains to deal with the case $t = -\frac{1}{2}$ in (i). Let $g \in H^{-\frac{1}{2}}_{\partial\Omega}$. Since $H^{-\frac{1}{2}}_{\partial\Omega} \subset H^t_{\partial\Omega}$ for $-\frac{3}{2} < t < -\frac{1}{2}$, then $g = \gamma^* v$ for some $v \in H^{t+\frac{1}{2}}(\partial\Omega)$ $\forall t \in (-\frac{3}{2}, -\frac{1}{2})$, and $\|g\|_{H^t_{\partial\Omega}} = \|\gamma^* v\|_{H^t_{\partial\Omega}} \geqslant C'\sqrt{C_{-t}}\|v\|_{H^{\frac{1}{2}+t}(\partial\Omega)}$ owing to Lemma 2.4. Since $C_{-t} \to \infty$ as $t \nearrow -\frac{1}{2}$, this means $\|v\|_{H^{\frac{1}{2}+t}(\partial\Omega)} \to 0$ as $t \nearrow -\frac{1}{2}$ implying $v = 0$. □

Combining (2.19) and (2.20) we have the following useful statement.

**Corollary 2.11.** *Let $\Omega$ be a Lipschitz domain in $\mathbb{R}^n$. If $g \in H^t_{\partial\Omega}$ with $-\frac{3}{2} < t < -\frac{1}{2}$, then $g = \gamma^*\gamma^*_{-1}g$ for any choice of $\gamma^*_{-1}$.*

**Theorem 2.12.** *Let $\Omega$ be a Lipschitz domain in $\mathbb{R}^n$ and $s \leqslant \frac{1}{2}$. Then $\mathcal{D}(\Omega)$ is dense in $H^s(\Omega)$, i.e., $H^s(\Omega) = H^s_0(\Omega)$.*

**Proof.** The proof for $0 \leqslant s \leqslant \frac{1}{2}$ is available in [13, Theorem 3.40(i)]. To prove the statement for any $s \leqslant \frac{1}{2}$ we remark that if $w \in H^s(\Omega)^* = \tilde{H}^{-s}(\Omega)$ satisfies $\langle w, \phi\rangle = 0$ for all $\phi \in \mathcal{D}(\Omega)$, then $w \in H^{-s}_{\partial\Omega}$ and Theorem 2.10(i) implies $w = 0$. Hence, $\mathcal{D}(\Omega)$ is dense in $H^s(\Omega)$, i.e., $H^s(\Omega) = H^s_0(\Omega)$. □

Theorem 2.12 implies that for any $u \in \mathcal{D}(\overline{\Omega})$ and $s \leqslant \frac{1}{2}$ there exists a sequence $\{\phi_k\} \in \mathcal{D}(\Omega)$ converging to $u$ in $H^s(\Omega)$. Evidently $\phi_k|_{\partial\Omega}$ converges to 0 in $H^{\sigma}(\partial\Omega)$ for any $\sigma$ since $\phi_k|_{\partial\Omega} = 0$. On the other hand, $u \in \mathcal{D}(\overline{\Omega})$ is the limit in $H^s(\Omega)$ of the sequence $\{\phi'_k\} = u$, meaning that $\phi'_k|_{\partial\Omega}$ converges in $H^{\sigma}(\partial\Omega)$ to $u|_{\partial\Omega}$, which is generally non-zero. This leads to the following conclusion of non-existence.

**Corollary 2.13.** *For $s \leqslant \frac{1}{2}$ the trace operators $\gamma^{\pm} : H^s(\Omega^{\pm}) \to H^{\sigma}(\partial\Omega)$, understood as in Definition 2.2, do not exist for any $\sigma$.*

**Remark 2.14.** (i) Evidently, Corollary 2.13 holds also if the space $H^{\sigma}(\partial\Omega)$ is replaced with any Banach space of distributions on $\partial\Omega$.

(ii) The trace operator $\gamma^{\pm} : B(\Omega^{\pm}) \to H^{\sigma}(\partial\Omega)$ can, of course, still exist on some Banach subspaces on $\Omega^{\pm}$, $B(\Omega^{\pm}) \subset H^s(\Omega^{\pm})$, $s \leqslant \frac{1}{2}$, with the norms stronger than the norm in $H^s(\Omega^{\pm})$, particularly on $H^t(\Omega^{\pm})$, $t > \frac{1}{2}$.

The following two statements give conditions when distributions from $H^s(\Omega)$ can be extended to distributions from $\tilde{H}^s(\Omega)$ and when the extension can be written in terms of a linear bounded operator. The first of them can be considered as a counterpart of Theorem 2.7 for negative $s$.

**Lemma 2.15.** *Let $\Omega$ be a Lipschitz domain, $s < \frac{1}{2}$, $s \neq \frac{1}{2} - k$ for any integer $k > 0$. Then for any $g \in H^s(\Omega)$ there exists $\tilde{g} \in \tilde{H}^s(\Omega)$ such that $g = \tilde{g}|_{\Omega}$ and $\|\tilde{g}\|_{\tilde{H}^s(\Omega)} \leqslant C\|g\|_{H^s(\Omega)}$, where $C > 0$ does not depend on $g$.*

**Proof.** Any distribution $g \in H^s(\Omega)$ is a bounded linear functional on $\tilde{H}^{-s}(\Omega)$. On the other hand, for any $v \in H_0^{-s}(\Omega) \subset H^{-s}(\Omega)$ its zero extension $\tilde{v} = E_0 v$ belongs to $\tilde{H}^{-s}(\Omega)$ with

$$\|\tilde{v}\|_{\tilde{H}^{-s}(\Omega)} \leqslant C\|v\|_{H^{-s}(\Omega)} \tag{2.21}$$

for $s \leqslant 0$, $s \neq \frac{1}{2} - k$, by Theorem 2.7. This holds true also for $0 < s < \frac{1}{2}$ since then $\tilde{H}^{-s}(\Omega) = [H^s(\Omega)]^* = [H_0^s(\Omega)]^* = [\tilde{H}^s(\Omega)]^* = H^{-s}(\Omega)$ by Theorems 2.12 and 2.7, while the extension $\tilde{v} \in \tilde{H}^{-s}(\Omega)$ is defined as

$$\langle \tilde{v}, w\rangle := \langle v, E_0 w\rangle \quad \forall w \in H^s(\Omega), \quad 0 < s < \frac{1}{2},$$

and by Theorems 2.12 and 2.7,

$$\begin{aligned}\|\tilde{v}\|_{\tilde{H}^{-s}(\Omega)} &= \sup_{w\in H^s(\Omega)\setminus\{0\}} \frac{|\langle \tilde{v}, w\rangle|}{\|w\|_{H^s(\Omega)}} = \sup_{w\in H^s(\Omega)\setminus\{0\}} \frac{|\langle v, E_0 w\rangle|}{\|w\|_{H^s(\Omega)}}\\ &\leqslant C \sup_{w\in H^s(\Omega)\setminus\{0\}} \frac{|\langle v, E_0 w\rangle|}{\|E_0 w\|_{\tilde{H}^s(\Omega)}} \leqslant C\|v\|_{H^{-s}(\Omega)}\end{aligned}$$

giving estimate (2.21).

Thus the functional $g \in H^s(\Omega)$ continuous on $\tilde{H}^{-s}(\Omega)$ and thus on $H_0^{-s}(\Omega)$ can be extended by the Hahn–Banach theorem to a functional $\tilde{g} \in \tilde{H}^s(\Omega)$ continuous on $H^{-s}(\Omega)$ such that $\|\tilde{g}\|_{\tilde{H}^s(\Omega)} = \|\tilde{g}\|_{[H^{-s}(\Omega)]^*} = \|g\|_{[H_0^{-s}(\Omega)]^*}$. Then by estimate (2.21) for $s < \frac{1}{2}$, $s \neq \frac{1}{2} - k$, we have,

$$\|g\|_{[H_0^{-s}(\Omega)]^*} = \sup_{v\in H_0^{-s}(\Omega)\setminus\{0\}} \frac{|\langle g, v\rangle|}{\|v\|_{H_0^{-s}(\Omega)}} \leqslant C \sup_{\tilde{v}\in \tilde{H}^{-s}(\Omega)\setminus\{0\}} \frac{|\langle g, \tilde{v}\rangle|}{\|\tilde{v}\|_{\tilde{H}^{-s}(\Omega)}} \leqslant C\|g\|_{[\tilde{H}^{-s}(\Omega)]^*} = C\|g\|_{H^s(\Omega)},$$

which completes the proof. □

**Theorem 2.16.** *Let $\Omega$ be a Lipschitz domain and $-\frac{3}{2} < s < \frac{1}{2}$, $s \neq -\frac{1}{2}$. There exists a bounded linear extension operator $\tilde{E}^s : H^s(\Omega) \to \tilde{H}^s(\Omega)$, such that $\tilde{E}^s g|_\Omega = g$, $\forall g \in H^s(\Omega)$. For $-\frac{1}{2} < s < \frac{1}{2}$ the extension operator is unique, $(\tilde{E}^s)^* = \tilde{E}^{-s}$ and*

$$\left\|\tilde{E}^s g\right\|_{\tilde{H}^s(\Omega)} \leqslant C\|g\|_{H^s(\Omega)}, \tag{2.22}$$

*where $C$ depends only on $s$ and on the maximum of the Lipschitz constants of the representation functions $\zeta_j$ for the boundary $\partial\Omega$, see Section* 2.1.

**Proof.** If $0 \leqslant s < \frac{1}{2}$, then $\tilde{H}^s(\Omega) = \{E_0 u,\ u \in H^s(\Omega)\}$, which implies that one can take $\tilde{E}^s = E_0$, where the operator $E_0 : H^s(\Omega) \to \tilde{H}^s(\Omega)$ of extension by zero is continuous by the Theorems 2.7 and 2.12 with the estimate (2.22) following from estimate (2.15).

If $-\frac{1}{2} < s < 0$, we define $\tilde{E}^s$ as

$$\left\langle \tilde{E}^s g, v\right\rangle_\Omega := \langle g, E_0 v\rangle_\Omega, \quad \forall g \in H^s(\Omega),\ \forall v \in H^{-s}(\Omega),$$

i.e., $\tilde{E}^s = E_0^* = (\tilde{E}^{-s})^*$, which is continuous with the estimate (2.22) following from the previous paragraph.

Theorem 2.10 implies that the extension operator $\tilde{E}^s : H^s(\Omega) \to \tilde{H}^s(\Omega)$ is unique for $-\frac{1}{2} < s < \frac{1}{2}$.

Let now $-\frac{3}{2} < s < -\frac{1}{2}$. For $s$ in this range, the trace operator $\gamma^+ : H^{-s}(\Omega) \to H^{-s-\frac{1}{2}}(\partial\Omega)$ is bounded due to [4, Lemma 3.6] (see also [13, Theorem 3.38]), and there exists a bounded right inverse to the trace operator $\gamma_{-1} : H^{-s-\frac{1}{2}}(\partial\Omega) \to H^{-s}(\Omega)$, see Lemma 2.6. Then $(I - \gamma_{-1}\gamma^+)$ is a bounded projector from $H^{-s}(\Omega)$ to $H_0^{-s}(\Omega) = \tilde{H}^{-s}(\Omega)$ due to Theorem 2.7. Thus any functional $v \in H^s(\Omega)$ can be continuously mapped into the functional $\tilde{v} \in \tilde{H}^s(\Omega)$ such that $\langle \tilde{v}, u\rangle = \langle v, E_0(I - \gamma_{-1}\gamma^+)u\rangle$ for any $u \in H^{-s}(\Omega)$. Since $\tilde{v}u = vu$ for any $u \in \tilde{H}^{-s}(\Omega)$, we have,

$$\tilde{E}^s := \left[E_0\left(I - \gamma_{-1}\gamma^+\right)\right]^* : H^s(\Omega) \to \tilde{H}^s(\Omega)$$

is a bounded extension operator. □

Since the extension operator $\tilde{E}^s : H^s(\Omega) \to \tilde{H}^s(\Omega)$ is unique for $-\frac{1}{2} < s < \frac{1}{2}$, we will call it *canonical extension operator* (as opposite to other possible extensions from $H^s(\Omega)$ to $\tilde{H}^\sigma(\Omega)$, $\sigma < -\frac{1}{2}$). For $-\frac{3}{2} < s < -\frac{1}{2}$, on the other hand, the operator $\gamma_{-1} : H^{-s-\frac{1}{2}}(\partial\Omega) \to H^{-s}(\Omega)$ in the proof of Theorem 2.16 is not unique, implying non-uniqueness of $\tilde{E}^s : H^s(\Omega) \to \tilde{H}^s(\Omega)$.

We will later need the following two results.

**Lemma 2.17.** *Let $\Omega$ and $\Omega' \subset \Omega$ be open sets, and $s \leqslant 0$. If $u \in H^s(\Omega)$, then $\|u\|_{H^s(\Omega')} \to 0$ as the Lebesgue measure of $\Omega'$ tends to zero.*

**Proof.** Let $\phi \in \mathcal{D}(\overline{\Omega})$. Then

$$\|u\|_{H^s(\Omega')} \leqslant \|u-\phi\|_{H^s(\Omega')} + \|\phi\|_{H^s(\Omega')} \leqslant \|u-\phi\|_{H^s(\Omega)} + \|\phi\|_{L_2(\Omega')}.$$

For any $\epsilon > 0$ we can chose $\phi$ such that $\|u-\phi\|_{H^s(\Omega)} < \epsilon/2$ due to the density of $\mathcal{D}(\overline{\Omega})$ in $H^s(\Omega)$ and then chose $\Omega'$ with sufficiently small measure so that $\|\phi\|_{L_2(\Omega')} < \epsilon/2$. $\square$

**Lemma 2.18.** *Let $\Omega_k \subset \Omega$ be a sequence of Lipschitz domains converging to a Lipschitz domain $\Omega$ and $-\frac{1}{2} < s < 1/2$. If $u \in H^s(\Omega)$ and $\tilde{u}_k = \tilde{E}^s u|_{\Omega_k}$, then there exists a constant $C$ independent of $u$ and $k$ such that $\|\tilde{u}_k\|_{\tilde{H}^s(\Omega_k)} \leqslant C\|u\|_{H^s(\Omega)}$ for all sufficiently large $k$.*

**Proof.** By Theorem 2.16,

$$\|\tilde{u}_k\|_{\tilde{H}^s(\Omega_k)} \leqslant C_k \|u|_{\Omega_k}\|_{H^s(\Omega_k)} \leqslant C_k\|u\|_{H^s(\Omega)},$$

where $C_k$ depend only on $s$ and on the maximum of the Lipschitz constants of the representation functions $\zeta_{jk}$ for the boundaries $\partial\Omega_k$. By (2.4), the Lipschitz constants are bounded and henceforth so are $C_k$. $\square$

## 3. Partial differential operator extensions and co-normal derivatives for infinitely smooth coefficients

Let us consider in $\Omega$ a system of $m$ complex linear differential equations of the second order with respect to $m$ unknown functions $\{u_i\}_{i=1}^m = u : \Omega \to \mathbb{C}^m$, which for sufficiently smooth $u$ has the following strong form,

$$Au(x) := -\sum_{i,j=1}^{n} \partial_i\big[a_{ij}(x)\partial_j u(x)\big] + \sum_{j=1}^{n} b_j(x)\partial_j u(x) + c(x)u(x) = f(x), \quad x \in \Omega, \tag{3.1}$$

where $f : \Omega \to \mathbb{C}^m$, $\partial_j := \partial/\partial x_j$ $(j = 1, 2, \dots, n)$, $a(x) = \{a_{ij}(x)\}_{i,j=1}^n = \{\{a_{ij}^{kl}(x)\}_{k,l=1}^m\}_{i,j=1}^n$, $b(x) = \{\{b_i^{kl}(x)\}_{k,l=1}^m\}_{i=1}^n$ and $c(x) = \{c^{kl}(x)\}_{k,l=1}^m$, i.e., $a_{ij}, b_i, c : \Omega \to \mathbb{C}^{m\times m}$ for fixed indices $i$, $j$. If $m = 1$, then (3.1) is a scalar equation. In this paper we assume that $a, b, c \in C^\infty(\overline{\Omega})$; the case of non-smooth coefficients is addressed in [14], see also [18].

The operator $A$ is (uniformly) strongly elliptic in an open domain $\Omega$ if there exists a bounded $m \times m$ matrix-valued function $\theta(x)$ such that

$$\operatorname{Re}\left\{\bar{\zeta}^\top \theta(x) \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j\zeta\right\} \geqslant C|\xi|^2|\zeta|^2$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{C}^m$, where $C$ is a positive constant, see e.g. [7, Definition 3.6.1] and references therein. We say that the operator $A$ is uniformly strongly elliptic in a closed domain $\overline{\Omega}$ if its is uniformly strongly elliptic in an open domain $\Omega' \supset \overline{\Omega}$. We will need the strong ellipticity in relation with the solution regularity, starting from Theorem 3.11.

### 3.1. *Partial differential operator extensions and generalized co-normal derivative*

For $u \in H^s(\Omega)$, $f \in H^{s-2}(\Omega)$, $s \in \mathbb{R}$, equation system (3.1) is understood in the distribution sense as

$$\langle Au, v\rangle_\Omega = \langle f, v\rangle_\Omega \quad \forall v \in \mathcal{D}(\Omega),$$

where $v : \Omega \to \mathbb{C}^m$ and

$$\langle Au, v\rangle_\Omega := \mathcal{E}(u, v) \quad \forall v \in \mathcal{D}(\Omega), \tag{3.2}$$

$$\mathcal{E}(u, v) = \mathcal{E}_\Omega(u, v) := \sum_{i,j=1}^{n} \langle a_{ij}\partial_j u, \partial_i v\rangle_\Omega + \sum_{j=1}^{n}\langle b_j\partial_j u, v\rangle_\Omega + \langle cu, v\rangle_\Omega. \tag{3.3}$$

Bilinear form (3.3) is well defined for any $v \in \mathcal{D}(\Omega)$ and moreover, the bilinear functional $\mathcal{E} : \{H^s(\Omega), \tilde{H}^{2-s}(\Omega)\} \to \mathbb{C}$ is bounded for any $s \in \mathbb{R}$. Since the set $\mathcal{D}(\Omega)$ is dense in $\tilde{H}^{2-s}(\Omega)$, expression (3.2) defines then a bounded linear operator $A : H^s(\Omega) \to H^{s-2}(\Omega) = [\tilde{H}^{2-s}(\Omega)]^*$, $s \in \mathbb{R}$,

$$\langle Au, v\rangle_\Omega := \mathcal{E}(u, v) \quad \forall v \in \tilde{H}^{2-s}(\Omega). \tag{3.4}$$

Let now $\frac{1}{2} < s < \frac{3}{2}$. In addition to the operator $A$ defined by (3.4), let us consider also the *aggregate* partial differential operator $\check{A}$, defined as,

$$\langle \check{A}u, v\rangle_\Omega := \check{\mathcal{E}}(u, v) \quad \forall v \in H^{2-s}(\Omega), \tag{3.5}$$

where

$$\check{\mathcal{E}}(u,v)=\check{\mathcal{E}}_\Omega(u,v):=\sum_{i,j=1}^{n}\big\langle\tilde{E}^{s-1}(a_{ij}\partial_j u),\partial_i v\big\rangle_\Omega+\sum_{j=1}^{n}\big\langle\tilde{E}^{s-1}(b_j\partial_j u),v\big\rangle_\Omega+\big\langle\tilde{E}^{s-1}(cu),v\big\rangle_\Omega \tag{3.6}$$

and $\tilde{E}^{s-1}: H^{s-1}(\Omega)\to\tilde{H}^{s-1}(\Omega)$ is a bounded extension operator, which is unique by Theorem 2.16. Note that by (2.2) one can rewrite (3.5) also as

$$(\check{A}u,v)_\Omega:=\Phi(u,v)\quad\forall v\in H^{2-s}(\Omega),$$

where $\Phi(u,v)=\overline{\check{\mathcal{E}}(u,\bar{v})}$ is the sesquilinear form.

If $s=1$, i.e. $u,v\in H^1(\Omega)$, evidently

$$\check{\mathcal{E}}(u,v)=\mathcal{E}(u,v)=\int_\Omega\Bigg[\sum_{i,j=1}^{n}(a_{ij}\partial_j u)\cdot\partial_i v+\sum_{j=1}^{n}(b_j\partial_j u)\cdot v+cu\cdot v\Bigg]dx.$$

The aggregate operator $\check{A}: H^s(\Omega)\to\tilde{H}^{s-2}(\Omega)=[H^{2-s}(\Omega)]^*$ is bounded since $\partial_i v\in H^{1-s}(\Omega)$, $v\in H^{2-s}(\Omega)\subset H^{1-s}(\Omega)$. For any $u\in H^s(\Omega)$, the functional $\check{A}u$ belongs to $\tilde{H}^{s-2}(\Omega)$ and is an extension of the functional $Au\in H^{s-2}(\Omega)$ from the domain of definition $\tilde{H}^{2-s}(\Omega)\subset H^{2-s}(\Omega)$ to the domain of definition $H^{2-s}(\Omega)$.

The functional $\check{A}u$ is not the only possible extension of the functional $Au$, and any functional of the form

$$\check{A}u+g,\quad g\in H^{s-2}_{\partial\Omega} \tag{3.7}$$

gives another extension. On the other hand, any extension of the domain of definition of the functional $Au$ from $\tilde{H}^{2-s}(\Omega)$ to $H^{2-s}(\Omega)$ has evidently form (3.7). The existence of such extensions is provided by Lemma 2.15.

For $u\in H^s(\Omega)$, $s>\frac{3}{2}$, the strong (classical) co-normal derivative operator

$$T_c^+u(x):=\sum_{i,j=1}^{n}a_{ij}(x)\gamma^+\big[\partial_j u(x)\big]\nu_i(x) \tag{3.8}$$

is well defined on $\partial\Omega$ in the sense of traces. Here $\gamma^+[\partial_j u]\in H^{s-\frac{3}{2}}(\partial\Omega)\subset L_2(\partial\Omega)$ if $\frac{3}{2}<s<\frac{5}{2}$, while the outward (to $\Omega$) unit normal vector $\nu(x)$ at the point $x\in\partial\Omega$ belongs to $L_\infty(\partial\Omega)$ for the Lipschitz boundary $\partial\Omega$, implying $T_c^+u\in L_2(\partial\Omega)$. Note that for Lipschitz domains one can not generally expect that $T_c^+u$ belongs to $H^s(\partial\Omega)$, $s>0$, even for infinitely smooth $u$.

We can extend the definition of the generalized co-normal derivative, given in [13, Lemma 4.3] for $s=1$ (cf. also [11, Lemma 2.2] for the generalized co-normal derivative on a manifold boundary), to a range of Sobolev spaces as follows.

**Definition 3.1.** Let $\Omega$ be a Lipschitz domain, $\frac{1}{2}<s<\frac{3}{2}$, $u\in H^s(\Omega)$, and $Au=\tilde{f}|_\Omega$ in $\Omega$ for some $\tilde{f}\in\tilde{H}^{s-2}(\Omega)$. Let us define *the generalized co-normal derivative* $T^+(\tilde{f},u)\in H^{s-\frac{3}{2}}(\partial\Omega)$ as

$$\big\langle T^+(\tilde{f},u),w\big\rangle_{\partial\Omega}:=\check{\mathcal{E}}(u,\gamma_{-1}w)-\langle\tilde{f},\gamma_{-1}w\rangle_\Omega=\langle\check{A}u-\tilde{f},\gamma_{-1}w\rangle_\Omega\quad\forall w\in H^{\frac{3}{2}-s}(\partial\Omega), \tag{3.9}$$

where $\gamma_{-1}: H^{\frac{3}{2}-s}(\partial\Omega)\to H^{2-s}(\Omega)$ is a bounded right inverse to the trace operator.

The notation $T^+(\tilde{f},u)$ corresponds to the notation $\tilde{T}^+(\tilde{f},u)$ in [17].

**Theorem 3.2.** *Under the hypotheses of Definition* 3.1*, the generalized co-normal derivative $T^+(\tilde{f},u)$ is independent of the operator $\gamma_{-1}$, the estimate*

$$\big\|T^+(\tilde{f},u)\big\|_{H^{s-\frac{3}{2}}(\partial\Omega)}\leqslant C_1\|u\|_{H^s(\Omega)}+C_2\|\tilde{f}\|_{\tilde{H}^{s-2}(\Omega)} \tag{3.10}$$

*takes place, and the first Green identity holds in the following form,*

$$\big\langle T^+(\tilde{f},u),\gamma^+v\big\rangle_{\partial\Omega}=\check{\mathcal{E}}(u,v)-\langle\tilde{f},v\rangle_\Omega=\langle\check{A}u-\tilde{f},v\rangle_\Omega\quad\forall v\in H^{2-s}(\Omega). \tag{3.11}$$

**Proof.** For $s=1$ the theorem proof is available in [13, Lemma 4.3], which idea is extended here to the whole range $\frac{1}{2}<s<\frac{3}{2}$.

By Lemma 2.6, a bounded operator $\gamma_{-1}: H^{\frac{3}{2}-s}(\partial\Omega)\to H^{2-s}(\Omega)$ does exist. Then estimate (3.10) follows from (3.9).

To prove independence of the co-normal derivative $T^+(\tilde{f},u)$ of $\gamma_{-1}$, let us consider two co-normal derivatives generated by two different operators $\gamma'_{-1}$ and $\gamma''_{-1}$. Then their difference is

$$\big\langle T'^+(\tilde{f},u)-T''^+(\tilde{f},u),w\big\rangle_{\partial\Omega}=\big\langle\check{A}u-\tilde{f},\gamma'_{-1}w-\gamma''_{-1}w\big\rangle_\Omega\quad\forall w\in H^{\frac{3}{2}-s}(\partial\Omega).$$

By definition, $\check{A}u - \tilde{f} \in H^{s-2}_{\partial\Omega}$, which by Corollary 2.11 implies that

$$\begin{aligned}\langle \check{A}u - \tilde{f}, \gamma'_{-1}w - \gamma''_{-1}w\rangle_\Omega &= \langle \check{A}u - \tilde{f}, \gamma'_{-1}w - \gamma''_{-1}w\rangle_{\mathbb{R}^n} = \langle \gamma^*\gamma^*_{-1}(\check{A}u - \tilde{f}), \gamma'_{-1}w - \gamma''_{-1}w\rangle_{\mathbb{R}^n}\\ &= \langle \gamma^*_{-1}(\check{A}u - \tilde{f}), \gamma\gamma'_{-1}w - \gamma\gamma''_{-1}w\rangle_{\partial\Omega}\\ &= \langle \gamma^*_{-1}(\check{A}u - \tilde{f}), w - w\rangle_{\partial\Omega} = 0 \quad \forall w \in H^{\frac{3}{2}-s}(\partial\Omega).\end{aligned}$$

To prove (3.11), let $V \in H^{2-s}(\mathbb{R}^n)$ be such that $v = V|_\Omega$ implying $\gamma^+ v = \gamma V$. Taking again into account that $\check{A}u - \tilde{f} \in H^{s-2}_{\partial\Omega}$, we have by Corollary 2.11,

$$\begin{aligned}\langle T^+(\tilde{f}, u), \gamma^+ v\rangle_{\partial\Omega} &= \langle \check{A}u - \tilde{f}, \gamma_{-1}\gamma^+ v\rangle_\Omega = \langle \check{A}u - \tilde{f}, \gamma_{-1}\gamma V\rangle_{\mathbb{R}^n}\\ &= \langle \gamma^*\gamma^*_{-1}(\check{A}u - \tilde{f}), V\rangle_{\mathbb{R}^n} = \langle \check{A}u - \tilde{f}, V\rangle_{\mathbb{R}^n} = \langle \check{A}u - \tilde{f}, v\rangle_\Omega\end{aligned}$$

as required. □

Because of the involvement of $\tilde{f}$, the generalized co-normal derivative $T^+(\tilde{f}, u)$ is generally *non-linear* in $u$. It becomes linear if a linear relation is imposed between $u$ and $\tilde{f}$ (including behavior of the latter on the boundary $\partial\Omega$), thus fixing an extension of $\tilde{f}|_\Omega$ into $\tilde{H}^{s-2}(\Omega)$. For example, $\tilde{f}|_\Omega$ can be extended as $\check{f} := \check{A}u$, which generally does not coincide with $\tilde{f}$. Then obviously, $T^+(\check{f}, u) = T^+(\check{A}u, u) = 0$, meaning that the co-normal derivatives associated with any other possible extension $\tilde{f}$ appears to be aggregated in $\check{f}$ as

$$\langle \check{f}, v\rangle_\Omega = \langle \tilde{f}, v\rangle_\Omega + \langle T^+(\tilde{f}, u), \gamma^+ v\rangle_{\partial\Omega} \tag{3.12}$$

due to (3.11). This justifies the term *aggregate* for the extension $\check{f}$, and thus for the operator $\check{A}u$.

As follows from Definition 3.1, the generalized co-normal derivative is still linear with respect to the couple $(\tilde{f}, u)$, i.e.,

$$T^+(\alpha_1\tilde{f}_1, \alpha_1 u_1) + T^+(\alpha_2\tilde{f}_2, \alpha_2 u_2) = T^+(\alpha_1\tilde{f}_1 + \alpha_2\tilde{f}_2, \alpha_1 u_1 + \alpha_2 u_2)$$

for any complex numbers $\alpha_1, \alpha_2$.

In fact, for a given function $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, any distribution $\tau \in H^{s-\frac{3}{2}}(\partial\Omega)$ may be nominated as a co-normal derivative of $u$, by an appropriate extension $\tilde{f}$ of the distribution $Au \in H^{s-2}(\Omega)$ into $\tilde{H}^{s-2}(\Omega)$. This extension is again given by the second Green formula (3.11) re-written as follows (cf. [2, Section 2.2, item 4] for $s = 1$),

$$\langle \tilde{f}, v\rangle_\Omega := \check{\mathcal{E}}(u, v) - \langle \tau, \gamma^+ v\rangle_{\partial\Omega} = \langle \check{A}u - \gamma^{+*}\tau, v\rangle_\Omega \quad \forall v \in H^{2-s}(\Omega). \tag{3.13}$$

Here the operator $\gamma^{+*} : H^{s-\frac{3}{2}}(\partial\Omega) \to \tilde{H}^{s-2}(\Omega)$ is adjoined to the trace operator, $\langle \gamma^{+*}\tau, v\rangle_\Omega := \langle \tau, \gamma^+ v\rangle_{\partial\Omega}$ for all $\tau \in H^{s-\frac{3}{2}}(\partial\Omega)$ and $v \in H^{2-s}(\Omega)$. Evidently, the distribution $\tilde{f}$ defined by (3.13) belongs to $\tilde{H}^{s-2}(\Omega)$ and is an extension of the distribution $Au$ into $\tilde{H}^{s-2}(\Omega)$ since $\gamma^+ v = 0$ for $v \in \tilde{H}^{2-s}(\Omega)$.

For $u \in C^1(\overline{\Omega}) \subset H^1(\Omega)$, one can take $\tau$ equal to the strong co-normal derivative, $T^+_c u \in L_\infty(\partial\Omega)$, and relation (3.13) can be considered as the *classical extension* of $f = Au \in H^{-1}(\Omega)$ to $\tilde{f}_c \in \tilde{H}^{-1}(\Omega)$, which is evidently linear.

### 3.2. Boundary value problems

Consider the BVP weak settings for PDE system (3.1) on Lipschitz domain for $\frac{1}{2} < s < \frac{3}{2}$.

*The Dirichlet problem*: for $f \in H^{s-2}(\Omega)$ and $\varphi_0 \in H^{s-\frac{1}{2}}(\partial\Omega)$, find $u \in H^s(\Omega)$ such that

$$\langle Au, v\rangle_\Omega = \langle f, v\rangle_\Omega \quad \forall v \in \tilde{H}^{2-s}(\Omega), \tag{3.14}$$

$$\gamma^+ u = \varphi_0 \quad \text{on } \partial\Omega. \tag{3.15}$$

*The Neumann problem*: for $\check{f} \in \tilde{H}^{s-2}(\Omega)$, find $u \in H^s(\Omega)$ such that

$$\langle \check{A}u, v\rangle_\Omega = \langle \check{f}, v\rangle_\Omega \quad \forall v \in H^{2-s}(\Omega). \tag{3.16}$$

Here $Au$ and $\check{A}u$ are defined by (3.4) and (3.5), respectively.

To set the mixed problem, let $\partial_D\Omega$ and $\partial_N\Omega = \partial\Omega \setminus \overline{\partial_D\Omega}$ be nonempty, open sub-manifolds of $\partial\Omega$, and $H^s_0(\Omega, \partial_D\Omega) = \{w \in H^s(\Omega): \ \gamma^+ w = 0 \text{ on } \partial_D\Omega\}$. We introduce the *mixed aggregate* operator $\check{A}_{\partial_D\Omega} : H^s(\Omega) \to [H^{2-s}_0(\Omega, \partial_D\Omega)]^*$, defined as

$$\langle \check{A}_{\partial_D\Omega} u, v\rangle_\Omega := \langle \check{A}u, v\rangle_\Omega = \check{\mathcal{E}}(u, v) \quad \forall v \in H^{2-s}_0(\Omega, \partial_D\Omega).$$

The mixed operator $\check{A}_{\partial_D\Omega}$ is bounded by the same argument as the aggregate operator $\check{A}$. For any $u \in H^s(\Omega)$, the distribution $\check{A}_{\partial_D\Omega} u$ belongs to $[H^{2-s}_0(\Omega, \partial_D\Omega)]^*$ and is an extension of the functional $Au \in H^{s-2}(\Omega)$ from the domain of definition $\tilde{H}^{2-s}(\Omega) = H_0^{2-s}(\Omega) \subset H_0^{2-s}(\Omega, \partial_D\Omega)$ to the domain of definition $H_0^{2-s}(\Omega, \partial_D\Omega)$, and a restriction of the functional $\check{A}u \in \tilde{H}^{s-2}(\Omega)$ from the domain of definition $H^{2-s}(\Omega) \supset H_0^{2-s}(\Omega, \partial_D\Omega)$ to the domain of definition $H_0^{2-s}(\Omega, \partial_D\Omega)$.

For $v \in H_0^{2-s}(\Omega, \partial_D\Omega)$, the trace $\gamma^+ v$ belongs to $\tilde{H}^{3/2-s}(\partial_N\Omega)$. If $Au = \tilde{f}|_\Omega$ in $\Omega$ for some $\tilde{f} \in \tilde{H}^{s-2}(\Omega)$, then the first Green identity (3.11) gives,

$$\langle \check{A}_{\partial_D\Omega} u, v\rangle_\Omega = \langle \check{f}_m, v\rangle_\Omega,$$
$$\langle \check{f}_m, v\rangle_\Omega = \langle \tilde{f}, v\rangle_\Omega + \big\langle T^+(\tilde{f}, u), \gamma^+ v\big\rangle_{\partial_N\Omega} \quad \forall v \in H_0^{2-s}(\Omega, \partial_D\Omega), \tag{3.17}$$

where, evidently, $\check{f}_m \in [H_0^{2-s}(\Omega, \partial_D\Omega)]^*$. This leads to the following weak setting.

*The mixed (Dirichlet–Neumann) problem*: for $\check{f}_m \in [H_0^{2-s}(\Omega, \partial_D\Omega)]^*$ and $\varphi_0 \in H^{s-\frac{1}{2}}(\partial_D\Omega)$, find $u \in H^s(\Omega)$ such that

$$\langle \check{A}_{\partial_D\Omega} u, v\rangle_\Omega = \langle \check{f}_m, v\rangle_\Omega \quad \forall v \in H_0^{2-s}(\Omega, \partial_D\Omega), \tag{3.18}$$
$$\gamma^+ u = \varphi_0 \quad \text{on } \partial_D\Omega. \tag{3.19}$$

The Neumann and the mixed problems are formulated in terms of the aggregate right-hand sides $\check{f}$ and $\check{f}_m$, respectively, prescribed on their own, i.e., without necessary splitting them into the right-hand side inside the domain $\Omega$ and the part related with the prescribed co-normal derivative. If a right-hand side extension $\tilde{f}$ and an associated non-zero generalized co-normal derivative $T^+(\tilde{f}, u)$ are prescribed instead, then $\check{f}$ and $\check{f}_m$ can be expressed through them by relations (3.12), (3.17). Thus the co-normal derivative does not enter, in fact, the weak settings of the Dirichlet, Neumann or mixed problem, implying that the non-uniqueness of $T^+(\tilde{f}, u)$ for a given function $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, does not influence the BVP weak settings (cf. [2, Section 2.2, item 4] for $s = 1$). On the other hand, for a given $u \in H^s(\Omega)$ the aggregate right-hand sides $\check{f}$ and $\check{f}_m$ are uniquely determined by (3.16), (3.18), as are, of course, $f$ and $\varphi_0$ by (3.14), (3.15)/(3.19).

Note that one can take $v = \overline{w}$ to make the settings (3.14)–(3.15), (3.16) and (3.18)–(3.19) look closer to the usual variational formulations, cf. e.g. [12].

### 3.3. Canonical co-normal derivative

As we have seen above, for an arbitrary $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, the co-normal derivative $T^+(\tilde{f}, u)$ is generally non-uniquely determined by $u$. An exception is $T^+(\check{A}u, u) \equiv 0$ but such co-normal derivative evidently differs from the strong co-normal derivative $T_c^+ u$, given by (3.8) for sufficiently smooth $u$. Another one way of making generalized co-normal derivative unique in $u \in H^1(\Omega)$ was presented in [7, Lemma 5.1.1] and is in fact associated with an extension of $Au \in H^{-1}(\Omega)$ to $\tilde{f} \in \tilde{H}^{-1}(\Omega)$, such that $\tilde{f}$ is orthogonal in $H^{-1}(\mathbb{R}^n)$ to $H^{-1}_{\partial\Omega} \subset H^{-1}(\mathbb{R}^n)$. However it appears (see Lemma A.1), that even for infinitely smooth functions $f$ such extension $\tilde{f}$ does not generally belong to $L_2(\mathbb{R}^n)$, which implies that the so-defined co-normal derivative operator $\tau$ from [7, Lemma 5.1.1] is not a bounded extension of the strong co-normal derivative operator.

Nevertheless, it is still possible to point out some subspaces of $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, where a unique definition of the co-normal derivative by $u$ is possible and leads to the strong co-normal derivative for sufficiently smooth $u$. We define below one such sufficiently wide subspace.

**Definition 3.3.** Let $s \in \mathbb{R}$ and $A_* : H^s(\Omega) \to \mathcal{D}^*(\Omega)$ be a linear operator. For $t \geqslant -\frac{1}{2}$, we introduce a space $H^{s,t}(\Omega; A_*) := \{g\colon\ g \in H^s(\Omega),\ A_* g|_\Omega = \tilde{f}_g|_\Omega,\ \tilde{f}_g \in \tilde{H}^t(\Omega)\}$ equipped with the graphic norm, $\|g\|^2_{H^{s,t}(\Omega;A_*)} := \|g\|^2_{H^s(\Omega)} + \|\tilde{f}_g\|^2_{\tilde{H}^t(\Omega)}$.

The distribution $\tilde{f}_g \in \tilde{H}^t(\Omega)$, $t \geqslant -\frac{1}{2}$, in the above definition is an extension of the distribution $A_* g|_\Omega \in H^t(\Omega)$, and the extension is unique (if it does exist), since otherwise the difference between any two extensions belongs to $H^t_{\partial\Omega}$ but $H^t_{\partial\Omega} = \{0\}$ for $t \geqslant -\frac{1}{2}$ due to the Theorem 2.10. The uniqueness implies that the norm $\|g\|_{H^{s,t}(\Omega;A_*)}$ is well defined. Note that another subspace of such kind, where $A_* g|_\Omega$ belongs to $L_p(\Omega)$ instead of $H^t(\Omega)$, was presented in [6, p. 59]. A particular case, $H^{s,0}(\Omega; A_*)$, was extensively employed in [4].

If $s_1 \leqslant s_2$ and $t_1 \leqslant t_2$, then we have the embedding, $H^{s_2,t_2}(\Omega; A_*) \subset H^{s_1,t_1}(\Omega; A_*)$.

**Remark 3.4.** If $s \in \mathbb{R}$, $-\frac{1}{2} < t < \frac{1}{2}$, and $A_* : H^s(\Omega) \to H^t(\Omega)$ is a linear continuous operator, then $H^{s,t}(\Omega; A_*) = H^s(\Omega)$ by Theorem 2.16.

**Lemma 3.5.** *Let $s \in \mathbb{R}$. If a linear operator $A_* : H^s(\Omega) \to \mathcal{D}^*(\Omega)$ is continuous, then the space $H^{s,t}(\Omega; A_*)$ is complete for any $t \geqslant -\frac{1}{2}$.*

**Proof.** Let $\{g_k\}$ be a Cauchy sequence in $H^{s,t}(\Omega; A_*)$. Then there exists a Cauchy sequence $\{\tilde{f}_k\}$ in $\tilde{H}^t(\Omega)$ such that $\tilde{f}_k|_\Omega = A_* g_k|_\Omega$. Since $H^s(\Omega)$ and $\tilde{H}^t(\Omega)$ are complete, there exist elements $g_0 \in H^s(\Omega)$ and $\tilde{f}_0 \in \tilde{H}^t(\Omega)$ such that

$\|g_k - g_0\|_{H^s(\Omega)} \to 0$, $\|\tilde f_k - \tilde f_0\|_{\tilde H^t(\Omega)} \to 0$ as $k \to \infty$. On the other hand, continuity of $A_*$ implies that $|\langle A_*(g_k - g_0), \phi\rangle| \to 0$ for any $\phi \in \mathcal{D}(\Omega)$. Taking into account that $A_* g_k|_\Omega = \tilde f_k|_\Omega$, we obtain

$$\left|\langle \tilde f_0 - A_* g_0, \phi\rangle\right| \leqslant \left|\langle \tilde f_0 - \tilde f_k, \phi\rangle\right| + \left|\langle \tilde f_k - A_* g_0, \phi\rangle\right| \leqslant \|\tilde f_0 - \tilde f_k\|_{\tilde H^t(\Omega)} \|\phi\|_{H^{-t}(\Omega)} + \left|\langle A_*(g_k - g_0), \phi\rangle\right| \to 0,$$
$$k \to \infty, \ \forall \phi \in \mathcal{D}(\Omega),$$

i.e., $A_* g_0|_\Omega = \tilde f_0|_\Omega \in H^t(\Omega)$, which implies $A_* g_0$ is extendable to $\tilde f_0 \in \tilde H^t(\Omega)$ and thus $g_0 \in H^{s,t}(\Omega; A_*)$. □

We will further use the space $H^{s,t}(\Omega; A_*)$ for the case when the operator $A_*$ is the operator $A$ from (3.2) or the operator $A^*$ formally adjoined to it (see Section 4).

**Definition 3.6.** Let $s \in \mathbb{R}$, $t \geqslant -\frac{1}{2}$. The operator $\tilde A$ mapping functions $u \in H^{s,t}(\Omega; A)$ to the extension of the distribution $Au \in H^t(\Omega)$ to $\tilde H^t(\Omega)$ will be called *the canonical extension* of the operator $A$.

**Remark 3.7.** If $s \in \mathbb{R}$, $t \geqslant -\frac{1}{2}$, then $\|\tilde A u\|_{\tilde H^t(\Omega)} \leqslant \|u\|_{H^{s,t}(\Omega; A)}$ by definition of the space $H^{s,t}(\Omega; A)$, i.e., the linear operator $\tilde A : H^{s,t}(\Omega; A) \to \tilde H^t(\Omega)$ is continuous. Moreover, if $-\frac{1}{2} < t < \frac{1}{2}$, then by Theorem 2.16 and uniqueness of the extension of $H^t(\Omega)$ to $\tilde H^t(\Omega)$, we have the representation $\tilde A := \tilde E^t A$.

As in [17, Definition 3] for scalar PDE, let us define the *canonical* co-normal derivative operator. This extends [6, Theorem 1.5.3.10] and [4, Lemma 3.2] where co-normal derivative operators acting on functions from $H^{1,0}_p(\Omega; \Delta)$ and $H^{1,0}(\Omega; A)$, respectively, were defined.

**Definition 3.8.** For $u \in H^{s,-\frac{1}{2}}(\Omega; A)$, $\frac{1}{2} < s < \frac{3}{2}$, we define the *canonical co-normal derivative* as $T^+u := T^+(\tilde A u, u) \in H^{s-\frac{3}{2}}(\partial\Omega)$, i.e.,

$$\langle T^+ u, w\rangle_{\partial\Omega} := \check{\mathcal{E}}(u, \gamma_{-1} w) - \langle \tilde A u, \gamma_{-1} w\rangle_\Omega = \langle \check A u - \tilde A u, \gamma_{-1} w\rangle_\Omega \quad \forall w \in H^{\frac{3}{2}-s}(\partial\Omega),$$

where $\gamma_{-1} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)$ is a bounded right inverse to the trace operator.

Theorem 3.2 for the generalized co-normal derivative and Definition 3.3 imply the following statement.

**Theorem 3.9.** *Under the hypotheses of Definition* 3.8*, the canonical co-normal derivative $T^+u$ is independent of the operator $\gamma_{-1}$, the operator $T^+ : H^{s,-\frac{1}{2}}(\Omega; A) \to H^{s-\frac{3}{2}}(\partial\Omega)$ is continuous, and the first Green identity holds in the following form,*

$$\langle T^+ u, \gamma^+ v\rangle_{\partial\Omega} = \langle T^+(\tilde A u, u), \gamma^+ v\rangle_{\partial\Omega} = \check{\mathcal{E}}(u, v) - \langle \tilde A u, v\rangle_\Omega = \langle \check A u - \tilde A u, v\rangle_\Omega \quad \forall v \in H^{2-s}(\Omega).$$

Thus unlike the generalized co-normal derivative, the canonical co-normal derivative is uniquely defined by the function $u$ and the operator $A$ only, uniquely fixing an extension of the latter on the boundary.

Definitions 3.1 and 3.8 imply that the generalized co-normal derivative of $u \in H^{s,-\frac{1}{2}}(\Omega; A)$, $\frac{1}{2} < s < \frac{3}{2}$, for any other extension $\tilde f \in \tilde H^{s-2}(\Omega)$ of the distribution $Au|_\Omega \in H^{-\frac{1}{2}}(\Omega)$ can be expressed as

$$\langle T^+(\tilde f, u), w\rangle_{\partial\Omega} = \langle T^+ u, w\rangle_{\partial\Omega} + \langle \tilde A u - \tilde f, \gamma_{-1} w\rangle_\Omega \quad \forall w \in H^{\frac{3}{2}-s}(\partial\Omega).$$

Note that the distributions $\check A u - \tilde f$, $\check A u - \tilde A u$ and $\tilde A - \tilde f$ belong to $H^{2-s}_{\partial\Omega}$ since $\tilde A u$, $\check A u$, $\tilde f$ belong to $\tilde H^{2-s}(\Omega)$, while $\tilde A u|_\Omega = \check A u|_\Omega = \tilde f|_\Omega = Au|_\Omega \in H^{s-2}(\Omega)$.

Since by Theorem 3.9 the canonical co-normal derivative does not depend on the extension operator $\gamma_{-1}$, the latter can be always chosen such that $\gamma_{-1} w$ has a support only near the boundary, which means that the co-normal derivative $T^+u$ is determined by the behavior of $u$ near the boundary. We can formalize this in the following statement.

**Theorem 3.10.** *Let $\Omega$ and $\Omega' \subset \Omega$ be bounded or unbounded open Lipschitz domains, $\partial\Omega \subset \partial\Omega'$, $u \in H^{s,-\frac{1}{2}}(\Omega; A)$, $u \in H^{s,-\frac{1}{2}}(\Omega'; A)$, $\frac{1}{2} < s < \frac{3}{2}$, while $T^+u$ and $T'^+u$ be the canonical co-normal derivatives on $\partial\Omega$ and $\partial\Omega'$ respectively. Then $T^+u = r_{\partial\Omega} T'^+ u$.*

**Proof.** By the definition of the restriction operator $r_{\partial\Omega}$ and Definition 3.8 we have,

$$\langle T'^+ u, w\rangle_{\partial\Omega'} := \check{\mathcal{E}}_{\Omega'}(u, \gamma'_{-1} w) - \langle \tilde A_{\Omega'} u, \gamma'_{-1} w\rangle_{\Omega'} \quad \forall w \in H^{\frac{3}{2}-s}(\partial\Omega'):\ r_{\partial\Omega'\setminus\partial\Omega} w = 0,$$

where $\gamma'_{-1}: H^{s-\frac{1}{2}}(\partial\Omega') \to H^s(\Omega')$ is a bounded right inverse to the trace operator. Since $\gamma\gamma'_{-1}w = 0$ on $\partial\Omega' \setminus \partial\Omega$, we can extend $\gamma'_{-1}w$ by zero on $\Omega \setminus \Omega'$ to $\gamma_{-1}w$. The operator $\gamma_{-1}: H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)$ is continuous, and we arrive at

$$\langle T'^{+}u, w\rangle_{\partial\Omega} = \check{\mathcal{E}}_\Omega(u, \gamma_{-1}w) - \langle \tilde{A}_{\Omega'}u, \gamma_{-1}w\rangle_\Omega = \check{\mathcal{E}}_\Omega(u, \gamma_{-1}w) - \langle \tilde{A}_\Omega u, \gamma_{-1}w\rangle_\Omega = \langle T^{+}u, w\rangle_{\partial\Omega}$$
$$\forall w \in H^{\frac{3}{2}-s}(\partial\Omega). \qquad \square$$

Theorem 3.10 can be considered as an alternative definition of the canonical co-normal derivative, where the domain $\Omega'$ can be chosen arbitrarily small, and particularly can be taken bounded when $\Omega$ is unbounded (with compact boundary). Note that similar reasoning holds also for the generalized co-normal derivative.

To give conditions when the canonical co-normal derivative $T^+u$ coincides with the strong co-normal derivative $T_c^+u$, if the latter does exist in the trace sense, we prove in Lemma 3.12 below that $\mathcal{D}(\overline{\Omega})$ is dense in $H^{s,t}(\Omega; A)$. The proof is based on the following local regularity theorem well known for the case of infinitely smooth coefficients, see e.g. [20,1,12].

**Theorem 3.11.** *Let $\Omega$ be an open set in $\mathbb{R}^n$, $s_1 \in \mathbb{R}$, function $u \in H^{s_1}_{loc}(\Omega)^m$, $m \geqslant 1$, satisfy strongly elliptic system* (3.1) *in $\Omega$ with $f \in H^{s_2}_{loc}(\Omega)^m$, $s_2 > s_1 - 2$, and infinitely smooth coefficients. Then $u \in H^{s_2+2}_{loc}(\Omega)^m$.*

Now we are in the position to prove the density theorem.

**Theorem 3.12.** *If $\Omega$ is a bounded Lipschitz domain, $s \in \mathbb{R}$, $-\frac{1}{2} \leqslant t < \frac{1}{2}$ and the operator $A$ is strongly elliptic on $\overline{\Omega}$, then $\mathcal{D}(\overline{\Omega})$ is dense in $H^{s,t}(\Omega; A)$.*

**Proof.** We modify appropriately the proof from [6, Lemma 1.5.3.9] given for another space of such kind associated with the Laplace operator.

For every continuous linear functional $l$ on $H^{s,t}(\Omega; A)$ there exist distributions $\tilde{h} \in \tilde{H}^{-s}(\Omega)$ and $g \in H^{-t}(\Omega)$ such that

$$l(u) = \langle \tilde{h}, u\rangle_\Omega + \langle g, \tilde{A}u\rangle_\Omega.$$

To prove the lemma claim, it suffice to show that any $l$, which vanishes on $\mathcal{D}(\overline{\Omega})$, will vanish on any $u \in H^{s,t}(\Omega; A)$. Indeed, if $l(\phi) = 0$ for any $\phi \in \mathcal{D}(\overline{\Omega})$, then

$$\langle \tilde{h}, \phi\rangle_\Omega + \langle g, \tilde{A}\phi\rangle_\Omega = 0. \tag{3.20}$$

Let us consider the case $-\frac{1}{2} < t < \frac{1}{2}$ first and extend $g$ outside $\Omega$ to $\tilde{g} = \tilde{E}^{-t}g \in \tilde{H}^{-t}(\Omega)$. Eq. (3.20) gives by Theorem 2.16,

$$\begin{aligned}\langle \tilde{h}, \phi\rangle_{\Omega'} + \langle \tilde{g}, A\phi\rangle_{\Omega'} &= \langle \tilde{h}, \phi\rangle_\Omega + \langle \tilde{g}, A\phi\rangle_\Omega = \langle \tilde{h}, \phi\rangle_\Omega + \langle \tilde{E}^{-t}g, A\phi\rangle_\Omega \\ &= \langle \tilde{h}, \phi\rangle_\Omega + \langle g, \tilde{E}^{t}A\phi\rangle_\Omega = \langle \tilde{h}, \phi\rangle_\Omega + \langle g, \tilde{A}\phi\rangle_\Omega = 0\end{aligned}$$

for any $\phi \in \mathcal{D}(\Omega')$ on some domain $\Omega' \supset \overline{\Omega}$, where the operator $A$ is still strongly elliptic. This means

$$A^*\tilde{g} = -\tilde{h} \quad \text{in } \Omega' \tag{3.21}$$

in the sense of distributions, where $A^*$ is the operator formally adjoint to $A$. If $t \leqslant s-2$, then evidently $\tilde{g} \in \tilde{H}^{2-s}(\Omega)$. If $t > s-2$, then (3.21) and Theorem 3.11 imply $\tilde{g} \in H^{2-s}_{loc}(\Omega')$ and consequently $\tilde{g} \in \tilde{H}^{2-s}(\Omega)$.

In the case $t = -\frac{1}{2}$, one can extend $g \in H^{\frac{1}{2}}(\Omega)$ outside $\overline{\Omega}$ by zero to $\tilde{g} \in \tilde{H}^{\frac{1}{2}-\epsilon}(\Omega)$, $0 < \epsilon$, and prove as in the previous paragraph that $\tilde{g} \in \tilde{H}^{2-s}(\Omega)$.

If $-\frac{1}{2} < t < \frac{1}{2}$ or $[t = -\frac{1}{2},\ s \leqslant \frac{3}{2}]$ then for any $u \in H^{s,t}(\Omega; A)$, we have,

$$l(u) = \langle -A^*\tilde{g}, u\rangle_\Omega + \langle g, \tilde{A}u\rangle_\Omega = -\langle \tilde{g}, Au\rangle_\Omega + \langle \tilde{g}, Au\rangle_\Omega = 0.$$

Thus $l$ is identically zero.

On the other hand, if $t = -\frac{1}{2}$, $s > \frac{3}{2}$, let $\{\tilde{g}_k\} \in \mathcal{D}(\Omega)$ be a sequence converging, as $k \to \infty$, to $g$ in $H_0^{\frac{1}{2}}(\Omega) = H^{\frac{1}{2}}(\Omega)$, cf. Theorem 2.12, and thus to $\tilde{g}$ in $\tilde{H}^{2-s}(\Omega)$. Then for any $u \in H^{s,\frac{1}{2}}(\Omega; A)$, we have,

$$l(u) = \langle -A^*\tilde{g}, u\rangle_\Omega + \langle g, \tilde{A}u\rangle_\Omega = \lim_{k\to\infty}\{\langle -A^*\tilde{g}_k, u\rangle_\Omega + \langle \tilde{g}_k, \tilde{A}u\rangle_\Omega\} = \lim_{k\to\infty}\{-\langle \tilde{g}_k, Au\rangle_\Omega + \langle \tilde{g}_k, Au\rangle_\Omega\} = 0,$$

which completes the proof. $\square$

**Lemma 3.13.** *Let $u \in H^{s,-\frac{1}{2}}(\Omega; A)$, $\frac{1}{2} < s < \frac{3}{2}$, and $\{u_k\} \in \mathcal{D}(\overline{\Omega})$ be a sequence such that*

$$\|u_k - u\|_{H^{s,-\frac{1}{2}}(\Omega;A)} \to 0 \quad \text{as } k \to \infty. \tag{3.22}$$

*Then $\|T_c^+u_k - T^+u\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \to 0$ as $k \to \infty$.*

**Proof.** Using the definition of $T^+u$ and the classical first Green identity for $u_k$, we have for any $w \in H^{\frac{3}{2}-s}(\partial\Omega)$,

$$\left|\left\langle T^+u - T_c^+u_k, w\right\rangle_{\partial\Omega}\right| = \left|\check{\mathcal{E}}(u-u_k, \gamma_{-1}w) - \left\langle \tilde{A}(u-u_k), \gamma_{-1}w\right\rangle_\Omega\right| \leqslant C\|u-u_k\|_{H^{s,-\frac{1}{2}}(\Omega;A)}\|w\|_{H^{\frac{3}{2}-s}(\partial\Omega)}.$$

This implies

$$\left\|T_c^+u_k - T^+u\right\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \leqslant \|u-u_k\|_{H^{s,-\frac{1}{2}}(\Omega;A)} \to 0 \quad \text{as } k \to \infty. \qquad \square$$

Note that a sequence satisfying (3.22) does always exist for bounded Lipschitz domains by Theorem 3.12.

The following statement gives the equivalence of the classical co-normal derivative (in the trace sense) and the canonical co-normal derivative, for functions from $H^s(\Omega)$, $s > \frac{3}{2}$.

**Corollary 3.14.** *If $u \in H^s(\Omega)$, $s > \frac{3}{2}$, then $T^+u = T_c^+u \in L_2(\partial\Omega)$.*

**Proof.** If $u \in H^s(\Omega)$, $\frac{3}{2} < s < \frac{5}{2}$, then $\gamma^+[\partial_j u] \in H^{s-\frac{3}{2}}(\partial\Omega)$, $T_c^+u \in L_2(\partial\Omega)$ and $u \in H^{s,s-2}(\Omega;A) \subset H^{s,-\frac{1}{2}}(\Omega;A) \subset H^{1,-\frac{1}{2}}(\Omega;A)$ by Remark 3.4. Let $\{u_k\} \in \mathcal{D}(\overline{\Omega})$ be a sequence such that $\|u_k - u\|_{H^s(\Omega)} \to 0$ and thus

$$\|u_k - u\|_{H^{1,-\frac{1}{2}}(\Omega;A)} \leqslant \|u_k - u\|_{H^{s,-\frac{1}{2}}(\Omega;A)} \leqslant C\|u_k - u\|_{H^s(\Omega)} \to 0, \quad k \to \infty.$$

Then

$$\left\|T^+u - T_c^+u\right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leqslant \left\|T^+u - T_c^+u_k\right\|_{H^{-\frac{1}{2}}(\partial\Omega)} + \left\|T_c^+(u_k - u)\right\|_{H^{-\frac{1}{2}}(\partial\Omega)},$$

where the first norm in the right-hand side vanishes as $k \to \infty$ by Lemma 3.13, while for the second norm we have,

$$\begin{aligned}\left\|T_c^+(u_k - u)\right\|_{H^{-\frac{1}{2}}(\partial\Omega)} &\leqslant \left\|\sum_{i,j=1}^n a_{ij}\gamma^+\left[\partial_j(u_k - u)\right]n_j\right\|_{L_2(\partial\Omega)} \leqslant C_1\|a\|_{L_\infty(\partial\Omega)}\left\|\gamma^+\nabla(u_k - u)\right\|_{L_2(\partial\Omega)} \\ &\leqslant C_2\|a\|_{L_\infty(\partial\Omega)}\|u_k - u\|_{H^s(\Omega)} \to 0, \quad k \to \infty.\end{aligned}$$

For $s \geqslant \frac{5}{2}$ the corollary follows by imbedding. $\square$

For a Lipschitz domain $\Omega$, the membership $u \in H^{s,t}_{loc}(\Omega;A)$ with $\frac{1}{2} < s < \frac{3}{2}$, $-\frac{1}{2} < t < \frac{1}{2}$ implies by Theorem 3.11 that $u \in H^{t+2}_{loc}(\Omega)$. Thus $u \in H^{t+2}_{loc}(\overline{\Omega}_1)$ for any Lipschitz subdomain $\Omega_1$ of $\Omega$ such that $\overline{\Omega}_1 \subset \Omega$. On $\partial\Omega_1$ then $T^+u = T_c^+u \in L_2(\partial\Omega_1)$ by Corollary 3.14.

**Lemma 3.15.** *Let $\Omega$ and $\{\Omega_k\}$ be Lipschitz domains such that $\overline{\Omega}_k \subset \Omega$ and $\Omega_k \to \Omega$ as $k \to \infty$ (cf. Definition 2.1). If $u \in H^{s,t}_{loc}(\overline{\Omega};A)$ for some $s \in (\frac{1}{2}, \frac{3}{2})$ and $t \in (-\frac{1}{2}, \frac{1}{2})$, then $\langle T^+u, v^+\rangle_{\partial\Omega} = \lim_{k\to\infty}\langle T_c^+u, v^+\rangle_{\partial\Omega_k}$ for any $v \in H^{2-s}(\Omega^+)$.*

**Proof.** By Theorem 3.10 it suffice to consider only a bounded domain $\Omega$. Let $\Omega'_k := \Omega \setminus \overline{\Omega_k}$ be the layer between $\partial\Omega$ and $\partial\Omega_k$. By Theorem 3.11, $u \in H^{t+2}_{loc}(\Omega)$, which by Corollary 3.14 implies $T^+u = T_c^+u \in L_2(\partial\Omega_k)$ on $\partial\Omega_k$. Then

$$\left\langle T^+u, v^+\right\rangle_{\partial\Omega} - \left\langle T_c^+u, v^+\right\rangle_{\partial\Omega_k} = \left\langle T^+u, v^+\right\rangle_{\partial\Omega'_k} = \check{\mathcal{E}}_{\Omega'_k}(u,v) - \langle \tilde{A}_{\Omega'_k}u, v\rangle_{\Omega'_k} = \check{\mathcal{E}}_{\Omega'_k}(u,v) - \langle Au, \tilde{v}_{\Omega'_k}\rangle_{\Omega'_k}, \tag{3.23}$$

where $\tilde{A}_{\Omega'_k}u = \tilde{E}^t_{\Omega'_k}r_{\Omega'_k}Au \in \tilde{H}^t(\Omega'_k)$ and $\tilde{v}_{\Omega'_k} = \tilde{E}^{-t}_{\Omega'_k}r_{\Omega'_k}v \in \tilde{H}^{-t}(\Omega'_k)$ are the unique extensions of $r_{\Omega'_k}Au \in H^t(\Omega'_k)$ and $r_{\Omega'_k}v \in H^{2-s}(\Omega'_k) \subset H^{-t}(\Omega'_k)$, respectively.

By (3.6) and Theorem 2.16 we have for the first term in the right-hand side of (3.23),

$$\begin{aligned}\left|\check{\mathcal{E}}_{\Omega'_k}(u,v)\right| \leqslant C&\sum_{i,j=1}^n \|a_{ij}\|_{L_\infty(\Omega'_k)}\|\partial_j u\|_{H^{s-1}(\Omega'_k)}\|\partial_i v\|_{H^{1-s}(\Omega'_k)} \\ &+ C\sum_{j=1}^n \|b_j\|_{L_\infty(\Omega'_k)}\|\partial_j u\|_{H^{s-1}(\Omega'_k)}\|v\|_{H^{1-s}(\Omega'_k)} + C\|c\|_{L_\infty(\Omega'_k)}\|u\|_{H^{s-1}(\Omega'_k)}\|v\|_{H^{1-s}(\Omega'_k)},\end{aligned}$$

where $C$ does not depend on $k$ for sufficiently large $k$. Then for $\frac{1}{2} < s \leqslant 1$,

$$\begin{aligned}\left|\check{\mathcal{E}}_{\Omega'_k}(u,v)\right| &\leqslant C\sum_{i,j=1}^{n}\|a_{ij}\|_{L_\infty(\Omega)}\|\partial_j u\|_{H^{s-1}(\Omega'_k)}\|\partial_i v\|_{H^{1-s}(\Omega)}\\ &\quad + C\sum_{j=1}^{n}\|b_j\|_{L_\infty(\Omega)}\|\partial_j u\|_{H^{s-1}(\Omega'_k)}\|v\|_{H^{1-s}(\Omega)} + C\|c\|_{L_\infty(\Omega)}\|u\|_{H^{s-1}(\Omega'_k)}\|v\|_{H^{1-s}(\Omega)}\\ &\leqslant \left\{C_1\|\nabla u\|_{H^{s-1}(\Omega'_k)} + C_2\|u\|_{H^{s-1}(\Omega'_k)}\right\}\|v\|_{H^{2-s}(\Omega)} \to 0, \quad k\to\infty\end{aligned}$$

by Lemma 2.17 since the Lebesgue measure of $\Omega'_k$ tends to zero. For $1 < s < \frac{3}{2}$ similarly,

$$\begin{aligned}\left|\check{\mathcal{E}}_{\Omega'_k}(u,v)\right| &\leqslant C\sum_{i,j=1}^{n}\|a_{ij}\|_{L_\infty(\Omega)}\|\partial_j u\|_{H^{s-1}(\Omega)}\|\partial_i v\|_{H^{1-s}(\Omega'_k)}\\ &\quad + C\sum_{j=1}^{n}\|b_j\|_{L_\infty(\Omega)}\|\partial_j u\|_{H^{s-1}(\Omega)}\|v\|_{H^{1-s}(\Omega'_k)} + C\|c\|_{L_\infty(\Omega)}\|u\|_{H^{s-1}(\Omega)}\|v\|_{H^{1-s}(\Omega'_k)}\\ &\leqslant \left\{C_3\|\nabla v\|_{H^{1-s}(\Omega'_k)} + C_4\|v\|_{H^{1-s}(\Omega'_k)}\right\}\|u\|_{H^{s}(\Omega)} \to 0, \quad k\to\infty.\end{aligned}$$

For the last term in (3.23) we have by Lemmas 2.18 and 2.17,

$$\begin{aligned}\left|\langle Au, \tilde{v}_{\Omega'_k}\rangle_{\Omega'_k}\right| &\leqslant \|Au\|_{H^t(\Omega'_k)}\|\tilde{v}_{\Omega'_k}\|_{\tilde{H}^{-t}(\Omega'_k)} \leqslant C\|Au\|_{H^t(\Omega'_k)}\|v\|_{H^{-t}(\Omega)}\\ &\leqslant C\|Au\|_{H^t(\Omega'_k)}\|v\|_{H^{2-s}(\Omega)} \to 0, \quad k\to\infty,\end{aligned}$$

if $-\frac{1}{2} < t \leqslant 0$. On the other hand, if $0 < t < \frac{1}{2}$, then again by Lemmas 2.18 and 2.17,

$$\left|\langle Au, \tilde{v}_{\Omega'_k}\rangle_{\Omega'_k}\right| = \left|\langle \tilde{A}_{\Omega'_k}u, v\rangle_{\Omega'_k}\right| \leqslant \|\tilde{A}_{\Omega'_k}u\|_{\tilde{H}^t(\Omega'_k)}\|v\|_{H^{-t}(\Omega'_k)} \leqslant C\|Au\|_{H^t(\Omega)}\|v\|_{H^{-t}(\Omega'_k)} \to 0, \quad k\to\infty. \qquad \square$$

Lemma 3.15 allows to show that the classical and canonical co-normal derivatives coincide also in another case (apart from the one from Corollary 3.14). First note, that $C^1(\overline{\Omega}) \subset H^1(\Omega)$ for bounded domain $\Omega$ and $C^1(\overline{\Omega'}) \subset H^1(\Omega')$ for any bounded subdomain $\Omega'$ of unbounded domain $\Omega$, but $C^1(\overline{\Omega})$ is not a subset of $H^{1,t}_{loc}(\overline{\Omega}; A)$. For $u \in C^1(\overline{\Omega})$, evidently, $\lim_{k\to\infty}\langle T^+_c u, v^+\rangle_{\partial\Omega_k} = \langle T^+_c u, v^+\rangle_{\partial\Omega}$ for any $v \in H^{2-s}(\Omega^+)$ if $\Omega_k \to \Omega$ as $k\to\infty$, $\overline{\Omega}_k \subset \Omega$. This immediately implies the following statement.

**Theorem 3.16.** *If $\Omega$ is a Lipschitz domain and $u \in C^1(\overline{\Omega}) \cap H^{1,t}_{loc}(\overline{\Omega}; A)$ for some $t \in (-\frac{1}{2}, \frac{1}{2})$, then $T^+u = T^+_c u \in L_\infty(\partial\Omega)$.*

## 4. Formally adjoined PDE system and the second Green identity

The PDE system formally adjoined to (3.1) is given in the strong form as

$$A^*v(x) := -\sum_{i,j=1}^{n}\partial_i\left[\bar{a}^\top_{ji}(x)\partial_j v(x)\right] - \sum_{j=1}^{n}\partial_j\left[\bar{b}^\top_j(x)v(x)\right] + \bar{c}^\top(x)v(x) = f(x), \quad x\in\Omega.$$

Similar to the operator $A$, for any $v \in H^{2-s}(\Omega)$, $s\in\mathbb{R}$, the weak form of the operator $A^*$ is

$$\langle A^*v, u\rangle_\Omega := \mathcal{E}^*(v,u) \quad \forall u \in \tilde{H}^s(\Omega),$$

where

$$\mathcal{E}^*(v,u) = \overline{\mathcal{E}(\bar{u},\bar{v})}$$

is the bilinear form and so defined operator $A^*: H^{2-s}(\Omega) \to H^{-s}(\Omega) = [\tilde{H}^s(\Omega)]^*$ is bounded for any $s\in\mathbb{R}$.

For $\frac{1}{2} < s < \frac{3}{2}$ let us consider also the *aggregate* operator $\check{A}^*: H^{2-s}(\Omega) \to \tilde{H}^{-s}(\Omega) = [H^s(\Omega)]^*$, defined as,

$$\langle \check{A}^*v, u\rangle_\Omega := \check{\mathcal{E}}^*(v,u) \quad \forall u \in H^s(\Omega), \tag{4.1}$$

where by (3.6),

$$\check{\mathcal{E}}^*(v,u) = \overline{\check{\mathcal{E}}(\bar{u},\bar{v})} = \Phi(\bar{u},v) = \sum_{i,j=1}^{n}\langle \bar{a}_{ij}\partial_j u, \tilde{E}^{1-s}\partial_i v\rangle_\Omega + \sum_{j=1}^{n}\langle \bar{b}_j\partial_j u, \tilde{E}^{1-s}v\rangle_\Omega + \langle \bar{c}u, \tilde{E}^{1-s}v\rangle_\Omega \tag{4.2}$$

which implies that $\check{A}^*: H^{2-s}(\Omega) \to \tilde{H}^{-s}(\Omega)$ is bounded. For any $v \in H^{2-s}(\Omega)$, the distribution $\check{A}^*v$ belongs to $\tilde{H}^{-s}(\Omega)$ and is an extension of the functional $A^*v \in H^{-s}(\Omega)$ from the domain of definition $\tilde{H}^s(\Omega)$ to the domain of definition $H^s(\Omega)$.

Relations (4.1), (4.2) and (3.5) lead to the *aggregate second Green identity*,

$$\langle \check{A}u, \bar{v}\rangle_\Omega = \langle u, \overline{\check{A}^*v}\rangle_\Omega, \quad u \in H^s(\Omega),\ v \in H^{2-s}(\Omega), \quad \frac{1}{2} < s < \frac{3}{2}. \tag{4.3}$$

For a sufficiently smooth function $v$, let

$$T^+_{*c}v(x) := \sum_{i,j=1}^n \bar{a}^\top_{ji}(x)\gamma^+\big[\partial_j v(x)\big]\nu_i(x) + \sum_{i=1}^n \bar{b}^\top_i(x)\gamma^+ v(x)\nu_i$$

be the strong (classical) modified co-normal derivative (it corresponds to $\mathfrak{B}_\nu v$ in [13]), associated with the operator $A^*$.

If $v \in H^{2-s}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, and $A^*v = \tilde{f}_*|_\Omega$ in $\Omega$ for some $\tilde{f}_* \in \tilde{H}^{-s}(\Omega)$, we define *the generalized modified co-normal derivative* $T^+_*(\tilde{f}_*, v) \in H^{\frac{1}{2}-s}(\partial\Omega)$, associated with the operator $A^*$, similar to Definition 3.1, as

$$\langle T^+_*(\tilde{f}_*, v), w\rangle_{\partial\Omega} := \check{\mathcal{E}}^*(v, \gamma_{-1}w) - \langle \tilde{f}_*, \gamma_{-1}w\rangle_\Omega \quad \forall w \in H^{s-\frac{1}{2}}(\partial\Omega).$$

As in Theorem 3.2, this leads to the following first Green identity for the function $v$,

$$\langle T^+_*(\tilde{f}_*, v), u^+\rangle_{\partial\Omega} = \check{\mathcal{E}}^*(v, u) - \langle \tilde{f}_*, u\rangle_\Omega \quad \forall u \in H^s(\Omega), \tag{4.4}$$

which by (4.2) implies

$$\langle u^+, \overline{T^+_*(\tilde{f}_*, v)}\rangle_{\partial\Omega} = \check{\mathcal{E}}(u, \bar{v}) - \langle u, \bar{\tilde{f}}_*\rangle_\Omega \quad \forall u \in H^s(\Omega). \tag{4.5}$$

If, in addition, $Au = \tilde{f}|_\Omega$ in $\Omega$ with some $\tilde{f} \in \tilde{H}^{s-2}(\Omega)$, then combining (4.5) and the first Green identity (3.11) for $u$, we arrive at the following generalized second Green identity,

$$\langle \tilde{f}, \bar{v}\rangle_\Omega - \langle u, \bar{\tilde{f}}_*\rangle_\Omega = \langle u^+, \overline{T^+_*(\tilde{f}_*, v)}\rangle_{\partial\Omega} - \langle T^+(\tilde{f}, u), \overline{v^+}\rangle_{\partial\Omega}. \tag{4.6}$$

Taking in mind (4.4), (4.1) and (3.11), (3.5), this, of course, leads to the aggregate second Green identity (4.3).

If $\frac{1}{2} < s < \frac{3}{2}$ and $v \in H^{2-s,-\frac{1}{2}}(\Omega; A^*)$, then similar to Definitions 3.6 and 3.8 we can introduce the *canonical extension* $\tilde{A}^*$ of the operator $A^*$, and the *canonical modified co-normal derivative* $T^+_*v := T^+_*(\tilde{A}^*v, v) \in H^{\frac{1}{2}-s}(\partial\Omega)$, i.e.,

$$\langle T^+_*v, w\rangle_{\partial\Omega} := \check{\mathcal{E}}^*(v, \gamma_{-1}w) - \langle \tilde{A}^*v, \gamma_{-1}w\rangle_\Omega \quad \forall w \in H^{s-\frac{1}{2}}(\partial\Omega).$$

Then the first Green identity (4.5) becomes,

$$\langle u^+, \overline{T^+_*v}\rangle_{\partial\Omega} = \check{\mathcal{E}}(u, \bar{v}) - \langle u, \overline{\tilde{A}^*v}\rangle_\Omega \quad \forall u \in H^s(\Omega).$$

For $u \in H^s(\Omega)$, $Au = \tilde{f}|_\Omega$ in $\Omega$, where $\tilde{f} \in \tilde{H}^{s-2}(\Omega)$, the second Green identity (4.6) takes form,

$$\langle \tilde{f}, \bar{v}\rangle_\Omega - \langle u, \overline{\tilde{A}^*v}\rangle_\Omega = \langle u^+, \overline{T^+_*v}\rangle_{\partial\Omega} - \langle T^+(\tilde{f}, u), \overline{v^+}\rangle_{\partial\Omega}. \tag{4.7}$$

This form was a starting point in formulation and analysis of the extended boundary-domain integral equations in [15].

If, moreover, $u \in H^{s,-\frac{1}{2}}(\Omega; A)$, we obtain from (4.7) the second Green identity for the canonical extensions and canonical co-normal derivatives,

$$\langle \tilde{A}u, \bar{v}\rangle_\Omega - \langle u, \overline{\tilde{A}^*v}\rangle_\Omega = \langle u^+, \overline{T^+_*v}\rangle_{\partial\Omega} - \langle T^+u, \overline{v^+}\rangle_{\partial\Omega}. \tag{4.8}$$

Particularly, if $u, v \in H^{1,0}(\Omega; A)$, then (4.8) takes the familiar form, cf. [4, Lemma 3.4],

$$\int_\Omega \big[\overline{v(x)}Au(x) - u(x)\overline{A^*v(x)}\big]\,dx = \langle u^+, \overline{T^+_*v}\rangle_{\partial\Omega} - \langle T^+u, \overline{v^+}\rangle_{\partial\Omega}.$$

## Appendix A

**Lemma A.1.** *There exist a distribution $w \in H^{-1}_{\partial\Omega}$ and a function $f \in L_2(\mathbb{R}^n)$, $f = 0$ on $\Omega^-$, such that $(w, f)_{H^{-1}(\mathbb{R}^n)} \neq 0$.*

**Proof.** Under the definition (2.3) of the inner product in $H^s(\mathbb{R}^n)$,

$$(w, f)_{H^{-1}(\mathbb{R}^n)} = \langle \overline{w}, \mathcal{J}^{-2}f\rangle_{\mathbb{R}^n}. \tag{A.1}$$

By Theorem 2.10, for any distribution $\overline{w} \in H^{-1}_{\partial\Omega}$ there exists a distribution $v \in H^{-1/2}(\partial\Omega)$ such that

$$\langle \overline{w}, \mathcal{J}^{-2}f\rangle_{\mathbb{R}^n} = \langle v, \gamma\mathcal{J}^{-2}f\rangle_{\partial\Omega}, \tag{A.2}$$

where $\gamma$ is the trace operator.

Denoting $\Phi = \mathcal{J}^{-2} f \in H^2(\mathbb{R}^n)$, we have, $\mathcal{J}^2\Phi = f$ in $\mathbb{R}^n$, and taking in mind the explicit representation for the operator $\mathcal{J}^2$, the latter equation can be rewritten as

$$\mathcal{J}^2\Phi \equiv -\frac{1}{4\pi^2}\Delta\Phi + \Phi = f \quad \text{in } \mathbb{R}^n \tag{A.3}$$

and its solution as

$$\mathcal{J}^{-2} f(y) = \Phi(y) = \mathcal{P}f := \int_\Omega F(x, y) f(x)\,dx, \quad y \in \mathbb{R}^n.$$

Here $\mathcal{P}$ is the Newton volume potential and $F(x, y)$ is the well known fundamental solution of Eq. (A.3). For example, for $n = 3$,

$$F(x, y) = C\frac{e^{-2\pi|x-y|}}{|x-y|}. \tag{A.4}$$

Then (A.1), (A.2) give,

$$(w, f)_{H^{-1}(\mathbb{R}^n)} = \langle v, \gamma\mathcal{J}^{-2} f\rangle_{\partial\Omega} = \langle v, \gamma\mathcal{P}f\rangle_{\partial\Omega}. \tag{A.5}$$

If we assume $(w, f)_{H^{-1}(\mathbb{R}^n)} = 0$ for any $w \in H^{-1}_{\partial\Omega}$, then (A.5) implies $\gamma\mathcal{P}f = 0$, which is not the case for arbitrary $f \in L_2(\Omega)$ and particularly for $f = 1$ in $\Omega$ due to (A.4). □

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