(24)

ISSN 0025-6544

MECHANICS OF SOLIDS

(Mekhanika Tverdogo Tela)

Vol. 27, No. 1

ALLERTON PRESS, INC.

CERTAIN BOUNDARY INTEGRAL EQUATIONS OF THE PLANE PROBLEM OF ELASTICITY THEORY FOR NONSINGLY CONNECTED BODIES WITH ONE-DIMENSIONAL ELASTIC BORDERS AND ANGLE POINTS

S. E. Mikhailov

Vol. 27, No. 1, pp. 36-47, 1992

UDC 539.3.01

Boundary integral equations (BIE) of the plane problem of elasticity theory are obtained using double-layer elastic potentials of the first and second kind for nonsingly connected bodies with closed elastic borders that possess variable longitudinal and bending rigidities, curvature, and thickness. In addition to the terms that appear in the corresponding BIE for problems with specified boundary displacements, these equations also contain Volterra and finite-dimensional operators. The spectral properties of the BIE and the relation of these properties to the smoothness of the boundary and the connectedness of the region are investigated and used to propose methods for solution of the BIE. The asymptotic behavior of the BIE solutions near boundary nodal points is presented, and its relation to the corresponding solution asymptotics of the original boundary-value problem* is indicated. In [1], the problem for an infinite plane with a circular hole reinforced by a bar of zero thickness was reduced to a different integral equation with the aid of complex Kolosov-Muskhelishvili potentials, and the same approach was applied in [2] to the case of a noncircular hole, whose exterior was conformally mapped onto a circle with the aid of a linear-fraction function.

1. BOUNDARY-VALUE PROBLEM

Let D be a finite or infinite region of a plane with a boundary $\partial D = U^{m}_{\beta=0}\partial_\beta$, formed by a set of simple closed, nonintersecting piecewise-Lyapunov contours ∂_β with a finite number of angle points where the interior angles $\omega \neq 2\pi, 0$. Contour ∂_0 encompasses all of the other contours ∂_β, and if it is absent, region D is infinite. Using the arc length s we parametrize the contours ∂_β in such a way that region D will be on the left at all times during a positive circuit. Then $k_i(s) = y_i'(s)$ is the unit tangent and $n_i(s) = e_{ij}k_j(s)$ is the outer normal to ∂D. Here and below, the prime indicates differentiation with respect to the arc length s, $e_{ij}=\delta_{1i}\delta_{2j}-\delta_{1j}\delta_{2i}$ is an alternating symbol, δ_{ij} is the Kronecker delta, and summation from 1 to 2 over repeating indices is understood unless otherwise stated.

Let us assume that region D is joined along contours ∂_β to one-dimensional elastic boundary borders L_β that possess variable longitudinal and bending rigidities and thickness (by bars for the case of the plane stressed state or by cylindrical shells for the case of plane deformation). We denote the set of median lines (lines of section centers of gravity) of the borders by $L = U^{m}_{\beta=0}L_\beta$ and parametrize them with the arc length τ in the positive direction around D. We then have the relations [3]: $\tau=\tau(s)$, $ds=\vartheta(\tau)d\tau$, $\vartheta(\tau) := \{[1+\chi(\tau)h(\tau)]^2+\dot{h}^2(\tau)\}^{1/2}$ and $x_i(s)=x_i^\circ[\tau(s)]+h[\tau(s)]n_i^\circ[\tau(s)]$ where $k_i^\circ(\tau):=\dot{x}_i^\circ(\tau)$ is the tangent and $n_i^\circ(\tau):=e_{ij}k_j^\circ(\tau)$ the normal to the median line of bar L, $h(\tau)$ is the distance along the normal n_i° from points $x_i^\circ(\tau)$ on L to points $x_i[\tau(s)]$ on ∂D, with $h < 0$ if ∂D lies to the left of L during a positive circuit; for a bar of zero thickness $h(\tau)=0$, $\tau=s$, $\vartheta(\tau)=1$, $n_i^\circ(\tau)=n_i(\tau)^2$.** Here and below, the symbol := means equality by definition; the overdot indicates differentiation with respect to τ; $\chi(\tau)=k_i^\circ(\tau)\dot{n}_i^\circ(\tau)$ is the curvature of L ($\chi(u) > 0$) if the center of

*Some of the results described here were reported in the paper by S. E. Mikhailov and I. V. Namestnikova, "Plane problems for nonsingly connected elastically constrained plates," in: *Mechanics of Inhomogeneous Structures. Abstracts of Papers at Second All-Union Conference* [in Russian], vol. 2, pp. 207-208, L'vov, 1987.

**Here the parameters L, s, and h are used instead of the respective parameters L_0, s_*, h_* of [4].

the circle tangent to L at point τ lies to the left of the bar during a positive circuit).

Let us consider the system of Lame equations in region D:

$$c_{ijkl}u_{k,lj}(x)=0, \quad (x\in D) \tag{1.1}$$

$$c_{ijkl}:=\Lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk}) \tag{1.2}$$

Here $\Lambda > 0$, $\mu > 0$ are the Lame constants for the case of plane deformation; for the plane stressed state that figures in (1.2) and what follows, the constant Λ must be replaced by $\Lambda^*:=2\Lambda\mu/(\Lambda+2\mu)$. Also, let $\varkappa:=(\Lambda+3\mu)/(\Lambda+\mu)>1$ be the constant of plane elasticity theory.

The interaction of elastic region D with the elastic borders L is described by the boundary conditions of problem (r) [3]:

$$\begin{gathered}\delta_i(s;u)=v_{0i}[\tau(s),h;p^{(e)}]\ (s\in\partial D,\ i=1-2)\\ \delta_i(s;u):=u_i(s)-v_{0i}[\tau(s),h;p^{(p)}]-C_i^{(\beta)}-C_3^{(\beta)}e_{ij}x_j(s)\ \ (s\in\partial_\beta)\\ v_{0i}(\tau,h)=v_{0i}^\circ(\tau)-hk_i^\circ n_j^\circ v_{0j}^{\prime\circ}(\tau)\end{gathered} \tag{1.3}$$

Here the u_i are the unknown displacements in region D and on its boundary; $C_i^{(\beta)}$ ($i = 1$-3, $\beta = 0$-m) are arbitrary constants subject to determination and characterize the displacement of the bar as a rigid whole; $v_{0i}^\circ[\tau, h; p^{(p)}](\tau\in\partial_\beta)$ are the displacements of the central axis of a closed bar L_β placed at a certain point $\tau_1^{(\beta)} \in \partial_\beta$ that are caused by the set $p^{(p)}=\{p_i^{(p)}, m^{(p)}\}$ of distributed forces $p_i^{(p)}(\tau; u)$ and moments $m^{(p)}(\tau; u)$: produced by the unknown contact stresses $\sigma_{ij}(s; u)$:

$$\begin{gathered}v_{0i}^\circ[\tau;p^{(p)}]=\int_{\tau_1^{(\beta)}}^{\tau}[\Pi_{ij}p_j^{(p)}+\Pi_i^{(m)}m^{(p)}](\tau_0)\,d\tau_0\ (\tau\in L_\beta,\ \beta=0-m)\\ p_i^{(p)}(\tau(s);u):=-\sigma_{ij}(s;u)n_j(s)\vartheta(\tau)H_0\\ m^{(p)}(\tau(s);u):=-\sigma_{ij}(s;u)n_j(s)\vartheta(\tau)H_0h(\tau)k_i^\circ(\tau)\end{gathered} \tag{1.4}$$

Similarly, $v_{0i}^\circ[\tau; p^{(e)}, m^{(e)}]$ is the displacement of the bar central axis under the action of the set $p^{(e)}=\{p_i^{(e)}, m^{(e)}, F_{Ri}^{(\beta\alpha)}, M_R^{(\beta\alpha)}\}$, the specified forces and moments $p_i^{(e)}(\tau)$, $m^{(e)}(\tau)$ distributed along L, and the concentrated forces and moments $F_{Ri}^{(\beta\alpha)}$, $M_R^{(\beta\alpha)}$ applied at the points $\tau_R^{(\beta\alpha)}$ $(\alpha=1-N_{R\beta},\ \beta=0-m)$:

$$\begin{gathered}v_{0i}^\circ[\tau;p^{(e)}]=\int_{\tau_1^{(\beta)}}^{\tau}\Big\{[\Pi_{ij}p_j^{(e)}+\Pi_i^{(m)}m^{(e)}]_i(\tau_0)+\sum_{\alpha=1}^{N_{R\beta}}[[\Pi_{ij}^1(\tau_0,\tau_R^{(\beta\alpha)})F_{Rj}^{(\beta\alpha)}+\\ +\Pi_i^{(1m)}(\tau_0,\tau_R^{(\beta\alpha)})M_R^{(\beta\alpha)}]H(\tau_0-\tau_R^{(\beta\alpha)})+\Pi_{ij}^\circ(\tau_0,\tau_R^{(\beta\alpha)})F_{Rj}^{(\beta\alpha)}+\\ +\Pi_i^{(0m)}(\tau_0,\tau_R^{(\beta\alpha)})M_R^{(\beta\alpha)}]\Big\}d\tau_0\quad(\tau\in L_\beta,\ \beta=0-m)\end{gathered} \tag{1.5}$$

The integral operators Π_{ij} and $\Pi_i^{(m)}$, which are the sums of Volterra and finite-dimensional (degenerate) operators and the functions Π_{ij}^1, $\Pi_{ij}^{(1m)}$, Π_{ij}°, $\Pi_{ij}^{(0m)}$ were represented in [3] in terms of the rigidity and geometric parameters of the borders; $\tau_i^{(\beta)}$ are arbitrary reference points on the borders L_β at which no specified concentrated forces or moments are applied, H_0 is the thickness of the plate for the plane state of stress and $H_0 = 1$ for the case of plane deformation, and H(s) is the Heaviside function.

The boundary conditions (1.3) must always be supplemented with integral equilibrium conditions for each connected part of the border:

$$-\int_{L_\beta}p_i^{(p)}(\tau)\,d\tau=F_i^{(e\beta)}:=\int_{L_\beta}p_i^{(e)}(\tau)\,d\tau+\sum_{\alpha=1}^{N_{R\beta}}F_{Ri}^{(\beta\alpha)}\qquad(i=1-2) \tag{1.6}$$

$$- \int [m^{(p)}(\tau) - p_i^{(p)}(\tau) e_{ij} x_j^{\circ}(\tau)] d\tau = M^{(e\beta)} := \\ = \int_{L_\beta} [m^{(e)}(\tau) - p_i^{(e)}(\tau) e_{ij} x_j^{\circ}(\tau)] d\tau + \\ + \sum_{\alpha=1}^{N_{R\beta}} [M_R^{(\beta\alpha)} + F_{Ri}^{(\beta\alpha)} e_{ij} x_j^{\circ}(\tau_R^{(\beta\alpha)}) \quad (\beta = 0 - m) \tag{1.7}$$

As indicated in [3], these conditions guarantee unique solution of problem (r) accurate to the rigid displacement of the reinforced body (without rotation for the infinite-body case).

Let us assume further that for a finite region D:

$$\sum_{\beta=0}^{m} F_i^{(e\beta)} = 0, \quad \sum_{\beta=0}^{m} M^{(e\beta)} = 0 \tag{1.8}$$

while for an infinite region

$$\sum_{\beta=1}^{m} F_i^{(e\beta)} = 0$$

It was shown in [3] that these conditions are necessary for the existence of a solution of problem (1.1)-(1.7) (problem (r)) in the class of functions u_i that have limited energy for the finite region and are also bounded at infinity for the infinite region. These conditions are zero principal vector and zero moment of the external forces applied to the borders (and hence also to the body).

So that the principal part of the boundary conditions will have the same order of smoothness as the boundary stresses, we differentiate (1.3) with respect to s and convert to conditions that are equivalent to (1.3) by virtue of the arbitrariness of $C_i^{(\beta)}$:

$$\delta_i'(s_0; u) := u_i'(s_0) - v_{0i}^{\cdot}[\tau(s_0), h; p^{(p)}]/\vartheta(\tau(s_0)) - C_3^{(\beta)} n_i(s_0) = \\ = v_{0i}^{\cdot}[\tau(s_0), h; p^{(e)}]/\vartheta(\tau(s_0)) \tag{1.9}$$

$$C_i^{(\beta)} = u_i(s_1^{(\beta)}) - C_3^{(\beta)} e_{ij} x_j(s_1^{(\beta)}) \quad (s_0, s_1^{(\beta)} \in \partial_\beta, \ \beta = 0 - m) \tag{1.10}$$

which are also supplemented with conditions (1.6), (1.7).

2. DERIVATION OF BOUNDARY INTEGRAL EQUATIONS

We denote by $G(\tau)$ and $G_1(\tau)$ the longitudinal and bending rigidities of the reinforcement and let $a(\tau) := \chi(\tau) G^{-1}(\tau)$, $b(\tau) := G_1^{-1}(\tau) + \chi^2(\tau) G^{-1}(\tau)$, $g(\tau) := G^{-1}(\xi)$. If

$$|a(\tau)|,\ b(\tau),\ g(\tau),\ |h^{\cdot}(\tau)| < \infty \quad (\tau \in L) \tag{2.1}$$

then, as we see from [3], the $v_{0i}^{\cdot}$ terms on the left in the integrodifferential boundary conditions (1.9), which are related to the displacements of the bar under the action of contact forces from the plate, are of low order compared to u_i'. Thus, the principal part of boundary conditions (1.9) agrees with the differentiated boundary conditions for displacements specified on boundary ∂D. Therefore, as in the problem with specified displacements,* we shall henceforth express the unknown solution in terms of double-layer elastic potentials of the first or second kind, U^{II} and U^{IV} respectively, which satisfy system (1.1) in D [4]:

*See S. E. Mikhailov and Yu. I. Kotov, "Integral equations of plane problems of elasticity theory for regions with holes and corners," VINITI File No. 6695-V86, Moscow, 17 September 1986.

$$U_i^{II}(x):=\pi^{-1}\int_{\partial D}\Gamma_{i\alpha,l}[z]c_{j\eta\alpha l}q_j(s)n_\eta(s)ds$$

$$U_i^{IV}(x):=\pi^{-1}\int_{\partial D}\Gamma_{i\alpha,l}[z]c_{j\eta\alpha l}^{(N)}q_j(s)n_\eta(s)ds \tag{2.2}$$

$$q_j(s):=\int_{s_1^{(\beta)}}^{s}Q_j(s_0)ds_0 \quad (s,s_1^{(\beta)}\in\partial_\beta)$$

$$\int_{\partial_\beta}Q_j(s)ds=0 \qquad (\beta=0-m), \quad Q_j(s)\in L_p(\partial D) \qquad (1<p<\infty) \tag{2.3}$$

Here $z_j=x_j-y_j$, $y\in\partial D$, $r^2=z_jz_j$. The tensor $\Gamma_{ij}(z)=[-\varkappa\delta_{ij}\ln(r)+z_iz_j/r^2]/[\mu(\varkappa+1)]$ is the fundamental solution of system (1.1): $c_{i\gamma kl}\Gamma_{kj,l\gamma}(z)=-2\pi\delta_{ij}\delta(z)$.

With conditions (2.3), the potentials $U_i^{II}(\infty)=U_I^{IV}(\infty)=0$; $|U_{i,j}^{II}(x)|,|U_{i,j}^{IV}(x)|<C_0R^2$ for large R. If $u_i^*\in C^1(D)\cap C^2(D)$ is any solution of the Lame system (1.1) (that satisfies the regularity condition $|u_i^*(\infty)|<\infty$, $|u_{i,j}^*(x)|<C_0R^2$ for large R in the case of the infinite region) and the density Q_j of potential U_i^{II} or U_i^{IV} satisfies (2.3), the Green-Betti formula applies for the sum $u_i^* + U_i^{II}$ or $u_i^* + U_i^{IV}$, the elastic energy form this sum is bounded, the potentials U^{II}, U^{IV} are continuously continuable onto the boundary, and the boundary stresses belong to $L_p(\partial D)$. If therefore follows under conditions (2.1) that the total energy of the plate D and its reinforcing bars, which have the displacement $v_{0i}[\tau(s),\ h;\ p^{(p)}(u_i^*+U_i)]$, is also bounded. Therefore the existence and uniqueness theorems formulated in [3] for problem (r) apply in the class of such combinations.

Let $x_i^{(\beta)}$ be arbitrary fixed points inside the contours $\partial_\beta(\beta=1-m)$, $r^{(\beta)2}:=[x_i-x_i^{(\beta)}]^2$; $\Gamma_{ij}^{(\beta)}(x):=\Gamma_{ij}[x-x^{(\beta)}]$, $W_i^{(\beta)}(x):=e_{ij}[x_j-x_j^{(\beta)}]/r^{(\beta)2}$ are the fields of the displacements from the forces and moments concentrated at point $x_j^{(\beta)}$, respectively.

Then for $y_i(s_0)\in\partial D$ we have $W_i^{(\beta)\prime}[y(s_0)]:=n_i(s_0)/r^{(\beta)2}-2[y_i(s_0)-x_i^{(\beta)}][y_j(s_0)-x_j^{(\beta)}]n_j(s_0)/r^{(\beta)4}$

We shall first find the solution of problem (r) in the form

$$u_i(x)=U_i^{II}(x)+\sum\nolimits_{\beta=1}^{m}\{A_j^{(\beta)}\Gamma_{ij}^{(\beta)}(x)+B^{(\beta)}W_i^{(\beta)}(x)\} \tag{2.4}$$

$$A_j^{(\beta)}=F_j^{(e\beta)}/(2\pi H_0), \quad B^{(\beta)}=M^{(e\beta)}/(4\pi\mu H_0) \tag{2.5}$$

Applying condition (1.9), we arrive at BIE II$^{(r)}$ (λ = 1):

$$Q_i(s_0)-\lambda[K_{ij}^{IIr}Q_j](s_0)=h_i^{IIr}(s_0), \qquad K_{ij}^{IIr}:=K_{ij}^{II}+K_{ij}^{\sim IIr}$$

$$[K_{ij}^{II}Q_j](s_0):=\int_{\partial D}[\pi r^2(1+\varkappa)]^{-1}\{[4z_iz_jr^{-2}+(\varkappa-1)\delta_{ij}]z_ln_l(s_0)+ \tag{2.6}$$

$$+(\varkappa-1)[n_i(s_0)z_j-n_j(s_0)z_i]\}Q_j(s)ds$$

$$[K_{ij}^{\sim IIr}Q_j](s_0):=v_{0i}^{\cdot}\{\tau(s_0),h;p^{(p)}[U^{II}(Q)]\}/\vartheta(\tau(s_0))$$

$$h_i^{IIr}(s_0):=-\sum\nolimits_{\gamma=1}^{m}[A_j^{(\gamma)}\delta_i'(s_0;\Gamma_{\cdot j}^{(\gamma)})+B^{(\gamma)}\delta_i'(s_0;W^{(\gamma)})]+$$

$$+v_{0i}^{\cdot}[\tau(s_0),h;p^{(e)}]/\vartheta(\tau(s_0))+C_3^{(\beta)}n_i(s_0) \qquad (s_0\in\partial_\beta,\ \beta=0-m) \tag{2.7}$$

We note that for reinforcements of zero thickness (h = 0), the operator $K_{ij}^{\sim IIv}$ can be presented in the form

$$[K_{ij}^{\sim IIr}Q_j](s_0)=\{\Pi_{ij}p_j^{(p)}[U^{II}(Q)]\}(s_0)+\{\Pi_i^{(m)}m^{(p)}[U^{II}(Q)]\}(s_0) \tag{2.8}$$

Because the constants $A_j^{(\beta)}$, $B^{(\beta)}$ were taken in the form (2.5), conditions (1.6) and (1.7) become

$$\int_{\partial_\beta} \sigma_{ij}[U^{II}(Q)](s)n_j(s)ds=0, \qquad \int_{\partial_\beta} \sigma_{ij}[U^{II}(Q)](s)n_j(s)e_{i\gamma}x_\gamma(s)ds=0 \tag{2.9}$$

$$(\beta=1-m)$$

For a finite region D we also have with (1.8):

$$\int_{\partial D} \sigma_{ij}[U^{II}(Q)](s)n_j(s)ds=0, \qquad \int_{\partial D} \sigma_{ij}[U^{II}(Q)](s)n_j(s)e_{i\gamma}x_\gamma(s)ds=0 \tag{2.10}$$

If the density Q of potential $U^{II}(Q)$ satisfies (2.3), conditions (2.10) for a finite region D are also satisfied. Recognizing the continuity of the force vector $\sigma_{ij}(U^{II})n_j$ when ∂D is crossed, we find that (2.9) reduce to the same conditions for the boundary stresses inside finite regions D_β^- (β = 1-m) that lie within ∂_β. But these conditions are satisfied on satisfaction of (2.3).

Thus, conditions (2.9)-(2.10) and hence also (1.6) and (1.7) are satisfied for the representations (2.4) and (2.5) with any densities Q_j that satisfy conditions (2.3) and, in particular, are solutions of BIE II^r. The open constants $C_3^{(\beta)}$ in (2.7) will be determined by the solvability conditions of BIE II^r.

The solution of the same problem (r) can be sought in the form

$$u_i(x)=U_i^{VI}(x)+\sum_{\beta=1}^{m} A_j^{(\beta)}\Gamma_{ij}^{(\beta)}(x) \tag{2.11}$$

where the $A_j^{(\beta)}$ have the form of (2.5). Then after substitution into conditions (1.9) we arrive at BIE IV^r (λ = 1):

$$\begin{aligned} &Q_i(s_0)-\lambda[K_{ij}^{IVr}Q_j](s_0)=h_i^{IVr}(s_0), \qquad K_{ij}^{IVr}:=K_{ij}^{IV}+\tilde{K}_{ij}^{IVr} \\ &[K_{ij}^{IV}Q_j](s_0):=\int_{\partial D}[\pi r^2\varkappa]^{-1}[2z_iz_jr^{-2}+(\varkappa-1)\delta_{ij}]z_ln_l(s_0)Q_j(s)ds \\ &h_i^{IV}(s_0):=-\sum_{\gamma=1}^{m} A_j^{(\gamma)}\delta_i'(s_0;\Gamma_{\cdot j}^{(\gamma)})+v_{0i}{}^{\cdot}[\tau(s_0),h;p^{(\cdot)}]/\vartheta(\tau(s_0))+ \\ &\qquad +C_3^{(\beta)}n_i(s_0) \qquad (s_0\in\partial_\beta,\ \beta=0-m) \end{aligned} \tag{2.12}$$

The operator $\tilde{K}_{ij}^{IVr}$ is given by the right side of expression (2.6) after substitution of U^{IV} for U^{II}. When h = 0 it becomes (2.8). Because the constants $A^{(\beta)}$ were chosen in the form (2.5), conditions (1.6) and (1.7) are transformed to

$$\int_{\partial_\beta} \sigma_{ij}[U^{IV}(Q)](s)n_j(s)ds=0, \qquad \int_{\partial_\beta} \sigma_{ij}[U^{IV}(Q)](s)n_j(s)e_{i\gamma}x_\gamma(s)ds=M^{(e\beta)}/H_0 \tag{2.13}$$

$$(\beta=1-m)$$

With (1.8), we also have for a finite region D

$$\int_{\partial D} \sigma_{ij}[U^{IV}(Q)](s)n_j(s)ds=0, \qquad \int_{\partial D} \sigma_{ij}[U^{IV}(Q)](s)n_j(s)e_{i\gamma}x_\gamma(s)ds=0 \tag{2.14}$$

As above, conditions (2.14) are satisfied by virtue of the zero principal vector and moment of the boundary conditions for all densities Q_j of the potentials U^{IV} that satisfy (2.3).

Recognizing that

$$\sigma_{ij}[U^{IV}(Q)](s)n_j(s) = -2\mu(\varkappa-1)\varkappa^{-1}e_{ij}Q_j + \sigma_{ij}^{-}[U^{IV}(Q)](s)n_j(s)$$

and that the left sides of conditions (2.3) from the stresses σ_{ij}^{-} in finite regions D_β^{-} (β = 1-m) are equal to zero, we obtain from (2.13)

$$\int_{\partial_\beta} Q_j(s)ds = 0 \tag{2.15}$$

$$\int_{\partial_\beta} Q_j(s)[x_j(s) - x_j^{(\beta)}]ds = -\varkappa M^{(e\beta)}[2\mu(\varkappa-1)H_0]^{-1} \quad (\beta = 1-m) \tag{2.16}$$

Since conditions (2.15) are satisfied when (2.3) are satisfied, we find that conditions (1.6) are satisfied for representations (2.11) and (2.5) with all densities Q_j that satisfy (2.3). Conditions (1.7) reduce to (2.16), and their satisfaction is ensured, as will be shown below, by appropriate selection of the constants $C_3^{(\beta)}$ in (2.12).

3. SPECTRAL PROPERTIES, PERTURBATION, AND METHODS OF SOLUTION OF BIE II^r AND IV^r

Thus, we have derived a new class of boundary integral equations of elasticity theory II^r and IV^r. The kernels of these equations have strong stationary singularities at angle points, and kernel K_{ij}^{IIr} also has a Cauchy-type singularity. Remembering that the operators $K_{ij}^{\sim IIr}$ and $K_{ij}^{\sim IVr}$ are sums of Volterra and finite-dimensional operators and that, as follows from [3], are compact in L_p ($1 < p < \infty$) under condition (2.1), we find that the Fredholm regions of the operators K^{IIr} and K^{IVr} in L_p are the same as those of the previously studied operators K^{II}, K^{IV}.* In particular, there exist for BIE II^r numbers $p_\pm^{II} > 2$ and for BIE I^r numbers $p_\pm^{IV} > 2$ such that the corresponding equations are Fredholm in the Lebesgue spaces $L_p(\partial D)$ ($1 < p < p_\pm$) for $\lambda = \pm 1$. On the other hand, there exist for these BIE real numbers p_0^{II}, $p_0^{IV} > 1$ such that the corresponding BIE are Fredholm in space $L_p(\partial D)$ ($1 < p < p_0$) in the closed unit circle $|\lambda| = 1$. We note that $p_+^{IV} = p_-^{IV} = p_0^{IV}$.

Using the uniqueness theorem for problem (r) that was proven in [3] and reasoning as for BIE II and IV, it can be shown that the number $\lambda = 1$ is characteristic for BIE II^r for both finite and infinite regions. Let $\lambda = 1$. If region D is infinite, HGIE [Homogeneous Boundary Integral Equation] II^r has m solutions in $L_p(\partial D)$ ($1 < p < p_+^{II}$):

$$Q_i^{(\beta)}(s) = n_i(s) \quad (s \in \partial_\beta), \quad Q_i^{(\beta)}(s) = 0 \quad (s \notin \partial_\beta) \quad \beta = 1-m \tag{3.1}$$

while the inhomogeneous BIE II^r can be solved if the right side $h_i^{IIr}(s)$m is subjected to the conditions

$$\int_{\partial D} h_i(s)Q_i^{*(\beta)}(s)ds = 0 \quad (\beta = 1-m) \tag{3.2}$$

which are generated by the solutions $Q_i^{*(\beta)}$ (β = 1-m) of the conjugate HBIE II^r and are not generally written out explicitly. If region D is finite, then HBIE II^r has (m + 2) solutions, m of which are given by relations (3.1), while the inhomogeneous BIE II^r can be solved if the right side h_i^{IIr}(m + 2) is subject to conditions two of which are given by

$$\int_{\partial D} h_i(s)ds = 0 \quad (i = 1-2) \tag{3.3}$$

and the other m by relations (3.2), where the $Q_i^{*(\beta)}$ are not written out explicitly.

For BIE IV^r, the number $\lambda = 1$ is characteristic only for finite regions. In this case, the HBIE IV^r has two solutions, and the inhomogeneous BIE IV^r can be solved if conditions (3.3) are imposed on the right-side.

---

*See publication cited on p. 35.

Using the Green-Betti formula [3] for problem (r) and reasoning as for BIE II, it can also be shown that in the Fredholm region all characteristic numbers of the BIE II^r are real, are simple poles of the resolvent, and are absent in the interval $(\lambda_-^{II}, 1)$ where $\lambda_-^{II} < 0$, and, consequently, the estimate $\lambda_-^{II} \leqslant -1/\|K\|_p$ $(1<p<p_0)$ applies.

The presence of the added operators K^{-IIr}, K^{-IVr} in BIE II^r and IV^r makes it difficult to obtain more complete information on the positions of the characteristic numbers of these BIE in the unit circle $|\lambda| \leq 1$ for the general case. But for infinitely rigid reinforcements (or inclusions), when $G = G_1 = \infty$, the added operators drop out and the BIE II^r and IV^r become BIE II and IV, respectively. In particular, $\lambda_-^{II} = -1$ and the characteristic numbers of BIE IV are also simple poles of the resolvent $(I - \lambda K^{IV})^{-1}$ that are real and absent in the interval (-1, 1). The number $\lambda = -1$ is not characteristic for BIE IV only in the case of a finite singly connected region D.

To satisfy the solvability conditions (3.2) of BIE II^r when $\lambda = 1$ in the case of a nonsingly connected or infinite region D ($m > 0$), we use the still undetermined constants $C_3^{(\beta)}$ in h^{IIr} and find them in the form

$$C_3^{(0)}=0, \quad C_3^{(\beta)}=(\lambda/l_\beta)\int_{\partial_\beta} Q_j(s)n_j(s)ds \quad (\beta=1-m), \quad l_\beta:=\int_{\partial_\beta} ds \tag{3.4}$$

Then after substitution into BIE II^r we arrive at the BIE

$$\begin{gathered} Q_i(s_0)-\lambda\int_{\partial D}[K_{ij}^{IIr}(s,s_0)+K_{ij}^{(2)}(s,s_0)]Q_j(s)ds=h_i^{IIr0}(s_0), \\ K_{ij}^{(2)}(s,s_0):=-\sum_{k=1}^{m}\varphi_i^{(k)}(s_0)\varphi_j^{(k)}(s)/l_k, \\ \varphi_i^{(k)}(s):=\{n_i(s)\ (s\in\partial_k),\ 0\ (s\notin\partial_k)\} \\ h_i^{IIr0}(s_0):=-\sum_{\gamma=1}^{m}[A_j^{(\gamma)}\delta_i'(s_0;\Gamma_j^{(\gamma)})+B^{(\gamma)}\delta_i'(s_0;W^{(\gamma)})]+ \\ +v_{0i}'[\tau(s_0),h;p^{(e)}]/\vartheta(\tau(s_0)) \end{gathered} \tag{3.5}$$

If region D is infinite, this BIE, which is perturbed compared to II^r, will, as follows from § 3 of [5], now be unconditional and uniquely solvable when $\lambda = 1$; this confirms that our choice of the constants $C_3^{(\beta)}$ in the form (3.4) was correct.

If region D is finite, the conditions (3.3) of solvability of BIE II^r at $\lambda = 1$ are satisfied because the $h_i(s)$ are derivatives of functions that are continuous on ∂_β. However, these conditions may be violated on numerical solution by rounding and discretization errors. To eliminate instability to such errors we follow the procedure of § 3 of [5] and, for the case of a finite region of BIE II^r, perturb not only $K_{ij}^{(2)}$ but also the finite-dimensional operator $K_{ij}^{(0)}$ with kernel $K_{ij}^{(0)}(s, s_0)=-\delta_{ij}/l$, where $l = \int ds$ is the length of ∂D, and arrive at the BIE:

$$Q_i(s_0)-\lambda\int_{\partial D}[K_{ij}^{IIr}(s,s_0)+K_{ij}^{(0)}(s,s_0)+K_{ij}^{(2)}(s,s_0)]Q_j(s)ds=h_i^{IIr0}(s_0) \tag{3.6}$$

As follows from § 3 of [5], this BIE will also be unconditional and uniquely solvable for $\lambda = 1$ which confirms that we were correct in choosing $C_3^{(\beta)}$ in the form (3.4) even for a finite region D. Given satisfaction of conditions (3.3), it will give one of the solutions of the BIE (3.5) such that

$$\int_{\partial D} Q_i(s)ds=0 \quad (i=1-2) \tag{3.7}$$

Further, the perturbed BIE (3.5) for an infinite region D and (3.6) for a finite region do not acquire new characteristic numbers in a finite region of the λ-plane as compared to the original BIE II^r.

The number $\lambda = 1$ is characteristic even for the finite-region BIE IV^r. Then, proceeding as above, we arrive in the case of a finite region D at the BIE

$$Q_i(s_0)-\lambda\int_{\partial D}[K_{ij}^{IVr}(s,s_0)+K_{ij}^{(0)}(s,s_0)]Q_j(s)\,ds=h_i^{IVr0}(s_0) \tag{3.8}$$

It will also be unconditional and uniquely solvable for $\lambda = 1$ and, on satisfaction of (3.3), will give one solution of the BIE IV^r that satisfies (3.7).

Thus, it has been possible to reduce problem (r) to a BIE that is unconditionally and uniquely solvable for $\lambda = 1$ in L_p: either, if $1 < p < p_+^{II}$, to perturbed analogs of BIE II^r (3.5) for an infinite region and (3.6) for a finite region, or, if $1 < p > p_+^{IV}$, to BIE IV^r for an infinite region and its perturbed analog (3.8) in a finite region.

It is easily shown by direct integration of these BIE over s_0 that their solutions for $\lambda = 1$ satisfy conditions (2.3) for $\beta = 1\text{-}m$, since the same conditions are satisfied for the right-hand sides h_i, and, if the region is finite, these conditions are also satisfied for $\beta = 0$ by virtue of the chosen perturbations and relations (3.7). Therefore solutions of these BIE satisfy conditions (2.3) and they can actually be used to construct single-sheet representations of solutions of problem (r) with the aid of (2.2), (2.4), and (2.11). Here conditions (1.6), (1.7) for the solution of problem (r) obtained with (2.4), (2.5) are definitely satisfied. Conditions (1.6) are definitely satisfied for the solution obtained with (2.11), (2.5), and conditions (1.7) reduce to (2.16) and must be satisfied by appropriate selection of the constants $C_3^{(\beta)}$ ($\beta = 1\text{-}m$).

Let us show that this choice is always possible and single-valued. Since BIE IV^r for an infinite region and BIE (3.8) for a finite region are unconditionally and uniquely solvable, their solutions can be presented in the form

$$Q_i=Q_i^{(0)}+\sum\nolimits_{\beta=1}^{m} C_3^{(\beta)}Q_i^{(\beta)} \tag{3.9}$$

where $Q_i^{(\beta)}$ ($\beta = 1\text{-}m$) are the solutions of BIE with right-hand sides $\varphi_i^{(\beta)}(s_0)=\{n_i(s_0)(s_0\in\partial_\beta),\ 0\ (s_0\notin\partial_\beta)\}$ and $Q_i^{(0)}$ is the solution of the BIE with the right-hand side (2.12) with $C_3^{(\beta)} = 0$. Substitution of (3.9) into (2.16) results in a system of linear algebraic equations for $C_3^{(\beta)}$:

$$\sum\nolimits_{\beta=1}^{m} C_3^{(\beta)}\int_{\partial_\alpha} Q_i^{(\beta)}(s)[x_i(s)-x_i^{(\alpha)}]\,ds=-\varkappa M^{(e\alpha)}[2\mu(\varkappa-1)H_0]^{-1}-\int_{\partial_\alpha} Q_i^{(0)}(s)[x_i(s)-x_i^{(\alpha)}]\,ds \quad (\alpha=1-m) \tag{3.10}$$

The matrix of this system is nondegenerate. Indeed, otherwise the $C_{30}^{(\beta)}$ would be nontrivial solutions of the corresponding homogeneous system, and the density $Q_{i0}=\sum\nolimits_{\beta=1}^{m} C_{30}^{(\beta)}Q_i^{(\beta)}$ could be used with (2.2) to construct the potential U_i^{IV}, which is the solution of problem (r) with boundary conditions (1.3)-(1.7), where $v_{0i}[p^{(e)}]=0,\ C_3^{(\beta)}=C_{30}^{(\beta)}\ (\beta=1-m),\ C_3^{(0)}0,\quad F_i^{(e\beta)}=M^{(\beta)}=0\ (\beta=0-m)$. But by virtue of the uniqueness theorem, the solution of this problem is

$$u_i=C_i^{(0)}+C_3^{(0)}e_{ij}x_j,\quad C_i^{(\beta)}=C_i^{(0)},\quad C_3^{(\beta)}=C_3^{(0)}\quad (\beta=1-m)$$

and $C_3^{(0)} = 0$ for the infinite region. Hence $C_{30}^{(\beta)}=C_3^{(\beta)}=C_3^{(0)}=0\ (\beta=1-m)$ which proves the necessary theorem.

The forces and moments in each cross section of the border can be obtained from expressions (1.14) of [3] with (1.4) after solution of the BIE and determination of the boundary stresses σ_{ij}^+ on ∂D from this solution with the aid of (2.4), (2.9). Then these forces and moments can be used with (1.4), (1.5) to obtain the displacements of the borders, the constants $C_i^{(\beta)}$ ($i = 1\text{-}2$) for which are given by (1.10).

Various methods can be proposed for solution of the above BIE. Solutions of the perturbed analogs of BIE II^r (3.5), (3.6) with consideration of their spectral properties can, as in [6], be represented by a modified Neumann series:

$$Q_i=(1+d)^{-1}\sum\nolimits_{p=0}^{\infty}[(d+K)/(1+d)]_{ij}^{p}h_j \tag{3.11}$$

Here K should be understood as the operator $K^{IIr} + K^{(2)}$ for the infinite region and $K^{IIr} + K^{(0)} + K^{(2)}$ as the operator for the finite region, while $h_j = h_j^{IIr0}$. This series, which was obtained by a substitution of variable, will, as follows from [7,8], converge for $d > -(1+1/\lambda_-^{II})/2$. In practical calculations, it makes sense from the standpoint of optimal convergence of the series [6] to choose d in the interval $-(1+1/\lambda_-^{II})/2 < d \leqslant -1/\lambda_-^{II}$, and, for the perturbed BIE (3.6) in the finite region, in the still narrower interval $-1(1+1/\lambda_-^{II})/2 < d \leqslant -1/(2\lambda_-^{II})$.

The information available on the spectral properties of BIE IV^r for the infinite region and its perturbed analog (3.8) for the finite region is generally inadequate for construction of iterative methods that can be guaranteed to converge. After discretization, this BIE can be solved by a direct method. However, if the reinforcements are infinitely rigid and BIE IV^r degenerates into BIE IV, it can be solved by using the Neumann series (3.11), where d is chosen in the intervals indicated above with -1 substituted for λ_- (for a finite singly connected region it is even possible to take d = 0, i.e., the standard Neumann series). The constants $C_3^{(\beta)}$ that appear in these equations for nonsingly connected or infinite regions can be determined with the aid of representation (3.9) and subsequent solution of system (3.10). Then to find $Q_i^{(0)}$, $Q_i^{(\beta)}$ one BIE must be solved m + 1 times with different right-hand sides. It is also possible to determine the constants $C_3^{(\beta)}$ by transposing them to the left side of the BIE and, after discretization, solving the expanded system of equations obtained by adding relations (2.16) to the BIE.

4. ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF BIE II^r AND IV^r

Asymptotic forms of the solutions of BIE II-IV for problems with boundary conditions and displacements will be found in [9,10].

To analyze the asymptotic behavior of solutions of BIE II^r and IV^r, we shall assume that conditions (2.1) are satisfied around an angle point s_* of contour ∂D with interior angle $0 < \omega < 2\pi$ and that the functions that figure in (2.1) satisfy Hoelder's condition in the left-hand ($\tau < \tau_*$) and right-hand ($\tau > \tau_*$) neighborhoods of the point $\tau_* = \tau(s_*)$. Then, applying [3], we find that the operators $K_{ij}^{\sim IIr}$, $K_{ij}^{\sim IVr}$ act from $L_p(\partial D)$ $(1 < p < \infty)$ into the space of Hoelder functions in left and right neighborhoods of s_*. Transposing the terms with these operators to the right-hand sides of the BIE, we obtain BIE II and IV, respectively, with fictitiously specified right-hand sides h_{i0}^{II}, h_{i0}^{IV} that belong to the Hoelder space in the neighborhoods of s_*. The asymptotics of solutions of these BIE were studied in [10], from which we find that solutions of BIE II^r and BIE IV^r have the respective forms

$$Q_i^{II}(s_* \mp \rho) = \sum_{k=1}^{4} A_k(\omega, \varkappa)\, d_{ki}^{II\mp}(\omega, \varkappa)\, \rho^{-\gamma_k^{II}} + \pi\{\sin\omega[\sin\omega + (2\pi - \omega)\cos\omega]\}^{-1} \times A_0^{II}(\omega, \varkappa)\, n_i^{\mp} + Q_i^{II*}(s_* \mp \rho) \tag{4.1}$$

$$Q_i^{IV}(s_* \mp \rho) = \sum_{k=1}^{4} A_k^{IV}(\omega, \varkappa)\, d_{ki}^{IV\mp}(\omega, \varkappa)\, \rho^{-\gamma_k^{IV}} + Q_i^{IV*}(s_* \mp \rho) \tag{4.2}$$

Here A_k^{II}, A_k^{IV} are density intensity factors yet to be determined, $d_{ki}^{II\mp}$, $d_{ki}^{IV\mp}$ are the bounded vectors represented in [10], $n_i^{\mp}$ is the outer normal in the neighborhoods of angle point s_*, the residual terms $Q_i^{II}(s_* \mp \rho)$, $Q_i^{IV}(s_* \mp \rho)$ satisfy Hoelder's condition in the neighborhoods s_* and $Q_i^{II}(s_*) = Q_i^{IV}(s_*) = 0$; $\gamma_k^{II}(\omega)$ are the largest roots for $Re(\gamma_k^{II}) < 1$ of the respective equations

$$\Delta_*(\varkappa, \omega, \gamma_1^{II}) = 0, \quad \Delta_*(-\varkappa, \omega, \gamma_2^{II}) = 0$$
$$\Delta_*(1, 2\pi - \omega, \gamma_3^{II}) = 0, \quad \Delta_*(-1, 2\pi - \omega, \gamma_4^{II})/\gamma_4^{II} = 0$$

and the $\gamma_k^{IV}(\omega)$ are the largest rots for $Re(\gamma_k^{IV}) < 1$ of the respective equations

$$\Delta_*(\varkappa, \omega, \gamma_1^{IV})=0, \quad \Delta_*(-\varkappa, \omega, \gamma_2^{IV})=0$$
$$\Delta_*(\varkappa, 2\pi-\omega, \gamma_3^{IV})=0, \quad \Delta_*(-\varkappa, 2\pi-\omega, \gamma_4^{IV})=0$$
$$\Delta_*(\varkappa, \omega, \gamma):=\varkappa \sin [(\gamma-1)\omega]+(\gamma-1)\sin \omega$$

It is therefore clear that $\gamma_1^{II}(\omega)$, $\gamma_2^{II}(\omega)<0$ if $\omega<\pi$; $\gamma_3^{II}(\omega)<0$ if $\omega>\pi$; $\gamma_4^{II}(\omega)<0$ if $\omega>\omega_{00}=2\pi+\operatorname{tg}\omega_{00}\approx 1{,}790$. In these intervals of the angle ω, the terms with the corresponding cofactors $\rho^{-\gamma_k^{II}}$ can be combined with the residual term $Q_i^{II*}(s_*\mp\rho)$, i.e., they can be omitted from the explicit notation (4.1). Similarly, $\gamma_1^{IV}(\omega)$, $\gamma_2^{IV}(\omega)<0$ if $\omega<\pi$; $\gamma_3^{IV}(\omega)$, $\gamma_4^{IV}(\omega)<0$ if $\omega>\pi$. In these ranges of ω, the terms with corresponding cofactors $\rho^{-\gamma_k^{IV}}$ can be combined with the residual term $Q_i^{IV*}(s_*\mp\rho)$, i.e., omitted from the explicit notation (4.2).

Let us write out the asymptotic form of the density of BIE II^r for critical values of the angles $\omega = \omega_{00}, \pi$, when some of the coefficients in (4.1) become unbounded.

For $\omega = \omega_{00} \approx 102.5°$ we have

$$\gamma_4^{II}(\omega_{00})=0; \quad \gamma_3^{II}(\omega_{00})>0; \quad \gamma_1^{II}(\omega_{00}), \quad \gamma_2^{II}(\omega_{00})<0$$

$$\begin{aligned} Q_i^{II}(s_*\mp\rho) = {} & A_3(\omega_{00},\varkappa)\, d_{3i}^{II\mp}(\omega_{00},\varkappa)\,\rho^{-\gamma_3^{II}} + \{\pi[2\pi-\omega_{00})\sin(\omega_{00})]^{-2}\times \\ & \times A_0^{II}(\omega_{00},\varkappa)(2\ln\rho-1)+A_4^{II0}(\omega_{00},\varkappa)\}\, n_i^{\mp} \pm \pi[(2\pi-\omega_{00})\sin^2(\omega_{00})]^{-1}\times \\ & \times A_0^{II}(\omega_{00},\varkappa)\, n_j^{\mp} + Q_i^{II*}(s_*\mp\rho) \end{aligned} \tag{4.3}$$

We obtain for $\omega = \pi$

$$\gamma_1(\pi)=\gamma_2(\pi)=\gamma_3(\pi)=0, \quad \gamma_4(\pi)<0$$

$$\begin{aligned} Q_i^{II}(s_*\mp\rho) = {} & -(\varkappa^2-1)(4\pi\varkappa)^{-1}[h_{j0}^{II-}(s_*)-h_{j0}^{II+}(s_*)]\,e_{ij}\ln\rho \pm \\ & \pm(\varkappa+1)^2(8\varkappa)^{-1}[h_{j0}^{II-}(s_*)-h_{j0}^{II+}(s_*)]-A_1^{II0}(\pi,\varkappa)(\varkappa-1)(\varkappa+1)^{-1}n_i- \\ & -[A_2^{II0}(\varkappa-1)(\varkappa+1)^{-1}+A_3^{II0}]\,e_{ip}n_p Q_i^{II*}(s_*\mp\rho) \end{aligned} \tag{4.4}$$

We have for the density of BIE IV^r at $\omega = \pi$

$$\begin{aligned} Q_i^{IV}(s_*\mp\rho) = {} & \pm[h_i^{IV-}(s_*)-h_i^{IV+}(s_*)]/2+A_4^{IV}(\pi,\varkappa)\,n_i- \\ & -A_3^{IV}(\pi,\varkappa)\,e_{ip}n_p+Q_i^{IV*}(s_*\mp\rho) \end{aligned} \tag{4.5}$$

In these asymptotic forms, the parameters A_k^{II0} are, like A_k^{II}, A_k^{IV}, subject to determination from the complete solution of BIE II^r, IV^r. In the asymptotic forms (4.4), (4.5) for $\omega = \pi$, i.e., at the points of smoothness of contour ∂D, we also have discontinuities of the fictitiously specified right-hand sides $[h_{i0}^+ - h_{i0}^-]$, which, generally speaking, are not defined a priori, but, using an analysis similar to that made in [3] for displacement and stress asymptotics, we find that these discontinuities are nonzero if there are discontinuities of any of the functions of (2.1) at τ_* or concentrated forces F_{Ri}^* or a moment M_R^* are applied at point τ_*. In particular, if the functions that figure in (2.1) are continuous and the thickness $h(s_*) = 0$, these discontinuities can be expressed in terms of the concentrated forces and moments

$$h_{i0}^{II-}(s_*)-h_{i0}^{II+}(s_*)=h_{i0}^{IV-}(s_*)-h_{i0}^{IV+}(s_*)=$$
$$=[F_{Rj}^* k_j^0(\tau)+\chi_h(\tau_*)M_R^*]\,g(\tau_*)\,k_i(s_*)$$
$$\chi_h(\tau)=\chi(\tau)+G(\tau)h(\tau)/[G_1(\tau)\vartheta(\tau)]$$

We note further that the stress intensity factors K_i and K_6 in the asymptotic form of the solution of problem (r) [3] can be expressed in terms of the density intensity factors A_i^{II}, A_i^{II0}, A_i^{IV}, A_i^{IV0} of BIE II^r, IV^r using the same formulas as for the problem with displacements specified on the boundary [10].

REFERENCES

1. A. A. Syas'kii, "Thermoelastic problem for a plate with a partially reinforced circular hole," Prikladnaya Mekhanika, vol. 20, no. 12, pp. 75-79, 1984.
2. T. L. Martynovich and A. A. Syas'kii, "Determination of stresses in a plate with an open stiffening rib," Izvestiya VUZ. Stroitel'stvo i Arkhitektura, no. 8, pp. 32-34, 1985.
3. S. E. Mikhailov and I. V. Namestnikova, "Boundary-value problems of elasticity theory for plane regions with one-dimensional elastic reinforcements," Prikl. Mekh. Tekh. Fiz., no. 1, pp. 103-114, 1991.
4. N. S. Kakhniashvili, "Investigation of plane problems of elasticity theory by a potential-theory method," Trudy Tbilisskogo Univ., vol. 50, pp. 23-58, 1953.
5. S. E. Mikhailov, "Integral equations of certain boundary-value problems for harmonic functions in plane multiply connected regions with irregular boundaries," Matem. Sbornik, vol. 121, no. 4, pp. 533-544, 1983.
6. S. E. Mikhailov, "Spectral properties and solution methods of certain integral equations of elasticity theory for plane nonsingly connected bodies with angle points and forces specified on the boundary," Izv. AN SSSR. MTT [Mechanics of Solids], no. 5, pp. 52-62, 1989.
7. V. N. Kublanovskaya, "Uses of analytic continuation by substitution of variables in numerical analysis," Trudy Mat. Inst. Akad. Nauk SSSR, vol. 53, pp. 145-185, 1959.
8. L. N. Kantorovich and V. K. Krylov, Approximate Methods of Higher Analysis [in Russian], Fizmatgiz, Leningrad, 1962.
9. S. E. Mikhailov, "Asymptotic forms of solutions of certain integral equations and plane problems of elasticity theory near angles with forces specified on the boundary," Izv. AN SSSR. MTT [Mechanics of Solids], no. 3, pp. 33-43, 1989.
10. S. E. Mikhailov, "Asymptotic forms of solutions of certain integral equations and plane problems of elasticity theory near angles with displacements specified on the boundary, Izv. AN SSSR. MTT [Mechanics of Solids], no. 2, pp. 28-40, 1991.

19 January 1990 Moscow