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GENERAL AND FUNDAMENTAL SOLUTIONS OF AXISYMMETRIC TORSION EQUATION FOR A CYLINDRICALLY ANISOTROPIC ELASTIC MEDIUM

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In this paper, we set up a general solution of the axisymmetric torsion equation for an elastic cylindrically anisotropic medium, by reducing it to the corresponding equation for an isotropic medium. Using this approach, and the available fundamental solution for the analogous isotropic problem, we obtain the fundamental solution of the axisymmetric torsion problem for the medium under consideration, i.e., the solution of the elasticity problem for an infinite cylindrically anisotropic space acted upon by forces concentrated on an annular line, uniformly distributed over this line, and oriented along the tangent to it. Limiting relations for characteristic parameter values are given. As one of the limiting cases, we obtain the fundamental solution of the antiplane problem for an elastic rectilinearly anisotropic medium.

1. BASIC EQUATIONS AND GENERAL SOLUTION

Numerical solution methods for boundary value problems of the mechanics of deformable solids, based on the use of fundamental solutions (i.e., solutions describing the reaction of an infinite space or plane to a concentrated action or stimulus) have gained widespread currency. These methods include the direct and indirect methods of boundary integral equations (see, e.g., [1,2]), and also the source method (compensating-load method [3,4]), in which the solution of the boundary value problem is set up by superposition of concentrated actions in space, located on some surface that envelops the region under consideration.

In an isotropic elastic medium, the fundamental solution is provided by the Kelvin-Somigliana matrix. There are also three-dimensional fundamental solutions of the elasticity equations for a medium with rectilinear anisotropy [5]. Paper [6] gives axisymmetric fundamental solutions of the heat equations for a cylindrically anisotropic medium.

Assume that (r, θ, z) is a cylindrical coordinate system. Let us consider the basic equations of elasticity of an anisotropic solid with axial symmetry [7]. In contrast to the customary coordinate numbering, we will set up the following correspondence of indexes: $r \leftrightarrow 1$, $z \leftrightarrow 2$, $\theta \leftrightarrow 3$. Then the equilibrium equations will have the form

$$\sigma_{p\alpha,\alpha}+r^{-1}(\sigma_{p1}-\delta_{p1}\sigma_{33}+\delta_{p3}\sigma_{13})=-F_p(r,\ z) \tag{1.1}$$

while Hooke's law and the relationship between the deformations and displacements will be respectively

$$\sigma_{pj}=c_{pjkl}\varepsilon_{kl} \tag{1.2}$$

$$\varepsilon_{kl}=(u_{k,l}+u_{l,k})/2+r^{-1}[\delta_{k3}\delta_{l3}u_1-(\delta_{k1}\delta_{l3}+\delta_{k3}\delta_{l1})u_3/2] \tag{1.3}$$

Here and henceforth, subscripts following commas denote derivatives with respect to the corresponding coordinate; summation from 1 to 2 is understood over repeating Greek-letter indexes, and from 1 to 3 over repeating Latin indexes, unless otherwise specified (except for r, θ, z, over which no summation is assumed); δ_{pj} is the Kronecker delta; and F are specified volume forces. In (1.1)-(1.3) it should be borne in mind that $u_{k,3} = 0$ because of the axial symmetry of the solution.

If the medium exhibits cylindrical anisotropy whose axis coincides with the z axis, and is homogeneous, then the components of the elasticity tensor c_{pjkl} in this coordinate system are constant. Then, substituting (1.2) into (1.1), with allowance for (1.3), we arrive at a Lame system in u_k:

$$c_{p\alpha k\beta}u_{k,\beta\alpha}+r^{-1}\overset{(1)}{c}_{p\alpha k}u_{k,\alpha}+r^{-2}\overset{(2)}{c}_{pk}\,u_k=-F_p$$

$$\overset{(1)}{c}_{p\alpha k}:=c_{p\alpha 33}\delta_{k1}-c_{p\alpha 13}\delta_{k3}+c_{p1k\alpha}-\delta_{p1}c_{33k\alpha}+\delta_{p3}c_{13k\alpha} \tag{1.4}$$

$$\overset{(2)}{c}_{pk}:=-c_{3333}\delta_{p1}\delta_{k1}-c_{1313}\delta_{p3}\delta_{k3}+c_{1333}(\delta_{k1}\delta_{p3}+\delta_{p3}\delta_{k1})$$

In what follows, we will assume that the meridian planes passing through the axis of symmetry are planes of symmetry of the elastic properties (see, e.g., [7]). Then $c_{3\alpha\gamma\beta}=\overset{(1)}{c}_{3\alpha\gamma}=\overset{(1)}{c}_{\gamma\alpha 3}=\overset{(2)}{c}_{3\gamma}=\overset{(2)}{c}_{\gamma 3}=0$ and system (1.4) breaks down into a system of axisymmetric equations without torsion with respect to u_γ:

$$c_{\omega\alpha\gamma\beta}u_{\gamma,\beta\alpha}+r^{-1}\overset{(1)}{c}_{\omega\alpha\gamma}u_{\gamma,\alpha}+r^{-2}\overset{(2)}{c}_{\omega\gamma}\,u_\gamma=-F_\omega(r,z)$$

and an axisymmetric torsion equation with respect to $u \equiv u_3 \equiv u_\theta$:

$$c_{3131}[u_{,rr}+(u/r)_{,r}]+c_{3232}u_{,zz}+c_{3132}(2u_{,rz}+u_{,z}/r)=-F_\theta(r,\ z) \tag{1.5}$$

For an isotropic solid, $c_{3131} = c_{3232} = G$ and Eq. (1.5) becomes

$$G[u_{,rr}+(u/r)_{,r}+u_{,zz}]=-F_0(r,\ z) \tag{1.6}$$

By analogy with [6], in (1.5) we replace the variables (r, z) by (r, z*), where

$$z^*=(z-\chi r)/\xi,\quad \chi=c_{3132}/c_{3131},\quad \xi=[c_{3232}/c_{3131}-\chi^2]^{1/2} \tag{1.7}$$

We set $u^*(r,\ z^*)=u(r,\ \xi z^*+\chi r)=u(r,\ z)$. Taking into account that

$$u_{,r}=\overset{*}{u}_{,r}-\chi\xi^{-1}\overset{*}{u}_{,z^*},\quad u_{,z}=\xi^{-1}\overset{*}{u}_{,z^*},\quad u_{,zz}=\xi^{-2}\overset{*}{u}_{,z^*z^*}$$

$$u_{,rr}=\overset{*}{u}_{,rr}-2\chi\xi^{-1}\overset{*}{u}_{,rz^*}+\chi^2\xi^{-2}\overset{*}{u}_{,z^*z^*},\quad u_{,rz}=\xi^{-1}\overset{*}{u}_{,rz^*}-\chi\xi^{-2}\overset{*}{u}_{,z^*z^*}$$

after substituting into (1.5) we arrive at the axisymmetric torsion equation for isotropic medium (1.6) with respect to u* in terms of the independent variables (r, z*) with right side $F_\theta{}^*(r,\ z^*)=F_\theta(r,\ \xi z^*+\chi r)$ and $G=c_{3131}$.

Thus, we have been able to express the general solution of the axisymmetric torsion equation (1.5) for our cylindrically anisotropic medium in terms of the solution u* of the analogous equation for an isotropic medium.

2. FUNDAMENTAL SOLUTION

We will seek solution (1.5) with right side $F_\theta(r,\ z)=F_0\delta(r_\Delta)\delta(z_\Delta)$, where δ is a Dirac delta function; F_0 is a constant; $r_\Delta=r-r_0$, $z_\Delta=z-z_0$, $(r_0,\ z_0)$ are the coordinates of the point of application of the concentrated force.

The solution of the corresponding equation (1.6) for an isotropic medium with the same right side is known (see, e.g., [2]); it has the form*

$$\begin{aligned} u(r,\ z;\ r_0,\ z_0)&=(2\pi G)^{-1}(r_0/r)^{1/2}F_0Q_{1/2}(a/b)=\\ &=(2\pi G)^{-1}(r_0/r)^{1/2}F_0[ab^{-1}\mu K(\mu)-2\mu^{-1}E(\mu)] \end{aligned} \tag{2.1}$$

*We should note that reference [2] evidently uses the definition $K(m)=\int_0^{\pi/2}(1-m\sin^2\theta)^{-1/2}\,d\theta$, for the elliptic integral, and therefore the parameter m = μ^2 appears instead of μ in the expression of type (2.1).

where $Q_{1/2}$ is a Legendre function of the second kind;

$$K(\mu)=\int_0^{\pi/2}(1-\mu^2\sin^2\theta)^{-1/2}d\theta,\quad E(\mu)=\int_0^{\pi/2}(1-\mu^2\sin^2\theta)^{1/2}d\theta$$

are complete normal elliptic integrals of the first and second kind respectively; $a=r^2+r_0^2+z_\Delta^2$, $b=2rr_0$, $\mu=[2b/(a+b)]^{1/2}$.

Assume, furthermore, that $z_0^*=(z_0-\chi r_0)/\xi$, $z_\Delta^*=z^*-z_0^*$, $R^*=(r_\Delta^2+z_\Delta^{*2})^{1/2}$, $a^*=r^2+r_0^2+z_\Delta^{*2}$, $\mu^*=[2b/(a^*+b)]^{1/2}$.

Allowing for the fact that, after change of variables (1.7), the original equation (1.5) reduces to (1.6) with respect to the independent variables (r, z*) with right side $-F^*(r,\ z^*;\ r_0,\ z_0^*)=-F_0\delta(r_\Delta)\delta(\xi z_\Delta^*+\chi r_\Delta)$, and with G replaced by c_{3131}, and using (2.1) as the kernel of the volume potential with density F*, we obtain the desired fundamental solution of the static equation of axisymmetric torsion:

$$\begin{aligned}u(r,\ z;\ r_0,\ z_0)&=(2\pi\xi c_{3131})^{-1}(r_0/r)^{1/2}F_0Q_{1/2}(a^*/b)=\\&=(2\pi\xi c_{3131})^{-1}(r_0/r)^{1/2}F_0[a^*b^{-1}\mu^*K(\mu^*)-2\mu^{*-1}E(\mu^*)]\end{aligned}\qquad(2.2)$$

Taking account of the properties of Legendre functions of the second kind and their relationship to elliptic integrals [8,9], we can obtain expressions for the gradient of the fundamental solution from (2.2):

$$\begin{aligned}\nabla u=u_{,\alpha}(r,\ z;\ r_0,\ z_0)e_\alpha&=(2\pi\xi c_{3131})^{-1}(r_0/r)^{1/2}F_0\{(a^{*2}-b^2)^{-1}[a^*Q_{1/2}(a^*/b)-\\&-bQ_{-1/2}(a^*/b)][(r-r_0a^*/b-z_\Delta^*\chi/\xi)e_r+(z_\Delta^*/\xi)e_z]-(2r)^{-1}Q_{1/2}(a^*/b)e_r\}=\\&=(2\pi\xi c_{3131})^{-1}(r_0/r)^{1/2}F_0\{[b^{-1}\mu^*K(\mu^*)-2a^*(a^{*2}-b^2)^{-1}\mu^{*-1}E(\mu^*)]\times\\&\times[(r-r_0a^*/b-z_\Delta^*\chi/\xi)e_r+(z_\Delta^*/\xi)e_z]-(2r)^{-1}\mu^*K(\mu^*)e_r\}\end{aligned}\qquad(2.3)$$

Substitution of these expressions into (1.3) and (1.2) makes it possible to calculate the nonzero components of the deformations $\epsilon_{r\theta}$, $\epsilon_{z\theta}$ and stresses $\sigma_{r\theta}$, $\sigma_{z\theta}$ of the fundamental solution.

Let us also consider the limiting properties of the fundamental solution as $r/r_0\to0$, $r_0/R^*\to0$, $R^*/r_0\to0$. We take into account [8] that

$$\begin{aligned}&Q_{1/2}(\eta)\to\pi2^{-5/2}\eta^{-3/2},\quad Q_{-1/2}(\eta)\to\pi2^{-1/2}\eta^{-1/2}\quad(\eta\to\infty)\\&Q_\nu(1+\eta)\to-(1/2)\ln(\eta/2)-\gamma-\Psi(1+\nu)+O(\eta\ln\eta)\quad(\eta\to0)\end{aligned}$$

where Ψ is a logarithmic derivative gamma function, while γ is the Euler-Mascheroni constant. Then, after some manipulations, we will have from (2.2) and (2.3)

$$\begin{aligned}&r/r_0\to0\Rightarrow u\to(4\xi c_{3131})^{-1}F_0[1+(z_\Delta^*/r_0)^2]^{-3/2}r/r_0\\&\nabla u\to(4\xi c_{3131}r_0)^{-1}F_0[1+(z_\Delta^*/r_0)^2]^{-3/2}e_r\\&r_0/R^*\to0\Rightarrow u\to(4\xi c_{3131})^{-1}F_0r_0^2r_\Delta R^{*-3}\\&\nabla u\to(4\xi c_{3131})^{-1}r_0^2R^{*-3}F_0(e_r+(r_\Delta/R^*))[3(\chi\xi^{-1}z_\Delta^*/R^*-r_\Delta/R^*)e_r-\\&-3\xi^{-1}(z_\Delta^*/R^*)e_z]\}\\&R^*/r_0\to0\Rightarrow u\to(2\pi\xi c_{3131})^{-1}F_0[-\ln R^*+\ln(2r_0)-\gamma-\Psi(3/2)]+\\&+O[R^*r_0^{-1}\ln(R^*r_0^{-1})]\end{aligned}\qquad(2.4)$$

$$\nabla u\to-(2\pi\xi c_{3131})^{-1}F_0[(r_\Delta-\chi\xi^{-1}z_\Delta^*)e_r+\xi^{-1}z_\Delta^*e_z]/R^{*2}\qquad(2.5)$$

Expressions (2.4) and (2.5) not only yield the principal terms of the fundamental solution for the case of small distance between the concentrated force and observation point, but also make it possible to obtain the fundamental solution $u^{(a)}$ in the antiplane problem with rectilinear anisotropy, to which cylindrical anisotropy degenerates as r, $r_0 \to \infty$. It should be borne in mind, however, that in the axisymmetric problem the value of u is reckoned from its value at infinity, whereas in plane problems the fundamental solution at infinity can be unbounded. Therefore we will reckon the value of the function from its value at some fixed point r_1, z_1:

$$u^{(a)}(r_\Delta, z_\Delta) = \lim\ [u(r_0+r_\Delta, z; r_0, z_0) - u(r_0+r_{1\Delta}, z_1; r_0, z_0)] \quad (r_0 \to \infty)$$

Taking account of (2.4), then, the fundamental solution of the antiplane strain problem for a rectilinearly anisotropic medium with plane of symmetry of the elastic properties has the form $u^{(a)}(r_\Delta, z_\Delta) = -(2\pi\xi c_{3131})^{-1}F_0 \ln R^* + C_0$, , where the constant $C_0 = (2\pi\xi c_{3131})^{-1}F_0 \ln(r_{1\Delta}^2 + z_{1\Delta}^{*2})$ and can be discarded. Representations for $\nabla u^{(a)}$ are given by the right side of (2.5).

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