Chapter 18
On United Boundary-Domain Integro-Differential Equations for Variable Coefficient Dirichlet Problem with General Right-Hand Side

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18.1 Introduction

In this paper, the Dirichlet boundary value problem (BVP) for the linear stationary diffusion partial differential equation with a variable coefficient is considered. The PDE right-hand side belongs to the Sobolev spaces $H^{-1}(\Omega)$, when neither classical nor canonical co-normal derivatives are well defined. Using an appropriate parametrix (Levi function) the problem is reduced to a direct boundary-domain integro-differential equation (BDIDE) or to a domain integral equation supplemented by the original boundary condition thus constituting a boundary-domain integro-differential problem (BDIDP). Solvability, solution uniqueness, and equivalence of the BDIDE/BDIDP to the original BVP are analysed in Sobolev (Bessel potential) spaces.

Let Ω be a bounded open three-dimensional region of $\mathbb{R}^3$. For simplicity, we assume that the boundary $\partial\Omega$ is a simply connected, closed, infinitely smooth surface. Let $a \in C^\infty(\overline{\Omega})$, $a(x) > 0$ for $x \in \overline{\Omega}$.

We consider the scalar elliptic differential equation, which for sufficiently smooth u has the following strong form

$$Au(x) := A(x, \partial_x)u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x), \quad x \in \Omega \qquad (18.1)$$

where u is an unknown function and f is a given function in Ω.

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In what follows $\mathscr{D}(\Omega) := C^{\infty}_{comp}(\Omega)$ denotes the space of Schwartz test functions, $H^s(\Omega) = H^s_2(\Omega)$, $H^s(\partial\Omega) = H^s_2(\partial\Omega)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ (see, e.g., [LiMa72, Mc00]). We recall that H^s coincide with the Sobolev-Slobodetski spaces W^s_2 for any non-negative s. We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^3)$,

$$\widetilde{H}^s(\Omega) := \{g : g \in H^s(\mathbb{R}^3),\ \operatorname{supp} g \subset \overline{\Omega}\}.$$

And the space $H^s(\Omega)$ denotes the space of restriction on Ω of distributions from $H^s(\mathbb{R}^3)$,

$$H^s(\Omega) = \{r_{\Omega} g : g \in H^s(\mathbb{R}^3)\}$$

where r_{Ω} denotes the restriction operator on Ω.

18.2 Co-normal Derivatives and the Boundary Value Problem

For $u \in H^1(\Omega)$, the partial differential operator A is understood in the sense of distributions,

$$\langle Au, v\rangle_{\Omega} := -\mathscr{E}(u, v) \quad \forall v \in \mathscr{D}(\Omega) \tag{18.2}$$

where

$$\mathscr{E}(u, v) := \int_{\Omega} a(x)\nabla u(x) \cdot \nabla v(x) dx$$

and the duality brackets $\langle g, \cdot\rangle_{\Omega}$ denote the value of a linear functional (distribution) g, extending the usual L_2 dual product.

Since the set $\mathscr{D}(\Omega)$ is dense in $\widetilde{H}^1(\Omega)$, formula (18.2) defines (cf. e.g. [Mi11, Section 3.1]) the continuous linear operator $A : H^1(\Omega) \to H^{-1}(\Omega) = [\widetilde{H}^1(\Omega)]^*$, where

$$\langle Au, v\rangle_{\Omega} := -\mathscr{E}(u, v) \quad \forall v \in \widetilde{H}^1(\Omega).$$

Let us also consider the different operator, $\check{A} : H^1(\Omega) \to \widetilde{H}^{-1}(\Omega) = [H^1(\Omega)]^*$

$$\begin{aligned}\langle \check{A}u, v\rangle_{\Omega} = -\mathscr{E}(u, v) &= -\int_{\Omega} a(x)\nabla u(x) \cdot \nabla v(x) dx \\ &= -\int_{\mathbb{R}^3} \mathring{E}[a\nabla u](x) \cdot \nabla V(x) dx\end{aligned}$$

$$= \langle \nabla \cdot \mathring{E}[a\nabla u], V\rangle_{\mathbb{R}^3}$$
$$= \langle \nabla \cdot \mathring{E}[a\nabla u], v\rangle_{\Omega}, \quad \forall u \in H^1(\Omega),\ v \in H^1(\Omega), \tag{18.3}$$

which is evidently continuous and can be written as

$$\check{A}u := \nabla \cdot \mathring{E}[a\nabla u].$$

Here $V \in H^1(\mathbb{R}^3)$ is such that $r_\Omega V = v$ and $\mathring{E}$ denotes the operator of extension of functions, defined in Ω, by zero outside Ω in $\mathbb{R}^3$. For any $u \in H^1(\Omega)$, the functional $\check{A}u$ belongs to $\widetilde{H}^{-1}(\Omega)$ and is an extension of the functional $Au \in H^{-1}(\Omega)$ which domain is thus extended from $\widetilde{H}^1(\Omega)$ to the domain $H^1(\Omega)$ for $\check{A}u$.

From the trace theorem (see, e.g., [LiMa72, DaLi90, Mc00]) for $u \in H^1(\Omega)$, it follows that $\gamma^+u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^+ := \gamma^+_{\partial\Omega}$ is the trace operator on $\partial\Omega$ from Ω. Let also $\gamma^{-1} : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega)$ denote a (non-unique) continuous right inverse to the trace operator γ^+, i.e., $\gamma^+\gamma^{-1}w = w$ for any $w \in H^{\frac{1}{2}}(\partial\Omega)$, and $(\gamma^{-1})^* : \widetilde{H}^{-1}(\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is the continuous operator dual to $\gamma^{-1} : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega)$, i.e., $\langle(\gamma^{-1})^*\tilde{f}, w\rangle_{\partial\Omega} := \langle\tilde{f}, \gamma^{-1}w\rangle_\Omega$ for any $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and $w \in H^{\frac{1}{2}}(\partial\Omega)$.

For $u \in H^2(\Omega)$, we can denote by T^{c+} the corresponding classical (strong) co-normal derivative operator on $\partial\Omega$ in the sense of traces,

$$T^{c+}u(x) := \sum_{i=1}^{3} a(x)n_i^+(x)\gamma^+\left(\frac{\partial u(x)}{\partial x_i}\right) = a(x)\gamma^+\left(\frac{\partial u(x)}{\partial n(x)}\right),$$

where $n^+(x)$ is the outward (to Ω) unit normal vectors at the point $x \in \partial\Omega$. However the classical co-normal derivative operator is generally not well defined if $u \in H^1(\Omega)$ (cf. an example in [Mi15, Appendix A]).

Definition 1 Let $u \in H^1(\Omega)$ and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$. Then the *formal co-normal derivative* $T^+(\tilde{f}, u) \in H^{-\frac{1}{2}}(\partial\Omega)$ is defined as

$$\langle T^+(\tilde{f}, u), w\rangle_{\partial\Omega} := \langle\tilde{f}, \gamma^{-1}w\rangle_\Omega + \mathscr{E}(u, \gamma^{-1}w)$$
$$= \langle\tilde{f} - \check{A}u, \gamma^{-1}w\rangle_\Omega \quad \forall w \in H^{\frac{1}{2}}(\partial\Omega).$$

that is,

$$T^+(\tilde{f}, u) := (\gamma^{-1})^*(\tilde{f} - \check{A}u) = (\gamma^{-1})^*\tilde{f} + T^+(0, u). \tag{18.4}$$

If, in addition, $Au = r_\Omega\tilde{f}$ in Ω, then $T^+(\tilde{f}, u)$ becomes the *generalised co-normal derivative*, cf. Definition 3.1 in [Mi11] and Definition 5.2 in [Mi13]. Note that the formal co-normal derivative generally depends on the chosen right inverse, γ^{-1}, of the trace operator; however, the generalised co-normal derivative does not. Some other properties of the generalised conormal derivative also hold true for the formal conormal derivative. In particular, similarly to [Mc00, Lemma 4.3], [Mi11, Theorem 5.3], we have the estimate

$$\left\| T^+(\tilde{f}, u) \right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leqslant C_1 \left\| u \right\|_{H^1(\Omega)} + C_2 \left\| \tilde{f} \right\|_{\widetilde{H}^{-1}(\Omega)}.$$

The first Green identity holds in the following form for $u \in H^1(\Omega)$ such that $Au = r_\Omega \tilde{f}$ in Ω for some $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$,

$$\langle T^+(\tilde{f}, u), \gamma^+ v\rangle_{\partial\Omega} = \langle \tilde{f}, v\rangle_\Omega + \mathscr{E}(u, v) = \langle \tilde{f} - \check{A}u, v\rangle_\Omega \quad \forall v \in H^1(\Omega). \tag{18.5}$$

As follows from Definition 1, the formal and generalised co-normal derivatives are non-linear with respect to u for a fixed $\tilde{f}$, but still linear with respect to the couple $(\tilde{f}, u)$.

We will consider the following Dirichlet boundary value problem:

Find a function $u \in H^1(\Omega)$ satisfying the conditions

$$Au = f \quad \text{in } \Omega, \tag{18.6}$$

$$\gamma^+ u = \varphi_0 \quad \text{on } \partial\Omega, \tag{18.7}$$

where $f \in H^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$.

Equation (18.6) is understood in the distributional sense (18.2) and the Dirichlet boundary condition (18.7) in the trace sense.

The following assertion is well-known and can be proved, e.g., using variational settings and the Lax-Milgram lemma.

Theorem 1 *The Dirichlet problem (18.6)–(18.7) is uniquely solvable in $H^1(\Omega)$. The solution is $u = (A^D)^{-1}(f, \varphi_0)^T$ where the inverse operator $(A^D)^{-1} : H^{\frac{1}{2}}(\partial\Omega) \times H^{-1}(\Omega) \longrightarrow H^1(\Omega)$ to the left-hand side operator, $A^D : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega) \times H^{-1}(\Omega)$, of the Dirichlet problem (18.6)–(18.7) is continuous.*

18.3 Parametrix and Potential Type Operators

We will say, a function $P(x, y)$ of two variables $x, y \in \Omega$ is a parametrix (the Levi function) for the operator $A(x, \partial_x)$ in $\mathbb{R}^3$ if (see, e.g., [Mi02, Mi70, Po98a, Po98b])

$$A(x, \partial_x)P(x, y) = \delta(x - y) + R(x, y), \tag{18.8}$$

where $\delta(.)$ is the Dirac distribution and $R(x, y)$ possesses a weak (integrable) singularity at $x = y$, i.e.,

$$R(x, y) = \mathscr{O}(|x - y|^{-\varkappa}) \text{ with } \varkappa < 3. \tag{18.9}$$

It is easy to see that for the operator $A(x, \partial_x)$ given by the left-hand side in (18.1), the function

$$P(x, y) = \frac{-1}{4\pi a(y)|x - y|}, \qquad x, y \in \mathbb{R}^3, \tag{18.10}$$

is a parametrix and the corresponding remainder function is

$$R(x, y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi a(y)|x - y|^3} \frac{\partial a(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^3, \tag{18.11}$$

and satisfies estimate (18.9) with $\varkappa = 2$, due to smoothness of the function $a(x)$. Evidently, the parametrix $P(x, y)$ given by (18.10) is related with the fundamental solution to the operator $A(y, \partial_x) := a(y)\Delta(\partial_x)$ with the "frozen" coefficient $a(x) = a(y)$ and $A(y, \partial_x)P(x, y) = \delta(x - y)$.

Let $a \in C^\infty(\mathbb{R}^3)$ and $a > 0$ a.e. in $\mathbb{R}^3$. For scalar functions g, for which the integrals have sense, the parametrix-based volume potential operator and the remainder potential operator, corresponding to parametrix (18.10) and remainder (18.11) are defined as

$$\mathbf{P}g(y) := \int_{\mathbb{R}^3} P(x, y)g(x)dx, \ \ y \in \mathbb{R}^3$$

$$\mathscr{P}g(y) := \int_{\Omega} P(x, y)g(x)dx, \ \ y \in \Omega$$

$$\mathscr{R}g(y) := \int_{\Omega} R(x, y)g(x)dx, \ \ y \in \Omega$$

The single and double layer surface potential operators are defined as

$$Vg(y) := -\int_{\partial\Omega} P(x, y)g(x)dS_x, \ \ y \notin \partial\Omega \tag{18.12}$$

$$Wg(y) := -\int_{\partial\Omega} [T(x, n(x), \partial_x)P(x, y)]g(x)dS_x, \ \ y \notin \partial\Omega \tag{18.13}$$

where the integrals are understood in the distributional sense if g is not integrable.

The corresponding boundary integral (pseudodifferential) operators of direct surface values of the single layer potential $\mathscr{V}$ and of the double layer potential $\mathscr{W}$, and the co-normal derivatives of the single layer potential $\mathscr{W}'$ and of the double layer potential $\mathscr{L}^+$, for $y \in \partial\Omega$ are

$$\mathscr{V}g(y) := -\int_{\partial\Omega} P(x,y)g(x)dS_x, \tag{18.14}$$

$$\mathscr{W}g(y) := -\int_{\partial\Omega} [T_x^+ P(x,y)]g(x)dS_x \tag{18.15}$$

$$\mathscr{W}'g(y) := -\int_{\partial\Omega} [T_y^+ P(x,y)]g(x)dS_x, \tag{18.16}$$

$$\mathscr{L}^+ g(y) := T^+ Wg(y). \tag{18.17}$$

When integrals in (18.12)–(18.16) are not well defined, they can be understood, e.g., as pseudo-differential operators or dual forms.

From definitions (18.10), (18.12), (18.13) one can obtain representations of the parametrix-based potential operators in terms of their counterparts for $a = 1$ (i.e. associated with the Laplace operator Δ), which we equip with the subscript Δ, cf. [CMN09],

$$\mathbf{P}g = \frac{1}{a}\mathbf{P}_\triangle g, \quad \mathscr{P}g = \frac{1}{a}\mathscr{P}_\triangle g, \quad \mathscr{R}g = -\frac{1}{a}\sum_{i=1}^{3}\partial_i \mathscr{P}_\triangle[g(\partial_i a)], \tag{18.18}$$

$$Vg = \frac{1}{a}V_\triangle g, \quad Wg = \frac{1}{a}W_\triangle(ag), \tag{18.19}$$

$$\mathscr{V}g = \frac{1}{a}\mathscr{V}_\triangle g, \quad \mathscr{W}g = \frac{1}{a}\mathscr{W}_\triangle(ag), \tag{18.20}$$

$$\mathscr{W}'g = \mathscr{W}'_\triangle g + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]\mathscr{V}_\triangle g, \tag{18.21}$$

$$\mathscr{L}^\pm g = \mathscr{L}_\triangle(ag) + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]\mathscr{W}_\triangle^\pm(ag). \tag{18.22}$$

Hence

$$\Delta(aVg) = 0, \quad \Delta(aWg) = 0 \text{ in } \Omega, \quad \forall g \in H^s(\partial\Omega) \ \forall s \in \mathbb{R},$$

$$\Delta(a\mathscr{P}g) = g \text{ in } \Omega, \ \forall g \in \widetilde{H}^s(\Omega) \ \forall s \in \mathbb{R}$$

The jump relations as well as mapping properties of potentials and operators are well known for the case $a = \text{const}$. They were extended to the case of variable coefficient $a(x)$ in [CMN09].

18.4 The Third Green Identity and Integral Relations

For $u \in H^1(\Omega)$ and $v(x) = P(x, y)$, where the parametrix $P(x, y)$ is given by (18.10), the following *generalised third Green identity* can be obtained from (18.5), (18.3), (18.8), see [Mi15, Theorem 4.1], [Mi18, Theorem 4.1],

$$u + \mathscr{R}u + W\gamma^+ u = \mathscr{P}\check{A}u \quad \text{in} \quad \Omega,$$

where

$$\mathscr{P}\check{A}u(y) := \langle \check{A}u, P(., y)\rangle_\Omega = -\mathscr{E}(u, P(., y)) = -\int_\Omega a(x)\nabla u(x) \cdot \nabla_x P(x, y)dx.$$

If $r_\Omega Au = \tilde{f}$ in Ω, where $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, then the generalised third Green identity takes the following form,

$$u + \mathscr{R}u - VT^+(\tilde{f}, u) + W\gamma^+ u = \mathscr{P}\tilde{f} \quad \text{in} \quad \Omega, \tag{18.23}$$

For some functions $\tilde{f}$, Ψ and Φ, let us consider a more general "indirect" integral relation associated with Eq. (18.23),

$$u + \mathscr{R}u - V\Psi + W\Phi = \mathscr{P}\tilde{f} \quad \text{in } \Omega \tag{18.24}$$

The following statement proved in [Mi15, Lemma 4.2] (see also [Mi18, Lemma 4.2] for Lipschitz domains and more general spaces and coefficients) extends Lemma 4.1 from [CMN09], where the corresponding assertion was proved for $\tilde{f} \in L_2(\Omega)$.

Lemma 1 *Let* $u \in H^1(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$, and $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ *satisfy Eq.* (18.24). *Then*

$$Au = r_\Omega \tilde{f} \ \text{ in } \ \Omega, \tag{18.25}$$

$$r_\Omega V(\Psi - T^+(\tilde{f}, u)) - r_\Omega W(\Phi - \gamma^+ u) = 0 \quad \text{in} \quad \Omega. \tag{18.26}$$

The following statement was proved in [CMN09, Lemma 4.2].

Lemma 2

(i) If $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ *and* $r_\Omega V\Psi^* = 0$ in Ω, *then* $\Psi^* = 0$.
(ii) If $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$ *and* $r_\Omega W\Phi^* = 0$ in Ω, *then* $\Phi^* = 0$.

Let us now generalise Theorem 5.1 from [Mi06] to the right-hand side $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$.

Theorem 2 *Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$. A function $u \in H^1(\Omega)$ is a solution of PDE $Au = r_\Omega \tilde{f}$ in Ω if and only if it is a solution of boundary-domain integro-differential equation* (18.23).

Proof If $u \in H^1(\Omega)$ solves PDE $Au = r_\Omega \tilde{f}$ in Ω, then it satisfies (18.23). On the other hand, if $u \in H^1(\Omega)$ solves boundary-domain integro-differential Eq. (18.23), then using Lemma 1 with $\Psi = T^+(\tilde{f}, u)$ and $\Phi = \gamma^+ u$, we obtain that u satisfies (18.25), which completes the proof. □

18.5 United Boundary-Domain Integro-Differential Equations

Let us consider reduction of the Dirichlet problem (18.6)–(18.7) with $f \in H^{-1}(\Omega)$, for $u \in H^1(\Omega)$, to a united boundary-domain integro-differential problem or to a united boundary-domain integro-differential equation. Formulations for the mixed problem for $u \in H^{1,0}(\Omega; \Delta)$ with $f \in L_2(\Omega)$ were introduced and analysed in [Mi06]. Let $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ be an extension of $f \in H^{-1}(\Omega)$ (i.e., $f = r_\Omega \tilde{f}$), which always exists, see [Mi11, Lemma 2.15 and Theorem 2.16].

18.5.1 United Boundary-Domain Integro-Differential Problem

Supplementing BDIDE (18.23) in the domain Ω, where we take into account (18.4), with the original Dirichlet condition (18.7) on the boundary $\partial\Omega$, we arrive at the following united boundary-domain integro-differential problem, BDIDP, for u in Ω,

$$\mathscr{G}^D u = \mathscr{F}^D \tag{18.27}$$

where

$$\mathscr{G}^D u = \begin{bmatrix} u + \mathscr{R}u - VT^+(0, u) + W\gamma^+ u \\ \gamma^+ u \end{bmatrix}, \quad \mathscr{F}^D = \begin{bmatrix} \mathscr{P}\tilde{f} + V(\gamma^{-1})^* \tilde{f} \\ \varphi_0 \end{bmatrix} \tag{18.28}$$

and we invoked representation (18.4). Note also that by (18.12),

$$V(\gamma^{-1})^* \tilde{f}(y) = -\langle \gamma P(\cdot, y), (\gamma^{-1})^* \tilde{f} \rangle_{\partial\Omega} = -\langle \gamma^{-1}\gamma P(\cdot, y), \tilde{f} \rangle_\Omega$$
$$= -\langle P(\cdot, y), \gamma^*(\gamma^{-1})^* \tilde{f} \rangle_{\partial\Omega} = -\mathscr{P}\gamma^*(\gamma^{-1})^* \tilde{f}.$$

BDIDP (18.27) is equivalent to the Dirichlet boundary value problem (18.6)–(18.7) in Ω, in the following sense.

Theorem 3 *Let* $f \in H^{-1}(\Omega), \varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, *and* $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ *be such that* $r_\Omega \tilde{f} = f$. *A function* $u \in H^1(\Omega)$ *solves the Dirichlet BVP* (18.6)–(18.7) *in* Ω *if and only if* u *solves BDIDP* (18.27). *Such solution does exist and is unique.*

Proof A solution of BVP (18.6)–(18.7) does exist and is unique due to Theorem 1 and provides a solution to BDIDP (18.27) due to Theorem 2. On the other hand, due to the same Theorem 2, any solution of BDIDP (18.27) satisfies also BVP (18.6)–(18.7), which is unique. □

Due to the mapping properties of operators V, W, $\mathscr{P}$ and $\mathscr{R}$, cf. [CMN09], we have $\mathscr{F}^D \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ and the operator $\mathscr{G}^D : H^1(\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is continuous. It is also injective due to Theorem 3.

18.5.2 United Boundary-Domain Integro-Differential Equation

Substituting the Dirichlet boundary condition (18.7) and relation (18.4) into (18.23), we arrive at the following boundary-domain integro-differential equation, BDIDE, for $u \in H^1(\Omega)$:

$$\mathscr{G}^2 u := u + \mathscr{R}u - VT^+(0, u) = \mathscr{F}^2 \quad in \quad \Omega \tag{18.29}$$

where

$$\mathscr{F}^2 = \mathscr{P}\tilde{f} + V(\gamma^{-1})^*\tilde{f} - W\varphi_0 \tag{18.30}$$

Let us prove the equivalence of BDIDE (18.29) to BVP (18.6)–(18.7).

Theorem 4 *Let* $f \in H^{-1}(\Omega)$, $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, *and* $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ *be such that* $r_\Omega \tilde{f} = f$. *A function* $u \in H^1(\Omega)$ *solves the Dirichlet BVP* (18.6)–(18.7) *in* Ω *if and only if* u *solves BDIDE* (18.29) *with right-hand side* (18.30). *Such solution does exist and is unique.*

Proof Any solution of BVP (18.6)–(18.7) solves BDIDE (18.29) due to the third Green formula (18.23). On the other hand, if u is a solution of BDIDE (18.29) then Lemma 1 implies that u satisfies Eq. (18.6) and $r_\Omega W(\varphi_0 - \gamma^+ u) = 0$ in Ω. Lemma 2 (ii) then implies that $\varphi_0 - \gamma^+ u = 0$, i.e., the Dirichlet boundary condition (18.7) is satisfied. Thus any solution of BDIDE (18.29) satisfies BVP (18.6)–(18.7). The unique solvability of BVP (18.6)–(18.7) and hence of BDIDE (18.29) is implied by Theorem 1. □

The mapping properties of operators V, W, $\mathscr{P}$ and $\mathscr{R}$ imply the membership $\mathscr{F}^2 \in H^1(\Omega)$ and continuity of the operator $\mathscr{G}^2$ in $H^1(\Omega)$, while Theorem 4 implies its injectivity.

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Note that Theorems 3 and 4 imply that the non-uniqueness of extension of $f \in H^{-1}(\Omega)$ to $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$ and the non-uniqueness of the right inverse to the trace operator, γ^{-1}, involved in the definition of $T^{+}(\tilde{f}, u)$, do not compromise the uniqueness of solutions u of BDIDP (18.27) and BDIDE (18.29).

18.6 Conclusion

A Dirichlet BVP for a variable-coefficient second order PDE with general right-hand side function from $H^{-1}(\Omega)$ and with the Dirichlet data from the space $H^{\frac{1}{2}}(\partial\Omega)$ was considered in this paper. It was shown that the BVP can be equivalently reduced to a united boundary-domain integro-differential problem, or to a united boundary-domain integro-differential equation of the second kind.

Similarly one can also consider the united BDIEs for the Neumann and mixed problems in interior and exterior domains for the general right-hand side as well as the united versions of other BDIEs formulated and analysed in [AyMi11, ADM17, Mi02, Mi06].

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References

[AyMi11] Ayele, T.G. and Mikhailov, S.E.: Analysis of two-operator boundary-domain integral equations for variable-coefficient mixed BVP. *Eurasian Math. J.,* **2(3)**: 20–41 (2011).

[ADM17] Ayele, T.G., Dufera, T.T. and Mikhailov, S.E.: Analysis of boundary-domain integral equations for variable-coefficient Neumann BVP in 2D. In: *Integral Methods in Science and Engineering, Vol.1 Theoretical Techniques. C. Constanda et al, eds*, Chapter 3, Springer (Birkhäuser), Boston (2017).

[CMN09] Chkadua, O., Mikhailov, S.E. and Natroshvili, D.: Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient, I: Equivalence and Invertibility. *J. Integral Equat. and Appl.* **21**: 499–543 (2009).

[DaLi90] Dautray, R. and Lions, J.: *Mathematical Analysis and Numerical Methods for Science and Technology. Integral Equations and Numerical Methods*, Vol. 4. Springer: Berlin, Heidelberg-New York (1990).

[LiMa72] Lions, J.L., Magenes, E.: *Non-Homogeneous Boundary Value Problems and Applications,* Vol. 1. Springer, Berlin, Heidelberg, New York (1972).

[Mc00] McLean, W.: *Strongly Elliptic Systems and Boundary Integral Equations.* Cambridge University Press, UK (2000).

[Mi02] Mikhailov, S.E.: Localized boundary-domain integral formulation for problems with variable coefficients. *Internat. J. of Engineering Analysis with Boundary Elements,* **26**: 681–690 (2002).

sergey.mikhailov@brunel.ac.uk

[Mi06] Mikhailov, S.E.: Analysis of united boundary domain integral and integro differential equations for a mixed BVP with variable coefficients. *Math. Meth. Appl. Sci.* **29**: 715–739 (2006).

[Mi11] Mikhailov, S.E.: Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains. *J. Math. Analysis and Appl.,* **378**: 324–342 (2011).

[Mi13] Mikhailov, S.E.: Solution regularity and co-normal derivatives for elliptic systems with non-smooth coefficients on Lipschitz domains. *J. Math. Anal. Appl.,* **400**: 48–67 (2013).

[Mi15] Mikhailov, S.E.: Analysis of segregated Boundary-Domain Integral Equations for Variable-Coefficient Dirichlet and Neumann Problems with General Data. *ArXiv,* 1509.03501:1–32 (2015).

[Mi18] Mikhailov, S.E.: Analysis of segregated boundary-domain integral equations for BVPs with non-smooth coefficients on Lipschitz domains. Boundary Value Problems, **2018**:87 (2018) 1–52.

[Mi70] Miranda, C.: *Partial differential equations of elliptic type,* Second revised edition. Springer-Verlag, Berlin (1970).

[Po98a] Pomp, A.: *The boundary-domain integral method for elliptic systems. With applications in shells.* Lecture notes in Mathematics. Vol. 1683, Springer, Berlin-Heidelberg- New York (1998).

[Po98b] Pomp, A.: Levi functions for linear elliptic systems with variable coefficients including shell equations. *Comput. Mech.,* **22**: 93–99 (1998).

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