Chapter 16
Periodic Solutions in $\mathbb{R}^n$ for Stationary Anisotropic Stokes and Navier-Stokes Systems

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16.1 Introduction

Analysis of Stokes and Navier-Stokes equations is an established and active field of research in the applied mathematical analysis, see, e.g., [CF88, Ga11, RRS16, Se15, So01, Te95, Te01] and references therein. In [KMW20, KMW21a, KMW21b] this field has been extended to the transmission and boundary-value problems for stationary Stokes and Navier-Stokes equations of anisotropic fluids, particularly, with relaxed ellipticity condition on the viscosity tensor. In this chapter, we present some further results in this direction considering periodic solutions to the stationary Stokes and Navier-Stokes equations of anisotropic fluids, with an emphasis on solution regularity.

First, the solution uniqueness and existence of a stationary, anisotropic (linear) Stokes system with constant viscosity coefficients in a compressible framework are analysed on n-dimensional flat torus in a range of periodic Sobolev (Bessel-potential) spaces. By employing the Leray-Schauder fixed point theorem, the linear results are used to show existence of solution to the stationary anisotropic (non-linear) Navier-Stokes incompressible system on torus in a periodic Sobolev space for $n \in \{2, 3\}$. Then the solution regularity results for stationary anisotropic Navier-Stokes system on torus are established for $n \in \{2, 3\}$.

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16.2 Anisotropic Stokes and Navier-Stokes Systems

Let $\mathfrak{L}$ denote a second order differential operator in the component-wise divergence form,

$$(\mathfrak{L}\mathbf{u})_k := \partial_\alpha\left(a_{kj}^{\alpha\beta} E_{j\beta}(\mathbf{u})\right), \quad k = 1, \ldots, n,$$

were $\mathbf{u} = (u_1, \ldots, u_n)^\top$, $E_{j\beta}(\mathbf{u}) := \frac{1}{2}(\partial_j u_\beta + \partial_\beta u_j)$ are the entries of the symmetric part $\mathbb{E}(\mathbf{u})$ of $\nabla\mathbf{u}$ (the gradient of $\mathbf{u}$), and $a_{kj}^{\alpha\beta}$ are constant components of the tensor viscosity coefficient $\mathbb{A} := \left(a_{kj}^{\alpha\beta}\right)_{1\le i,j,\alpha,\beta\le n}$, cf. [Duf78].

Here and further on, the Einstein summation convention in repeated indices from 1 to n is used unless stated otherwise.

The following symmetry conditions are assumed (see [OSY92, (3.1),(3.3)]),

$$a_{kj}^{\alpha\beta} = a_{\alpha j}^{k\beta} = a_{k\beta}^{\alpha j}. \tag{16.1}$$

In addition, we require that tensor $\mathbb{A}$ satisfies the (relaxed) ellipticity condition in terms of all *symmetric* matrices in $\mathbb{R}^{n\times n}$ with *zero matrix trace*, see [KMW21a, KMW21b]. Thus, we assume that there exists a constant $C_{\mathbb{A}} > 0$ such that,

$$a_{kj}^{\alpha\beta}\zeta_{k\alpha}\zeta_{j\beta} \ge C_{\mathbb{A}}^{-1}|\boldsymbol{\zeta}|^2, \ \forall\, \boldsymbol{\zeta} = (\zeta_{k\alpha})_{k,\alpha=1,\ldots,n} \in \mathbb{R}^{n\times n}$$
$$\text{such that } \boldsymbol{\zeta} = \boldsymbol{\zeta}^\top \text{ and } \sum_{k=1}^{n}\zeta_{kk} = 0, \tag{16.2}$$

where $|\boldsymbol{\zeta}|^2 = \zeta_{k\alpha}\zeta_{k\alpha}$, and the superscript $\top$ denotes the transpose of a matrix.

The tensor $\mathbb{A}$ is endowed with the norm

$$\|\mathbb{A}\| := \max\left\{|a_{kj}^{\alpha\beta}| : k, j, \alpha, \beta = 1\ldots, n\right\}.$$

Symmetry conditions (16.1) lead to the following equivalent form of the operator $\mathfrak{L}$

$$(\mathfrak{L}\mathbf{u})_k = \partial_\alpha\left(a_{kj}^{\alpha\beta}\partial_\beta u_j\right), \quad k = 1, \ldots, n. \tag{16.3}$$

Let us also define the Stokes operator $\mathcal{L}$ as

$$\mathscr{L}(\mathbf{u}, p) := \mathfrak{L}\mathbf{u} - \nabla p. \tag{16.4}$$

Let $\mathbf{u}$ be an unknown vector field, p be an unknown scalar field, $\mathbf{f}$ be a given vector field and g be a given scalar field defined in $\mathbb{T}$. Then the equations

$$-\mathscr{L}(\mathbf{u}, p) = \mathbf{f}, \ \operatorname{div}\mathbf{u} = g \text{ in } \mathbb{T} \tag{16.5}$$

determine the *anisotropic stationary Stokes system with viscosity tensor coefficient* $\mathbb{A} = \left(A^{\alpha\beta}\right)_{1\le\alpha,\beta\le n}$ *in a compressible framework.*

In addition, the following nonlinear system

$$-\boldsymbol{\mathscr{L}}(\mathbf{u}, p) + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}\,,\quad \operatorname{div}\mathbf{u} = g \text{ in } \mathbb{T} \tag{16.6}$$

is called the *anisotropic stationary Navier-Stokes system with viscosity tensor coefficient* $\mathbb{A} = \left(A^{\alpha\beta}\right)_{1\le\alpha,\beta\le n}$ *in a compressible framework*. If $g = 0$ in (16.5) and (16.6), then these equations are reduced, respectively, to the *incompressible anisotropic stationary Stokes and Navier-Stokes systems*.

In the *isotropic case*, the tensor $\mathbb{A}$ reduces to

$$a_{kj}^{\alpha\beta} = \lambda\delta_{k\alpha}\delta_{j\beta} + \mu\left(\delta_{\alpha j}\delta_{\beta k} + \delta_{\alpha\beta}\delta_{kj}\right),\ 1 \le i, j, \alpha, \beta \le n\,, \tag{16.7}$$

where λ and μ are real constant parameters with $\mu > 0$ (cf., e.g., Appendix III, Part I, Section 1 in [Te01]), and (16.3) becomes

$$\mathfrak{L}\mathbf{u} = (\lambda + \mu)\nabla\operatorname{div}\mathbf{u} + \mu\Delta\mathbf{u}. \tag{16.8}$$

Then it is immediate that condition (16.2) is fulfilled (cf. [KMW21b]) and thus our results apply also to the Stokes and Navier-Stokes systems in the *isotropic case*. Assuming $\lambda = 0$, $\mu = 1$ we arrive at the classical mathematical formulations of isotropic Stokes and Navier-Stokes systems.

16.3 Some Function Spaces on Torus

Let us introduce some function spaces on torus and periodic function spaces (see, e.g., [Agm65, p.26], [Agr15], [McL91], [RT10, Chapter 3], [RRS16, Section 1.7.1], and [Te95, Chapter 2], for more details).

Let $n \ge 1$ be an integer and $\mathbb{T}$ be the n-dimensional flat torus that can be parametrized as the semi-open cube $\mathbb{T} = [0, 1)^n \subset \mathbb{R}^n$, cf. [Zy02, p. 312]. In what follows, $\mathcal{D}(\mathbb{T}) = \mathcal{C}^\infty(\mathbb{T})$ denotes the space of infinitely smooth real or complex functions on the torus. As usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ the set of natural numbers complemented by 0, and $\mathbb{Z}$ the set of integers.

Let $\boldsymbol{\xi} \in \mathbb{Z}^n$ denote the n-dimensional vector with integer components. We will further need also the set

$$\dot{\mathbb{Z}}^n := \mathbb{Z}^n \setminus \{\mathbf{0}\}.$$

Extending the torus parametrisation to $\mathbb{R}^n$, it is often useful to identify $\mathbb{T}$ with the quotient space $\mathbb{R}^n \setminus \mathbb{Z}^n$. Then the space of functions $\mathcal{C}^\infty(\mathbb{T})$ on the torus can be identified with the space of $\mathbb{T}$-periodic (1-periodic) functions $\mathcal{C}^\infty_\# = \mathcal{C}^\infty_\#(\mathbb{R}^n)$ that consists of functions $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\phi(\mathbf{x}+\boldsymbol{\xi}) = \phi(\mathbf{x}) \quad \forall \boldsymbol{\xi} \in \mathbb{Z}^n.$$

Similarly, the Lebesgue space on the torus $L_p(\mathbb{T})$, $1 \le p \le \infty$, can be identified with the periodic Lebesgue space $L_{p\#} = L_{p\#}(\mathbb{R}^n)$ that consists of functions $\phi \in L_{p,\mathrm{loc}}(\mathbb{R}^n)$, which satisfy the periodicity condition for a.e. $\mathbf{x}$.

The space dual to $\mathcal{D}(\mathbb{T})$, i.e., the space of linear bounded functionals on $\mathcal{D}(\mathbb{T})$, called the space of torus distributions is denoted by $\mathcal{D}'(\mathbb{T})$ and can be identified with the space of periodic distributions $\mathcal{D}'_\#$ acting on $\mathcal{C}^\infty_\#$.

The toroidal/periodic Fourier transform mapping a function $g \in \mathcal{C}^\infty_\#$ to a set of its Fourier coefficients $\hat{g}$ is defined as (see, e.g., [RT10, Definition 3.1.8])

$$\hat{g}(\boldsymbol{\xi}) = [\mathcal{F}_\mathbb{T} g](\boldsymbol{\xi}) := \int_\mathbb{T} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} g(\mathbf{x}) d\mathbf{x}, \quad \boldsymbol{\xi} \in \mathbb{Z}^n.$$

and can be generalised to the Fourier transform acting on a distribution $g \in \mathcal{D}'_\#$.

For any $\boldsymbol{\xi} \in \mathbb{Z}^n$, let $|\boldsymbol{\xi}| := (\sum_{j=1}^n \xi_j^2)^{1/2}$ be the Euclidean norm in $\mathbb{Z}^n$ and let us denote

$$\rho(\boldsymbol{\xi}) := (1+|\boldsymbol{\xi}|^2)^{1/2}.$$

Evidently,

$$\frac{1}{2}\rho(\boldsymbol{\xi})^2 \le |\boldsymbol{\xi}|^2 \le \rho(\boldsymbol{\xi})^2 \quad \forall \boldsymbol{\xi} \in \dot{\mathbb{Z}}^n. \tag{16.9}$$

Similar to [RT10, Definition 3.2.2], for $s \in \mathbb{R}$ we define the *periodic/toroidal Sobolev (Bessel-potential) spaces* $H^s_\# := H^s_\#(\mathbb{R}^n) := H^s(\mathbb{T})$, which consist of the torus distributions $g \in \mathcal{D}'(\mathbb{T})$, for which the norm

$$\|g\|_{H^s_\#} := \|\rho^s \widehat{g}\|_{\ell_2} := \left(\sum_{\xi\in\mathbb{Z}^n} \rho(\boldsymbol{\xi})^{2s} |\widehat{g}(\boldsymbol{\xi})|^2\right)^{1/2} \tag{16.10}$$

is finite, i.e., the series in (16.10) converges. Here $\|\cdot\|_{\ell_2}$ is the standard norm in the space of square summable sequences. By Ruzhansky and Turunen [RT10, Proposition 3.2.6], $H^s_\#$ are Hilbert spaces.

For $g \in H^s_\#$, $s \in \mathbb{R}$, and $m \in \mathbb{N}_0$, let us consider the partial sums

$$g_m(\mathbf{x}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n, |\boldsymbol{\xi}|\le m} \hat{g}(\boldsymbol{\xi}) e^{2\pi i \mathbf{x}\cdot\boldsymbol{\xi}}.$$

Evidently, $g_m \in \mathcal{C}_\#^\infty$, $\hat{g}_m(\boldsymbol{\xi}) = \hat{g}(\boldsymbol{\xi})$ if $|\boldsymbol{\xi}| \le m$ and $\hat{g}_m(\boldsymbol{\xi}) = 0$ if $|\boldsymbol{\xi}| > m$. This implies that $\|g - g_m\|_{H_\#^s} \to 0$ as $m \to \infty$ and hence we can write

$$g(\mathbf{x}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n} \hat{g}(\boldsymbol{\xi}) e^{2\pi i \mathbf{x}\cdot\boldsymbol{\xi}}, \tag{16.11}$$

where the Fourier series converges in the sense of norm (16.10). Moreover, since g is an arbitrary distribution from $H_\#^s$, this also implies that the space $\mathcal{C}_\#^\infty$ is dense in $H_\#^s$ for any $s \in \mathbb{R}$ (cf. [RT10, Exercise 3.2.9]).

There holds the compact embedding $H_\#^t \hookrightarrow H_\#^s$ if $t > s$, embeddings $H_\#^s \subset \mathcal{C}_\#^m$ if $m \in \mathbb{N}_0$, $s > m + n/2$, and moreover, $\bigcap_{s\in\mathbb{R}} H_\#^s = \mathcal{C}_\#^\infty$ (cf. [RT10, Exercises 3.2.10, 3.2.10 and Corollary 3.2.11]). Note also that the torus norms on $H_\#^s$ are equivalent to the corresponding standard (non-periodic) Bessel potential norms on $\mathbb{T}$ as a cubic domain, see, e.g., [Agr15, Section 13.8.1].

By (16.10), $\|g\|^2_{H_\#^s} = |\widehat{g}(\mathbf{0})|^2 + |g|^2_{H_\#^s}$, where

$$|g|_{H_\#^s} := \|\rho^s \widehat{g}\|_{\dot{\ell}_2} := \left(\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \rho(\boldsymbol{\xi})^{2s} |\widehat{g}(\boldsymbol{\xi})|^2 \right)^{1/2}$$

is the seminorm in $H_\#^s$.

For any $s \in \mathbb{R}$, let us also introduce the space $\dot{H}_\#^s := \{g \in H_\#^s : \langle g, 1\rangle_\mathbb{T} = 0\}$. The definition implies that if $g \in \dot{H}_\#^s$, then $\widehat{g}(\mathbf{0}) = 0$ and

$$\|g\|_{\dot{H}_\#^s} = \|g\|_{H_\#^s} = |g|_{H_\#^s} = \|\rho^s \widehat{g}\|_{\dot{\ell}_2}\,. \tag{16.12}$$

Denoting $\dot{\mathcal{C}}_\#^\infty := \{g \in \mathcal{C}_\#^\infty : \langle g, 1\rangle_\mathbb{T} = 0\}$, then $\bigcap_{s\in\mathbb{R}} \dot{H}_\#^s = \dot{\mathcal{C}}_\#^\infty$.

The corresponding spaces of n-component vector functions/distributions are denoted as $\mathbf{H}_\#^s := (H_\#^s)^n$, etc.

Note that the norm $\|\nabla(\cdot)\|_{\mathbf{H}_\#^0}$ is an equivalent norm in $\dot{H}_\#^1$. Indeed, by (16.11)

$$\nabla g(\mathbf{x}) = 2\pi i \sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \boldsymbol{\xi} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} \hat{g}(\boldsymbol{\xi}), \quad \widehat{\nabla g}(\boldsymbol{\xi}) = 2\pi i \boldsymbol{\xi} \hat{g}(\boldsymbol{\xi})$$

and then (16.9) and (16.12) imply

$$\begin{aligned} 2\pi^2 \|g\|^2_{H_\#^1} &= 2\pi^2 \|g\|^2_{\dot{H}_\#^1} = 2\pi^2 |g|^2_{H_\#^1} \le \|\nabla g\|^2_{\mathbf{H}_\#^0} \\ &\le 4\pi^2 |g|^2_{H_\#^1} = 4\pi^2 \|g\|^2_{\dot{H}_\#^1} = 4\pi^2 \|g\|^2_{H_\#^1} \quad \forall\, g \in \dot{H}_\#^1. \end{aligned} \tag{16.13}$$

The vector counterpart of (16.13) takes form

$$2\pi^2\|\mathbf{v}\|^2_{\mathbf{H}^1_\#} = 2\pi^2\|\mathbf{v}\|^2_{\dot{\mathbf{H}}^1_\#} \le \|\nabla\mathbf{v}\|^2_{(H^0_\#)^{n\times n}} \le 4\pi^2\|\mathbf{v}\|^2_{\dot{\mathbf{H}}^1_\#} = 4\pi^2\|\mathbf{v}\|^2_{\mathbf{H}^1_\#} \quad \forall\,\mathbf{v}\in\dot{\mathbf{H}}^1_\#. \tag{16.14}$$

We will further need also the first Korn inequality

$$\|\nabla\mathbf{v}\|^2_{(L_{2\#})^{n\times n}} \le 2\|\mathbb{E}(\mathbf{v})\|^2_{(L_{2\#})^{n\times n}} \quad \forall\,\mathbf{v}\in\mathbf{H}^1_\# \tag{16.15}$$

that can be easily proved by adapting, e.g., the proof in [McL00, Theorem 10.1]) to the periodic Sobolev space.

Let us define the Sobolev spaces of divergence-free functions/distributions,

$$\dot{\mathbf{H}}^s_{\#\sigma} := \left\{\mathbf{w}\in\dot{\mathbf{H}}^s_\# : \operatorname{div}\mathbf{w}=0\right\}, \quad s\in\mathbb{R},$$

endowed with the same norm (16.10).

16.4 Stationary Anisotropic Stokes System on Flat Torus

In this section, we generalise to the isotropic and anisotropic (linear) Stokes systems in compressible framework and to a range of Sobolev spaces the analysis, available in [Te95, Section 2.2]

For the unknowns $(\mathbf{u}, p)\in\dot{\mathbf{H}}^s_\#\times\dot{H}^{s-1}_\#$ and the given data $(\mathbf{f}, g)\in\dot{\mathbf{H}}^{s-2}_\#\times\dot{H}^{s-1}_\#$, $s\in\mathbb{R}$, let us consider the Stokes system

$$-\mathscr{L}(\mathbf{u}, p) = \mathbf{f}, \tag{16.16}$$

$$\operatorname{div}\mathbf{u} = g, \tag{16.17}$$

that should be understood in the sense of distributions, i.e.,

$$-\langle\mathscr{L}(\mathbf{u}, p), \boldsymbol{\phi}\rangle_{\mathbb{T}} = \langle\mathbf{f}, \boldsymbol{\phi}\rangle_{\mathbb{T}} \quad \forall\,\boldsymbol{\phi}\in(\mathcal{C}^\infty_\#)^n, \tag{16.18}$$

$$\langle\operatorname{div}\mathbf{u}, \phi\rangle_{\mathbb{T}} = \langle g, \phi\rangle_{\mathbb{T}} \quad \forall\,\phi\in\mathcal{C}^\infty_\#. \tag{16.19}$$

For $\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n$, let us employ $\bar{e}_{\boldsymbol{\xi}}(\mathbf{x}) = e^{-2\pi i x\cdot\boldsymbol{\xi}}$ as ϕ in (16.19) and $\bar{e}_{\boldsymbol{\xi}}(\mathbf{x})$, multiplied by the unit coordinate vector, as $\boldsymbol{\phi}$ in (16.18). Then recalling (16.3) and (16.4), we arrive for each $\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n$ at the following algebraic system for the Fourier coefficients, $\hat{u}_j(\boldsymbol{\xi})$, $k=1,2,\ldots,n$, and $\hat{p}(\boldsymbol{\xi})$.

$$4\pi^2\xi_\alpha a^{\alpha\beta}_{kj}\xi_\beta\hat{u}_j(\boldsymbol{\xi}) + 2\pi i\xi_k\hat{p}(\boldsymbol{\xi}) = \hat{f}_k(\boldsymbol{\xi}) \quad \forall\,\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n,\ k=1,2,\ldots,n \tag{16.20}$$

$$2\pi i\xi_j\hat{u}_j(\boldsymbol{\xi}) = \hat{g}(\boldsymbol{\xi}) \quad \forall\,\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n. \tag{16.21}$$

The $(n+1)\times(n+1)$ matrix, $\mathfrak{S}(\boldsymbol{\xi})$, of system (16.20)–(16.21) is in fact the principal symbol of the anisotropic Stokes system (16.16)–(16.17) that was analysed in [KMW21b, Lemma 15] to prove that the Stokes system is elliptic in the sense of Agmon–Douglis–Nirenberg. It was, particularly proved that the matrix $\mathfrak{S}$ is nonsingular if $\boldsymbol{\xi}\neq 0$ and hence the solution of system (16.20) and (16.21) can be represented in terms of the inverse matrix $\mathfrak{S}^{-1}(\boldsymbol{\xi})$ as

$$\begin{pmatrix}\widehat{\mathbf{u}}(\boldsymbol{\xi})\\ \hat{p}(\boldsymbol{\xi})\end{pmatrix}=\mathfrak{S}^{-1}(\boldsymbol{\xi})\begin{pmatrix}\widehat{\mathbf{f}}(\boldsymbol{\xi})\\ \hat{g}(\boldsymbol{\xi})\end{pmatrix}\quad\forall\,\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n. \tag{16.22}$$

Moreover, using the estimates for the matrix, obtained in that lemma proof, and implementing to the algebraic system the variant of Babuska–Brezzi theory given in Theorem 2.34 and Remark 2.35(i) in [EG04], see also [KMW21b, Theorem 10], we obtain the following estimates for the solution of the algebraic system (16.20)–(16.21),

$$|\widehat{\mathbf{u}}(\boldsymbol{\xi})|\le C_{uf}\frac{|\widehat{\mathbf{f}}(\boldsymbol{\xi})|}{|2\pi\boldsymbol{\xi}|^2}+C_{ug}\frac{|\hat{g}(\boldsymbol{\xi})|}{2\pi|\boldsymbol{\xi}|}, \tag{16.23}$$

$$|\hat{p}(\boldsymbol{\xi})|\le C_{pf}\frac{|\widehat{\mathbf{f}}(\boldsymbol{\xi})|}{2\pi|\boldsymbol{\xi}|}+C_{pg}|\hat{g}(\boldsymbol{\xi})|\quad\forall\,\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n, \tag{16.24}$$

where $\quad C_{uf}=2C_{\mathbb{A}},\ C_{ug}=C_{pf}=1+2C_{\mathbb{A}}\|\mathbb{A}\|,\ C_{pg}=\|\mathbb{A}\|(1+2C_{\mathbb{A}}\|\mathbb{A}\|).$

Remark 16.1 For the isotropic case (16.7), due to (16.8), system (16.20)–(16.21) reduces to

$$4\pi^2\left[(\lambda+\mu)\boldsymbol{\xi}(\boldsymbol{\xi}\cdot\widehat{\mathbf{u}}(\boldsymbol{\xi}))+\mu|\boldsymbol{\xi}|^2\widehat{\mathbf{u}}(\boldsymbol{\xi})\right]+2\pi i\boldsymbol{\xi}\,\hat{p}(\boldsymbol{\xi})=\widehat{\mathbf{f}}(\boldsymbol{\xi}),\quad\forall\,\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n, \tag{16.25}$$

$$2\pi i\boldsymbol{\xi}\cdot\widehat{\mathbf{u}}(\boldsymbol{\xi})=\hat{g}(\boldsymbol{\xi})\quad\forall\,\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n. \tag{16.26}$$

Taking scalar product of Eq. (16.25) with $\boldsymbol{\xi}$ and employing (16.26), we obtain

$$\hat{p}(\boldsymbol{\xi})=\frac{\boldsymbol{\xi}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})}{2\pi i|\boldsymbol{\xi}|^2}+(\lambda+2\mu)\hat{g}(\boldsymbol{\xi}),\quad\forall\,\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n, \tag{16.27}$$

and substituting this back to (16.25), we get

$$\widehat{\mathbf{u}}(\boldsymbol{\xi})=\frac{1}{4\pi^2\mu|\boldsymbol{\xi}|^2}\left[\widehat{\mathbf{f}}(\boldsymbol{\xi})-\boldsymbol{\xi}\frac{\boldsymbol{\xi}\cdot\widehat{\mathbf{f}}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}\right]+\boldsymbol{\xi}\frac{\hat{g}(\boldsymbol{\xi})}{2\pi i|\boldsymbol{\xi}|^2},\quad\forall\,\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n \tag{16.28}$$

(cf. [Te95, Section 2.2] for the case $s=1$, $g=0$, $\lambda=0$, and $\mu=1$). Expressions (16.27) and (16.28) evidently satisfy estimates (16.23) and (16.24). □

The anisotropic Stokes system (16.16) and (16.17) can be re-written as

$$S\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}=\begin{pmatrix}\mathbf{f}\\ g\end{pmatrix},$$

where

$$S\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}:=\begin{pmatrix}-\boldsymbol{\mathscr{L}}(\mathbf{u},p)\\ \operatorname{div}\mathbf{u}\end{pmatrix},$$

and for any $s\in\mathbb{R}$,

$$S:\dot{\mathbf{H}}_{\#}^{s}\times\dot{H}_{\#}^{s-1}\to\dot{\mathbf{H}}_{\#}^{s-2}\times\dot{H}_{\#}^{s-1}\tag{16.29}$$

is a linear continuous operator.

Now we are in the position to prove the following assertion.

Theorem 16.11 *Let condition* (16.2) *hold.*

(i) For any $(\mathbf{f},g)\in\dot{\mathbf{H}}_{\#}^{s-2}\times\dot{H}_{\#}^{s-1}$, $s\in\mathbb{R}$, *the anisotropic Stokes system* (16.16)–(16.17) *in torus* $\mathbb{T}$ *has a unique solution* $(\mathbf{u},p)\in\dot{\mathbf{H}}_{\#}^{s}\times\dot{H}_{\#}^{s-1}$, *where*

$$\mathbf{u}(\mathbf{x})=\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n}e^{2\pi i x\cdot\boldsymbol{\xi}}\widehat{\mathbf{u}}(\boldsymbol{\xi}),\quad p(\mathbf{x})=\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n}e^{2\pi i x\cdot\boldsymbol{\xi}}\hat{p}(\boldsymbol{\xi})\tag{16.30}$$

with $\widehat{\mathbf{u}}(\boldsymbol{\xi})$ *and* $\hat{p}(\boldsymbol{\xi})$ *given by* (16.22). *In addition, there exists a constant* $C=C(C_{\mathbb{A}},n)>0$ *such that*

$$\|\mathbf{u}\|_{\dot{\mathbf{H}}_{\#}^{s}}+\|p\|_{\dot{H}_{\#}^{s-1}}\le C\left(\|\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{s-2}}+\|g\|_{\dot{H}_{\#}^{s-1}}\right)\tag{16.31}$$

and operator (16.29) *is an isomorphism.*

(ii) Moreover, if $(\mathbf{f},g)\in(\dot{\mathcal{C}}_{\#}^{\infty})^n\times\dot{\mathcal{C}}_{\#}^{\infty}$ *then* $(\mathbf{u},p)\in(\dot{\mathcal{C}}_{\#}^{\infty})^n\times\dot{\mathcal{C}}_{\#}^{\infty}$.

Proof

(i) Expressions (16.22) supplemented by the relations $\widehat{\mathbf{u}}(\mathbf{0})=\mathbf{0}$, $\hat{p}(\mathbf{0})=0$ imply the uniqueness. From estimates (16.23) and (16.24) we obtain the estimate

$$\begin{aligned}\|\mathbf{u}\|_{\dot{\mathbf{H}}_{\#}^{s}}&=\left(\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n}\rho(\boldsymbol{\xi})^{2s}|\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2\right)^{1/2}\\ &\le\frac{C_{uf}}{4\pi^2}\left(\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n}\rho(\boldsymbol{\xi})^{2s}\frac{|\widehat{\mathbf{f}}(\boldsymbol{\xi})|^2}{|\boldsymbol{\xi}|^4}\right)^{1/2}+\frac{C_{ug}}{2\pi}\left(\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n}\rho(\boldsymbol{\xi})^{2s}\frac{|\hat{g}(\boldsymbol{\xi})|^2}{|\boldsymbol{\xi}|^2}\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{C_{uf}}{4\pi^2}\left(\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \rho(\boldsymbol{\xi})^{2(s-2)}|\widehat{\mathbf{f}}(\boldsymbol{\xi})|^2 \frac{\rho(\boldsymbol{\xi})^4}{|\boldsymbol{\xi}|^4}\right)^{1/2} \\
&\quad + \frac{C_{ug}}{2\pi}\left(\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \rho(\boldsymbol{\xi})^{2(s-1)}|\hat{g}(\boldsymbol{\xi})|^2 \frac{\rho(\boldsymbol{\xi})^2}{|\boldsymbol{\xi}|^2}\right)^{1/2} \\
&\le \frac{C_{uf}}{2\pi^2}\|\mathbf{f}\|_{\dot{\mathbf{H}}_\#^{s-2}} + \frac{C_{ug}}{2\pi}\sqrt{2}\|g\|_{\dot{H}_\#^{s-1}}
\end{aligned}
$$

and the similar estimate for $\|p\|_{\dot{H}_\#^{s-1}}$, which imply (16.31) and hence inclusions in the corresponding spaces.

(ii) The inclusion $(\mathbf{f}, g) \in (\dot{\mathcal{C}}_\#^\infty)^n \times \dot{\mathcal{C}}_\#^\infty$ implies that $(\mathbf{f}, g) \in \dot{\mathbf{H}}_\#^{s-2} \times \dot{H}_\#^{s-1}$ for any $s \in \mathbb{R}$. Then by item (i), $(\mathbf{u}, p) \in \dot{\mathbf{H}}_\#^s \times \dot{H}_\#^{s-1}$ for any $s \in \mathbb{R}$ and hence $(\mathbf{u}, p) \in (\dot{\mathcal{C}}_\#^\infty)^n \times \dot{\mathcal{C}}_\#^\infty$.

□

If $g = 0$ in (16.17), we can re-formulate the Stokes system (16.16)–(16.17) as one vector equation

$$
-\mathscr{L}(\mathbf{u}, p) = \mathbf{f} \tag{16.32}
$$

for the unknowns $(\mathbf{u}, p) \in \dot{\mathbf{H}}_{\#\sigma}^s \times \dot{H}_\#^{s-1}$ and the given data $\mathbf{f} \in \dot{\mathbf{H}}_\#^{s-2}$, $s \in \mathbb{R}$. Then Theorem 16.11 implies the following assertion.

Corollary 16.1 *Let condition* (16.2) *hold.*

(i) For any $\mathbf{f} \in \dot{\mathbf{H}}_\#^{s-2}$, $s \in \mathbb{R}$, *the anisotropic Stokes equation* (16.32) *in torus* $\mathbb{T}$ *has a unique incompressible solution* $(\mathbf{u}, p) \in \dot{\mathbf{H}}_{\#\sigma}^s \times \dot{H}_\#^{s-1}$, *with* $\widehat{\mathbf{u}}(\boldsymbol{\xi})$ *and* $\hat{p}(\boldsymbol{\xi})$ *given by* (16.22) *and* (16.30) *(and particularly by* (16.28), (16.27), *and* (16.30) *for the isotropic case* (16.7)) *with* $g = 0$. *In addition, there exists a constant* $C = C(C_{\mathbb{A}}, n) > 0$ *such that*

$$
\|\mathbf{u}\|_{\dot{\mathbf{H}}_\#^s} + \|p\|_{\dot{H}_\#^{s-1}} \le C\|\mathbf{f}\|_{\dot{\mathbf{H}}_\#^{s-2}}
$$

and the operator

$$
\mathscr{L} : \dot{\mathbf{H}}_{\#\sigma}^s \times \dot{H}_\#^{s-1} \to \dot{\mathbf{H}}_\#^{s-2}
$$

is an isomorphism.

(ii) Moreover, if $\mathbf{f} \in (\dot{\mathcal{C}}_\#^\infty)^n$ *then* $(\mathbf{u}, p) \in (\dot{\mathcal{C}}_\#^\infty)^n \times \dot{\mathcal{C}}_\#^\infty$.

16.5 Stationary Anisotropic Navier-Stokes System with Constant Coefficients on Torus

16.5.1 Existence of a Weak Solution to Anisotropic Incompressible Navier-Stokes System on Torus

In this section, we show the existence of a weak solution of the anisotropic Navier-Stokes system in the incompressible case with general data in L^2-based Sobolev spaces on the torus $\mathbb{T}$, for $n \in \{2, 3\}$. We use the well-posedness result established in Theorem 16.11 for the Stokes system on a torus and the following variant of the *Leray-Schauder fixed point theorem* (see, e.g., [GT01, Theorem 11.3]).

Theorem 16.2 *Let B denote a Banach space and $T : B \to B$ be a continuous and compact operator. If there exists a constant $M_0 > 0$ such that $\|x\|_B \leq M_0$ for every pair $(\mathbf{x}, \theta) \in B \times [0, 1]$ satisfying $\mathbf{x} = \theta T\mathbf{x}$, then the operator T has a fixed point $\mathbf{x}_0$ (with $\|x_0\|_B \leq M_0$).*

Let us consider the Navier-Stokes system

$$-\mathscr{L}(\mathbf{u}, p) = \mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u}, \tag{16.33}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{16.34}$$

for the couple of unknowns $(\mathbf{u}, p) \in \dot{\mathbf{H}}^1_\# \times \dot{H}^0_\#$ and the given data $\mathbf{f} \in \dot{\mathbf{H}}^{-1}_\#$. As for the Stokes system, the Navier-Stokes system (16.33) and (16.34) can be re-written as one vector equation

$$-\mathscr{L}(\mathbf{u}, p) = \mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u} \tag{16.35}$$

for the unknowns $(\mathbf{u}, p) \in \dot{\mathbf{H}}^1_{\#\sigma} \times \dot{H}^0_\#$ and the given data $\mathbf{f} \in \dot{\mathbf{H}}^{-1}_\#$.

Let us denote the nonlinear operator as $\boldsymbol{B}$, i.e.,

$$\boldsymbol{B}\mathbf{w} := (\mathbf{w} \cdot \nabla)\mathbf{w}, \quad \forall\, \mathbf{w} \in \mathbf{H}^s_\#, \; s \in \mathbb{R}. \tag{16.36}$$

Theorem 16.3 *Let the operator $\boldsymbol{B} : \mathbf{w} \mapsto \boldsymbol{B}\mathbf{w}$ be defined by (16.36) and let $n \geq 2$.*

(i) If $0 < s < n/2$ then

$$\boldsymbol{B} : \dot{\mathbf{H}}^s_{\#\sigma} \to \dot{\mathbf{H}}^{2s-1-n/2}_\# \tag{16.37}$$

is a well defined, continuous and bounded quadratic operator, i.e., there exists $C_{n,s} > 0$ such that

$$\|\boldsymbol{B}\mathbf{w}\|_{\mathbf{H}^{2s-1-n/2}_\#} \leq C_{n,s} \|\mathbf{w}\|^2_{\mathbf{H}^s_\#} \quad \forall\, \mathbf{w} \in \mathbf{H}^s_\#. \tag{16.38}$$

(ii) If $s > n/2$ then

$$\boldsymbol{B} : \dot{\mathbf{H}}^s_{\#\sigma} \to \dot{\mathbf{H}}^{s-1}_{\#} \tag{16.39}$$

is well defined, continuous and bounded quadratic operator, i.e., there exists $C_{n,s} > 0$ such that

$$\|\boldsymbol{B}\mathbf{w}\|_{\mathbf{H}^{s-1}_{\#}} \le C_{n,s}\|\mathbf{w}\|^2_{\mathbf{H}^s_{\#}} \quad \forall\, \mathbf{w} \in \mathbf{H}^s_{\#}. \tag{16.40}$$

Proof If a function $\mathbf{w}$ is periodic, then evidently the function $\boldsymbol{B}\mathbf{w}$ is periodic as well.

(i) Let $0 < s < n/2$. Due to Theorem 1(iii) in Section 4.6.1 of [RS96] and equivalence of the Bessel potential norms on square and norms (16.10) for the Sobolev spaces on torus, we have,

$$\|(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2\|_{\mathbf{H}^{2s-1-n/2}_{\#}} \le C_{n,s}\|\mathbf{v}_1\|_{\mathbf{H}^s_{\#}}\|\mathbf{v}_2\|_{\mathbf{H}^s_{\#}}, \quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{H}^s_{\#}. \tag{16.41}$$

for some constant $C_{n,s} > 0$. This particularly implies estimate (16.38).

Further, if $\mathbf{u} \in \dot{\mathbf{H}}^s_{\#\sigma}$ then

$$\langle \boldsymbol{B}\mathbf{u}, 1\rangle_{\mathbb{T}} = \langle \mathbf{u} \cdot \nabla)\mathbf{u}, 1\rangle_{\mathbb{T}} = -\langle (\operatorname{div}\mathbf{u})\mathbf{u}, 1\rangle_{\mathbb{T}} = \mathbf{0}$$

since $\operatorname{div}\mathbf{u} = 0$. Together with estimate (16.38) this implies that quadratic operator (16.37) is well defined and bounded.

Let $\mathbf{w}, \mathbf{w}' \in \dot{\mathbf{H}}^1_{\#\sigma}$. Then by (16.41) we obtain

$$\begin{aligned}\left\|\boldsymbol{B}\mathbf{w} - \boldsymbol{B}\mathbf{w}'\right\|_{\mathbf{H}^{2s-1-n/2}_{\#}} &\le \left\|(\mathbf{w} \cdot \nabla)\mathbf{w} - (\mathbf{w}' \cdot \nabla)\mathbf{w}'\right\|_{\mathbf{H}^{2s-1-n/2}_{\#}} \\ &\le \left\|((\mathbf{w} - \mathbf{w}') \cdot \nabla)\mathbf{w} + (\mathbf{w}' \cdot \nabla)(\mathbf{w} - \mathbf{w}')\right\|_{\mathbf{H}^{2s-1-n/2}_{\#}} \\ &\le C_{n,s}\left\|\mathbf{w} - \mathbf{w}'\right\|_{\mathbf{H}^s_{\#}}\left(\|\mathbf{w}\|_{\mathbf{H}^s_{\#}} + \|\mathbf{w}'\|_{\mathbf{H}^s_{\#}}\right).\end{aligned}$$

This estimate shows that operator (16.37) is continuous.

(ii) Let $s > n/2$. Due to Theorem 1(i) in Section 4.6.1 of [RS96] and equivalence of the Bessel potential norms and norms (16.10) for the Sobolev spaces on torus, we have,

$$\|(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2\|_{\mathbf{H}^{s-1}_{\#}} \le C_{n,s}\|\mathbf{v}_1\|_{\mathbf{H}^s_{\#}}\|\mathbf{v}_2\|_{\mathbf{H}^s_{\#}}, \quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{H}^s_{\#}.$$

for some constant $C_{n,s} > 0$. This particularly implies estimate (16.40) and then the boundedness of operator (16.39). By the same arguments as in item (i), one can prove that this operator is also well defined and continuous.

□

Corollary 16.2 *Let $n \in \{2, 3\}$. Then the quadratic operator*

$$\boldsymbol{B} : \dot{\mathbf{H}}^1_{\#\sigma} \to \mathbf{H}^{-1}_{\#} \tag{16.42}$$

is well defined, continuous, bounded and compact.

Proof Let $n = 3$. Due to Theorem 16.3(i), the operator $\boldsymbol{B} : \dot{\mathbf{H}}^1_{\#\sigma} \to \dot{\mathbf{H}}^{-1/2}_{\#}$ is well defined, continuous and bounded. On the other hand, the compactness of embedding $H^{-1/2}_{\#} \hookrightarrow H^{-1}_{\#}$ implies the compactness of embedding $\dot{H}^{-1/2}_{\#} \hookrightarrow \dot{H}^{-1}_{\#}$ and hence gives the compactness of operator (16.42) and thus the corollary claim for $n = 3$.

Let now $n = 2$. Then by Theorem 16.3(i), the operator $\boldsymbol{B} : \dot{\mathbf{H}}^s_{\#\sigma} \to \dot{\mathbf{H}}^{2s-2}_{\#}$ is well defined, continuous and bounded for any $s \in (1/2, 1)$. In addition, for $s \in (1/2, 1)$ we also have the compact embeddings $\dot{H}^1_{\#\sigma} \hookrightarrow \dot{H}^s_{\#\sigma}$ and $\dot{H}^{2s-2}_{\#} \hookrightarrow \dot{H}^{-1}_{\#}$ that lead to the corollary claim for $n = 2$. □

Next we show the existence of a weak solution of the Navier-Stokes equation.

Theorem 16.4 *Let $n \in \{2, 3\}$ and suppose that condition (16.2) holds. If $\mathbf{f} \in \dot{\mathbf{H}}^{-1}_{\#}$, then the anisotropic Navier-Stokes equation (16.35) has a solution $(\mathbf{u}, p) \in \dot{\mathbf{H}}^1_{\#\sigma} \times \dot{H}^0_{\#}$.*

Proof We will reduce the analysis of the nonlinear equation (16.35) to the analysis of a nonlinear operator in the Hilbert space $\dot{\mathbf{H}}^1_{\#\sigma}$ and show that this operator has a fixed-point due to the Leray-Schauder Theorem.

Nonlinear equation (16.35) can be re-written as

$$-\mathscr{L}(\mathbf{u}, p) = \mathbf{f} - \boldsymbol{B}\mathbf{u}. \tag{16.43}$$

By Corollary 16.1, the linear operator

$$-\mathscr{L} : \dot{\mathbf{H}}^1_{\#\sigma} \times \dot{H}^0_{\#} \to \dot{\mathbf{H}}^{-1}_{\#} \tag{16.44}$$

is an isomorphism. Its inverse operator, $-\mathscr{L}^{-1}$, can be split into two operator components,

$$-\mathscr{L}^{-1} = \begin{pmatrix} \mathscr{U} \\ \mathcal{P} \end{pmatrix}$$

where $\mathscr{U} : \dot{\mathbf{H}}^{-1}_{\#} \to \dot{\mathbf{H}}^1_{\#\sigma}$ and $\mathcal{P} : \dot{\mathbf{H}}^{-1}_{\#} \to \dot{H}^0_{\#}$ are linear continuous operators such that

$$-\mathscr{L}\begin{pmatrix} \mathscr{U}\,\mathscr{F} \\ \mathcal{P}\,\mathscr{F} \end{pmatrix} = \mathscr{F}$$

for any $\mathscr{F} \in \dot{\mathbf{H}}_{\#}^{-1}$. Applying the inverse operator, $-\mathscr{L}^{-1}$, to Eq. (16.43), we reduce it to the equivalent nonlinear system

$$\mathbf{u} = \mathbf{U}\mathbf{u}, \tag{16.45}$$

$$p = P\mathbf{u}, \tag{16.46}$$

where $\mathbf{U} : \dot{\mathbf{H}}_{\#\sigma}^1 \to \dot{\mathbf{H}}_{\#\sigma}^1$ and $P : \dot{\mathbf{H}}_{\#\sigma}^1 \to \dot{H}_{\#}^0$ are the nonlinear operators defined as

$$\mathbf{U}\mathbf{w} := \mathscr{U}(\mathbf{f} - \boldsymbol{B}\mathbf{w}), \tag{16.47}$$

$$P\mathbf{w} := \mathcal{P}(\mathbf{f} - \boldsymbol{B}\mathbf{w}) \tag{16.48}$$

for the fixed $\mathbf{f}$.

Since p is not involved in (16.45), we will first prove the existence of a solution $\mathbf{u} \in \dot{\mathbf{H}}_{\#\sigma}^1$ to this equation. Then we use (16.46) as a representation formula for p, which gives the existence of the pressure field $p \in \dot{H}_{\#}^0$. In order to show the existence of a fixed point of the operator $\mathbf{U}$ and, thus, the existence of a solution of Eq. (16.45), we employ Theorem 16.2.

By Corollary 16.2, for $n \in \{2, 3\}$ the operator $\boldsymbol{B} : \dot{\mathbf{H}}_{\#\sigma}^1 \to \mathbf{H}_{\#}^{-1}$ is bounded, continuous and compact. Since $\mathbf{f} \in \mathbf{H}_{\#}^{-1}$ is fixed and the operator $\mathscr{U} : \mathbf{H}_{\#}^{-1} \to \dot{\mathbf{H}}_{\#\sigma}^1$ is linear and continuos, definition (16.47) implies that the operator $\mathbf{U} : \dot{\mathbf{H}}_{\#\sigma}^1 \to \dot{\mathbf{H}}_{\#\sigma}^1$ is also bounded, continuous, and compact.

Next, we show that there exists a constant $M_0 > 0$ such that if $\mathbf{w} \in \dot{\mathbf{H}}_{\#\sigma}^1$ satisfies the equation

$$\mathbf{w} = \theta\mathbf{U}\mathbf{w} \tag{16.49}$$

for some $\theta \in [0, 1]$, then $\|\mathbf{w}\|_{\dot{\mathbf{H}}_{\#\sigma}^1} \leq M_0$. Let us denote

$$q := \theta P\mathbf{w}. \tag{16.50}$$

By applying the operator $-\mathscr{L}$ to Eqs. (16.49) and (16.50) and by using relations (16.47) and (16.48), we deduce that whenever the pair $(\mathbf{w}, \theta) \in \dot{\mathbf{H}}_{\#\sigma}^1 \times \mathbb{R}$ satisfies Eq. (16.49), then the equation

$$-\mathscr{L}(\mathbf{w}, q) = \theta(\mathbf{f} - \boldsymbol{B}\mathbf{w}),$$

is also satisfied due to the isomorphism property of operator (16.44). This equation should be understood in the sense of distribution, i.e.,

$$\begin{aligned}\langle -\mathscr{L}(\mathbf{w}, q), \boldsymbol{\phi}\rangle_{\mathbb{T}} &= \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\boldsymbol{\phi})\right\rangle_{\mathbb{T}} - \langle q, \operatorname{div} \boldsymbol{\phi}\rangle_{\mathbb{T}} \\ &= \theta\langle \mathbf{f} - \boldsymbol{B}\mathbf{w}, \boldsymbol{\phi}\rangle_{\mathbb{T}} \quad \forall \boldsymbol{\phi} \in (\mathcal{C}_{\#}^{\infty})^n. \end{aligned} \tag{16.51}$$

Taking into account that the space $(\mathcal{C}^\infty_\#)^n$ is dense in $\mathbf{H}^1_\#$ and the continuity of the dual products in (16.51) with respect to $\boldsymbol{\phi} \in \mathbf{H}^1_\#$, Eq. (16.51) should hold also for $\boldsymbol{\phi} = \mathbf{w} \in \dot{\mathbf{H}}^1_{\#\sigma}$. Then we obtain

$$\left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{w}) \right\rangle_{\mathbb{T}} = \theta \langle \mathbf{f} - \boldsymbol{B}\mathbf{w}, \mathbf{w} \rangle_{\mathbb{T}}. \tag{16.52}$$

Since $\mathbf{w} \in \dot{\mathbf{H}}^1_{\#\sigma}$, relation (16.55) implies that $\langle \boldsymbol{B}\mathbf{w}, \mathbf{w} \rangle_{\mathbb{T}} = \langle (\mathbf{w} \cdot \nabla)\mathbf{w}, \mathbf{w} \rangle_{\mathbb{T}} = 0$. Then by using the norm equivalence (16.14), the Korn first inequality (16.15), the ellipticity condition (16.2), Eq. (16.52), and the Hölder inequality, we obtain for $\theta \geq 0$ that

$$\begin{aligned}
\|\mathbf{w}\|^2_{\dot{\mathbf{H}}^1_\#} &\leq \frac{1}{2\pi^2} \|\nabla \mathbf{w}\|^2_{(L_{2\#})^{n\times n}} \leq \frac{1}{\pi^2} \|\mathbb{E}(\mathbf{w})\|^2_{(L_{2\#})^{n\times n}} \\
&\leq \frac{1}{\pi^2} C_{\mathbb{A}} \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{w}) \right\rangle_{\mathbb{T}} \leq \frac{\theta}{\pi^2} C_{\mathbb{A}} \|\mathbf{f}\|_{\dot{\mathbf{H}}^{-1}_\#} \|\mathbf{w}\|_{\dot{\mathbf{H}}^1_\#}.
\end{aligned}$$

Hence, for $\theta \in [0, 1]$,

$$\|\mathbf{w}\|_{\dot{\mathbf{H}}^1_\#} \leq M_0 := \frac{1}{\pi^2} C_{\mathbb{A}} \|\mathbf{f}\|_{\dot{\mathbf{H}}^{-1}_\#}.$$

Therefore, the operator $\mathbf{U} : \dot{\mathbf{H}}^1_{\#\sigma} \to \dot{\mathbf{H}}^1_{\#\sigma}$ satisfies the hypothesis of Theorem 16.2 (with $B = \dot{\mathbf{H}}^1_{\#\sigma}$), and hence it has a fixed point $\mathbf{u} \in \dot{\mathbf{H}}^1_{\#\sigma}$, that is, $\mathbf{u} = \mathbf{U}\mathbf{u}$. Then with $p \in \dot{H}^0_\#$ as in (16.46), we obtain that the couple $(\mathbf{u}, p) \in \dot{\mathbf{H}}^1_{\#\sigma} \times \dot{H}^0_\#$ satisfies the nonlinear equation (16.35). □

16.5.2 Solution Regularity for the Stationary Anisotropic Navier-Stokes System

In this section, using the bootstrap argument we show that the regularity of a solution of the anisotropic incompressible Navier-Stokes system on $\mathbb{T}^n$, $n \in \{2, 3\}$, is completely determined by the regularity of its right-hand side, as for the Stokes system. To prove this we use the inclusions of the nonlinear term $\boldsymbol{B}\mathbf{u}$ given by Theorem 16.3 and the unique solvability of corresponding (linear) Stokes system.

Theorem 16.5 *Let condition (16.2) hold. Let $n \geq 2$ and $n/2 - 1 < s_1 < s_2$.*

(i) If $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s_1}_{\#\sigma} \times \dot{H}^{s_1 - 1}_\#$ is a solution of the anisotropic Navier-Stokes equation (16.35) with a right hand side $\mathbf{f} \in \dot{\mathbf{H}}^{s_2 - 2}_\#$, then $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s_2}_{\#\sigma} \times \dot{H}^{s_2 - 1}_\#$.

(ii) Moreover, if $\mathbf{f} \in (\dot{\mathcal{C}}^\infty_\#)^n$ then $(\mathbf{u}, p) \in (\dot{\mathcal{C}}^\infty_\#)^n \times \dot{\mathcal{C}}^\infty_\#$.

Proof

(i) Let $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s_1}_{\#\sigma} \times \dot{H}^{s_1-1}_{\#}$ be a solution of (16.35) with $\mathbf{f} \in \dot{\mathbf{H}}^{s_2-2}_{\#}$. Then by Theorem 16.3, for the nonlinear term we have the inclusion $\boldsymbol{B}\mathbf{u} \in \dot{\mathbf{H}}^{t_1}_{\#}$ with $t_1 = 2s_1 - 1 - n/2$ if $s_1 < n/2$, with $t_1 = s_1 - 1$ if $s_1 > n/2$, and with any $t_1 \in (s_1 - 2, s_1 - 1)$ (and we can further use $t_1 = s_1 - 3/2$ for certainty) if $s_1 = n/2$. Hence the couple $(\mathbf{u}, p)$ satisfies the equation

$$-\mathscr{L}(\mathbf{u}, p) = \mathbf{f}^{(1)} \tag{16.53}$$

with $\mathbf{f}^{(1)} := \mathbf{f} - \boldsymbol{B}\mathbf{u} \in \dot{\mathbf{H}}^{s^{(1)}-2}_{\#}$, where $s^{(1)} = \min\{s_2, t_1 + 2\}$. By Corollary 16.1(i), the linear equation (16.53) has a unique solution in $\dot{\mathbf{H}}^{s}_{\#\sigma} \times \dot{H}^{s-1}_{\#}$ for any $s \leq s^{(1)}$ and thus $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s^{(1)}}_{\#\sigma} \times \dot{H}^{s^{(1)}-1}_{\#}$. If $s^{(1)} = s_2$, which we call the case (a), this proves item (i) of the theorem.

Otherwise we have the case (b), when $s^{(1)} < s_2$, i.e., $s^{(1)} = t_1 + 2$, by the definition of $s^{(1)}$ and the theorem condition $s_1 > n/2 - 1$. Then we arrange an iterative process by replacing s_1 with $s^{(1)} = t_1 + 2$ on each iteration until we arrive at the case (a), thus proving item (i) of the theorem. Note that in the case (b),

$$s^{(1)} - s_1 \geq \delta := \min\{s_1 + 1 - n/2, 1, 1/2\} > 0$$

in the first iteration, and δ can only increase in the next iterations due to the increase of s_1. This implies that the iteration process will reach the case (a) and stop after a finite number of iterations.

(ii) If $\mathbf{f} \in (\dot{\mathcal{C}}^{\infty}_{\#})^n$, then for any $s_2 \in \mathbb{R}$ we have $\mathbf{f} \in \dot{\mathbf{H}}^{s_2-2}_{\#}$ and item (i) implies that $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s_2}_{\#\sigma} \times \dot{H}^{s_2-1}_{\#}$. Hence $(\mathbf{u}, p) \in (\dot{\mathcal{C}}^{\infty}_{\#})^n \times \dot{\mathcal{C}}^{\infty}_{\#}$.

□

Combining Theorems 16.4 and 16.5, we obtain the following assertion on existence and regularity of solution to the Navier-Stokes system on torus.

Theorem 16.6 *Let $n \in \{2, 3\}$ and condition* (16.2) *hold.*

(i) If $\mathbf{f} \in \dot{\mathbf{H}}^{s-2}_{\#}$, $s \geq 1$, then the anisotropic Navier-Stokes equation (16.35) *has a solution $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s}_{\#\sigma} \times \dot{H}^{s-1}_{\#}$.*

(ii) Moreover, if $\mathbf{f} \in (\dot{\mathcal{C}}^{\infty}_{\#})^n$ then (16.35) *has a solution $(\mathbf{u}, p) \in (\dot{\mathcal{C}}^{\infty}_{\#})^n \times \dot{\mathcal{C}}^{\infty}_{\#}$.*

Note that in the *isotropic case* (16.7) with $\lambda = 0$, similar results for the Navier-Stokes system in torus as well as in domains of $\mathbb{R}^n$ are available, e.g., in [Ga11, RRS16, Se15, So01, Te01].

16.6 Some Auxiliary Results

The dense embedding of the space $(\mathcal{C}^\infty_\#)^n$ into $\mathbf{H}^1_\#$ and the divergence theorem imply the following identity for any $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{H}^1_\#$.

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_{\mathbb{T}} = \int_{\mathbb{T}} \nabla\cdot(\mathbf{v}_1(\mathbf{v}_2\cdot\mathbf{v}_3))\,d\mathbf{x} - \langle(\nabla\cdot\mathbf{v}_1)\mathbf{v}_3 + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_{\mathbb{T}}$$
$$= -\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_{\mathbb{T}} - \langle(\nabla\cdot\mathbf{v}_1)\mathbf{v}_3, \mathbf{v}_2\rangle_{\mathbb{T}}\,. \quad (16.54)$$

In view of (16.54) we obtain the identity

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_{\mathbb{T}} = -\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_{\mathbb{T}} \quad \forall\, \mathbf{v}_1 \in \mathbf{H}^1_{\#\sigma},\ \mathbf{v}_2,\ \mathbf{v}_3 \in \mathbf{H}^1_\#\,,$$

and hence the well known formula

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2, \mathbf{v}_2\rangle_{\mathbb{T}} = 0 \quad \forall\, \mathbf{v}_1 \in \mathbf{H}^1_{\#\sigma},\ \mathbf{v}_2 \in \mathbf{H}^1_\#. \quad (16.55)$$

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