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Chapter 19
A New Family of Boundary-Domain Integral Equations for the Mixed Exterior Stationary Heat Transfer Problem with Variable Coefficient

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19.1 Introduction

In this paper, a mixed boundary value problem for the stationary heat transfer partial differential equation with variable coefficient in an exterior domain is reduced to a system of direct segregated parametrix-based boundary-domain integral equations.

We use the parametrix different from the one employed in [Mi02, CMN09, CMN13] but coinciding with the one implemented in [MiPo15a] for interior domains.

It leads to a parametrix-based integral potentials involving the variable PDE coefficient that depends on the variable of integration.

This makes the relations between the parametrix-based potentials and their counterparts for constant coefficients more complicated than in [Mi02, CMN09, CMN13]. Notwithstanding, the mapping properties in *weighted Sobolev spaces* similar to the ones in [CMN13] still hold. Therefore, it is possible to prove equivalence and invertibility for a Boundary-Domain Integral Equation system derived from the original boundary value problem.

Unlike for the case of bounded domains, the Rellich compactness embeding theorem is not available for Sobolev spaces defined over unbounded domains. Nevertheless we are able to prove that the remainder operator is compact. Therefore, we can still benefit from the Fredholm alternative theory to prove the BDIE system operator invertibility.

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19.2 Preliminaries

Let $\Omega = \Omega^+$ be a unbounded exterior connected domain, $\Omega^- := \mathbb{R}^3 \smallsetminus \overline{\Omega}^+$ the complementary (bounded) subset of Ω. The boundary $S := \partial\Omega$ is simply connected, closed and infinitely differentiable, $S \in \mathscr{C}^\infty$. Furthermore, $S := \overline{S}_N \cup \overline{S}_D$ where both S_N and S_D are non-empty, connected disjoint manifolds of S. The border of these two submanifolds is also infinitely differentiable: $\partial S_N = \partial S_D \in \mathscr{C}^\infty$.

We consider the following partial differential equation:

$$\mathscr{A}u := \mathscr{A}(x)[u(x)] := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x), \qquad x \in \Omega, \tag{19.1}$$

where $u(x)$ is the unknown function, $a(x) \in \mathscr{C}^\infty$, $a(x) > 0$, is the variable coefficient and f is a given function on Ω. It is easy to see that if $a \equiv 1$, then the operator $\mathscr{A}$ becomes the Laplace operator Δ.

In what follows, $H^s(\Omega) = H^s_2(\Omega)$, $H^s(S) = H^s_2(S)$ denote the Bessel potential spaces (coinciding with the Sobolev–Slobodetski spaces if $s \geq 0$). For an open set Ω, we denote $\mathscr{D}(\Omega) = C^\infty_{comp}(\Omega)$, $\mathscr{D}^*(\Omega)$ is the Schwartz space of sequentially continuous functionals on $\mathscr{D}(\Omega)$, while $\mathscr{D}(\overline{\Omega})$ is the set of restrictions on $\overline{\Omega}$ of functions from $\mathscr{D}(\mathbb{R}^3)$. We also denote $\widetilde{H}^s(S_1) = \{g : \; g \in H^s(S), \; \text{supp}\, g \subset \overline{S_1}\}$, $H^s(S_1) = \{r_{S_1} g : \; g \in H^s(S)\}$, where S_1 is a proper submanifold of a closed surface S and r_{S_1} is the restriction operator on S_1.

Let $\omega(x) = (1+|x|^2)^{\frac{1}{2}}$ be the weight function and let

$$L_2(\omega;\Omega) := \{g : \omega g \in L_2(\Omega)\}, \quad L_2(\omega^{-1};\Omega) := \{g : \omega^{-1} g \in L_2(\Omega)\}$$

be the weighted Lebesgue spaces and $\mathscr{H}^1(\Omega)$ be the weighted Sobolev (Beppo-Levi) space,

$$\mathscr{H}^1(\Omega) := \{g \in L_2(\omega^{-1};\Omega) \; : \; \nabla g \in L_2(\Omega)\},$$
$$\|g\|^2_{\mathscr{H}^1(\Omega)} := \|\omega^{-1} g\|^2_{L_2(\Omega)} + \|\nabla g\|^2_{L_2(\Omega)},$$

cf. [Gr78, Ha71, CMN13] and references therein. Let us also define as $\widetilde{\mathscr{H}}^1(\Omega)$ a completion of $\mathscr{D}(\Omega)$ in $\mathscr{H}^1(\mathbb{R}^3)$, while $\widetilde{\mathscr{H}}^{-1}(\Omega) := [\mathscr{H}^1(\Omega)]^*$, $\mathscr{H}^{-1}(\Omega) := [\widetilde{\mathscr{H}}^1(\Omega)]^*$ are the corresponding dual spaces.

The operator $\mathscr{A}$ acting on $u \in \mathscr{H}^1(\Omega)$ is well defined in the distribution sense as long as the variable coefficient $a \in L^\infty(\Omega)$, as

$$\langle \mathscr{A}u, v\rangle = -\langle a\nabla u, \nabla v\rangle = -\mathscr{E}(u,v) \quad \forall v \in \mathscr{D}(\Omega).$$

$$\mathscr{E}(u,v) := \int_\Omega E(u,v)(x)dx; \qquad E(u,v)(x) := a(x)\nabla u(x)\nabla v(x). \tag{19.2}$$

Note that the functional $\mathscr{E}(u,v) : \mathscr{H}^1(\Omega) \times \widetilde{\mathscr{H}^1}(\Omega) \longrightarrow \mathbb{R}$ is continuous, thus by the density of $\mathscr{D}(\Omega)$ in $\widetilde{\mathscr{H}^1}(\Omega)$, also is the operator $\mathscr{A} : \mathscr{H}^1(\Omega) \longrightarrow \mathscr{H}^{-1}(\Omega)$ (19.2) which gives the week form of the operator $\mathscr{A}$ in (19.1).

From now on, we will assume $a(x) \in L^\infty(\Omega)$ and that there exist two positive constants, C_1 and C_2, such that:

$$0 < C_1 < a(x) < C_2 \quad \text{a.e.} \tag{19.3}$$

The trace operators on S from $\Omega^\pm$ are denoted by $\gamma^\pm$ and the operators $\gamma^\pm : H^1(\Omega^\pm \to H^{\frac{1}{2}}(S)$ and $\gamma^\pm : \mathscr{H}^1(\Omega^\pm \to H^{\frac{1}{2}}(S)$ are continuous (see, e.g., [McL00, Mi11, CMN13]).

For $u \in H^s(\Omega), s > 3/2$, we can define by $T^\pm$ the conormal derivative operator acting on S understood in the classical sense:

$$T^\pm[u(x)] := \sum_{i=1}^{3} a(x) n_i(x) \gamma^\pm \left(\frac{\partial u}{\partial x_i} \right) = a(x) \gamma^\pm \left(\frac{\partial u(x)}{\partial n(x)} \right), \tag{19.4}$$

where $n(x)$ is the exterior unit normal vector to the domain Ω at a point $x \in S$.

However, for $u \in \mathscr{H}^1(\Omega)$ (as well as for $u \in H^1(\Omega)$), the classical conormal derivative operator may not exist in the trace sense. We can overcome this difficulty by introducing the following function space for the operator $\mathscr{A}$, (cf. [CMN13, Gr78, Ha71])

$$\mathscr{H}^{1,0}(\Omega;\mathscr{A}) := \{g \in \mathscr{H}^1(\Omega) : \mathscr{A}g \in L^2(\omega;\Omega)\}$$

endowed with the norm

$$\| g \|^2_{\mathscr{H}^{1,0}(\Omega;\mathscr{A})} := \| g \|^2_{\mathscr{H}^1(\Omega)} + \| \omega \mathscr{A} g \|^2_{L^2(\Omega)} .$$

Now, if $u \in \mathscr{H}^{1,0}(\Omega;\mathscr{A})$ we can define the conormal derivative $T^+u \in H^{-\frac{1}{2}}(S)$ as hinted by the Green's formula, cf. [McL00, CMN13],

$$\langle T^+u, w\rangle_S := \pm \int_{\Omega^\pm} [(\gamma^+_{-1}\omega)\mathscr{A}u + E(u, \gamma^+_{-1}w)]\, dx; \text{ for all } w \in H^{\frac{1}{2}}(S),$$

where $\gamma^+_{-1} : H^{\frac{1}{2}}(S) \to \mathscr{H}^1(\Omega)$ is a continuous right inverse to the trace operator $\gamma^+ : \mathscr{H}^1(\Omega) \longrightarrow H^{\frac{1}{2}}(S)$, whereas the brackets $\langle u, v\rangle_S$ represent the duality brackets of the spaces $H^{\frac{1}{2}}(S)$ and $H^{\frac{-1}{2}}(S)$ which coincide with the scalar product in $L^2(S)$ when $u, v \in L^2(S)$.

The operator $T^+ : \mathscr{H}^{1,0}(\Omega;\mathscr{A}) \longrightarrow H^{\frac{-1}{2}}(S)$ is bounded and gives a continuous extension on $\mathscr{H}^{1,0}(\Omega;\mathscr{A})$ of the classical conormal derivative operator (19.4). We remark that when $a \equiv 1$, the operator T^+ becomes T^+_Δ, which is the continuous extension on $\mathscr{H}^{1,0}(\Omega;\Delta)$ of the classical normal derivative operator $\partial_n := n \cdot \nabla$.

In a similar manner as in the proof [McL00, Lemma 4.3] or [Co88, Lemma 3.2], the first Green identity holds for $u \in \mathscr{H}^{1,0}(\Omega;\mathscr{A})$, cf. Eq. (2.8) in [CMN13],

$$\langle T^+u, \gamma^+ v\rangle_S = \int_\Omega [v\mathscr{A}u + E(u,v)] dx \quad \forall v \in \mathscr{H}^1(\Omega). \tag{19.5}$$

Applying identity (19.5) to $u,v \in \mathscr{H}^{1,0}(\Omega;\mathscr{A})$ and then exhanging roles and subtracting the one from the other, we arrive to the following second Green identity (see, for example, Eq. (2.9) in [CMN13]).

$$\int_\Omega [v\mathscr{A}u - u\mathscr{A}v]\,dx = \int_S \left[\gamma^+ v T^+ u - \gamma^+ u T^+ v\right] dS(x). \tag{19.6}$$

19.3 Boundary Value Problem

We aim to derive a system of *Boundary-Domain Integral Equations* (BDIEs) equivalent to the following mixed boundary value problem defined in an exterior domain Ω.

Find $v \in \mathscr{H}^{1,0}(\Omega;\mathscr{A})$ such that:

$$\mathscr{A}u = f, \quad \text{in } \Omega; \tag{19.7}$$

$$r_{S_D}\gamma^+ u = \phi_0, \quad \text{on } S_D; \tag{19.8}$$

$$r_{S_N} T^+ u = \psi_0, \quad \text{on } S_N; \tag{19.9}$$

where $f \in L^2(\omega,\Omega)$, $\phi_0 \in H^{\frac{1}{2}}(S_D)$ and $\psi_0 \in H^{\frac{-1}{2}}(S_N)$.

Let us denote the left-hand side operator of the mixed problem as

$$\mathscr{A}_M : \mathscr{H}^{1,0}(\Omega;\mathscr{A}) \longrightarrow L^2(\omega,\Omega) \times H^{\frac{1}{2}}(S_D) \times H^{\frac{-1}{2}}(S_N). \tag{19.10}$$

By using variational settings and the Lax Milgram lemma, it is possible to prove (see [CMN13, Theorem A.6]) that the operator (19.10) is continuously invertible and thus the unique solvability of the BVP (19.7)-(19.9) follows.

19.4 Parametrices and remainders

Boundary Integral Equations (BIEs) are derived from BVPs with constant coefficients using an explicit fundamental solution. Although a fundamental solution may exist for the variable coefficient case, it is not always available explicitly. Therefore, we introduce a *parametrix* or Levi function, see [CMN09, CMN13, MiPo15a, MiPo15b] for more details.

In this chapter, we will use the same parametrix as in [MiPo15b]

$$P(x,y) = \frac{1}{a(x)} P_\Delta(x-y) = \frac{-1}{4\pi a(x)|x-y|}, \quad x,y \in \mathbb{R}^3,$$

whose corresponding remainder is

$$R(x,y) = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{1}{a(x)}\frac{\partial a(x)}{\partial x_i} P_\Delta(x,y)\right), \quad x,y \in \mathbb{R}^3.$$

Condition 1 *To obtain boundary-domain integral equations, we will assume the following condition further on, unless stated otherwise:*

$$a \in \mathscr{C}^1(\mathbb{R}^3) \quad \textit{and} \quad \omega \nabla a \in L^\infty(\mathbb{R}^3). \tag{19.11}$$

19.5 Surface and volume potentials

Here, we present the surface and volume potential type operators which will be involved in the BDIEs derived later on. We provide below the key mapping properties needed to prove the equivalence and invertibility theorems at the end of the chapter.

The mapping properties are presented in weighted Sobolev spaces. The analogous properties for the bounded domain case in standard Sobolev spaces can be consulted in [MiPo15b].

The parametrix-based single layer and double layer surface potentials are defined for $y \in \mathbb{R}^3 : y \notin S$, as

$$V\rho(y) := -\int_S P(x,y)\rho(x)\, dS(x) \quad y \notin S,$$
$$W\rho(y) := -\int_S T_x^+ P(x,y)\rho(x)\, dS(x) \quad y \notin S.$$

We also define the following pseudo-differential operators associated with direct values of the single and double layer potentials and with their conormal derivatives, for $y \in S$,

$$\mathscr{V}\rho(y) := -\int_S P(x,y)\rho(x)\, dS(x),$$
$$\mathscr{W}\rho(y) := -\int_S T_x^+ P(x,y)\rho(x)\, dS(x),$$
$$\mathscr{W}'\rho(y) := -T_y^\pm[\mathscr{V}\rho(y)],$$
$$\mathscr{L}^\pm\rho(y) := -T_y^\pm[\mathscr{W}\rho(y)].$$

Let us introduce now the parametrix-based Newtonian and remainder volume potentials which are defined, for $y \in \mathbb{R}^3$, as

$$\mathscr{P}\rho(y) := \int_\Omega P(x,y)\rho(x)\, dx,$$
$$\mathscr{R}\rho(y) := \int_\Omega R(x,y)\rho(x)\, dx.$$

Note that when, in the definition above, $\Omega = \mathbb{R}^3$ we will denote the operators $\mathscr{P}$ and $\mathscr{R}$ by $\mathbf{P}$ and $\mathbf{R}$ respectively

The following theorem presents the relationships between the parametrix-based volume and surface potentials and their counterparts for the Laplace equation ($a \equiv 1$). It is now rather simple to obtain, similar to [CMN13], the mapping properties, jump relations and invertibility results for the parametrix-based surface and volume potentials. The following relations coincide with their analogous relations for the potentials in bounded domains which appear in [MiPo15b].

Theorem 1. *The operators $V, W, \mathscr{V}, \mathscr{W}, \mathscr{W}'$ and $\mathscr{L}$ satisfy the following relations for their counterparts associated with the Laplace operator:*

$$\mathscr{P}\rho = \mathscr{P}_\Delta\left(\frac{\rho}{a}\right), \qquad \mathscr{R}\rho = -\nabla\cdot[\mathscr{P}_\Delta(\rho\nabla\ln a)],$$

$$V\rho = V_\Delta\left(\frac{\rho}{a}\right), \qquad \mathscr{V}\rho = \mathscr{V}_\Delta\left(\frac{\rho}{a}\right),$$

$$W\rho = W_\Delta\rho - V_\Delta\left(\frac{\partial \ln a}{\partial n}\rho\right), \qquad \mathscr{W}\rho = \mathscr{W}_\Delta\rho - \mathscr{V}_\Delta\left(\frac{\partial \ln a}{\partial n}\rho\right),$$

$$\mathscr{W}'\rho = a\mathscr{W}'_\Delta\left(\frac{\rho}{a}\right), \hat{\mathscr{L}}\rho := a\mathscr{L}_\Delta\rho, \qquad \mathscr{L}^\pm\rho = \hat{\mathscr{L}}\rho - a\gamma^\pm W_\Delta\left(\frac{\partial \ln a}{\partial n}\rho\right).$$

Remark 1 *The subscript Δ refers to the analogous surface potentials with $a \equiv 1$, i.e. $P|_{a=1} = P_\Delta$. Note that P_Δ is the fundamental solution of the Laplace equation. Furthermore, in virtue of the Lyapunov-Tauber theorem $\mathscr{L}_\Delta\rho = \mathscr{L}_\Delta^+\rho = \mathscr{L}_\Delta^-\rho$.*

One of the main differences with respect the bounded domain case is that the integrands of the operators V, W, $\mathscr{P}$ and $\mathscr{R}$ and their corresponding direct values and conormal derivatives do not always belong to L_1. In these cases, the integrals should be understood as the corresponding duality forms (or the limits of these forms for the infinitely smooth functions, existing due to their density in corresponding Sobolev spaces).

Condition 2 *In addition to conditions* (19.3) *and* (19.11)*, we will also sometimes assume the following condition:*

$$\omega^2\Delta a \in L^\infty(\Omega). \tag{19.12}$$

Remark 2 *Note as well that due to the essential boundedness of the function a and the continuity of the function $\ln a$, the components of $\nabla\ln a$ and $\Delta\ln a$ will be essentially bounded as well.*

Theorem 2. *The following operators are continuous under condition* (19.11)*:*

$$V : H^{\frac{-1}{2}}(S) \longrightarrow \mathscr{H}^1(\Omega),$$

$$W : H^{\frac{1}{2}}(S) \longrightarrow \mathscr{H}^1(\Omega).$$

Corollary 1 *The following operators are continuous under condition* (19.11) *and* (19.12)*,*

$$V : H^{\frac{-1}{2}}(S) \longrightarrow \mathscr{H}^{1,0}(\Omega;\mathscr{A}),$$
$$W : H^{\frac{1}{2}}(S) \longrightarrow \mathscr{H}^{1,0}(\Omega;\mathscr{A}).$$

Theorem 3. *The following operators are continuous under condition* (19.11)*,*

$$\boldsymbol{P} : \mathscr{H}^{-1}(\mathbb{R}^3) \longrightarrow \mathscr{H}^{1}(\mathbb{R}^3),$$
$$\boldsymbol{R} : L^2(\omega^{-1};\mathbb{R}^3) \longrightarrow \mathscr{H}^{1}(\mathbb{R}^3),$$
$$\mathscr{P} : \widetilde{\mathscr{H}}^{-1}(\Omega) \longrightarrow \mathscr{H}^{1}(\mathbb{R}^3).$$

Theorem 4. *The following operators are continuous under condition* (19.11) *and* (19.12)*,*

$$\mathscr{P} : L^2(\omega;\Omega) \longrightarrow \mathscr{H}^{1,0}(\mathbb{R}^3;\mathscr{A}),$$
$$\mathscr{R} : \mathscr{H}^{1}(\Omega) \longrightarrow \mathscr{H}^{1,0}(\Omega;\mathscr{A}).$$

19.6 Third Green identities and integral relations

Let $B_\varepsilon(y)$ be the ball centred at $y \in \Omega$ with radius ε sufficiently small. Then, $R(\cdot,y) \in L^2(\omega;\Omega \smallsetminus B_\varepsilon(y))$ and thus $P(\cdot,y) \in \mathscr{H}^{1,0}(\Omega \smallsetminus B_\varepsilon(y))$. Applying the second Green identity (19.6) with $v = P(\cdot,y)$ and any $u \in \mathscr{H}^{1,0}(\Omega;\mathscr{A})$ in $\Omega \smallsetminus B_\varepsilon(y)$ and using standard limiting procedures as $\varepsilon \to 0$ (cf.,[Mr70]) we obtain the third Green identity (*integral representation formula*) for the function $u \in \mathscr{H}^{1,0}(\Omega;\mathscr{A})$:

$$u + \mathscr{R}u - VT^+u + W\gamma^+u = \mathscr{P}\mathscr{A}u, \quad in \ \ \Omega. \tag{19.13}$$

If $u \in H^{1,0}(\Omega;\mathscr{A})$ is a solution of the partial differential equation (19.7), then, from (19.13), we obtain

$$u + \mathscr{R}u - VT^+(u) + W\gamma^+u = \mathscr{P}f, \ \ in \ \ \Omega. \tag{19.14}$$

Taking the trace of (19.14), we obtain

$$\frac{1}{2}\gamma^+u + \gamma^+\mathscr{R}u - \mathscr{V}T^+u + \mathscr{W}\gamma^+u = \gamma^+\mathscr{P}f, \ \ on \ \ S. \tag{19.15}$$

For some distributions f, Ψ and Ψ, we consider an indirect integral relation associated with the third Green identity (19.14)

$$u + \mathscr{R}u - V\Psi + W\Phi = \mathscr{P}f, \ \ in \ \ \Omega. \tag{19.16}$$

Appropriately modifying the proofs of Lemma 4.1 in [CMN13] and Lemma 9.4.1 in [MiPo15b], one can prove the following assertion.

Lemma 1. *Let $u \in \mathscr{H}^1(\Omega)$, $f \in L_2(\omega;\Omega)$, $\Psi \in H^{\frac{-1}{2}}(S)$ and $\Phi \in H^{\frac{1}{2}}(S)$ satisfy relation* (19.16). *Let conditions* (19.11) *and* (19.12) *hold. Then $u \in \mathscr{H}^{1,0}(\Omega,\mathscr{A})$, solves the equation $\mathscr{A}u = f$ in Ω and the following identity is satisfied*

$$V(\Psi - T^+u) - W(\Phi - \gamma^+u) = 0 \ \ in \ \ \Omega.$$

The following statement is the counterpart of Lemma [MiPo15b, Lemma 4.2] for exterior domains. The proof follows from the invertibility of the operator $\mathscr{V}_\Delta$, see [McL00, Corollary 8.13].

Lemma 2. *Let $\Psi^* \in H^{\frac{-1}{2}}(S)$. If $V\Psi^*(y) = 0$, $y \in \Omega$, then $\Psi^*(y) = 0$.*

19.7 Boundary-Domain Integral Equation System

Let the functions $\Phi_0 \in H^{\frac{1}{2}}(S)$ and $\Psi_0 \in H^{\frac{-1}{2}}(S)$ be continuous fixed extensions to S of the functions $\phi_0 \in H^{\frac{1}{2}}(S_D)$ and $\psi_0 \in H^{\frac{1}{2}}(S_N)$. Moreover, let $\phi \in \widetilde{H}^{\frac{1}{2}}(S_N)$ and $\psi \in \widetilde{H}^{-\frac{1}{2}}(S_D)$ be arbitrary functions formally segregated from u, cf. [CMN09, CMN13, MiPo15b].

We will derive a system of BDIEs for the BVP (19.7)-(19.9) substituting the functions

$$\gamma^+u = \Phi_0 + \phi, \qquad T^+u = \Psi_0 + \psi, \quad on \quad S; \tag{19.17}$$

in the third Green identities (19.14) and (19.15).

In what follows, we will denote by $\mathscr{X}$ the vector of unknown functions

$$\mathscr{X} = (u,\psi,\phi)^\top \in \mathbb{H} := \mathscr{H}^{1,0}(\Omega;\mathscr{A}) \times \widetilde{H}^{\frac{-1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \subset \mathbb{X}$$

where

$$\mathbb{X} := \mathscr{H}^1(\Omega) \times \widetilde{H}^{\frac{-1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N).$$

We substitute the functions (19.17) in (19.14) and (19.15) to obtain the following BDIE system (M12)

$$u + \mathscr{R}u - V\psi + W\phi = F_0 \ \ in \ \ \Omega, \tag{19.18a}$$

$$\frac{1}{2}\phi + \gamma^+\mathscr{R}u - \mathscr{V}\psi + \mathscr{W}\phi = \gamma^+F_0 - \Phi_0 \ \ on \ \ S, \tag{19.18b}$$

where $F_0 = \mathscr{P}f + V\Psi_0 - W\Phi_0$.

We denote by $\mathscr{M}^{12}$ the matrix operator that defines the system $(M12)$:

$$\mathscr{M}^{12} = \begin{bmatrix} I+\mathscr{R} & -V & W \\ \gamma^+\mathscr{R} & -\mathscr{V} & \frac{1}{2}I + \mathscr{W} \end{bmatrix},$$

and by $\mathscr{F}^{12}$ the right-hand side of the system:

$$\mathscr{F}^{12} = [F_0,\ \gamma^+ F_0 - \Phi_0]^\top.$$

The system (M12) can be expressed in terms of matrix notation as

$$\mathscr{M}^{12}\mathscr{X} = \mathscr{F}^{12}$$

If the conditions (19.11) and (19.12) hold, then due to the mapping properties of the potentials, $\mathscr{F}^{12} \in \mathbb{F}^{12} \subset \mathbb{Y}^{12}$, while operators $\mathscr{M}^{12} : \mathbb{H} \to \mathbb{F}^{12}$ and $\mathscr{M}^{12} : \mathbb{X} \to \mathbb{Y}^{12}$ are continuous. Here, we denote

$$\mathbb{F}^{12} := \mathscr{H}^{1,0}(\Omega, \mathscr{A}) \times H^{\frac{1}{2}}(S),$$
$$\mathbb{Y}^{12} := \mathscr{H}^{1}(\Omega) \times H^{\frac{1}{2}}(S).$$

A proof of the following assertion is similar to the proof of the corresponding Theorem 9.5.1 for the interior domain in [MiPo15b].

Theorem 5. *[Equivalence] Let $f \in L_2(\omega;\Omega)$, let $\Phi_0 \in H^{\frac{-1}{2}}(S)$ and let $\Psi_0 \in H^{\frac{-1}{2}}(S)$ be some fixed extensions of $\phi_0 \in H^{\frac{1}{2}}(S_D)$ and $\psi_0 \in H^{\frac{-1}{2}}(S_N)$ respectively. Let conditions* (19.11) *and* (19.12) *hold.*

i) If some $u \in \mathscr{H}^{1,0}(\Omega;\mathscr{A})$ solves the BVP (19.7)-(19.9)*, then the triplet $(u,\psi,\phi)^\top \in \mathscr{H}^{1,0}(\Omega;\mathscr{A}) \times \widetilde{H}^{\frac{-1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N)$ where*

$$\phi = \gamma^+ u - \Phi_0, \qquad \psi = T^+ u - \Psi_0, \qquad \text{on } S,$$

solves the BDIE system (M12).

ii) If a triple $(u,\psi,\phi)^\top \in \mathscr{H}^{1,0}(\Omega;\mathscr{A}) \times \widetilde{H}^{\frac{-1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N)$ solves the BDIE system (M12), then this solution is unique. Furthermore, u solves the BVP (19.7)-(19.9) *and the functions ψ,ϕ satisfy*

$$\phi = \gamma^+ u - \Phi_0, \qquad \psi = T^+ u - \Psi_0, \qquad \text{on } S.$$

19.8 Invertibility

In this section, we aim to prove the invertibility of the operator $\mathscr{M}^{12} : \mathbb{H} \to \mathbb{F}^{12}$ by showing first that the arbitrary right-hand side $\mathbb{F}^{12}$ from the respective spaces can be represented in terms of the parametrix-based potentials and using then the equivalence theorems.

The following result can be proved similar to its counterpart, Corollary 7.1 in [CMN13] with another parametrix. The analogous result for bounded domains can be found in [CMN09, Lemma 3.5].

Lemma 3. *Let*

$$(\mathscr{F}_0, \mathscr{F}_1) \in \mathscr{H}^{1,0}(\Omega; \mathscr{A}) \times H^{\frac{1}{2}}(\partial\Omega).$$

Then there exists a unique triplet (f_*, Ψ_*, Φ_*) *such that* $(f_*, \Psi_*, \Phi_*) = \mathscr{C}_*(\mathscr{F}_0, \mathscr{F}_1)^\top$, *where* $\mathscr{C}_* : \mathscr{H}^{1,0}(\Omega, \mathscr{A}) \times H^{\frac{1}{2}}(S) \to L^2(\omega; \Omega) \times H^{\frac{-1}{2}}(S) \times H^{\frac{1}{2}}(S)$ *is a linear bounded operator and* $(\mathscr{F}_0, \mathscr{F}_1)$ *are given by*

$$\mathscr{F}_0 = \mathscr{P} f_* + V\Psi_* - W\Phi_* \quad in \;\; \Omega$$
$$\mathscr{F}_1 = \gamma^+ \mathscr{F}_0 - \Phi_* \quad on \;\; \partial\Omega$$

Employing Lemma 3 and the arguments as in the proof of Theorem 7.1 in [CMN13], it is possible to prove one of the main results on the invertibility of the matrix operator of the BDIE system (M12).

Theorem 6. *If conditions* (19.11) *and* (19.12) *hold, then the following operator is continuous and continuously invertible:*

$$\mathscr{M}^{12} : \mathbb{H} \to \mathbb{F}^{12}.$$

Let us introduce the additional condition

$$\lim_{x \to \infty} \omega(x) \nabla a(x) = 0. \tag{19.19}$$

Now, in a similar fashion as in Lemma 7.4 of [CMN13], we we can prove the following assertion.

Lemma 4. *Let conditions* (19.11) *and* (19.19) *hold. Then, for any* $\varepsilon > 0$ *the operator* $\mathscr{R}$ *can be represented as* $\mathscr{R} = \mathscr{R}_s + \mathscr{R}_c$, *where* $\| \mathscr{R}_s \|_{\mathscr{H}^1(\Omega)} < \varepsilon$, *while* $\mathscr{R}_c : \mathscr{H}^1(\Omega) \to \mathscr{H}^1(\Omega)$ *is compact.*

Since the limit of any converging sequence of compact operators is also compact, Lemma 4 implies the following result.

Corollary 2 *Let conditions* (19.11) *and* (19.19) *hold. Then the operator* $\mathscr{R} : \mathscr{H}^1(\Omega) \to \mathscr{H}^1(\Omega)$ *is compact and the operator* $I + \mathscr{R} : \mathscr{H}^1(\Omega) \to \mathscr{H}^1(\Omega)$ *is Fredholm with zero index.*

Theorem 7. *If conditions* (19.11),(19.12) *and* (19.19) *hold, then the operator*

$$\mathscr{M}^{12} : \mathbb{X} \to \mathbb{Y}^{12},$$

is continuously invertible.

The theorem can be proved by implementing Corollary 2, Fredholm alternative and Theorem 5; cf. Theorem 7.4 in [CMN13].

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