CONSTRUCTION OF FUNDAMENTAL SOLUTIONS FOR THE THREE-DIMENSIONAL AND PLANE PROBLEMS FOR AN ANISOTROPIC HEREDITARY-ELASTIC AGING MEDIUM

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Recent years have witnessed widespread adoption of numerical solution methods for boundary value problems in the mechanics of deformable solids, based on the use of fundamental solutions, i.e., solutions that describe the reaction to a concentrated force of the infinite space (or plane) occupied by the medium under consideration. Methods of this type include the elastic-potential method [1, 2], the method of boundary integral equations (to which the Betti formulas lead [3-6]), and also the method of sources, in which the solution of the boundary value problem is constructed by superposition of concentrated forces in space that are located on some surface that encompasses the solid under consideration (see, e.g., [7]).

In anisotropic elastic medium, the fundamental solution is provided by the Kelvin-Somigliana matrix, and can be written in explicit form. When an elastic medium contains arbitrary anisotropy, the fundamental solution constructed in [8] can be reduced to an integral over a unit circle with a special orientation in space [2, 9, 10]. Analogous solutions for hereditary elasticity without aging [11] were given [4] in the isotropic case, and in [12, 13] in the anisotropic case. Paper [14] offers a fundamental solution for a homogeneous rectilinear-anisotropic aging medium [15] in the three-dimensional case for finite times.

In this paper we provide a three-dimensional fundamental solution in an anisotropic aging medium, and we analyze in detail the case of an isotropic hereditary aging medium. The three-dimensional fundamental solution yields the fundamental solutions of the problems of generalized plane strain and of the generalized plane stressed state for an anisotropic aging medium; we also investigate the behavior of two- and three-dimensional fundamental solutions for the case of small and large times, when the applied load becomes stabilized or becomes harmonic in time in the limit.

1. Assume that x_k, k = 1, 2, 3, are Cartesian coordinates. We will assume, furthermore (unless otherwise stipulated), that Latin subscripts vary from 1 to 3, while Greek subscripts vary from 1 to 2. Within these limits, we will assume that summation is performed over repeating indexes.

Consider an anisotropic homogeneous hereditary-elastic medium in which the stresses σ_{ij} and strains ε_{kl} are related by the equation [15]

$$\sigma_{ij}(t)=[\mathrm{C}_{ijkl}\varepsilon_{kl}](t),\quad \mathrm{C}_{ijkl}=C^{\mathrm{I}}_{ijkl}+\mathrm{C}^{\mathrm{II}}_{ijkl} \tag{1.1}$$

Here t is time; $C_{ijkl}^{I}(t)$ is the tensor of the instantaneous elastic moduli; and C_{ijkl}^{II} is an integral operator of the form

$$[\mathrm{C}^{\mathrm{II}}_{ijkl}\varepsilon_{kl}](t)=\int_0^t C_{ijkl}(t,\tau)\,\varepsilon_{kl}(\tau)\,d\tau$$

The operator C_{ijkl} satisfies the symmetry conditions $C_{ijkl} = C_{jikl} = C_{ijkl} = C_{klij}$.

Assume that the time dependence of the components of the concentrated force applied at the coordinate origin is specified by functions $f_i(t)$. The desired unknown solution satisfies the equilibrium equation

$$\sigma_{ij,j}(x, t) = -\delta_{(3)}(x) f_i(t). \tag{1.2}$$

where $\delta_{(3)} = \delta(x_1)\delta(x_2)\delta(x_3)$ is the Dirac delta function. Substituting (1.1) into (1.2) and allowing for the fact that $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$, we arrive at the Lame equations in the displacements:

$$[C_{ijkl} u_{k,jl}](x, t) = -\delta_{(3)}(x) f_i(t) \tag{1.3}$$

In the customary fashion, we introduce the direct and inverse Fourier transformations, using a carat to denote the transforms and transformation parameters:

$$u^{\vee}(x^{\vee}, t) = (2\pi)^{-3/2} \int_V u(x, t) \exp(i x_p^{\vee} x_p)\, dV$$
$$u(x, t) = (2\pi)^{-3/2} \int_V u^{\vee}(x^{\vee}, t) \exp(-i x_p^{\vee} x_p)\, dV^{\vee} \tag{1.4}$$
$$dV = dx_1 dx_2 dx_3 \quad dV^{\vee} = dx_1^{\vee} dx_2^{\vee} dx_3^{\vee}$$

Applying the Fourier transformation to the left and right sides of (1.3), we obtain

$$[C_{ik}^{\vee} u_k^{\vee}](x^{\vee}, t) = (2\pi)^{-3/2} f_i(t), \quad C_{ik}^{\vee} = x_j^{\vee} x_l^{\vee} C_{ijkl} \tag{1.5}$$

The solution of system (1.5) of linear integral Volterra equations has the form [14]

$$u_k^{\vee}(x^{\vee}, t) = (2\pi)^{-3/2} [C_{ki}^{\vee -1} f_i](\eta^{\vee}, t) r^{\vee -2}, \quad r^{\vee 2} = x_i^{\vee} x_i^{\vee}, \quad \eta_i^{\vee} = x_i^{\vee}/r^{\vee} \tag{1.6}$$

Assume that C[0, T] is a space of functions that are continuous on [0, T], while L_∞[0, T] is the space of functions $f(t)$ such that $\operatorname{ess\,sup} |f(t)| < \infty$, $0 \leqslant t \leqslant T$.

We will consider media with instantaneous moduli $C_{ijkl}^{I}(t)$ belonging to C[0, T] (or L_∞[0, T]) such that the instantaneous strain energy $C_{ijkl}^{I} \varepsilon_{ij} \varepsilon_{kl}/2 > 0$ for any $\varepsilon_{ij} \neq 0$. We will also assume that the operators C_{ijkl}^{II} are completely continuous in space C[0, T] (or L_∞[C, T]). Sufficient conditions for this, which should be imposed on the kernels $C_{ijkl}(t, \tau)$, are given, e.g., in [16].

In particular, functions C_{ijkl} may have the form $C_{ijkl}(t, \tau) = C_{ijkl}^{\circ}(t, \tau) \times (t-\tau)^{-\alpha}$, where functions $C_{ijkl}^{\circ}(t, \tau)$ are continuous for $0 \leq \tau \leq t \leq T$, while $\alpha < 1$. Under these conditions in C[0, T] (or in L_∞[, T]) there exists a continuous inverse operator $C_{ki}^{\vee -1}(\eta^{\vee})$, which can be conveniently represented in the form of a Neumann series that converges uniformly with respect to $\eta^{\vee}$ ($|\eta^{\vee}| = 1$) [14]:

$$C_{ki}^{\vee -1} = (C_{\cdot}^{\vee I})_{ki}^{-1} + \sum_{n=1}^{\infty} (-1)^n (C^{\vee III})_{kq}^n (C^{\vee I})_{qi}^{-1} \tag{1.7}$$

where $(C^{\vee III})_{kq}{}^n$ is the n-th degree of operator $C_{kq}^{\vee III}$, i.e., $(C^{\vee III})_{kq}{}^n = C_{ks}^{\vee III} (C^{\vee III})_{sq}{}^{n-1}$, $C_{kq}^{\vee III} = (C^{\vee I})_{ks}{}^{-1} C_{sq}{}^{II}$.

For the kernel $C_{kq}^{\vee III}$ of integral operator $C_{kq}^{\vee III}$ we have $C_{kq}^{\vee III}(t, \tau, \eta) = (C^{\vee I})_{ks}{}^{-1}(t, \eta) C_{sq}^{\vee}(t, \tau, \eta)$.

Performing the inverse Fourier transformation (1.6) in a manner similar to that employed for an anisotropic solid without the hereditary property [9], and understanding $C_{ki}^{\vee -1}$ to mean the series in (1.7), we obtain [14]

$$u_k(x, t) = \frac{1}{8\pi^2 |x|} \oint_{\zeta \in \Omega(x)} [C_{ki}^{\vee -1}(\zeta) f_i](t)\, |d\zeta| \tag{1.8}$$

A particular case of (1.8) is provided by the fundamental solution for media with kernels that depend on a difference of arguments, given in [12, 13].

Following [14], we change over to integration over a unit sphere in (1.8), using the Dirac delta function. After doing so, it is not difficult to calculate the derivatives of this expression with respect to the spatial coordinates, and then, substituting the result into the generalized Hooke's law (1.1), to obtain the stresses

$$\sigma_{ij}(x,t)=\frac{C_{ijkl}}{8\pi^2|x|^3}\Big\{-x_l\oint_{\zeta\in\Pi(x)}[\check{C}_{kq}^{-1}(\zeta)f_q](t)|d\zeta|+ + x_m\oint_{\zeta\in\Pi(x)}[\check{C}_{ks}^{-1}(\zeta)\check{C}_{sp,m}(\zeta)\check{C}_{pq}^{-1}(\zeta)f_q](t)\zeta_l|d\zeta|\Big\} \quad (1.9)$$

In (1.8) and (1.9), $\Pi(x)$ is a plane orthogonal to vector x. Vector ζ satisfies the condition $|\zeta|$ = 1, and thus the integral is evaluated along a unit circle lying in $\Pi(x)$:

$$C_{sp,m}(\check{\eta})=(\delta_{jm}\check{\eta}_l+\delta_{lm}\check{\eta}_j)C_{sjpl}$$

Let us consider in greater detail the case of an isotropic hereditary-elastic aging medium, for which

$$C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \quad (1.10)$$

where λ and μ are Volterra operators of general form, that describe the aging process,

$$[\lambda\varepsilon](t)=\lambda^I(t)\varepsilon(t)+\int_0^t\lambda(t,\tau)\varepsilon(\tau)d\tau$$

Functions $\lambda^I(t)$ and $\lambda(t, \tau)$ are characteristics of the medium. We can similarly determine the operator μ. In this case the solution of (1.5) can be expressed explicitly in terms of the operators λ and μ:

$$\check{u}_k(\check{x},t)=[(\lambda+2\mu)^{-1}\{-(\lambda+\mu)\check{\eta}_i\check{\eta}_k+(\lambda+2\mu)\delta_{ik}\}\mu^{-1}f_i](t)\check{r}^{-2}(2\pi)^{-3/2} \quad (1.11)$$

Fourier transformation of expression(1.11) in explicit form is also possible:

$$u_k(x,t)=\left[\frac{(\lambda+2\mu)^{-1}}{8\pi|x|}\left\{(\lambda+\mu)\frac{x_ix_k}{|x|^2}+(\lambda+3\mu)\delta_{ik}\right\}\mu^{-1}f_i(t)\right](t) \quad (1.12)$$

Differentiating (1.12), we obtain $u_{k,l}$; after substituting these into (1.1), we have for the stresses

$$\sigma_{pj}=-\frac{1}{|x|^2}\left[(x_p\delta_{ij}+x_j\delta_{ip}-x_i\delta_{pj})m+\frac{x_ix_px_j}{|x|^2}n\right]f_i$$
$$m=\mu(\lambda+2\mu)^{-1}/(4\pi)=(1-\nu)^{-1}(1-2\nu)/(8\pi) \quad (1.13)$$
$$n=3(\lambda+\mu)(\lambda+2\mu)^{-1}/(4\pi)=3(1-\nu)^{-1}/(8\pi),\quad \nu=\lambda(\lambda+\mu)^{-1}/2$$

Here ν is a right Poisson operator [17, 18], and the given order of action of the operators λ and $(\lambda+\mu)^{-1}$ is important in its definition.

If λ and μ in (1.10) are understood to mean the Lame moduli of an elastic isotropic medium (without the hereditary property), then expressions (1.12) and (1.13) yield the Kelvin-Somigliana solution [1, 2]. Solution (1.12), (1.13) for an isotropic medium can also be obtained by using the Volterra principle for hereditary-elastic aging solids [17, 18] from the Kelvin-Somigliana solution.

Thus, to calculate the fundamental solution in an aging hereditary-elastic isotropic medium for all orientations of the vector x, we need have only the two inverse operators $(\lambda+2\mu)^{-1}$ and μ^{-1}, which can be obtained (if not expressed explicitly) by summing up the corresponding Neumann series. In the case of general anisotropy, for this we require the inverse operator $C_{ik}^{-1}(\zeta)$ for all ζ values lying on the surface of the unit sphere $|\zeta| = 1$, i.e., we need to sum up Neumann series for each point ζ on the surface of this sphere.

2. Now let us obtain fundamental solutions for the two-dimensional problem of generalized plane strain, i.e., for an infinite space loaded by concentrated forces with components f_i ($i = 1, 2, 3$), that are uniformly distributed along some straight line (which can be assumed, without loss of generality, to coincide with the x_3 axis). For a medium without the hereditary property, solutions of this type were obtained using complex potentials in [19]. They depend on the values of the roots of the characteristic equation. In the case of a hereditary medium, these roots will be hereditary operators, but their explicit expression in terms of the original hereditary operators C_{ijkl} is difficult in the general case (particularly for an aging medium).

The fundamental solution of the two-dimensional problem of generalized plane strain can be obtained from the above fundamental solution of the three-dimensional problem by integrating the resultant expressions for the displacements $u_i(x - y, t)$ and stresses $\sigma_{ij}(x - y, t)$ with respect to y_3 under the condition $y_1 = y_2 = 0$ (here y is the point of application of the concentrated force). In constructing the fundamental solution for the three-dimensional problem, however, it was implicitly assumed that the displacements at infinity are zero. In other words, expressions (1.8) yield the displacement values relative to their values at infinity. It is known that in the fundamental solutions for two-dimensional problems of elasticity, the displacements at infinity have a logarithmic singularity. Thus, in attempting to determine the displacements at a finite point relative to the displacements at infinity, they prove to be unbounded. Therefore we will calculate the displacements $w_k(x, t)$ in the two-dimensional problem relative to some finite point with coordinates a_i. Then

$$w_k(x,t) = \int_{-\infty}^{\infty} [u_k(x-y,t) - u_k(a-y,t)]\,dy_3, \quad y_1 = y_2 = 0$$

Taking account of (1.4), we obtain

$$w_k = (2\pi)^{-3/2} \lim_{z\to\infty} \int_{-z}^{z} dy_3 \int_{V^\vee} u_k^\vee(x^\vee, t)\{\exp[-ix_p^\vee(x_p - y_p)] -$$
$$-\exp[-ix_p^\vee(a_p - y_p)]\}\,dV^\vee = (2\pi)^{-3/2} \lim_{z\to\infty} \int dx_3^\vee\, 2\,\frac{\sin(x_3^\vee z)}{x_3^\vee} \times$$
$$\times \int_{S^\vee} u_k^\vee(x^\vee, t)\{\exp(-ix_\alpha^\vee x_\alpha)\exp(-ix_3^\vee x_3) -$$
$$-\exp(-ix_\alpha^\vee a_\alpha)\exp(-ix_3^\vee a_3)\}\,dS^\vee$$

Here $S^\vee$ is a two-dimensional plane in the parameter space of the Fourier transformation, $dS^\vee = dx_1^\vee dx_2^\vee$. Taking into account that (see, e.g., [2])

$$\lim_{z\to\infty} \int_{-\infty}^{\infty} \psi(x_3^\vee)\frac{\sin(x_3^\vee z)}{x_3^\vee}\,dx_3^\vee = \frac{\pi}{2}[\psi(+0) + \psi(-0)]$$

we have

$$w_k = (2\pi)^{-1/2} \int_{S^\vee} w_k^\vee(x^\vee, t)[\exp(-ix_\alpha^\vee x_\alpha) - \exp(-ix_\alpha^\vee a_\alpha)]\,dS^\vee$$
$$w_k^\vee(x^\vee, t) = u_k^\vee(x^\vee, t)|_{x_3^\vee=0} = (2\pi)^{-3/2}[C_{ki}^{\vee -1}(x^\vee)|_{x_3^\vee=0} f_i](t)$$

On the $(x_1^\vee, x_2^\vee)$ plane we introduce a polar $(r^\vee, \varphi^\vee)$, coordinate system, where the angle $\varphi^\vee$ is reckoned from the x direction. Then $x_\alpha^\vee = r^\vee \eta_\alpha^\vee$, $\eta_\alpha^\vee \eta_\alpha^\vee = 1$, $x_\alpha^\vee x_\alpha = r^\vee r\cos\varphi^\vee$, $dS^\vee = r^\vee dr^\vee d\varphi^\vee$, $r^2 = x_\alpha x_\alpha$, $r_a^2 = a_\alpha a_\alpha$, $\cos\varphi_a = a_\alpha x_\alpha/(r_a r)$. Allowing for the fact that $w_k^\vee(\eta^\vee, t) = w_k^\vee(-\eta^\vee, t)$, we obtain [20]

$$w_k = (2\pi)^{-1/2} \int_0^{2\pi} d\varphi^\vee w_k^\vee(\eta^\vee, t) \int_0^{\infty} [\cos(r^\vee r\cos\varphi^\vee) - \cos(r^\vee r_a \cos\varphi_a)]\frac{dr^\vee}{r^\vee} =$$

$$= -(2\pi)^{-1/2}\int_0^{2\pi} d\varphi^{\vee} w_k^{\vee}(\eta^{\vee}, t) \lim_{R^{\vee}\to 0} [\mathrm{Ci}(R^{\vee} r|\cos\varphi|) - \mathrm{Ci}(R^{\vee} r_a|\cos\varphi_a|)]$$

where $\mathrm{Ci}(\psi)$ is the integral cosine, while C is the Euler-Mascheroni constant:

$$\mathrm{Ci}(\psi) = C + \ln\psi + g(\psi), \quad g(\psi) = \sum_{p=1}^{\infty}(-1)^p \frac{\psi^{2p}}{(2p)!\,2p}.$$

Allowing for the fact that g(0) = 0, we have

$$\begin{aligned} w_k &= -(2\pi)^{-1/2}\int_0^{2\pi} w_k^{\vee}(\eta^{\vee}(\varphi^{\vee}), t)\ln\frac{r|\cos\varphi^{\vee}|}{r_a|\cos\varphi_a^{\vee}|}\,d\varphi^{\vee} = \\ &= -(2\pi)^{-1/2}\oint_{\eta_\alpha^{\vee}\eta_\alpha^{\vee}=1} w_k^{\vee}(\eta^{\vee}, t)\ln|x_\alpha\eta_\alpha^{\vee}|\,|d\eta^{\vee}| + B \end{aligned} \qquad (2.1)$$

Here B denotes a constant independent of x, which can be omitted.

For the stresses we obtain, after differentiating (2.1),

$$\sigma_{ij} = C_{ijk\beta} w_{k,\beta} = -(2\pi)^{-1/2} C_{ijk\beta}\int_{\eta_\alpha^{\vee}\eta_\alpha^{\vee}=1} w_k^{\vee}(\eta^{\vee}, t)\frac{\eta_\beta|d\eta^{\vee}|}{x_\alpha\eta_\alpha^{\vee}} \qquad (2.2)$$

Substituting (1.7) into (2.2), changing over to the complex variable $\eta^{\circ} = \eta_1^{\vee} + i\eta_2^{\vee}$ and bounding the contour integral using residue theory, we can obtain an analog of the solution of [19] for a concentrated force in an infinite plane, in the form of a time series.

When the medium under consideration contains a plane of elastic symmetry that coincides with (x_1, x_2), part of the components of the tensor $C_{\alpha\beta\gamma\omega}$ are equal to zero; specifically, $C_{\alpha\beta\gamma3} = C_{333\beta} = 0$. Components that coincide with those given by virtue of the symmetry conditions are also equal to zero. Then we have $C_{\alpha3}|_{\hat{x}_3=0} = C_{3\alpha}|_{\hat{x}_3=0} = 0$ from (1.5). Consequently,

$$\begin{aligned} w_\alpha(x^{\vee}, t) &= (2\pi)^{-1/2}[C_{\alpha\beta}^{\vee -1}|_{x_3^{\vee}=0} f_\beta](x^{\vee}, t) \\ w_3^{\vee}(x^{\vee}, t) &= (2\pi)^{-1/2}[C_{33}^{\vee -1}|_{x_3^{\vee}=0} f_3](x^{\vee}, t) \end{aligned} \qquad (2.3)$$

Substitution of (2.3) into (2.1) and (2.2) shows that in this case the displacements u_α and stresses $\sigma_{\alpha\beta}$, σ_{33} depend only on two components of the force f_α, while the displacements u_3 and stresses $\sigma_{3\alpha}$ depend only on the third component f_3, i.e., the solution has been separated into plane and antiplane strain.

If $f_3 = 0$, these same results can be obtained by substituting the condition $u_3 = 0$ into (1.3) and using these equations for i = 1, 2. Here, if a plane of elastic symmetry exists, then instead of (1.2) we obtain

$$C_{\alpha\beta\gamma\omega} u_{\gamma,\alpha\omega} = -\delta_{(2)}(x) f_\alpha(t), \quad \delta_{(2)}(x) = \delta(x_1)\delta(x_2) \qquad (2.4)$$

Employing the Fourier transformation with respect to the coordinates x_1, x_2, and solving the resultant system of integral equations, after employing the inverse Fourier transformation we arrive at the desired result, namely expressions (2.1)-(2.3) for the displacements and the stresses.

Let us consider the problem of the generalized plane stressed state. In a plate that is loaded in its plane by a concentrated force, and that has force-free surfaces, we will calculate the average stresses and displacements over the thickness of the plate. Let us assume that the center plane (x_1, x_2) of the plate is the plane of symmetry. Then for $\langle\sigma_{33}\rangle = 0$ we obtain

$$\langle \varepsilon_{\alpha\beta}\rangle = b_{\alpha\beta\gamma\omega}\langle \sigma_{\alpha\omega}\rangle \qquad (2.5)$$

Here the bent brackets denote averaging over the thickness, while $b_{\alpha\beta\gamma\omega}$ denotes the Hooke's-law tensor, which is the inverse of C_{ijkl}, i.e., the solution of system (1.1) of linear Volterra equations (with respect to time) in the strains. Considering (2.5) as a system of four equations in the four unknowns $\langle\sigma_{\alpha\omega}\rangle$ (or, if we allow for the symmetry of the tensors, three equations in three unknowns), and solving it, we obtain

$$\sigma_{\alpha\beta} = C^{*}_{\alpha\beta\gamma\omega}\varepsilon_{\gamma\omega} \qquad (2.6)$$

Here and henceforth, the bent brackets are omitted. It can be shown that, when there is a plane of elastic symmetry,

$$C^{*}_{\alpha\beta\gamma\omega} = C_{\alpha\beta\gamma\omega} - C_{\alpha\beta33}(C_{3333})^{-1}C_{33\gamma\omega} \qquad (2.7)$$

Averaging the equilibrium equations (1.2) for i = 1, 2 (in view of the conditions $\sigma_{i3} = 0$ on the plate surfaces, the derivatives with respect to the coordinate x_3 disappear), and substituting (2.6) into the resultant expressions, we obtain

$$C^{*}_{\alpha\beta\gamma\omega}u_{\gamma,\alpha\omega} = -\delta_2(x)f_\alpha(t) \qquad (2.8)$$

Comparing (2.8) and (2.4), we see that these equations are identical, to within replacement of $C^{\#}_{\alpha\beta\gamma\omega}$ by $C_{\alpha\beta\gamma\omega}$. Consequently, expressions (2.1)-(2.3) yield the solutions of the corresponding problem of the plane stressed state if we replace $C_{\alpha\beta\gamma\omega}$ by $C^{\#}_{\alpha\beta\gamma\omega}$ in them, i.e.,

$$w_\varkappa = -(2\pi)^{-2}\oint_{\eta_\alpha^\vee\eta_\alpha^\vee=1}[C^{\vee *-1}_{\varkappa\beta}(\eta^\vee)f_\beta](t)\ln|x_\alpha\eta_\alpha^\vee|\,|d\eta^\vee|$$

$$\sigma_{\varkappa\omega} = -(2\pi)^{-2}C^{*}_{\varkappa\omega\gamma\beta}\oint_{\eta_\alpha^\vee\eta_\alpha^\vee=1}[C^{\vee *-1}_{\gamma\delta}(\eta^\vee)f_\delta](t)\frac{\eta_\beta^\vee}{x_\alpha\eta_\alpha}|d\eta^\vee|, \qquad (2.9)$$

$$C^{\vee *}_{\gamma\delta} = C^{*}_{\gamma\alpha\delta\beta}\eta_\alpha^\vee\eta_\beta^\vee$$

If there is no plane of elastic symmetry, then there is no corresponding formulation of the problems of the generalized stressed state, and the different formulations will depend on additional hypotheses that are introduced. If we assume that, in addition to $\langle\sigma_{33}\rangle = 0$, we have $\langle\sigma_{\alpha 3}\rangle = 0$, then the expressions for the strains will again have the form (2.5). The remaining formulas are maintained, except for (2.7), since the tensor $C^{\#}_{\alpha\beta\gamma\omega}$, which is the inverse matrix for system (2.5), will be expressed in terms of C_{ijkl} in somewhat more complicated fashion. However, the expressions for the displacements and stresses will have the same form (2.9).

In the problem of generalized plane strain for an isotropic hereditary-elastic aging medium, after substitution of (1.11) into (2.1) we obtain

$$w_k = -(2\pi)^{-2}\int_0^{2\pi}[(\lambda+2\mu)^{-1}\{-(\lambda+\mu)\eta_i^\vee(\varphi)\eta_k^\vee(\varphi)+(\lambda+2\mu)\delta_{ik}\}\mu^{-1}f_i]\times$$
$$\times[\ln r+\ln|\cos\varphi^\vee|]d\varphi^\vee$$

Allowing for the fact that $\eta_1^\vee = \cos(\varphi^\vee+\psi)$, $\eta_2^\vee = \sin(\varphi^\vee+\psi)$, $\eta_3^\vee = 0$, where ψ is the angle between vector x and the x_1 axis, and using the expressions [20]

$$\int_0^{\pi/2}\sin^2\varphi\ln(\cos\varphi)d\varphi = -\frac{\pi}{8}(1+2\ln 2)$$

$$\int_0^{\pi/2}\cos^2\varphi\ln(\cos\varphi)d\varphi = -\frac{\pi}{8}(-1+2\ln 2)$$

we obtain, after discarding terms independent of x,

$$w_\alpha = \frac{1}{4\pi}(\lambda+2\mu)^{-1}\left\{-(\lambda+3\mu)\delta_{\alpha\beta}\ln r + (\lambda+\mu)\frac{x_\alpha x_\beta}{r^2}\right\}\mu^{-1} f_\beta \quad (2.10)$$
$$w_3 = -{}^1/_2 \ln r \mu^{-1} f_3/\pi$$

Here as well, the solution breaks down into solutions for plane and antiplane strain. Differentiation of (2.10) yields

$$w_{\alpha,\gamma} = \frac{1}{4\pi r}(\lambda+2\mu)^{-1}\left\{-(\lambda+3\mu)\delta_{\alpha\beta}\frac{x_\gamma}{r} + (\lambda+\mu)\times\right.$$

$$\left.\times\left(\delta_{\alpha\gamma}\frac{x_\beta}{r} + \delta_{\beta\gamma}\frac{x_\alpha}{r} - 2\frac{x_\alpha x_\beta x_\gamma}{r^3}\right)\right\}\mu^{-1} f_\beta, \quad w_{3,\gamma} = -\frac{1}{2\pi}\frac{x_\gamma}{r^2}\mu^{-1} f_3$$

while the use of (1.1) and (1.10) enables us to determine the stresses:

$$\sigma_{\alpha\gamma} = -\frac{1}{r}\left[2\left(\frac{x_\alpha}{r}\delta_{\beta\gamma} + \frac{x_\gamma}{r}\delta_{\beta\alpha} - \frac{x_\beta}{r}\delta_{\alpha\gamma}\right)\mathrm{m} + \frac{4}{3}\frac{x_\alpha x_\beta x_\gamma}{r^3}\mathrm{n}\right] f_\beta$$
$$\sigma_{33} = -\frac{1}{2\pi r}\frac{x_\beta}{r}\nu(1-\nu)^{-1} f_\beta, \quad \sigma_{\alpha 3} = -\frac{1}{2\pi r}\frac{x_\alpha}{r} f_3 \quad (2.11)$$

In the case of the plane stressed state, we have from (2.7) and (1.10)

$$C^*_{\alpha\beta\gamma\omega} = [\lambda - \lambda(\lambda+2\mu)^{-1}\lambda]\delta_{\alpha\beta}\delta_{\gamma\omega} + \mu(\delta_{\alpha\gamma}\delta_{\beta\omega} + \delta_{\alpha\omega}\delta_{\beta\gamma})$$

Comparing this expression with (1.10), we obtain that $C^{*}_{\alpha\beta\gamma\omega}$ is generated from $C_{\alpha\beta\gamma\omega}$ by replacing λ by $\lambda^* = 2\lambda(\lambda+2\mu)^{-1}\mu$. Allowing for the fact that, as was shown, all the relations of the plane stressed state can be obtained from the corresponding relations of the plane strain state by replacing $C_{\alpha\beta\gamma\omega}$ by $C^{*}_{\alpha\beta\gamma\omega}$, we see that the expressions for the displacements ω_α and stresses $\sigma_{\alpha\beta}$ in the plane stressed state can be obtained from (2.10) and (2.11) after replacing λ by λ^*.

3. Let us consider some limiting cases. Let us assume that the load is applied at time $t = 0$, i.e., $f_i(t) = 0$ for $t < 0$. In what follows, we will employ two functional spaces on the semiaxis: $C[0, \infty]$, the space of functions that are continuous on the semiaxis and have a finite limit at infinity; and $C[0, \infty)$, the space of functions that are continuous and uniformly bounded on the semiaxis, which may not necessarily have a bounded limit at infinity. In both spaces we define the norm $\|g\| = \sup|g(t)|, 0 \leq t < \infty$.

3.1. Let $t \to +0$; then, as can be seen from (1.10), we arrive at the problem for an anisotropic medium without the hereditary property, with elastic moduli that coincide with the initial instantaneous moduli $C_{ijkl}{}^{I}(0)$. From (1.6) we obtain $u_k{}^{\sim}(x^{\sim}, 0) = (2\pi)^{-1/2}(C^{\sim I})_{ki}{}^{-1}(\eta^{\sim}, 0) f_i(0) r^{\sim -2}$ and for the stresses and displacements we arrive at expressions (1.8), (1.9), (2.1), (2.2), (2.9) with the operators C_{ijkl}, $C^*_{\alpha\beta\gamma\omega}$, $C^{\sim -1}_{ij}$, $C^{\sim * -1}_{\alpha\beta}$ replaced by matrices $C^{I}_{ijkl}(0)$, $C^{I*}_{\alpha\beta\gamma\omega}(0)$ $(C^{\sim I})_{ij}{}^{-1}(\eta^{\sim}, 0)$, $(C^{\sim I*})_{\alpha\beta}{}^{-1}(\eta^{\sim}, 0)$ respectively.

In the isotropic case, expressions (1.12), (1.13), (2.10), and (2.11) will be valid if the operators λ and μ are replaced by the constants $\lambda^I(0)$, $\mu^I(0)$.

3.2. Let us consider the behavior of the fundamental solution as $t \to \infty$ for the case in which $f_i(t) \in C[0, \infty]$. Assume, furthermore, that operators C_{ijkl} act and are bounded in $C[0, \infty]$. Conditions on the kernels $C_{ijkl}(t, \tau)$ sufficient for this were obtained in a lemma of [21]. This class will also include operators on which somewhat more stringent conditions are imposed ([15], Chapter 1, Sec. 4). The operator $C_{qk}{}^{\sim III}(\eta)$, $|\eta| = 1$ will also be bounded in $C[0, \infty]$. Henceforth, in this subsection and the next, we will consider materials for which the operator $C_{qk}{}^{\sim}(\eta^{\sim})$, $|\eta^{\sim}| = 1$ has a bounded inverse in $C[0, \infty]$; then, as follows from (1.8), as $t \to \infty$ we have $u_i(x, t) \to u_i(x, \infty) \neq \pm\infty$. This manifestly occurs if the norms of the integral operators C_{ijkl}^{II} are sufficiently small. More precise assertions are given in [22].

When the spectral radius ρ of operator $C^{\smile III}$ is less than 1, we can obtain $u_i(x, \infty)$, $\sigma_{ij}(x, \infty)$ by substituting operator $C_{ki}^{\smile -1}$ in the form of Neumann series (1.7) into expressions (1.8), (1.9), (2.1), (2.2), (1.6). On the semiaxis as well, it will converge uniformly with respect to η, |η| = 1. For $\rho(C^{\smile III})$ we can employ the bound [22]

$$\rho(C^{\smile III}) \leqslant \lim_{t^*\to\infty} \sup_{t^*\leqslant t<\infty} \int_{t^*}^{t} \| C^{\smile III}(t,\tau,\eta^{\smile}) \|_E \, d\tau, \quad \| C^{\smile III} \|_E \leqslant \| (C^{\smile I})^{-1} \|_E \cdot \| C^{\smile} \|_E$$
$$\| C^{\smile} \|_E = \| C_{ijkl}\eta_j{}^{\smile}\eta_l{}^{\smile} \| = \sup_{|\xi|=1} \{ C_{ijkl}\eta_j{}^{\smile}\eta_l{}^{\smile}\xi_k C_{ipqr}\xi_r\eta_p{}^{\smile}\eta_q{}^{\smile} \}^{1/2}$$

This will also obtain for the plane stressed state if $\rho(C^{\smile III*}) < 1$.

Assume, furthermore, that C_{ijkl} are bounded operators with attenuating memory, i.e., $C_{ijkl}^{I}(t) \in C[0, \infty)$:

$$\lim_{t\to\infty} \int_0^t C_{ijkl}(t,\tau)\, d\tau = C_{ijkl}^{1\infty} \neq \pm\infty, \quad \lim_{t\to\infty} \int_0^a | C_{ijkl}(t,\tau) | \, d\tau = 0$$

for any $a \in [0, \infty)$. This class includes all bounded operators with kernels that depend only on the difference of arguments. In particular, the operator moduli of a hereditary-elastic nonaging medium are of this type. Taking into account that, as a result of the presumed continuous invertibility in C[0, ∞) of the operator $C^{\smile}$ a solution of the problem exists as t → ∞, we obtain [21] that as t → ∞ in (1.1), (1.2), etc., it is necessary to replace the operators C_{ijkl} by constants $C_{ijkl}^{\infty} = C_{ijkl}^{I}(\infty) + C_{ijkl}^{1\infty}$.

Thus, for $u_i(x, \infty)$ we change over to problems in a nonhereditary anisotropic elastic medium, whose solution is given by expressions (1.8), (1.9), (2.1), (2.2), (2.9), where operator $C_{ki}^{\smile -1}$ must be replaced by matrix $(C^{\smile\infty})_{ki}^{-1}$, $C_{ij}^{\smile\infty} = C_{ijkl}^{\infty}\eta_j{}^{\smile}\eta_l{}^{\smile}$.

In isotropic materials, to obtain $u_i(x, \infty)$ it is necessary to replace λ, μ in (1.12), (1.13), (2.10), (2.11) by λ^∞, μ^∞, and λ^{-1}, μ^{-1} by $1/\lambda^\infty$, $1/\mu^\infty$.

3.3. Let us consider the case of steady-state oscillations, when the stimulus $f_i(t) = 0$ for $t<0$ $f_i(t) \in C[0, \infty)$ and as $t\to\infty$ $f(t) \to f_0\cos(\omega t+\alpha)$, while the constant $f_0 \neq \pm\infty$. Assume that K is is a Volterra operator such that

$$(Kg)(t) = K^I(t) g(t) + \int_0^t K(t,\tau) g(\tau)\, d\tau$$

acts and is bounded in spaces C[0, ∞] and C[0, ∞):

$$K^I(t) \in C[0,\infty]; \quad \int_0^t K(t,\tau)\, d\tau \to K^{1\infty} \neq \pm\infty, \quad \int_0^a K(t,\tau)\, d\tau \to 0$$

as t → ∞ for any $a \in [0, \infty)$, i.e., K is an operator with attenuating memory. As t → ∞, function K(t, τ) tends to $K^o(t - \tau)$ in the sense [15] that

$$\lim_{t^*\to\infty} \sup_{t^*\leqslant t<\infty} \int_{t^*}^{t} | K(t,\tau) - K^o(t-\tau) | \, d\tau = 0$$

while function $g(t) \in C[0, \infty)$ and as $t\to\infty$ $g(t) \to g_0 e^{i\omega t}$, where the constant $g_0 \neq \pm\infty$. It can be shown that, under these conditions, $[Kg](t) \to K^c g_0 e^{i\omega t}$ as t → ∞:

$$K^c = K^I(\infty) + \int_0^\infty K^o(\tau) e^{-i\omega\tau}\, d\tau$$

If, in addition, operator K is continuously invertible in C[0, ∞] and C[0, ∞), then $[K^{-1}g](t) \to g_0 e^{i\omega t}/K^c$ as $t \to \infty$.

From this is evident that if C_{ijkl} are bounded operators with attenuating memory in C[0, ∞], C[0, ∞), that go over as $t \to \infty$ into operators with kernels $C^{\circ}_{ijkl}(t - \tau)$, and, in addition, $C_{ik}^{\sim}(\eta^{\sim})$ is a continuously invertible operator in C[0, ∞] and C[0, ∞) for $|\eta^{\sim}|=1$, while the forces $f_i(t) \to f_i \cos(\omega t + \alpha)$ as $t \to \infty$, then $u_j \to \mathrm{Re}[u_j^c e^{i(\omega t+\alpha)}]$, $\sigma_{ij} = \mathrm{Re}[\sigma_{ij}^c e^{i(\omega t+\alpha)}]$ as $t \to \infty$.

Here u_j^c and σ_{ij}^c are the solutions of the corresponding problems for an anisotropic-elastic medium (without the hereditary property) with elastic moduli

$$C^c_{ijkl} = C^1_{ijkl}(\infty) + \int_0^\infty C^{\circ}_{ijkl}(\tau) e^{-i\omega\tau} d\tau$$

which are given by expressions (1.8), (1.9), (2.1), (2.2), (2.9) if we replace operator $C_{ki}^{\sim -1}$ by matrix $(C^{\sim c})_{ki}^{-1}$ in them. In isotropic materials, u_i^c and σ_{ij}^c are given by expressions (1.12), (1.13), (2.10), (2.11), in which λ and μ should be replaced by λ^c, μ^c, and λ^{-1}, μ^{-1} by $1/\lambda^c$, $1/\mu^c$.

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