Construction of a fundamental solution for an anisotropic aging medium of the hereditary type

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When reducing boundary-value problems to integral equations for media with linear determining equations (an anisotropic elastic medium, an aging medium) one must have a fundamental solution giving the fields of the displacements and stresses for an unbounded medium being deformed by a concentrated single force. In an isotropic medium such a solution is given by a Kelvin–Somigliana matrix and is written in explicit form. For arbitrary anisotropy in an elastic medium the fundamental solution constructed in Ref. 1 is reduced to an integral over the contour of a unit circle oriented in space in a special way.[2-4] Analogous solutions for hereditary elasticity without aging[5] are given in the isotropic case in Ref. 6 and in the anisotropic case in Refs. 7 and 8. In the present article the results of Refs. 1, 2, 7, and 8 are generalized to a homogeneous, aging, hereditary-elastic medium.[9,10]

Let x_k, $k = 1, 2, 3$, be Cartesian coordinates. Let us consider an anisotropic, homogeneous, hereditary-elastic, aging medium in which the stresses σ_{ij} and the deformations ε_{kl} are connected by the equation[9,10]

$$\sigma_{ij}(t) = [c_{ijkl}\,\epsilon_{kl}](t),\qquad c_{ijkl} = c^{I}_{ijkl} + c^{II}_{ijkl}. \tag{1}$$

Here t is the time, $c^{I}_{ijkl}(t)$ is the tensor of the instantaneous elastic moduli, and c^{II}_{ijkl} is an integral operator of the type

$$[c^{II}_{ijkl}\,\epsilon_{kl}](t) = \int_0^t c_{ijkl}(t,\tau)\,\epsilon_{kl}(\tau)\,d\tau.$$

The operator c_{ijkl} satisfies the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}.$$

Let the time dependence of the components of the force applied at the origin of coordinates be determined by functions $f_i(t)$. The solution being sought satisfies the equilibrium equation

$$\sigma_{ij,j}(x,t) = -\delta(x) f_i(t), \tag{2}$$

where $\delta(x)$ is a delta function. Substituting (1) into (2) and using $\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$, we arrive at the Lamé equations for the displacements:

$$[c_{ijkl}\,u_{k,jl}](x,t) = -\delta(x) f_i(t). \tag{3}$$

We introduce the direct and inverse Fourier transformations in the usual way, denoting the transforms and the parameters of the transformation by a tilde:

$$\tilde u(\tilde x, t) = (2\pi)^{-3/2}\int_V u(x,t)\exp(i\tilde x_p x_p)\,dV,$$
$$u(x,t) = (2\pi)^{-3/2}\int_{\tilde V} \tilde u(\tilde x,t)\exp(-i\tilde x_p x_p)\,d\tilde V,$$
$$dV = dx_1 dx_2 dx_3,\quad d\tilde V = d\tilde x_1 d\tilde x_2 d\tilde x_3. \tag{4}$$

Applying a Fourier transformation to the left- and right-hand sides of Eq. (3), we obtain

$$\tilde x_j \tilde x_l\,[c_{ijkl}\,\tilde u_k](\tilde x, t) = (2\pi)^{-3/2} f_i(t). \tag{5}$$

We shall consider media for which the instantaneous elastic moduli are such that the energy of elastic deformation is a positive-definite quadratic form of the deformation components. Then for $\tilde x \neq 0$ the square matrix

$$\tilde c^{I}_{ik}(\tilde x, t) = c^{I}_{ijkl}(t)\,\tilde x_j \tilde x_l$$

is invertible, and hence there always exists a matrix $(\tilde c^{I})^{-1}_{qi}(\tilde x, t)$ such that

$$(\tilde c^{I})^{-1}_{qi}(\tilde x, t)\,\tilde c^{I}_{ik}(\tilde x, t) = \delta_{qk},$$

where δ_{qk} is the Kronecker symbol. Multiplying (5) by $(\tilde c^{I})^{-1}_{qi}$, we obtain

$$[\delta_{qk} + \tilde c^{III}_{qk}]\,\tilde u_k = (\tilde c^{I})^{-1}_{qi} f_i (2\pi)^{-3/2}. \tag{6}$$

Here the operator $\tilde c^{III}_{qk}(\tilde x)$ has the form

$$\tilde c^{III}_{qk}(\tilde x) = (\tilde c^{I})^{-1}_{qi}(\tilde x, t)\,\tilde x_j \tilde x_l\, c^{II}_{ijkl}. \tag{7}$$

Thus, the problem of constructing the fundamental solution in transforms reduces to the inversion of the operator $[\delta_{qk} + \tilde c^{III}_{qk}]$. To accomplish this inversion, we represent the vector $\tilde x$ in the form $\tilde x_j = \tilde r \tilde\eta_j$, where $\tilde\eta_j\tilde\eta_j = 1$, while $0 \le \tilde r < \infty$. Since $(\tilde c^{I})^{-1}_{qi}(\tilde x, t)$ is a uniform function of $\tilde x$ of degree −2, we have

$$(\tilde c^{I})^{-1}_{qi}(\tilde x, t) = (\tilde c^{I})^{-1}_{qi}(\tilde\eta, t)\,\tilde r^{-2},$$

and on the basis of (7) we obtain

$$\tilde c^{III}_{qk}\,\tilde u_k = \int_0^t (\tilde c^{I})^{-1}_{qi}(\tilde\eta, t)\,\tilde\eta_j\tilde\eta_l\, c_{ijkl}(t,\tau)\,\tilde u_k(\tilde x, \tau)\,d\tau. \tag{8}$$

Below we shall consider operators c_{ijkl} such that the functions c^{I}_{ijkl} are continuous on the segment $0 \le t \le T$, while c^{II}_{ijkl} is a compact Volterra operator in the space of vector functions continuous on this segment. In particular, the kernels of the operators c^{II}_{ijkl} can have the form

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$$c_{ijkl}(t,\tau) = c^0_{ijkl}(t,\tau)(t-\tau)^{-\alpha},$$

where the functions $c^0_{ijkl}(t,\tau)$ are continuous for $0 \le \tau \le t \le T$, while $\alpha < 1$. Then, since $\tilde\eta_l\tilde\eta_l = 1$, while the matrix $(\tilde c^{\mathrm{I}})^{-1}_{ql}(\tilde\eta,t)$ is bounded, it is easy to show that $\tilde c^{\mathrm{III}}_{qk}(\tilde\eta,t)$ is also a compact Volterra operator, and hence (see Ref. 11, for example) the solution of Eq. (6) can be represented in the form of a Neumann series converging uniformly with respect to $\tilde\eta$ for $0 \le t \le T$:

$$\tilde u_k(\tilde x,t) = (2\pi)^{-3/2}[\tilde c^{-1}_{kl} f_l](\tilde\eta,t)\tilde r^{-2},$$
$$\tilde c^{-1}_{kl} = (\tilde c^{\mathrm{I}})^{-1}_{kl} + \sum_{n=1}^{\infty}(-1)^n d^{(n)}_{kq}(\tilde c^{\mathrm{I}})^{-1}_{ql}, \qquad (9)$$

where $d^{(n)}_{kq}$ is the n-th power of the operator $\tilde c^{\mathrm{III}}_{kq}$, i.e.,

$$d^{(n)}_{kq} = \tilde c^{\mathrm{III}}_{ks} d^{(n-1)}_{sq}, \quad d^{(1)}_{kq} = \tilde c^{\mathrm{III}}_{kq},$$

and if we denote the kernel of the integral operator $d^{(n)}_{kq}$ by $d^{(n)}_{kq}$, then

$$[d^{(n)}_{kq}(\tilde c^{\mathrm{I}})^{-1}_{ql} f_l](\tilde\eta,t) = \int_0^t d^{(n)}_{kq}(\tilde\eta,t,\tau)(\tilde c^{\mathrm{I}})^{-1}_{ql}(\tilde\eta,\tau) f_l(\tau)\,d\tau.$$

To obtain $u_k(x,t)$ we perform an inverse Fourier transformation of the series (9), in analogy with what is done in Ref. 2 in the absence of heredity. We introduce the solid-angle element $d\tilde\omega$ in the space of transforms. Then $dV = d\tilde\omega$, $\tilde r^2 d\tilde r$, with where $d\tilde\omega = -d\tilde\varphi\, d\tilde\nu$, while $\tilde\varphi$ and $\tilde\theta$ are the angles in the spherical coordinate system. We reckon the angle $\tilde\theta$ from the x direction. Then

$$\tilde\nu = \frac{\tilde\eta_p x_p}{|x|}.$$

After integration over $\tilde r$ from 0 to $\tilde R$, it is convenient to represent (4) in the form

$$u_k(x,t) = \lim_{\tilde R\to\infty}(2\pi)^{-3/2}\int_{\tilde\omega}\tilde u_k(\tilde\eta,t)\frac{\sin(\tilde R\tilde\eta_p x_p)}{\tilde\eta_p x_p}\,d\tilde\omega. \qquad (10)$$

Using the relation[2]

$$\lim_{\tilde R\to\infty}\int_{-\infty}^{\infty}\psi(z)\frac{\sin(\tilde R z)}{z}\,dz = \frac{\pi}{2}[\psi(+0)+\psi(-0)],$$

from (10) we obtain

$$u_k(x,t) = (2\pi)^{-3/2}|x|^{-1}\lim_{\tilde R\to\infty}\int_0^{2\pi}d\tilde\varphi\int_{-1}^{1}\tilde u_k(\tilde\nu,\tilde\varphi,t)\frac{\sin(\tilde R|x|\tilde\nu)}{\tilde\nu}\,d\tilde\nu$$
$$= (8\pi)^{-1/2}|x|^{-1}\int_0^{2\pi}\tilde u_k(0,\tilde\varphi,t)\,d\tilde\varphi. \qquad (11)$$

To obtain the stresses corresponding to this displacement field, we reduce (11) to a form convenient for calculating the partial derivatives $u_{k,l}(x,t)$:

$$\sqrt{8\pi}\,u_k(x,t) = \frac{1}{|x|}\int_0^{2\pi}d\tilde\varphi\int_{-1}^{1}u_k(\tilde\nu,\tilde\varphi,t)\,\delta(\tilde\nu)\,d\tilde\nu$$
$$= \frac{1}{|x|}\int_{\tilde\omega}\tilde u_k(\tilde\eta,t)\,\delta(\tilde\nu)\,d\tilde\omega = \frac{1}{|x|}\int_{\tilde\omega}\tilde u_k(\tilde\eta,t)\,\delta\left(\frac{x_\alpha\tilde\eta_\alpha}{|x|}\right)d\tilde\omega. \qquad (12)$$

Differentiation of (12) with respect to x_l yields

$$\sqrt{8\pi}\,|x|^3 u_{k,l}(x,t) = -x_l\int_{\tilde\omega}\tilde u_k(\tilde\eta,t)\,\delta\left(\frac{x_\alpha\tilde\eta_\alpha}{|x|}\right)d\tilde\omega$$
$$+ \int_{\tilde\omega}\tilde u_k(\tilde\eta,t)\,\delta'\left(\frac{x_\alpha\tilde\eta_\alpha}{|x|}\right)[\tilde\eta_l|x| - x_l x_\alpha\tilde\eta_\alpha|x|^{-1}]\,d\tilde\omega. \qquad (13)$$

Here δ' is the derivative of the delta function with respect to its argument. Integrating (13) over $\tilde\nu$ with allowance for the definition of the delta function and its derivatives,[12] and for the relation

$$\left.\frac{\partial\tilde\eta_m}{\partial\tilde\nu}\right|_{\nu=0} = \frac{x_m}{|x|},$$

we obtain

$$\sqrt{8\pi}\,u_{k,l}(x,t)$$
$$= -\frac{x_l}{|x|^3}\int_0^{2\pi}\tilde u_k(0,\tilde\varphi,t)\,d\tilde\varphi - \frac{x_m}{|x|^3}\int_0^{2\pi}\tilde u_{k,m}(0,\tilde\varphi,t)\,\tilde\eta_l\,d\tilde\varphi. \qquad (14)$$

To calculate $\tilde u_{k,m} = \partial\tilde u_k(\tilde\eta,t)/\partial\tilde\eta_m$, we set $\tilde x = \tilde\eta$ in Eq. (5), after which we differentiate it with respect to $\tilde\eta_m$. Then

$$\frac{\partial\tilde u_k(\tilde\eta,t)}{\partial\tilde\eta_m} = -\tilde c^{-1}_{kl}\,\tilde c_{ip,m}\,\tilde u_p(\tilde\eta,t),$$

where $\tilde c_{lp,m} = (\delta_{jm}\tilde\eta_l + \delta_{lm}\tilde\eta_j)\,c_{ijpl}$.

We substitute (14) into (1), after which we reduce the expressions for the displacements and stresses to a form convenient for programmed realization:

$$u_k(x,t) = \frac{1}{8\pi^2|x|}\oint_{\zeta\in\Pi(x)}[\tilde c^{-1}_{kl}(\zeta)f_l](t)\,|d\zeta|, \qquad (15)$$

$$\sigma_{ij}(x,t) = \frac{c_{ijkl}}{8\pi^2|x|^3}\left\{-x_l\oint_{\zeta\in\Pi(x)}[\tilde c^{-1}_{kq}(\zeta)f_q](t)\,|d\zeta| + x_m\oint_{\zeta\in\Pi(x)}[\tilde c^{-1}_{ks}(\zeta)\,\tilde c_{sp,m}(\zeta)\,\tilde c^{-1}_{pq}(\zeta)f_q](t)\,\zeta_l\,|d\zeta|\right\}. \qquad (16)$$

Here $\Pi(\mathbf{x})$ is a plane orthogonal to the vector $\mathbf{x}$. The vector ζ satisfies the condition $|\zeta| = 1$, and thus the integral is calculated along a unit circle lying in the plane $\Pi(\mathbf{x})$. The fundamental solutions for media with difference kernels, presented in Refs. 7 and 8, are a particular case of (15) and (16). Examples of the numerical realization of the algorithm for calculating the fundamental solution in the absence of heredity in media with rectilinear anisotropy are given in Refs. 4, 13, and 14.

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