Chapter 18
On Maximum Principles for Weak Solutions of Some Parabolic Systems

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18.1 Introduction

Maximum principles for solutions parabolic equations constitute a traditional part of PDE analysis. It is well developed for classical solutions of scalar parabolic equations with constant coefficients, and these results also generalized to weak solutions of scalar elliptic and parabolic equations with variable coefficients; see, e.g., [LaSoUr67, Chapter III, Theorem 7.2]. The estimates of the essential maximum of weak solutions of parabolic systems are also available although with a constant depending on the system coefficients, cf., e.g., [LaSoUr67, Chapter VII, Theorem 2.1].

In this paper, by employing special test functions, sharper versions of the maximum principle for weak solutions of several linear parabolic variable-coefficient systems have been proved. The considered systems include non-stationary convection-reaction-diffusion systems as well as the Stokes and Brinkman systems. The obtained maximum principles for weak solutions can be employed to prove global existence of solutions of some nonlinear parabolic systems, cf. [PoRo16], where a maximum principle for strong solutions of the Burgers system has been used for this.

We presented here maximum principles for the spatially periodic solutions, i.e., solutions on the n-dimensional flat torus; similar results can be re-stated also for solutions on bounded domains.

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18.2 Periodic Function Spaces

We will employ some function spaces on torus and periodic function spaces (see, e.g., [Ag65, p.26], [Ag15], [Mc91], [RuTu10, Chapter 3], [RoRoSa16, Section 1.7.1] for more details).

Let $n \geq 1$ be an integer and $\mathbb{T}$ be the n-dimensional flat torus that can be parametrized as the semi-open cube $\mathbb{T} = [0, 1)^n \subset \mathbb{R}^n$, cf. [Zy02, p. 312]. Let $\mathbb{Z}$ denote the set of integers and $\boldsymbol{\xi} \in \mathbb{Z}^n$ denote the n-dimensional vector with integer components. The Lebesgue space on the torus $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, can be identified with the periodic Lebesgue space $L_{p\#} = L_{p\#}(\mathbb{R}^n)$ that consists of functions $\phi \in L_{p,\mathrm{loc}}(\mathbb{R}^n)$, which satisfy the periodicity condition

$$\phi(\mathbf{x} + \boldsymbol{\xi}) = \phi(\mathbf{x}) \quad \forall \boldsymbol{\xi} \in \mathbb{Z}^n.$$

for a.e. $\mathbf{x} \in \mathbb{R}^n$. For $s \in \mathbb{R}$, let $H^s_\# := H^s_\#(\mathbb{R}^n) := H^s(\mathbb{T})$ denote the L_2-based *periodic/toroidal Sobolev (Bessel-potential) spaces*, cf. [RuTu10, Definition 3.2.2, Proposition 3.2.6]. For any $s \in \mathbb{R}$, the space $H^{-s}_\#$ is adjoint (dual) to $H^s_\#$, i.e., $H^{-s}_\# = (H^s_\#)^*$. Note that the torus/periodic norms on $H^s_\#$ are equivalent to the corresponding standard (non-periodic) Bessel potential norms on $\mathbb{T}$ as a cubic domain, see, e.g., [Ag15, Section 13.8.1].

For any $s \in \mathbb{R}$, let us also introduce the space

$$\dot{H}^s_\# := \{g \in H^s_\# : \langle g, 1 \rangle_{\mathbb{T}} = 0\}. \tag{18.1}$$

with the same norm as $H^s_\#$. Definition (18.1) and the Riesz theorem also imply that the space adjoin to $\dot{H}^s_\#$ can be expressed as $(\dot{H}^s_\#)^* = \dot{H}^{-s}_\#$.

The corresponding spaces of n-component vector functions/distributions are denoted as $\mathbf{L}_{q\#} := (L_{q\#})^n$, $\mathbf{H}^s_\# := (H^s_\#)^n$, etc. Let us also define the Sobolev spaces of divergence-free functions and distributions

$$\dot{\mathbf{H}}^s_{\#\sigma} := \left\{\mathbf{w} \in \dot{\mathbf{H}}^s_\# : \operatorname{div} \mathbf{w} = 0\right\}, \quad s \in \mathbb{R},$$

endowed with the same norm as $\mathbf{H}^s_\#$. Similarly, $\mathbf{L}_{q\#\sigma}$ denote the subspaces of divergence-free vector-functions from $\mathbf{L}_{q\#}$, etc.

Some more details about the periodic Sobolev spaces used here are available in [Mi22, Section 16.3].

For the evolution problems, we will use the spaces of Banach-valued functions $L_q(0, T; H^s_\#)$, $s \in \mathbb{R}$, $1 \leq q \leq \infty$, $0 < T < \infty$, which consists of functions that map $t \in (0, T)$ to a function or distribution from $H^s_\#$. For $1 \leq q < \infty$, the space $L_q(0, T; H^s_\#)$ is endowed with the norm

$$\|h\|_{L_q(0,T;H^s_\#)} = \left(\int_0^T \|h(\cdot, t)\|^q_{H^s_\#} dt\right)^{1/q} < \infty$$

and for $q = \infty$ with the norm

$$\|h\|_{L_\infty(0,T;H^s_\#)} = \operatorname{ess\,sup}_{t\in(0,T)} \|h(\cdot,t)\|_{H^s_\#} < \infty.$$

For a function (or distribution) $h(\mathbf{x},t)$, we will use the notation $h'(\mathbf{x},t) := \partial_t h(\mathbf{x},t) := \frac{\partial}{\partial t} h(\mathbf{x},t)$ for the partial derivative in the scalar variable $t \in \mathbb{R}$ and the notation $\partial_\alpha h(\mathbf{x},t) := \frac{\partial}{\partial x_\alpha} h(\mathbf{x},t)$ for the partial derivative in the space variable x_α.

Let X and Y be some Hilbert spaces. We will further need the space

$$H^1(0,T;X,Y) := \{u \in L_2(0,T;X) : u' \in L_2(0,T;Y)\}$$

endowed with the norm

$$\|u\|_{H^1(0,T;X,Y)} = (\|u\|^2_{L_2(0,T;X)} + \|u'\|^2_{L_2(0,T;Y)})^{1/2}.$$

Spaces of such type are considered in [LiMa72, Section 2.2], where they are denoted as $W(a,b)$. We will particularly need the space $H^1(0,T;H^s_\#, H^{s-2}_\#)$ and its vector counterparts.

18.3 Maximum Principles for Some Parabolic Systems

Let us consider the parabolic linear transport initial value PDE system

$$\partial_t \mathbf{u} - \operatorname{div}(\mu\nabla\mathbf{u}) + (\mathbf{U}\cdot\nabla)\mathbf{u} + \mathfrak{c}\mathbf{u} = \mathbf{0} \quad \text{in } \mathbb{T}\times(0,T]. \tag{18.2}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \quad \text{in } \mathbb{T}. \tag{18.3}$$

for unknown vector-function $\mathbf{u}(\mathbf{x},t) \in \mathbb{R}^n$, $(\mathbf{x},t) \in \mathbb{T}\times[0,T]$. The matrix $\mu = \{\mu^{\alpha\beta}(\mathbf{x},t)\}^n_{\alpha,\beta=1}$, the vector $\mathbf{U} = \{U_\alpha(\mathbf{x},t)\}^n_{\alpha=1}$, and the scalar $\mathfrak{c}(\mathbf{x},t)$ are some know functions of $\mathbf{x}$ and t. Appropriate periodic function spaces for these functions will be specified later on, when necessary. We denoted $\operatorname{div}(\mu\nabla\mathbf{u}) := \partial_\alpha(\mu^{\alpha\beta}\partial_\beta\mathbf{u})$, and the Einstein summation in repeated indices from 1 to n is assumed here and further on. Let us employ the notation $f^+ := \max\{f, 0\}$. We will also denote by $\langle\cdot,\cdot\rangle_\mathbb{T}$ the dual product on the periodicity cell $\mathbb{T}$.

Theorem 1 *Let $n \geq 2$ and the bilinear form $a(t;\mathbf{u},\mathbf{v})$ for vectors $\mathbf{u}$ and $\mathbf{v}$ be defined as*

$$a(t;\mathbf{u},\mathbf{v}) := \left\langle \mu^{\alpha\beta}(\cdot,t)\partial_\beta\mathbf{u}, \partial_\alpha\mathbf{v}\right\rangle_\mathbb{T} + \langle(\mathbf{U}(\cdot,t)\cdot\nabla)\mathbf{u},\mathbf{v}\rangle_\mathbb{T} + \langle\mathfrak{c}(\cdot,t)\mathbf{u},\mathbf{v}\rangle_\mathbb{T}\ \forall\, \mathbf{u},\mathbf{v}\in\mathbf{H}^1_\#.$$

Assume that $\mathfrak{c}(\mathbf{x},t) \geq 0$ and the form $a(t;\mathbf{u},\mathbf{v})$ is such that

$$a(t;\mathbf{v},\mathbf{v})\geq 0\quad\forall\,\mathbf{v}\in\mathbf{H}^1_\#\quad \textit{for a.e. } t\in[0,T].$$

Let $\mathbf{u}\in H^1(0,T;\mathbf{H}^1_\#,\mathbf{H}^{-1}_\#)$ *be a solution of the initial-variational problem*

$$\langle\partial_t\mathbf{u}(\cdot,t),\mathbf{v}\rangle_{\mathbb{T}}+a(\mathbf{u}(\cdot,t),\mathbf{v})=\mathbf{0}\quad\forall\,\mathbf{v}\in\mathbf{H}^1_\#,\tag{18.4}$$

$$\mathbf{u}|_{t=0}=\mathbf{u}^0\quad in\ \mathbb{T},\tag{18.5}$$

associated with (18.2)–(18.3)*, with* $\mathbf{u}^0\in\mathbf{L}_{\infty\#}$.

Then $\|\mathbf{u}\|_{L_\infty(0,T;\mathbf{L}_{\infty\#})}\leq\|\mathbf{u}^0\|_{\mathbf{L}_{\infty\#}}$.

Proof We will generalize to the parabolic system the idea applied to a scalar parabolic heat equation; see, e.g., [LaSoUr67, Chapter III, Theorem 7.1], [ErGu04, Proposition 6.12]).

Let $\mathbf{w}$ be a constant unit vector in $\mathbb{R}^n$ and let us denote $M:=\|\mathbf{u}^0\|_{\mathbf{L}_{\infty\#}}$. Then $(\mathbf{u}\cdot\mathbf{w})\mathbf{w}$ is the orthogonal projection of the variable vector $\mathbf{u}$ on the constant direction $\mathbf{w}$. Let us denote

$$\mathbf{v}_*(\mathbf{x},t):=((\mathbf{u}(\mathbf{x},t)\cdot\mathbf{w})-M)^+\,\mathbf{w}.\tag{18.6}$$

Then $\mathbf{v}_*\in H^1(0,T;\mathbf{H}^1_\#,\mathbf{H}^{-1}_\#)$ and we can use this vector as a test function $\mathbf{v}$ in (18.4). First, we have

$$\begin{aligned}
a(t;\mathbf{u},\mathbf{v}_*)&=a(t;\mathbf{u}-M\mathbf{w},(\mathbf{u}\cdot\mathbf{w}-M)^+\,\mathbf{w})+a(t;M\mathbf{w},(\mathbf{u}\cdot\mathbf{w}-M)^+\,\mathbf{w})\\
&=\left\langle\mu^{\alpha\beta}\partial_\beta(\mathbf{u}-M\mathbf{w}),\mathbf{w}\partial_\alpha\,(\mathbf{u}\cdot\mathbf{w}-M)^+\right\rangle_{\mathbb{T}}\\
&\quad+\left\langle(\mathbf{U}(\cdot,t)\cdot\nabla)(\mathbf{u}-M\mathbf{w}),(\mathbf{u}\cdot\mathbf{w}-M)^+\,\mathbf{w}\right\rangle_{\mathbb{T}}\\
&\quad+\langle\mathfrak{c}(\mathbf{u}-M\mathbf{w}),(\mathbf{u}\cdot\mathbf{w}-M)^+\,\mathbf{w}\rangle_{\mathbb{T}}+\langle\mathfrak{c}M\mathbf{w},(\mathbf{u}\cdot\mathbf{w}-M)^+\,\mathbf{w}\rangle_{\mathbb{T}}\\
&=\left\langle\mu^{\alpha\beta}(\cdot,t)\partial_\beta(\mathbf{u}\cdot\mathbf{w}-M),\partial_\alpha\,(\mathbf{u}\cdot\mathbf{w}-M)^+\right\rangle_{\mathbb{T}}\\
&\quad+\left\langle(\mathbf{U}(\cdot,t)\cdot\nabla)(\mathbf{u}\cdot\mathbf{w}-M),(\mathbf{u}\cdot\mathbf{w}-M)^+\right\rangle_{\mathbb{T}}\\
&\quad+\langle\mathfrak{c}(\mathbf{u}\cdot\mathbf{w}-M),(\mathbf{u}\cdot\mathbf{w}-M)^+\rangle_{\mathbb{T}}+\langle\mathfrak{c}M,(\mathbf{u}\cdot\mathbf{w}-M)^+\rangle_{\mathbb{T}}\\
&=a(t;(\mathbf{u}\cdot\mathbf{w}-M)^+\,\mathbf{w},(\mathbf{u}\cdot\mathbf{w}-M)^+\,\mathbf{w})+\langle\mathfrak{c}M,(\mathbf{u}\cdot\mathbf{w}-M)^+\rangle_{\mathbb{T}}\\
&=a(t;\mathbf{v}_*,\mathbf{v}_*)+\langle\mathfrak{c}M,(\mathbf{u}\cdot\mathbf{w}-M)^+\rangle_{\mathbb{T}}\geq 0.
\end{aligned}\tag{18.7}$$

Then, employing Lemma 1.3 from Chapter 3 of [Te02] for the final equality, we obtain

$$\begin{aligned}
\langle\partial_t\mathbf{u},\mathbf{v}_*\rangle_{\mathbb{T}}&=\langle\partial_t(\mathbf{u}-M\mathbf{w}),(\mathbf{u}\cdot\mathbf{w}-M)^+\,\mathbf{w}\rangle_{\mathbb{T}}\\
&=\langle\partial_t(\mathbf{u}\cdot\mathbf{w}-M),(\mathbf{u}\cdot\mathbf{w}-M)^+\rangle_{\mathbb{T}}
\end{aligned}$$

$$= \langle \partial_t (\mathbf{u}\cdot\mathbf{w} - M)^+, (\mathbf{u}\cdot\mathbf{w} - M)^+\rangle_{\mathbb{T}} = \frac{1}{2}\partial_t \|\mathbf{v}_*\|^2_{\mathbf{L}_{2\#}}. \tag{18.8}$$

Thus, (18.4) with $\mathbf{v} = \mathbf{v}_*$, (18.8), and (18.7) imply

$$\frac{1}{2}\partial_t \|\mathbf{v}_*\|^2_{\mathbf{L}_{2\#}} = -a(t; \mathbf{u}, \mathbf{v}_*) \le 0$$

Hence

$$\|(\mathbf{u}(\cdot, t)\cdot\mathbf{w} - M)^+\|^2_{\mathbf{L}_{2\#}} \le \|(\mathbf{u}^0\cdot\mathbf{w} - M)^+\|^2_{\mathbf{L}_{2\#}} = 0,$$

meaning that $(\mathbf{u}(\mathbf{x}, t)\cdot\mathbf{w} - M)^+ = 0$, that is, $\mathbf{u}(\mathbf{x}, t)\cdot\mathbf{w} \le M$ for a.e. $\mathbf{x}$, $\forall\, t \ge 0$. Since $\mathbf{w}$ is an arbitrary unit vector, this implies $|\mathbf{u}(\mathbf{x}, t)| \le M$ for a.e. $\mathbf{x}$, $\forall\, t \ge 0$. □

Note that Theorem 1 deals with a particular case of anisotropy. It is applicable also to the isotropic variable-coefficient case, $\mu^{\alpha\beta}(\mathbf{x}, t) = \mu(\mathbf{x}, t)\delta_{\alpha\beta}$.

The proof of Theorem 1 is not directly applicable if we replace $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1_\#$ by $\mathbf{u}, \mathbf{v} \in \dot{\mathbf{H}}^1_\#$ because the chosen test function $\mathbf{v}_*$ given by (18.6) does not belong to the corresponding dot-space. In the following assertion, we modify the proof accordingly.

Theorem 2 *Let $n \ge 2$ and the bilinear form $a(t; \mathbf{u}, \mathbf{v})$ for vectors $\mathbf{u}$ and $\mathbf{v}$ be defined as*

$$a(t; \mathbf{u}, \mathbf{v}) := \left\langle \mu^{\alpha\beta}(\cdot, t)\partial_\beta \mathbf{u}, \partial_\alpha \mathbf{v}\right\rangle_{\mathbb{T}} + \langle (\mathbf{U}(\cdot, t)\cdot\nabla)\mathbf{u}, \mathbf{v}\rangle_{\mathbb{T}} + \mathfrak{c}(t)\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{T}} \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1_\#. \tag{18.9}$$

Assume that $\mathfrak{c}(t) \ge 0$, $\operatorname{div}\mathbf{U}(\mathbf{x}, t)$ does not depend on $\mathbf{x}$ and the form $a(t; \mathbf{u}, \mathbf{v})$ is such that

$$a(t; \mathbf{v}, \mathbf{v}) \ge 0 \quad \forall\, \mathbf{v} \in \mathbf{H}^1_\# \quad \textit{for a.e. } t \in [0, T].$$

Let $\mathbf{u} \in H^1(0, T; \dot{\mathbf{H}}^1_\#, \dot{\mathbf{H}}^{-1}_\#)$ be a solution of the initial-variational problem

$$\langle \partial_t \mathbf{u}(\cdot, t), \mathbf{v}\rangle_{\mathbb{T}} + a(\mathbf{u}(\cdot, t), \mathbf{v}) = \mathbf{0} \quad \forall\, \mathbf{v} \in \dot{\mathbf{H}}^1_\#, \tag{18.10}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \quad \textit{in } \mathbb{T}, \tag{18.11}$$

associated with (18.2)–(18.3), where $\mathbf{u}^0 \in \dot{\mathbf{L}}_{\infty\#}$.

Then $\|\mathbf{u}\|_{L_\infty(0,T;\mathbf{L}_{\infty\#})} \le \|\mathbf{u}^0\|_{\mathbf{L}_{\infty\#}}$.

Proof Let, as in the proof of Theorem 1, $\mathbf{w}$ be a constant unit vector in $\mathbb{R}^n$ and let us denote $M := \|\mathbf{u}^0\|_{\mathbf{L}_{\infty\#}}$. Let us also denote

$$\mathbf{v}_*(\mathbf{x}, t) := ((\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{w}) - M)^+ \mathbf{w}, \quad \mathbf{v}_0(t) := \int_{\mathbb{T}} \mathbf{v}_*(\mathbf{x}, t) d\mathbf{x},$$
$$\widetilde{\mathbf{v}}(\mathbf{x}, t) := \mathbf{v}_*(\mathbf{x}, t) - \mathbf{v}_0(t).$$

Since $\mathbf{v}_0(t)$ is an average of $\mathbf{v}_*(\mathbf{x}, t)$ on the unite periodicity cell $\mathbb{T}$, it does not depend on $\mathbf{x}$. Moreover, $\widetilde{\mathbf{v}} \in H^1(0, T; \dot{\mathbf{H}}^1_\#, \dot{\mathbf{H}}^{-1}_\#)$, and we can use $\widetilde{\mathbf{v}}$ as a test function $\mathbf{v}$ in (18.10). First, we have

$$a(t; \mathbf{u}, \widetilde{\mathbf{v}}) = a(t; \mathbf{u}, \mathbf{v}_*) - a(t; \mathbf{u}, \mathbf{v}_0). \tag{18.12}$$

By (18.9), and taking into account that the average of $\mathbf{u}$ over $\mathbb{T}$ is zero and div $\mathbf{U}(\mathbf{x}, t)$ does not depend on $\mathbf{x}$, we obtain

$$a(t; \mathbf{u}, \mathbf{v}_0) = \left\langle \mu^{\alpha\beta}(\cdot, t)\partial_\beta \mathbf{u}, \partial_\alpha \mathbf{v}_0 \right\rangle_{\mathbb{T}} + \langle (\mathbf{U}(\cdot, t) \cdot \nabla)\mathbf{u}, \mathbf{v}_0 \rangle_{\mathbb{T}} + \mathfrak{c}(t)\langle \mathbf{u}, \mathbf{v}_0 \rangle_{\mathbb{T}}$$
$$= - \langle \operatorname{div} \mathbf{U}(\cdot, t), \mathbf{u} \cdot \mathbf{v}_0 \rangle_{\mathbb{T}} = 0. \tag{18.13}$$

Then from (18.12) and (18.13), we obtain similar to (18.7)

$$a(t; \mathbf{u}, \widetilde{\mathbf{v}}) = a(t; \mathbf{u}, \mathbf{v}_*) = a(t; \mathbf{v}_*, \mathbf{v}_*) + \mathfrak{c}(t)\langle M, (\mathbf{u} \cdot \mathbf{w} - M)^+ \rangle_{\mathbb{T}} \geq 0. \tag{18.14}$$

Further, $\mathbf{u} \in H^1(0, T; \dot{\mathbf{H}}^1_\#, \dot{\mathbf{H}}^{-1}_\#)$ implies that the average of $\partial_t \mathbf{u}$ over $\mathbb{T}$ is zero. Hence $\langle \partial_t \mathbf{u}, \mathbf{v}_0 \rangle_{\mathbb{T}} = 0$, meaning that $\langle \partial_t \mathbf{u}, \widetilde{\mathbf{v}} \rangle_{\mathbb{T}} = \langle \partial_t \mathbf{u}, \mathbf{v}_* \rangle_{\mathbb{T}}$. Thus, similar to (18.8)

$$\langle \partial_t \mathbf{u}, \widetilde{\mathbf{v}} \rangle_{\mathbb{T}} = \langle \partial_t \mathbf{u}, \mathbf{v}_* \rangle_{\mathbb{T}} = \frac{1}{2} \partial_t \|\mathbf{v}_*\|^2_{\mathbf{L}_{2\#}} \tag{18.15}$$

Thus (18.10) with $\mathbf{v} = \widetilde{\mathbf{v}}$, (18.15), and (18.14) implies

$$\frac{1}{2} \partial_t \|\mathbf{v}_*\|^2_{\mathbf{L}_{2\#}} = -a(t; \mathbf{u}, \widetilde{\mathbf{v}}) \leq 0.$$

Hence,

$$\|(\mathbf{u}(\cdot, t) \cdot \mathbf{w} - M)^+\|^2_{\mathbf{L}_{2\#}} \leq \|(\mathbf{u}^0 \cdot \mathbf{w} - M)^+\|^2_{\mathbf{L}_{2\#}} = 0$$

meaning that $(\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{w} - M)^+ = 0$, that is, $\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{w} \leq M$ for a.e. $\mathbf{x}$, $\forall\, t \geq 0$. Since $\mathbf{w}$ is an arbitrary unit vector, this implies $|\mathbf{u}(\mathbf{x}, t)| \leq M$ for a.e. $\mathbf{x}$, $\forall\, t \geq 0$. □

Let us now prove an analogue of the maximum principle given by Theorem 2 for the initial-variational problem defined on the *divergence-free* functions. Such setting is particularly associated with the Stokes, Oseen, and Brinkman problems that are parabolic in the sense of Solonnikov, cf. [So65, Section 1], [LaSoUr67, Chapter VII, Section 8, Definition 4], [Ei98, Definition I.4])

$$\partial_t \mathbf{u} - \operatorname{div}(\mu \nabla \mathbf{u}) + (\mathbf{U} \cdot \nabla)\mathbf{u} + \mathfrak{c}\mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \mathbb{T} \times (0, T], \tag{18.16}$$

$$\operatorname{div} \mathbf{u}(\mathbf{x}, t) = 0 \quad \text{in } \mathbb{T} \times (0, T], \tag{18.17}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \quad \text{in } \mathbb{T}, \tag{18.18}$$

where $p(\mathbf{x}, t)$ is another unknown function.

Theorem 3 *Let $n \geq 2$ and the bilinear form $a(t; \mathbf{u}, \mathbf{v})$ for vectors $\mathbf{u}$ and $\mathbf{v}$ be defined as*

$$a(t; \mathbf{u}, \mathbf{v}) := \mu^{\alpha\beta}(t) \left\langle \partial_\beta \mathbf{u}, \partial_\alpha \mathbf{v} \right\rangle_{\mathbb{T}} + \langle (\mathbf{U}(t) \cdot \nabla)\mathbf{u}, \mathbf{v} \rangle_{\mathbb{T}} + \mathfrak{c}(t) \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{T}} \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1_\#.$$

Assume that $\mathfrak{c}(t) \geq 0$, $\mathbf{U}(t)$ does not depend on $\mathbf{x}$ and the form $a(t; \mathbf{u}, \mathbf{v})$ is such that

$$a(t; \mathbf{v}, \mathbf{v}) \geq 0 \quad \forall\, \mathbf{v} \in \mathbf{H}^1_\# \quad \textit{for a.e. } t \in [0, T].$$

Let $\mathbf{u} \in H^1(0, T; \dot{\mathbf{H}}^1_{\#\sigma}, \dot{\mathbf{H}}^{-1}_{\#\sigma})$ be a solution of the initial-variational problem

$$\langle \partial_t \mathbf{u}(\cdot, t), \mathbf{v} \rangle_{\mathbb{T}} + a(\mathbf{u}(\cdot, t), \mathbf{v}) = \mathbf{0} \quad \forall\, \mathbf{v} \in \dot{\mathbf{H}}^1_{\sigma\#}, \tag{18.19}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \quad \textit{in } \mathbb{T}, \tag{18.20}$$

associated with (18.16)–(18.18), where $\mathbf{u}^0 \in \dot{\mathbf{L}}_{\infty\#\sigma}$.

Then $\|\mathbf{u}\|_{L_\infty(0,T;\mathbf{L}_{\infty\#})} \leq \|\mathbf{u}^0\|_{\mathbf{L}_{\infty\#}}$.

Proof Let $\mathbf{w}$ be a constant unit vector in $\mathbb{R}^n$ and let us denote $M := \|\mathbf{u}^0\|_{\mathbf{L}_{\infty\#}}$ and let, as in the proof of Theorem 2

$$\mathbf{v}_*(\mathbf{x}, t) := ((\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{w}) - M)^+ \mathbf{w}, \quad \mathbf{v}_0(t) := \int_{\mathbb{T}} \mathbf{v}_*(\mathbf{x}, t) d\mathbf{x},$$

$$\widetilde{\mathbf{v}}(\mathbf{x}, t) := \mathbf{v}_*(\mathbf{x}, t) - \mathbf{v}_0(t).$$

Since equation (18.19) is defined only for the divergence-free test functions $\mathbf{v}$, the function $\widetilde{\mathbf{v}}$ can be not used in this role and needs further modification. Let us define

$$\mathbf{v}_\sigma(\mathbf{x}, t) := \mathbb{P}\widetilde{\mathbf{v}}(\mathbf{x}, t),$$

where $\mathbb{P} : \dot{\mathbf{H}}^1_\# \to \dot{\mathbf{H}}^1_{\#\sigma}$ is the Leray projector and $\mathbf{v}_\sigma(\mathbf{x}, t) = \widetilde{\mathbf{v}}(\mathbf{x}, t) - \nabla q(x, t)$ with $q(\cdot, t) \in \dot{H}^1_\#$ (see, e.g., [RoRoSa16, Section 2.1]). Moreover, one can easily check the following orthogonality, cf. also [RoRoSa16, Corollary 2.5]

$$\langle \mathbf{u}, \nabla q \rangle_{\mathbb{T}} = 0 \quad \forall\, \mathbf{u} \in \dot{\mathbf{H}}^1_{\#\sigma},\ q \in L_{2\#}.$$

We can now use vector $\mathbf{v}_\sigma$ as a test function $\mathbf{v}$ in (18.19). Then

$$\mu^{\alpha\beta}(t)\left\langle\partial_\beta\mathbf{u},\,\partial_\alpha\mathbf{v}_\sigma\right\rangle_{\mathbb{T}} = \mu^{\alpha\beta}(t)\left\langle\partial_\beta\mathbf{u},\,\partial_\alpha\widetilde{\mathbf{v}}\right\rangle_{\mathbb{T}} - \mu^{\alpha\beta}(t)\left\langle\partial_\beta\mathbf{u},\,\partial_\alpha\nabla q\right\rangle_{\mathbb{T}}$$
$$= \mu^{\alpha\beta}(t)\left\langle\partial_\beta\mathbf{u},\,\partial_\alpha\widetilde{\mathbf{v}}\right\rangle_{\mathbb{T}} + \mu^{\alpha\beta}(t)\left\langle\partial_\beta\mathrm{div}\,\mathbf{u},\,\partial_\alpha q\right\rangle_{\mathbb{T}} = \mu^{\alpha\beta}(t)\left\langle\partial_\beta\mathbf{u},\,\partial_\alpha\widetilde{\mathbf{v}}\right\rangle_{\mathbb{T}},$$

$$\langle(\mathbf{U}(t)\cdot\nabla)\mathbf{u},\,\mathbf{v}_\sigma\rangle_{\mathbb{T}} = \langle(\mathbf{U}(t)\cdot\nabla)\mathbf{u},\,\widetilde{\mathbf{v}}\rangle_{\mathbb{T}} - \langle(\mathbf{U}(t)\cdot\nabla)\mathbf{u},\,\nabla q\rangle_{\mathbb{T}}$$
$$= \langle(\mathbf{U}(t)\cdot\nabla)\mathbf{u},\,\widetilde{\mathbf{v}}\rangle_{\mathbb{T}} + \langle(\partial_\alpha\mathbf{U}(t)\cdot\nabla)u_\alpha,\,q\rangle_{\mathbb{T}} + \langle(\mathbf{U}(t)\cdot\nabla)\mathrm{div}\,\mathbf{u},\,q\rangle_{\mathbb{T}}$$
$$= \langle(\mathbf{U}(t)\cdot\nabla)\mathbf{u},\,\widetilde{\mathbf{v}}\rangle_{\mathbb{T}},$$

and

$$\langle\mathbf{u},\,\mathbf{v}_\sigma\rangle_{\mathbb{T}} = \langle\mathbf{u},\,\widetilde{\mathbf{v}}\rangle_{\mathbb{T}} - \langle\mathbf{u},\,\nabla q\rangle_{\mathbb{T}} = \langle\mathbf{u},\,\widetilde{\mathbf{v}}\rangle_{\mathbb{T}}.$$

Hence

$$a(t;\mathbf{u},\mathbf{v}_\sigma) = a(t;\mathbf{u},\widetilde{\mathbf{v}}) = a(t;\mathbf{v}_*,\mathbf{v}_*) + \mathfrak{c}(t)\langle M,\,(\mathbf{u}\cdot\mathbf{w}-M)^+\rangle_{\mathbb{T}} \geq 0. \tag{18.21}$$

as in (18.14). Further,

$$\langle\partial_t\mathbf{u},\,\mathbf{v}_\sigma\rangle_{\mathbb{T}} = \langle\partial_t\mathbf{u},\,\widetilde{\mathbf{v}}\rangle_{\mathbb{T}} - \langle\partial_t\mathbf{u},\,\nabla q\rangle_{\mathbb{T}} = \langle\partial_t\mathbf{u},\,\mathbf{v}_*\rangle_{\mathbb{T}} - \langle\partial_t\mathbf{u},\,\mathbf{v}_0\rangle_{\mathbb{T}} + \langle\partial_t\mathrm{div}\,\mathbf{u},\,q\rangle_{\mathbb{T}}$$
$$= \langle\partial_t\mathbf{u},\,\mathbf{v}_*\rangle_{\mathbb{T}} = \frac{1}{2}\partial_t\|\mathbf{v}_*\|^2_{\mathbf{L}_{2\#}} \tag{18.22}$$

as in (18.8). Thus (18.19) with $\mathbf{v} = \mathbf{v}_\sigma$, (18.22), and (18.21) implies

$$\frac{1}{2}\partial_t\|\mathbf{v}_*\|^2_{\mathbf{L}_{2\#}} = -a(t;\mathbf{u},\mathbf{v}_\sigma) \leq 0.$$

Hence

$$\|(\mathbf{u}(\cdot,t)\cdot\mathbf{w}-M)^+\|^2_{\mathbf{L}_{2\#}} \leq \|(\mathbf{u}^0\cdot\mathbf{w}-M)^+\|^2_{\mathbf{L}_{2\#}} = 0,$$

meaning that $(\mathbf{u}(\mathbf{x},t)\cdot\mathbf{w}-M)^+ = 0$, that is, $\mathbf{u}(\mathbf{x},t)\cdot\mathbf{w} \leq M$ for a.e. $\mathbf{x}$, $\forall\, t \geq 0$. Since $\mathbf{w}$ is an arbitrary unit vector, this implies $|\mathbf{u}(\mathbf{x},t)| \leq M$ for a.e. $\mathbf{x}$, $\forall\, t \geq 0$. □

Remark 1 Note that the maximum principle for the divergence-free system (18.19)–(18.20) with the Oseen velocity, $\mathbf{U}(\mathbf{x},t)$, variable in $\mathbf{x}$ is not covered by Theorem 3.

Remark 2 In Theorems 1–3, we did not specify the conditions on the known functions μ, $\mathbf{U}$, and $\mathfrak{c}$ sufficient for the corresponding solutions $\mathbf{u}$ to exist and the quadratic form $a(t;\mathbf{w},\mathbf{w})$ to be non-negative. To satisfy these assumptions, the known functions should, of course, satisfy the appropriate conditions.

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