ONE PLANE PROBLEM FOR TWO BONDED ANISOTROPIC WEDGES

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The regular and reverse Mellin transformations are used to set up a solution for two bonded wedges that are made of different anisotropic materials and are under plane deformation or a generalized plane stressed state. Normal and tangential stresses are applied to the faces of the composite wedge. The degree of singularity of the stresses at the apex of the wedge is investigated.

Problems involving a composite orthotropic or homogeneous anisotropic wedge were examined in [1-9]. The present paper offers a generalization of the procedure proposed in those studies for the case of two bonded anisotropic wedges.

Assume that D' and D" are two wedges that are rigidly bonded along a common boundary. We introduce Cartesian and polar coordinates x, y and r, φ as shown in the figure. The boundary conditions $(\varphi_2 \leq 0 \leq \varphi_1,\ \varphi_1 - \varphi_2 \leq 2\pi)$:

$$\begin{gathered}\sigma_\varphi'(r,\varphi_1) = n'(r), \quad \tau_{r\varphi}'(r,\varphi_1) = t'(r), \quad \sigma_\varphi''(r,\varphi_2) = n''(r), \quad \tau_{r\varphi}''(r,\varphi_2) = t''(r) \\ \sigma_\varphi'(r,0) = \sigma_\varphi''(r,0), \quad \tau_{r\varphi}'(r,0) = \tau_{r\varphi}''(r,0), \quad u_r'(r,0) = u_r''(r,0), \quad u_\varphi'(r,0) = u_\varphi''(r,0)\end{gathered} \tag{1}$$

We differentiate the last two equations with respect to r:

$$\frac{\partial u_r'(r,0)}{\partial r} = \frac{\partial u_r''(r,0)}{\partial r}, \quad \frac{\partial u_\varphi'(r,0)}{\partial r} = \frac{\partial u_\varphi''(r,0)}{\partial r} \tag{2}$$

The elastic state of each wedge will be determined by the solutions of the following equation:

$$\beta_{22}\frac{\partial^4 f}{\partial x^4} - 2\beta_{26}\frac{\partial^4 f}{\partial x^3\,\partial y} + (2\beta_{12}+\beta_{66})\frac{\partial^4 f}{\partial x^2\,\partial y^2} - 2\beta_{16}\frac{\partial^4 f}{\partial x\,\partial y^3} + \beta_{11}\frac{\partial^4 f}{\partial y^4} = 0 \tag{3}$$

The characteristic equation for (3) has the form

$$\beta_{11}\mu^4 - \beta_{16}\mu^3 + (2\beta_{12}+\beta_{66})\mu^2 - 2\beta_{26} + \beta_{22} = 0 \tag{4}$$

Here $\beta_{km} = a_{km}$ for the generalized plane stressed state and $\beta_{km} = a_{km} - a_{k3}a_{m3}/a_{33}$ for the plane deformed state; a_{km} are the elastic constants of the material [10].

Let

$$z_j = x + \mu_j y, \quad D_j = d/dz_j, \quad a_j = \cos\varphi + \mu_j \sin\varphi, \quad b_j = \mu_j \cos\varphi - \sin\varphi$$

Then (see [10]), for the case of unequal roots μ_j of (4), we obtain

$$\begin{gathered}\sigma_r = \sum_{j=1}^{4} b_j^2 D_j^2 f_j, \quad \sigma_\varphi = \sum_{j=1}^{4} a_j^2 D_j^2 f_j, \quad \tau_{r\varphi} = -\sum_{j=1}^{4} a_j b_j D_j^2 f_j \\ \frac{\partial u_r}{\partial r} = \sum_{j=1}^{4} a_j (p_j \cos\varphi + q_j \sin\varphi) D_j^2 f_j, \quad \frac{\partial u_\varphi}{\partial r} = \sum_{j=1}^{4} a_j (q_j \cos\varphi - p_j \sin\varphi) D_j^2 f_j\end{gathered} \tag{5}$$

For the case of pairwise equal roots $\mu_j (\mu_1 = \mu_3,\ \mu_2 = \mu_4)$, we will have

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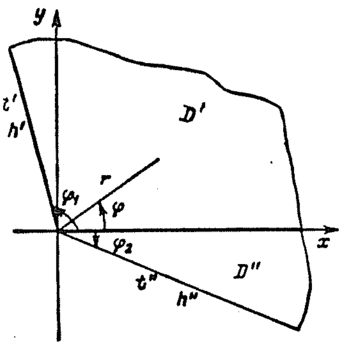
$$\sigma_r=\sum_{j=1}^{2}[b_j^2(D_j^2f_j+z_kD_j^2g_j)+2b_jb_kD_jg_j],\qquad \sigma_\varphi=\sum_{j=1}^{2}[a_j^2(D_j^2f_j+z_kD_j^2g_j)+2a_ja_kD_jg_j]$$

$$\tau_{r\varphi}=\sum_{j=1}^{2}[a_j^2b_j(D_j^2f_j+z_kD_j^2g_j)+(a_jb_k+a_kb_j)D_jg_j]$$

$$\frac{\partial u_r}{\partial r}=\sum_{j=1}^{2}[h_1a_j^2(D_j^2f_j+z_kD_j^2g_j)+(h_1+h_2)a_ja_kD_jg_j] \tag{6}$$

$$\frac{\partial u_\varphi}{\partial r}=\sum_{j=1}^{2}[h_1a_jb_j(D_j^2f_j+z_kD_j^2g_j)+(h_1a_kb_j+h_2a_jb_k)D_jg_j]$$



Here $k=3-j$, $\mu_1=\overline{\mu_2}$, $\mu_3=\overline{\mu_4}$; $f_j=f_j(z_j)$, $g_j=g_j(z_j)$ are analytic functions of their arguments, where

$$\overline{f_1(z_1)}=f_2(z_2),\quad \overline{f_3(z_3)}=f_4(z_4)$$
$$g_1(z_1)=g_2\overline{(z_2)}$$
$$p_j=\beta_{11}\mu_j+\beta_{12}-\beta_{16}\mu_j,\qquad q_j=\beta_{12}\mu_j+\beta_{22}\mu_j^{-1}-\beta_{26}$$
$$h_1=\beta_{12}-\mu_1\mu_2\beta_{11},\qquad h_2=h_1-\beta_{11}(\mu_1-\mu_2)^2$$

Consider the Mellin transformation of analytic function $v(z)=v(x+\mu y)$, where μ is some complex constant, while $\langle v(z)\rangle$ is the Mellin transform; then

$$\langle v(z)\rangle=\int_0^\infty v(x+\mu y)r^{s-1}\,dr=a^{-s}(\varphi)V(s)$$

$$a(\varphi)=\cos\varphi+\mu\sin\varphi,\qquad V(s)=\int_0^\infty v(z)z^{s-1}\,dz$$

But if v(z) is analytic in some sector $\psi_1<\varphi<\psi_2$, $0<r<\infty$; $|v(z)|=O(r^\xi)$, $r\to0$; $|v(z)|=O(r^\eta)$, $r\to\infty$; $\eta<\xi$, then V(s) exists in the strip $-\xi<\mathrm{Re}\,s<-\eta$ and is independent of the radius of integration, i.e., on the angle φ in the given sector. We define the inverse transformation in the customary fashion:

$$v(z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\langle v(z)\rangle r^{-s}\,ds$$

We will seek a solution that satisfies the following conditions:

$$|\sigma_{ij}|=O(r^{-1+\alpha}),\quad r\to0,\quad \alpha>0;\quad |\sigma_{ij}|=O(r^{-1-\beta}),\quad r\to\infty,\quad \beta\geqslant0 \tag{7}$$

Then a Mellin transformation exists in the strip $1-\alpha<\mathrm{Re}\,s<1+\beta$. We apply the Mellin transformation to expressions (5) and (6). For unequal roots μ_j we obtain

$$\langle\sigma_r\rangle=\sum_{j=1}^{4}a_j^{-s}b_j^2F_j,\qquad \langle\sigma_\varphi\rangle=\sum_{j=1}^{4}a_j^{2-s}F_j,\qquad \langle\tau_{r\varphi}\rangle=-\sum_{j=1}^{4}a_j^{1-s}b_jF_j$$

$$\left\langle\frac{\partial u_r}{\partial r}\right\rangle=\sum_{j=1}^{4}a_j^{-s}(p_j\cos\varphi+q_j\sin\varphi)F_j,\quad \left\langle\frac{\partial u_\varphi}{\partial r}\right\rangle=\sum_{j=1}^{4}a_j^{-s}(q_j\cos\varphi-p_j\sin\varphi)F_j \tag{8}$$

For the case of pairwise equal roots μ_j we will have

$$\langle\sigma_r\rangle=\sum_{j=1}^{2}a_j^{-s}b_j^2[F_j+(2b_j^{-1}b_k-sa_j^{-1}a_k)G_j],\qquad \langle\sigma_\varphi\rangle=\sum_{j=1}^{2}a_j^{2-s}[F_j+(2-s)a_j^{-1}a_kG_j]$$

$$\langle\tau_{r\varphi}\rangle=-\sum_{j=1}^{2}a_j^{1-s}b_j\{F_j+[(1-s)a_j^{-1}a_k+b_j^{-1}b_k]G_j\} \tag{9}$$

$$\left\langle\frac{\partial u_r}{\partial r}\right\rangle=\sum_{j=1}^{2}a_j^{2-s}\{h_1F_j+a_j^{-1}a_k[(1-s)h_1+h_2]G_j\}$$

$$\left\langle \frac{\partial u_\varphi}{\partial r} \right\rangle = \sum_{j=1}^{2} a_j^{1-s} b_j \{h_1 F_j + [(1-s) h_1 a_j^{-1} a_k + h_2 b_j^{-1} b_k] G_j\} \tag{9}$$

$$F_j = \int_0^\infty D_j^2 f_j(z_j) z_j^{s-1}\, dz_j, \quad G_j = \int_0^\infty D_j g_j(z_j) z_j^{s-1}\, dz_j$$

Let us consider three possible cases: both wedges consist of materials without equal roots μ_j; both wedges consist of materials with pairwise equal roots μ_j; one wedge (D') consists of a material with different μ_j, the other of a material with pairwise equal μ_j.

Applying the Mellin transformation to (1) and (2) and taking account of (8) and (9), for all three cases we obtain the system (the superscript corresponding to the number of the case)

$$\sum_{j=1}^{8} B_{ij}^k U_j^k = T_i \qquad (i=1,\ldots,8) \tag{10}$$

$$U_j^1 = [F_1', F_2', F_3', F_4', F_1'', F_2'', F_3'', F_4''],$$
$$U_j^2 = [F_1', F_2', G_1', G_2', F_1'', F_2'', G_1'', G_2'']$$
$$U_j^3 = [F_1', F_2', F_3', F_4', F_1'', F_2'', G_1'', G_2''],$$
$$T_j = [\langle n'\rangle, \langle -t'\rangle, \langle n''\rangle, \langle -t''\rangle, 0, 0, 0, 0]$$

The matrix of coefficients B_{ij}^1 is

$$\left\| \begin{matrix}
a_1^{2-s} & a_2^{2-s} & a_3^{2-s} & a_4^{2-s} & 0 & 0 & 0 & 0 \\
a_1^{1-s} b_1 & a_2^{1-s} b_2 & a_3^{1-s} b_3 & a_4^{1-s} b_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_5^{2-s} & a_6^{2-s} & a_7^{2-s} & a_8^{2-s} \\
0 & 0 & 0 & 0 & a_5^{1-s} b_5 & a_6^{1-s} b_6 & a_7^{1-s} b_7 & a_8^{1-s} b_8 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
\mu_1 & \mu_2 & \mu_3 & \mu_4 & -\mu_5 & -\mu_6 & -\mu_7 & -\mu_8 \\
p_1 & p_2 & p_3 & p_4 & -p_5 & -p_6 & -p_7 & -p_8 \\
q_1 & q_2 & q_3 & q_4 & -q_5 & -q_6 & -q_7 & -q_8
\end{matrix} \right\|$$

In view of their cumbersomeness, we will not write out matrices B_{ij}^2 and B_{ij}^3. Here and in what follows it is assumed that in region D' the indexes for μ_i, a_j, b_j vary from 1 to 4, while in D" they extend from 5 to 8; it is also assumed (only here) that $a_j = a_j(\varphi_1)$, $b_j = b_j(\varphi_1)$, $j = 1, \ldots, 4$; $a_j = a_j(\varphi_2)$, $b_j = b_j(\varphi_2)$, $j = 5, \ldots, 8$.

Assume that $\det(B_{ij}) = \Delta$, A_{ij} is the algebraic complement of B_{ij}, $a_j = a_j(\varphi)$, $b_j = b_j(\varphi)$, $k = 3 - j$. Determining F_j and G_j from (10) and substituting them into (8) and (9), we obtain expressions for the transforms of the stress components. In the first case, in wedge D',

$$\langle \sigma_r' \rangle = \sum_{i=1}^{4} T_i \left[\sum_{j=1}^{4} a_j^{-s} b_j^2 A_{ij} \right] \Delta^{-1}, \quad \langle \sigma_\varphi' \rangle = \sum_{i=1}^{4} T_i \left[\sum_{j=1}^{4} a_j^{2-s} A_{ij} \right] \Delta^{-1} \tag{11}$$

$$\langle \tau_{r\varphi}' \rangle = -\sum_{i=1}^{4} T_i \left[\sum_{j=1}^{4} a_j^{1-s} b_j A_{ij} \right] \Delta^{-1}$$

In wedge D" the transforms of the stress $\langle \sigma_{ij}'' \rangle$ can be similarly expressed. In the second case, in D',

$$\langle \sigma_r' \rangle = \sum_{i=1}^{4} T_i \left[\sum_{j=1}^{q} a_j^{-s} b_j^2 \{A_{ij} + (2 b_j^{-1} b_k - s a_j^{-1} a_k) A_{ij+2}\} \right] \Delta^{-1}$$

$$\langle \sigma_\varphi' \rangle = \sum_{i=1}^{4} T_i \left[\sum_{j=1}^{q} a_j^{2-s} \{A_{ij} + (2-s) a_j^{-1} a_k A_{ij+2}\} \right] \Delta^{-1} \tag{12}$$

$$\langle \tau_{r\varphi}' \rangle = -\sum_{i=1}^{4} T_i \left[\sum_{j=1}^{2} a_j^{1-s} b_j \{A_{ij} + ((1-s) a_j^{-1} a_k + b_j^{-1} b_k) A_{ij+2}\} \right] \Delta^{-1}$$

The transforms of the stress $\langle \sigma_{ij}'' \rangle$ in D" are similarly expressed. In the third case, in D',

$$\langle\sigma_r'\rangle=\sum_{i=1}^{4}T_i\left[\sum_{j=1}^{4}a_j^{-s}b_j^2A_{ij}\right]\Delta^{-1},\quad \langle\sigma_\varphi'\rangle=\sum_{i=1}^{4}T_i\left[\sum_{j=1}^{4}a_j^{2-s}A_{ij}\right]\Delta^{-1} \tag{13}$$

$$\langle\tau_{r\varphi}'\rangle=-\sum_{i=1}^{4}T_i\left[\sum_{j=1}^{4}a_j^{1-s}b_jA_{ij}\right]\Delta^{-1}$$

while in D'' $(k=11-j)$:

$$\langle\sigma_r''\rangle=\sum_{i=1}^{4}T_i\left[\sum_{j=5}^{8}a_j^{-s}b_j^2\{A_{ij}+(2b_j^{-1}b_k-sa_j^{-1}a_k)A_{ij+2}\}\right]\Delta^{-1}$$

$$\langle\sigma_\varphi''\rangle=\sum_{i=1}^{4}T_i\left[\sum_{j=5}^{8}a_j^{2-s}\{A_{ij}+(2-s)a_j^{-1}a_kA_{ij+2}\}\right]\Delta^{-1} \tag{14}$$

$$\langle\tau_{r\varphi}''\rangle=-\sum_{i=1}^{4}T_i\left[\sum_{j=5}^{8}a_j^{1-s}b_j\{A_{ij}+((1-s)a_j^{-1}a_k+b_j^{-1}b_k)A_{ij+2}\}\right]\Delta^{-1}$$

We find the stress field from the transforms $\langle\sigma_{ij}\rangle$, which are defined in the same region as the Mellin transformations of n(r) and r(t). We will consider n and t that satisfy the following conditions (we will use the example of t(r)): there exist $\delta_0>0, \delta_\infty>0$, such that

$$t(r)=t^\circ\ln r+\sum_{n=0}^{\infty}t_{n0}r^{\varkappa_n}\ \text{for}\ \ r<\delta_0;\quad t(r)=\sum_{n=0}^{\infty}t_{n\infty}r^{\gamma_n}\ \text{for}\ \ r>\delta_\infty;\quad t(r)\in L(\delta_0,\delta_\infty)$$

where $\{\varkappa_n\}$ is an increasing sequence, while $\{\gamma_n\}$ is a decreasing one ($\varkappa_n, \gamma_n$ are not necessarily integers).

If $\varkappa_0>\gamma_0$, then the Mellin transform of this function exists in a strip $-\varkappa_0<\operatorname{Re}s<-\gamma_0$ (for (7) to be satisfied, we will assume that $\varkappa_0>-1\geqslant\gamma_0$). It can be analytically continued onto the entire complex plane as follows:

$$\langle t\rangle=t^\circ\delta_0^s\left(\frac{\ln\delta_0}{s}-\frac{1}{s^2}\right)+\sum_{n=0}^{\infty}\frac{t_{n0}}{\varkappa_n+s}\delta_0^{\varkappa_n+s}+\int_{\delta_0}^{\delta_\infty}t(r)r^{s-1}\,dr+\sum_{n=0}^{\infty}\frac{t_{n\infty}}{\gamma_n+s}\delta_\infty^{\gamma_n+s} \tag{15}$$

If for T_i we substitute the transforms of the stresses on the boundary, continued in this fashion, we obtain the analytic continuation of $\langle\sigma_{ij}\rangle$, which is analytic everywhere except for points of poles of T_i and poles of cofactors for T_i in (11)-(14). The path of integration for the inverse transformation should like in the strip

$$x_1<c<x_2,\quad x_1=\max_{\operatorname{Re}s_k<1}(\operatorname{Re}s_k),\quad x_2=\min_{\operatorname{Re}s_k\geqslant1}(\operatorname{Re}s_k)$$

where s_k are the points of poles of $\langle\sigma_{ij}\rangle$. Otherwise condition (7) will not be satisfied.

The explicit form of the functions $\sigma_{ij}(r,\varphi)$ can be obtained by numerical integration, but for an asymptotic analysis of the stress field as $r\to0$ it is more convenient to employ residue theory. This requires that the $\langle\sigma_{ij}\rangle$ tend to zero uniformly with respect to arg(s) on some system of semicircles ρ_k such that $|s(\rho_k)|\to\infty, k\to\infty, \rho_k$ the ρ_k lying to the left of the path of integration c.

Since the multipliers for T_i in (11)-(14) represent a sum of exponents divided by another sum of exponents (in the second and third cases the exponents are multiplied by polynomials of s), they increase not more rapidly than some exponent when $s\to\infty$ along a radius $\arg(s)=\text{const}$. On radii that coincide with the imaginary axis, the numerator of this fraction increases not more rapidly than the denominator. But the T_i have the form (15), and since we are considering the stressed state only for small r, by varying the scale we can make δ_0 and δ_∞ large enough that the δ_0, δ_∞ decrease uniformly. Thus,

$$\sigma_{ij}=\sum_{(s_k)}\operatorname{res}_{s_k}(\langle\sigma_{ij}\rangle r^{-s})$$

where the s_k are poles of $\langle\sigma_{ij}\rangle$, taken in order to decreasing modulus, where $\operatorname{Re}s_k<1$.

If the $\langle\sigma_{ij}\rangle$ have first-order poles, then the corresponding terms will have the form $\operatorname{res}(\langle\sigma_{ij}\rangle) r^{-s_k}$; if the order of the pole is greater than the first, terms containing different powers of $\ln r$, multiplied by r^{-s_k} are added. Consequently, singular terms appear in the expressions for σ_{ij} if the $\langle\sigma_{ij}\rangle$ have poles in the strip $0<\operatorname{Re} s<1$. The maximum degree of singularity will be equal to the real part of the root that is closest to the straight line $\operatorname{Re} s=1$.

If the stresses on the boundary $t(r), n(r)\to 0$ as $r\to 0$, then the degree of singularity depends only on the zeros of $\Delta(s)$ in the strip $0<\operatorname{Re} s<1$ and the behavior of the terms in brackets in (11)-(14) at these points. Since the $\Delta(s)$ constitute a sum of exponents with different indexes multiplied by polynomials of s, in the strip in question the roots $\Delta(s)$ exist only in some rectangle $|\operatorname{Im} s|<\Gamma$, and the boundary Γ can be readily evaluated. This markedly simplifies the procedure for finding the roots by numerical methods.

It can be seen from (8)-(14) that in all three cases the σ_{ij} depend on a number of combinations of elastic constants β_{km} that is 2 less than the over-all number of these constants for two different bodies in the plane problem. For instance, for the first case, if the wedges consist of nonorthotropic materials, instead of 12 constants $\beta'_{km}, \beta''_{km}$ there are only 10 involved. These can be taken to be

$$\mu_1,\ \mu_3,\ \mu_5,\ \mu_7,\ \frac{2\beta_{12}''+\beta_{66}''}{2\beta_{12}'+\beta_{66}'},\ \frac{\beta_{12}'-\beta_{12}''}{2\beta_{12}'+\beta_{66}'}$$

where the μ_j are complex numbers. If both materials are orthotropic, i.e., $\beta_{26}=\beta_{16}=0$, then for $(2\beta_{12}+\beta_{66})^2>4\beta_{11}\beta_{22}$ we have $\arg(\mu_1)=\arg(\mu_3)=\arg(\mu_5)=\arg(\mu_7)=\pi/2$ and instead of 8 constants there are 6; for $(2\beta_{12}+\beta_{66})^2<4\beta_{11}\beta_{22}$

$$\left|\frac{\mu_3}{\mu_1}\right| = \left|\frac{\mu_7}{\mu_5}\right| = 1, \quad \arg(\mu_1)+\arg(\mu_3)=\arg(\mu_5)+\arg(\mu_7)=\frac{\pi}{2}$$

we again have 6 unknown constants, and so forth.

Similar results were obtained by Dundurs [5,6] for isotropic composite bodies. Unfortunately, the large number of constants that remain does not permit us to employ a tractible representation of the way in which they depend on the degree of singularity, as proposed in [5,6].

The method used in this paper can also be employed in the problem with specified displacements on the boundary and in the mixed problem.

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REFERENCES

1. D. B. Bogy, "Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading," Trans. ASME, Series E, J. Appl. Mech., vol. 35, no. 3, 1968.

2. D. B. Bogy, "On the problem of edge-bonded quarter-planes loaded at the boundary," Internat. J. Solids and Structures, vol. 6, no. 9, 1970.

3. D. B. Bogy, "Two edge-bonded elastic wedges of different materials and wedge angles under surface traction," Trans. ASME, Series E, J. Appl. Mech., vol. 38, no. 2, 1971.

4. V. L. Hein and F. Erdogan, "Stress singularities in a two-material wedge," Internat. J. Fracture Mech., vol. 7, no. 3, 1971.

5. J. Dundurs, "Effect of elastic constants on stress in a composite under plane deformation," J. Compos. Mater., vol. 1, no. 3, 1967.

6. J. Dundurs, Discussion on the paper by D. B. Bogy, "Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading," Trans. ASME, Series E, J. Appl. Mech., vol. 36, no. 3, 1969.

7. J. P. Benthem, "On the stress distribution in anisotropic infinite wedges," Quart. Appl. Math., vol. 21, no. 3, 1963.

8. D. B. Bogy, "The plane solution for anisotropic elastic wedges under normal and shear loading," Trans. ASME, Series E, J. Appl. Mech., vol. 39, no. 4, 1972.

9. R. K. Aleksanyan, "One class of solutions of the equations of the plane problem of elasticity for an anisotropic solid," Dokl. AN ArmSSR, vol. 61, no. 4, 1975.

10. S. G. Lekhnitskii, Theory of Elasticity of Anisotropic Solids [in Russian], Gostekhizdat, Moscow-Leningrad, 1950.

24 October 1977 Moscow