Spatially-Periodic Solutions for Evolution Anisotropic Variable-Coefficient Navier-Stokes Equations: I. Existence

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Abstract

We consider evolution (non-stationary) space-periodic solutions to the n-dimensional non-linear Navier-Stokes equations of anisotropic fluids with the viscosity coefficient tensor variable in space and time and satisfying the relaxed ellipticity condition. Employing the Galerkin algorithm with the basis constituted by the eigenfunctions of the periodic Bessel-potential operator, we prove the existence of a global weak solution.

Keywords. Partial differential equations; Evolution Navier-Stokes equations; Anisotropic Navier-Stokes; Spatially periodic solutions; Variable coefficients; Relaxed ellipticity condition.
MSC classes: 35A1, 35B10, 35K45, 35Q30, 76D05

1 Introduction

Analysis of Stokes and Navier-Stokes equations is an established and active field of research in the applied mathematical analysis, see, e.g., [5, 7, 14, 23, 26, 28, 29, 30] and references therein. In [9, 10, 11, 12, 19, 20] this field has been extended to the transmission and boundary-value problems for stationary Stokes and Navier-Stokes equations of anisotropic fluids, particularly, with relaxed ellipticity condition on the viscosity tensor.

In this paper, we consider evolution (non-stationary) space-periodic solutions in $\mathbb{R}^n$, $n \geq 2$, to the Navier-Stokes equations of anisotropic fluids with the viscosity coefficient tensor variable in space coordinates and time and satisfying the relaxed ellipticity condition. By the Galerkin algorithm with the basis constituted by the eigenfunctions of the periodic Bessel-potential operator, the solution existence is analysed in the spaces of Banach-valued functions mapping a finite time interval to periodic Sobolev (Bessel-potential) spaces on n-dimensional flat torus.

Anisotropic Stokes and Navier-Stokes PDE systems

Let $n \geq 2$ be an integer, $\mathbf{x} \in \mathbb{R}^n$ denote the space coordinate vector, and $t \in \mathbb{R}$ be time. Let $\boldsymbol{\mathfrak{L}}$ denote the second-order differential operator represented in the component-wise divergence form as

$$(\boldsymbol{\mathfrak{L}}\mathbf{u})_k := \partial_\alpha\big(a_{kj}^{\alpha\beta} E_{j\beta}(\mathbf{u})\big), \quad k = 1, \dots, n, \tag{1.1}$$

where $\mathbf{u} = (u_1, \dots, u_n)^\top$, $E_{j\beta}(\mathbf{u}) := \frac{1}{2}(\partial_j u_\beta + \partial_\beta u_j)$ are the entries of the symmetric part, $\mathbb{E}(\mathbf{u})$, of the gradient, $\nabla\mathbf{u}$, in space coordinates, and $a_{kj}^{\alpha\beta}(\mathbf{x}, t)$ are variable components of the tensor viscosity coefficient,

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cf. [6], $\mathbb{A}(\mathbf{x},t) = \left\{a_{kj}^{\alpha\beta}(\mathbf{x},t)\right\}_{1\le i,j,\alpha,\beta\le n}$, depending on the space coordinate vector $\mathbf{x}$ and time t. We also denoted $\partial_j = \dfrac{\partial}{\partial x_j}$, $\partial_t = \dfrac{\partial}{\partial t}$. Here and further on, the Einstein convention on summation in repeated indices from 1 to n is used unless stated otherwise.

The following symmetry conditions are assumed (see [21, (3.1),(3.3)]),

$$a_{kj}^{\alpha\beta}(\mathbf{x},t) = a_{\alpha j}^{k\beta}(\mathbf{x},t) = a_{k\beta}^{\alpha j}(\mathbf{x},t). \tag{1.2}$$

In addition, we require that tensor $\mathbb{A}$ satisfies the relaxed ellipticity condition in terms of all *symmetric* matrices in $\mathbb{R}^{n\times n}$ with *zero matrix trace*, see [10], [11]. Thus, we assume that there exists a constant $C_{\mathbb{A}} > 0$ such that,

$$C_{\mathbb{A}} a_{kj}^{\alpha\beta}(\mathbf{x},t)\zeta_{k\alpha}\zeta_{j\beta} \ge |\boldsymbol{\zeta}|^2, \quad \text{for a.e. } \mathbf{x}, t, \tag{1.3}$$
$$\forall\ \boldsymbol{\zeta} = \{\zeta_{k\alpha}\}_{k,\alpha=1,\dots,n} \in \mathbb{R}^{n\times n} \text{ such that } \boldsymbol{\zeta} = \boldsymbol{\zeta}^\top \text{ and } \sum_{k=1}^n \zeta_{kk} = 0,$$

where $|\boldsymbol{\zeta}| = |\boldsymbol{\zeta}|_F := (\zeta_{k\alpha}\zeta_{k\alpha})^{1/2}$ is the Frobenius matrix norm and the superscript $^\top$ denotes the transpose of a matrix. Note that in the more common, strong ellipticity condition, inequality (1.3) should be satisfied for all matrices (not only symmetric with zero trace), which makes it much more restrictive.

We assume that $a_{ij}^{\alpha\beta} \in L_\infty(\mathbb{R}^n \times [0,T])$, where $[0,T]$ is some finite time interval, and the tensor $\mathbb{A}$ is endowed with the norm

$$\|\mathbb{A}\| := \|\mathbb{A}\|_{L_\infty(\mathbb{R}^n\times[0,T]),F} := \left|\left\{\|a_{ij}^{\alpha\beta}\|_{L_\infty(\mathbb{T}\times[0,T])}\right\}_{\alpha,\beta,i,j=1}^n\right|_F < \infty, \tag{1.4}$$

where $\left|\left\{b_{ij}^{\alpha\beta}\right\}_{\alpha,\beta,i,j=1}^n\right|_F := \left(b_{ij}^{\alpha\beta} b_{ij}^{\alpha\beta}\right)^{1/2}$ is the Frobenius norm of a 4-th order tensor.

Symmetry conditions (1.2) lead to the following equivalent form of the operator $\boldsymbol{\mathfrak{L}}$

$$(\boldsymbol{\mathfrak{L}}\mathbf{u})_k = \partial_\alpha\big(a_{kj}^{\alpha\beta}\partial_\beta u_j\big), \quad k = 1,\dots,n. \tag{1.5}$$

Let $\mathbf{u}(\mathbf{x},t)$ be an unknown vector velocity field, $p(\mathbf{x},t)$ be an unknown (scalar) pressure field, and $\mathbf{f}(\mathbf{x},t)$ be a given vector field $\mathbb{R}^n$, where $t \in \mathbb{R}$ is the time variable. Then the linear PDE system

$$\partial_t\mathbf{u} - \boldsymbol{\mathfrak{L}}\mathbf{u} + \nabla p = \mathbf{f},\ \operatorname{div}\mathbf{u} = 0,$$

determines the *anisotropic evolution incompressible Stokes system*.

The nonlinear system

$$\partial_t\mathbf{u} - \boldsymbol{\mathfrak{L}}\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0$$

is the *evolution anisotropic incompressible Navier-Stokes system*, where we use the notation $(\mathbf{u}\cdot\nabla) := u_j\partial_j$.

In the *isotropic case*, the tensor $\mathbb{A}$ reduces to

$$a_{kj}^{\alpha\beta}(\mathbf{x},t) = \lambda(\mathbf{x},t)\delta_{k\alpha}\delta_{j\beta} + \mu(\mathbf{x},t)\left(\delta_{\alpha j}\delta_{\beta k} + \delta_{\alpha\beta}\delta_{kj}\right),\ 1 \le k,j,\alpha,\beta \le n,$$

where $\lambda, \mu \in L_\infty(\mathbb{R}^n \times [0,T])$, and $c_\mu^{-1} \le \mu(\mathbf{x},t) \le c_\mu$ for a.e. $\mathbf{x}$ and t, with some constant $c_\mu > 0$ (cf., e.g., Appendix III, Part I, Section 1 in [30]). Then it is immediate that condition (1.3) is fulfilled with $C_{\mathbb{A}} = c_\mu/2$ and thus our results apply also to the Stokes and Navier-Stokes systems in the *isotropic case*. Moreover, (1.1) becomes

$$\boldsymbol{\mathfrak{L}}\mathbf{u} = (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \mu\Delta\mathbf{u} + (\nabla\lambda)\operatorname{div}\mathbf{u} + 2(\nabla\mu)\cdot\mathbb{E}(\mathbf{u}).$$

Assuming $\lambda = 0$ and $\mu = 1$ we arrive at the classical mathematical formulations of isotropic, constant-coefficient Stokes and Navier-Stokes systems in the familiar form

$$\partial_t\mathbf{u} - \Delta\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0.$$

2 Periodic function spaces

Let us introduce some function spaces on torus, i.e., periodic function spaces (see, e.g., [1, p.26], [2], [17], [25, Chapter 3], [23, Section 1.7.1] [29, Chapter 2], for more details).

Let $n \geq 1$ be an integer and $\mathbb{T}$ be the n-dimensional flat torus that can be parametrized as the semi-open cube $\mathbb{T} = \mathbb{T}^n = [0,1)^n \subset \mathbb{R}^n$, cf. [31, p. 312]. In what follows, $\mathcal{D}(\mathbb{T}) = \mathcal{C}^\infty(\mathbb{T})$ denotes the (test) space of infinitely smooth real or complex functions on the torus. As usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ the set of natural numbers augmented by 0, and $\mathbb{Z}$ the set of integers.

Let $\boldsymbol{\xi} \in \mathbb{Z}^n$ denote the n-dimensional vector with integer components. We will further need also the set

$$\dot{\mathbb{Z}}^n := \mathbb{Z}^n \setminus \{\mathbf{0}\}.$$

Extending the torus parametrisation to $\mathbb{R}^n$, it is often useful to identify $\mathbb{T}$ with the quotient space $\mathbb{R}^n \setminus \mathbb{Z}^n$. Then the space of functions $\mathcal{C}^\infty(\mathbb{T})$ on the torus can be identified with the space of $\mathbb{T}$-periodic (1-periodic) functions $\mathcal{C}^\infty_\# = \mathcal{C}^\infty_\#(\mathbb{R}^n)$ that consists of functions $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\phi(\mathbf{x} + \boldsymbol{\xi}) = \phi(\mathbf{x}) \quad \forall\, \boldsymbol{\xi} \in \mathbb{Z}^n. \tag{2.1}$$

Similarly, the Lebesgue space on the torus $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, can be identified with the periodic Lebesgue space $L_{p\#} = L_{p\#}(\mathbb{R}^n)$ that consists of functions $\phi \in L_{p,\mathrm{loc}}(\mathbb{R}^n)$, which satisfy the periodicity condition (2.1) for a.e. $\mathbf{x}$.

The space dual to $\mathcal{D}(\mathbb{T})$, i.e., the space of linear bounded functionals on $\mathcal{D}(\mathbb{T})$, called the space of torus distributions, is denoted by $\mathcal{D}'(\mathbb{T})$ and can be identified with the space of periodic distributions $\mathcal{D}'_\#$ acting on $\mathcal{C}^\infty_\#$.

The toroidal/periodic Fourier transform mapping a function $g \in \mathcal{C}^\infty_\#$ to a set of its Fourier coefficients $\hat{g}$ is defined as (see, e.g., [25, Definition 3.1.8])

$$\hat{g}(\boldsymbol{\xi}) = [\mathcal{F}_\mathbb{T} g](\boldsymbol{\xi}) := \int_\mathbb{T} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} g(\mathbf{x}) d\mathbf{x}, \quad \boldsymbol{\xi} \in \mathbb{Z}^n,$$

and can be generalised to the Fourier transform acting on a distribution $g \in \mathcal{D}'_\#$.

For any $\boldsymbol{\xi} \in \mathbb{Z}^n$, let $|\boldsymbol{\xi}| := (\sum_{j=1}^n \xi_j^2)^{1/2}$ be the Euclidean norm in $\mathbb{Z}^n$ and let us denote

$$\varrho(\boldsymbol{\xi}) := 2\pi(1 + |\boldsymbol{\xi}|^2)^{1/2}.$$

Evidently,

$$\frac{1}{2}\varrho(\boldsymbol{\xi})^2 \leq |2\pi\boldsymbol{\xi}|^2 \leq \varrho(\boldsymbol{\xi})^2 \quad \forall\, \boldsymbol{\xi} \in \dot{\mathbb{Z}}^n. \tag{2.2}$$

Similar to [25, Definition 3.2.2], for $s \in \mathbb{R}$ we define the *periodic/toroidal Sobolev (Bessel-potential) spaces* $H^s_\# := H^s_\#(\mathbb{R}^n) := H^s(\mathbb{T})$ that consist of the torus distributions $g \in \mathcal{D}'(\mathbb{T})$, for which the norm

$$\|g\|_{H^s_\#} := \|\varrho^s \widehat{g}\|_{\boldsymbol{\ell}_2(\mathbb{Z}^n)} := \left(\sum_{\boldsymbol{\xi} \in \mathbb{Z}^n} \varrho(\boldsymbol{\xi})^{2s} |\widehat{g}(\boldsymbol{\xi})|^2\right)^{1/2} \tag{2.3}$$

is finite, i.e., the series in (2.3) converges. Here $\|\cdot\|_{\boldsymbol{\ell}_2(\mathbb{Z}^n)}$ is the standard norm in the space of square summable sequences with indices in $\mathbb{Z}^n$. By [25, Proposition 3.2.6], $H^s_\#$ is the Hilbert space with the inner (scalar) product in $H^s_\#$ defined as

$$(g, f)_{H^s_\#} := \sum_{\boldsymbol{\xi} \in \mathbb{Z}^n} \varrho(\boldsymbol{\xi})^{2s} \hat{g}(\boldsymbol{\xi}) \overline{\hat{f}(\boldsymbol{\xi})}, \quad \forall\, g, f \in H^s_\#,\ s \in \mathbb{R}, \tag{2.4}$$

where the bar denotes complex conjugate. Evidently, $H^0_\# = L_{2\#}$.

The dual product between $g \in H^s_\#$ and $f \in H^{-s}_\#$, $s \in \mathbb{R}$, is defined (cf. [25, Definition 3.2.8]) as

$$\langle g, f\rangle_{\mathbb{T}} := \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n} \hat{g}(\boldsymbol{\xi})\hat{f}(-\boldsymbol{\xi}). \tag{2.5}$$

If $s = 0$, i.e., $g, f \in L_{2\#}$, then (2.4) and (2.5) reduce to

$$\langle g, f\rangle_{\mathbb{T}} = \int_{\mathbb{T}} g(\mathbf{x})f(\mathbf{x})d\mathbf{x} = (g, \bar{f})_{L_{2\#}}.$$

For real function $g, f \in L_{2\#}$ we, of course, have $\langle g, f\rangle_{\mathbb{T}} = (g, f)_{L_{2\#}}$.

For any $s \in \mathbb{R}$, the space $H^{-s}_\#$ is Banach adjoint (dual) to $H^s_\#$, i.e., $H^{-s}_\# = (H^s_\#)^*$. Similar to, e.g., [18, p.76] one can show that

$$\|g\|_{H^s_\#} = \sup_{f\in H^{-s}_\#, f\neq 0} \frac{|\langle g, f\rangle_{\mathbb{T}}|}{\|f\|_{H^{-s}_\#}}.$$

For $g \in H^s_\#$, $s \in \mathbb{R}$, and $m \in \mathbb{N}_0$, let us consider the partial sums

$$g_m(\mathbf{x}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n, |\boldsymbol{\xi}|\leq m} \hat{g}(\boldsymbol{\xi})e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}.$$

Evidently, $g_m \in \mathcal{C}^\infty_\#$, $\hat{g}_m(\boldsymbol{\xi}) = \hat{g}(\boldsymbol{\xi})$ if $|\boldsymbol{\xi}| \leq m$ and $\hat{g}_m(\boldsymbol{\xi}) = 0$ if $|\boldsymbol{\xi}| > m$. This implies that $\|g - g_m\|_{H^s_\#} \to 0$ as $m \to \infty$ and hence we can write

$$g(\mathbf{x}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n} \hat{g}(\boldsymbol{\xi})e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}, \tag{2.6}$$

where the Fourier series converges in the sense of norm (2.3). Moreover, since g is an arbitrary distribution from $H^s_\#$, this also implies that the space $\mathcal{C}^\infty_\#$ is dense in $H^s_\#$ for any $s \in \mathbb{R}$ (cf. [25, Exercise 3.2.9]).

There holds the compact embedding $H^t_\# \hookrightarrow H^s_\#$ if $t > s$, embeddings $H^s_\# \subset \mathcal{C}^m_\#$ if $m \in \mathbb{N}_0$, $s > m + \frac{n}{2}$, and moreover, $\bigcap_{s\in\mathbb{R}} H^s_\# = \mathcal{C}^\infty_\#$ (cf. [25, Exercises 3.2.10, 3.2.10, and Corollary 3.2.11]).

Note that for each s, the periodic norm (2.3) is equivalent to the periodic norm that we used in [19, 20], which is obtained from (2.3) by replacing there $\varrho(\boldsymbol{\xi}) = 2\pi(1 + |\boldsymbol{\xi}|^2)^{1/2}$ with $\rho(\boldsymbol{\xi}) = (1 + |\boldsymbol{\xi}|^2)^{1/2}$. We employ here the norm (2.3) to simplify some norm estimates further in the paper. Note also that the periodic norms on $H^s_\#$ are equivalent to the corresponding standard (non-periodic) Bessel potential norms on $\mathbb{T}$ as a cubic domain, see, e.g., [2, Section 13.8.1].

By (2.3), $\|g\|^2_{H^s_\#} = |\widehat{g}(\mathbf{0})|^2 + |g|^2_{H^s_\#}$, where

$$|g|_{H^s_\#} := \|\varrho^s\widehat{g}\|_{\boldsymbol{\ell}_2(\dot{\mathbb{Z}}^n)} := \left(\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \varrho(\boldsymbol{\xi})^{2s}|\widehat{g}(\boldsymbol{\xi})|^2\right)^{1/2}$$

is the seminorm in $H^s_\#$.

For any $s \in \mathbb{R}$, let us also introduce the space

$$\dot{H}^s_\# := \{g \in H^s_\# : \langle g, 1\rangle_{\mathbb{T}} = 0\}.$$

The definition implies that if $g \in \dot{H}^s_\#$, then $\widehat{g}(\mathbf{0}) = 0$ and

$$\|g\|_{\dot{H}^s_\#} = \|g\|_{H^s_\#} = |g|_{H^s_\#} = \|\varrho^s\widehat{g}\|_{\boldsymbol{\ell}_2(\dot{\mathbb{Z}}^n)}\,. \tag{2.7}$$

The space $\dot{H}^s_\#$ is the Hilbert space with inner product inherited from (2.4), that is,

$$(g_1, g_2)_{\dot{H}^s_\#} := \sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \varrho(\boldsymbol{\xi})^{2s}\hat{g}_1(\boldsymbol{\xi})\overline{\hat{g}_2(\boldsymbol{\xi})}, \quad \forall\, g_1, g_2 \in \dot{H}^s_\#,\ s \in \mathbb{R}. \tag{2.8}$$

Due to the Riesz representation theorem, the dual product between $g_1 \in \dot H^s_\#$ and $f_2 \in (\dot H^s_\#)^*$, $s \in \mathbb{R}$, can be represented as

$$\langle g_1, f_2\rangle_{\mathbb{T}} := \sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \hat g_1(\boldsymbol{\xi})\hat f_2(-\boldsymbol{\xi}) = (g_1, g_2)_{\dot H^s_\#} = \sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \varrho(\boldsymbol{\xi})^{2s}\hat g_1(\boldsymbol{\xi})\overline{\hat g_2(\boldsymbol{\xi})}.$$

where

$$\hat f_2(\boldsymbol{\xi}) = \varrho(\boldsymbol{\xi})^{2s}\overline{\hat g_2(-\boldsymbol{\xi})}, \quad \hat g_2(\boldsymbol{\xi}) = \varrho(\boldsymbol{\xi})^{-2s}\overline{\hat f_2(-\boldsymbol{\xi})}, \quad \boldsymbol{\xi}\in\dot{\mathbb{Z}}^n$$

for some $g_2 \in \dot H^s_\#$. This implies that

$$f_2(\mathbf{x}) = \left(\Lambda^{2s}_\#\,\overline{g_2}\right)(\mathbf{x}), \tag{2.9}$$

where $\Lambda^r_\# : H^s_\# \to H^{s-r}_\#$ is the continuous periodic Bessel-potential operator of the order $r\in\mathbb{R}$ defined as

$$\left(\Lambda^r_\#\, g\right)(\mathbf{x}) := \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n} \varrho(\boldsymbol{\xi})^r\hat g(\boldsymbol{\xi})e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}} \quad \forall\, g\in H^s_\#,\ s\in\mathbb{R}, \tag{2.10}$$

see, e.g., [2, Section 13.8.1]. Note that (2.10) implies

$$\left(\Lambda^2_\#\, g\right)(\mathbf{x}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n}(2\pi)^2(1+|\boldsymbol{\xi}|^2)\hat g(\boldsymbol{\xi})e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}} = (2\pi)^2 g(\mathbf{x}) - \Delta^2 g(\mathbf{x}) \quad \forall\, g\in H^s_\#,\ s\in\mathbb{R}.$$

If $\hat g(\mathbf{0}) = 0$ then (2.10) implies that $\widehat{\Lambda^r_\#\, g}(\mathbf{0}) = 0$, and thus the operator

$$\Lambda^r_\# : \dot H^s_\# \to \dot H^{s-r}_\# \tag{2.11}$$

is continuous as well. Hence by (2.9) we conclude that $(\dot H^s_\#)^* = \dot H^{-s}_\#$.

Denoting $\dot{\mathcal{C}}^\infty_\# := \{g\in\mathcal{C}^\infty_\# : \langle g, 1\rangle_{\mathbb{T}} = 0\}$, then $\bigcap_{s\in\mathbb{R}}\dot H^s_\# = \dot{\mathcal{C}}^\infty_\#$.

The corresponding spaces of n-component vector functions/distributions are denoted as $\mathbf{L}_{q\#} := (L_{q\#})^n$, $\mathbf{H}^s_\# := (H^s_\#)^n$, etc.

Note that the norm $\|\nabla(\cdot)\|_{\mathbf{H}^{s-1}_\#}$ is an equivalent norm in $\dot H^s_\#$. Indeed, by (2.6)

$$\nabla g(\mathbf{x}) = 2\pi i\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n}\boldsymbol{\xi}e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}\hat g(\boldsymbol{\xi}), \quad \widehat{\nabla g}(\boldsymbol{\xi}) = 2\pi i\boldsymbol{\xi}\hat g(\boldsymbol{\xi}) \quad \forall\, g\in\dot H^s_\#,$$

and then (2.2) and (2.7) imply

$$\begin{gathered}\frac12|g|^2_{H^s_\#} \le \|\nabla g\|^2_{\mathbf{H}^{s-1}_\#} \le |g|^2_{H^s_\#} \quad \forall\, g\in H^s_\#,\\ \frac12\|g\|^2_{H^s_\#} = \frac12\|g\|^2_{\dot H^s_\#} = \frac12|g|^2_{H^s_\#} \le \|\nabla g\|^2_{\mathbf{H}^{s-1}_\#} \le |g|^2_{H^s_\#} = \|g\|^2_{\dot H^s_\#} = \|g\|^2_{H^s_\#} \quad \forall\, g\in\dot H^s_\#.\end{gathered} \tag{2.12}$$

The vector counterpart of (2.12) takes form

$$\frac12\|\mathbf{v}\|^2_{\mathbf{H}^s_\#} = \frac12\|\mathbf{v}\|^2_{\dot{\mathbf{H}}^s_\#} \le \|\nabla\mathbf{v}\|^2_{(H^{s-1}_\#)^{n\times n}} \le \|\mathbf{v}\|^2_{\dot{\mathbf{H}}^s_\#} = \|\mathbf{v}\|^2_{\mathbf{H}^s_\#} \quad \forall\,\mathbf{v}\in\dot{\mathbf{H}}^s_\#. \tag{2.13}$$

Note that the second inequalities in (2.12) and (2.13) are valid also in the more general cases, i.e., for $g\in H^s_\#$ and $\mathbf{v}\in\mathbf{H}^s_\#$, respectively.

We will further need the first Korn inequality

$$\|\nabla\mathbf{v}\|^2_{(L_{2\#})^{n\times n}} \le 2\|\mathbb{E}(\mathbf{v})\|^2_{(L_{2\#})^{n\times n}} \quad \forall\,\mathbf{v}\in\mathbf{H}^1_\# \tag{2.14}$$

that can be easily proved by adapting, e.g., the proof in [18, Theorem 10.1] to the periodic Sobolev space; cf. also [21, Theorem 2.8].

Let us also define the Sobolev spaces of divergence-free functions and distributions,

$$\dot{\mathbf{H}}^s_{\#\sigma} := \left\{ \mathbf{w} \in \dot{\mathbf{H}}^s_{\#} : \operatorname{div} \mathbf{w} = 0 \right\}, \quad s \in \mathbb{R},$$

endowed with the same norm (2.3). Similarly, $\mathbf{C}^\infty_{\#\sigma}$ and $\mathbf{L}_{q\#\sigma}$ denote the subspaces of divergence-free vector-functions from $\mathbf{C}^\infty_{\#}$ and $\mathbf{L}_{q\#}$, respectively, etc.

The space $\dot{\mathbf{H}}^s_{\#\sigma}$ is the Hilbert space with inner product inherited from (2.4) and (2.8), that is,

$$(\mathbf{g}_1, \mathbf{g}_2)_{\dot{\mathbf{H}}^s_{\#\sigma}} := \sum_{\boldsymbol{\xi} \in \dot{\mathbb{Z}}^n} \varrho(\boldsymbol{\xi})^{2s} \widehat{\mathbf{g}}_1(\boldsymbol{\xi}) \overline{\widehat{\mathbf{g}}_2(\boldsymbol{\xi})}, \quad \forall\, \mathbf{g}_1, \mathbf{g}_2 \in \dot{\mathbf{H}}^s_{\#\sigma}, \ s \in \mathbb{R}.$$

Due to the Riesz representation theorem, the dual product between $\mathbf{g}_1 \in \dot{\mathbf{H}}^s_{\#\sigma}$ and $\mathbf{f}_2 \in (\dot{\mathbf{H}}^s_{\#\sigma})^*$, $s \in \mathbb{R}$, can be represented as

$$\langle \mathbf{g}_1, \mathbf{f}_2 \rangle_{\mathbb{T}} := \sum_{\boldsymbol{\xi} \in \dot{\mathbb{Z}}^n} \widehat{\mathbf{g}}_1(\boldsymbol{\xi}) \widehat{\mathbf{f}}_2(-\boldsymbol{\xi}) = (\mathbf{g}_1, \mathbf{g}_2)_{\dot{\mathbf{H}}^s_{\#\sigma}} = \sum_{\boldsymbol{\xi} \in \dot{\mathbb{Z}}^n} \varrho(\boldsymbol{\xi})^{2s} \widehat{\mathbf{g}}_1(\boldsymbol{\xi}) \overline{\widehat{\mathbf{g}}_2(\boldsymbol{\xi})}.$$

where

$$\widehat{\mathbf{f}}_2(\boldsymbol{\xi}) = \varrho(\boldsymbol{\xi})^{2s} \overline{\widehat{\mathbf{g}}_2(-\boldsymbol{\xi})}, \quad \boldsymbol{\xi} \in \dot{\mathbb{Z}}^n$$

for some $\mathbf{g}_2 \in \dot{\mathbf{H}}^s_{\#\sigma}$. This implies that

$$\mathbf{f}_2(\mathbf{x}) = \left(\Lambda^{2s}_{\#} \, \overline{\mathbf{g}_2} \right)(\mathbf{x}), \tag{2.15}$$

where the operator

$$\Lambda^r_{\#} : \dot{\mathbf{H}}^s_{\#\sigma} \to \dot{\mathbf{H}}^{s-r}_{\#\sigma} \tag{2.16}$$

defined as in (2.10) is continuous. Hence by (2.15) we conclude that

$$(\dot{\mathbf{H}}^s_{\#\sigma})^* = \dot{\mathbf{H}}^{-s}_{\#\sigma}.$$

Let us also introduce the space

$$\dot{\mathbf{H}}^s_{\#g} := \left\{ \mathbf{w} = \nabla q, \ q \in \dot{H}^{s+1}_{\#} \right\}, \quad s \in \mathbb{R},$$

endowed with the norm (2.3).

Let $s \in \mathbb{R}$, $\mathbf{w} \in \dot{\mathbf{H}}^s_{\#g}$ and $\mathbf{v} \in \dot{\mathbf{H}}^s_{\#\sigma}$ By (2.4), for their inner product in $\dot{\mathbf{H}}^s_{\#}$ we obtain

$$(\mathbf{w}, \mathbf{v})_{H^s_{\#}} := \sum_{\boldsymbol{\xi} \in \mathbb{Z}^n} \varrho(\boldsymbol{\xi})^{2s} \widehat{\mathbf{w}}(\boldsymbol{\xi}) \cdot \overline{\widehat{\mathbf{v}}(\boldsymbol{\xi})} = \sum_{\boldsymbol{\xi} \in \mathbb{Z}^n} \varrho(\boldsymbol{\xi})^{2s} 2\pi i \boldsymbol{\xi} \widehat{q}(\boldsymbol{\xi}) \cdot \overline{\widehat{\mathbf{v}}(\boldsymbol{\xi})}$$
$$= - \sum_{\boldsymbol{\xi} \in \mathbb{Z}^n} \varrho(\boldsymbol{\xi})^{2s} \widehat{q}(\boldsymbol{\xi}) \overline{2\pi i \boldsymbol{\xi} \cdot \widehat{\mathbf{v}}(\boldsymbol{\xi})} = - \sum_{\boldsymbol{\xi} \in \mathbb{Z}^n} \varrho(\boldsymbol{\xi})^{2s} \widehat{q}(\boldsymbol{\xi}) \overline{\widehat{\operatorname{div} \mathbf{v}}(\boldsymbol{\xi})} = 0.$$

Hence $\dot{\mathbf{H}}^s_{\#g}$ and $\dot{\mathbf{H}}^s_{\#\sigma}$ are orthogonal subspaces of $\dot{\mathbf{H}}^s_{\#}$ in the sense of inner product.

On the other hand, if $s \in \mathbb{R}$, $\mathbf{w} \in \dot{\mathbf{H}}^s_{\#g}$ and $\mathbf{v} \in \dot{\mathbf{H}}^{-s}_{\#\sigma}$, then for their dual product we obtain

$$\langle \mathbf{w}, \mathbf{v} \rangle = \langle \nabla q, \mathbf{v} \rangle = -\langle q, \operatorname{div} \mathbf{v} \rangle = 0.$$

Hence the spaces $\dot{\mathbf{H}}^s_{\#g}$ and $\dot{\mathbf{H}}^{-s}_{\#\sigma}$ are orthogonal in the sense of dual product.

For $s \in \mathbb{R}$ and $\mathbf{F} \in \dot{\mathbf{H}}^s_\#$, let us introduce the operators $\mathbb{P}_g$ and $\mathbb{P}_\sigma$ as follows,

$$(\mathbb{P}_g \mathbf{F})(\mathbf{x}) := \sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \boldsymbol{\xi}\frac{\boldsymbol{\xi}\cdot\widehat{\mathbf{F}}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2} e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}},$$

$$(\mathbb{P}_\sigma \mathbf{F})(\mathbf{x}) := \sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \left(\widehat{\mathbf{F}}(\boldsymbol{\xi}) - \boldsymbol{\xi}\frac{\boldsymbol{\xi}\cdot\widehat{\mathbf{F}}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}\right) e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}.$$

Note that

$$\mathbf{F}(\mathbf{x}) = (\mathbb{P}_\sigma \mathbf{F})(\mathbf{x}) + (\mathbb{P}_g \mathbf{F})(\mathbf{x}) \quad \forall\, \mathbf{F} \in \dot{\mathbf{H}}^s_\#,\ s \in \mathbb{R}. \tag{2.17}$$

Evidently

$$(\mathbb{P}_g \mathbf{F})(\mathbf{x}) = \nabla q(\mathbf{x}),\ \text{where } q(\mathbf{x}) = \sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \frac{\boldsymbol{\xi}\cdot\widehat{\mathbf{F}}(\boldsymbol{\xi})}{2\pi i|\boldsymbol{\xi}|^2} e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}},$$

hence $q \in \dot{H}^{s+1}_\#$.

One can check that $\mathbb{P}_g(\mathbb{P}_g \mathbf{F}) = \mathbb{P}_g \mathbf{F}$ and thus $\mathbb{P}_g : \dot{\mathbf{H}}^s_\# \to \dot{\mathbf{H}}^s_{\#g}$ is a bounded projector. On the other hand, $\operatorname{div} \mathbb{P}_\sigma \mathbf{F} = 0$, $\mathbb{P}_\sigma(\mathbb{P}_\sigma \mathbf{F}) = \mathbb{P}_\sigma \mathbf{F}$ and hence $\mathbb{P}_\sigma : \dot{\mathbf{H}}^s_\# \to \dot{\mathbf{H}}^s_{\#\sigma}$ is also a bounded projector. Since $\dot{\mathbf{H}}^s_{\#g}$ and $\dot{\mathbf{H}}^s_{\#\sigma}$ are orthogonal subspaces of $\dot{\mathbf{H}}^s_\#$, the projectors $\mathbb{P}_g$ and $\mathbb{P}_\sigma$ are orthogonal in $\dot{\mathbf{H}}^s$. The projector $\mathbb{P}_\sigma$ is called the Leray projector (see, e.g., [23, Section 2.1]).

Decomposition (2.17) implies the representation $\dot{\mathbf{H}}^s_\# = \dot{\mathbf{H}}^s_{\#g} \oplus \dot{\mathbf{H}}^s_{\#\sigma}$ called the Helmholtz-Weyl decomposition. Note that the orthogonality of $\dot{\mathbf{H}}^s_{\#g}$ and $\dot{\mathbf{H}}^s_{\#\sigma}$ implies that for any $\mathbf{F} \in \dot{\mathbf{H}}^s_\#$, the representation $\mathbf{F} = \mathbf{F}_g + \mathbf{F}_\sigma$, where $\mathbf{F}_g \in \dot{\mathbf{H}}^s_{\#g}$ and $\mathbf{F}_\sigma \in \dot{\mathbf{H}}^s_{\#\sigma}$, is unique and hence is given by (2.17).

Summarising the obtained results, we arrive at the following assertion (cf., e.g., [23, Theorem 2.6], where a similar statement is proved for $s = 0$ and $n = 3$).

THEOREM 2.1. *Let $s \in \mathbb{R}$ and $n \geq 2$.*

(a) The space $\dot{\mathbf{H}}^s_\#$ has the Helmholtz-Weyl decomposition, $\dot{\mathbf{H}}^s_\# = \dot{\mathbf{H}}^s_{\#g} \oplus \dot{\mathbf{H}}^s_{\#\sigma}$, that is, any $\mathbf{F} \in \dot{\mathbf{H}}^s_\#$ can be uniquely represented as $\mathbf{F} = \mathbf{F}_g + \mathbf{F}_\sigma$, where $\mathbf{F}_g \in \dot{\mathbf{H}}^s_{\#g}$ and $\mathbf{F}_\sigma \in \dot{\mathbf{H}}^s_{\#\sigma}$.

(b) The spaces $\dot{\mathbf{H}}^s_{\#g}$ and $\dot{\mathbf{H}}^s_{\#\sigma}$ are orthogonal subspaces of $\dot{\mathbf{H}}^s_\#$ in the sense of inner product, i.e., $(\mathbf{w}, \mathbf{v})_{H^s_\#} = 0$ for any $\mathbf{w} \in \dot{\mathbf{H}}^s_{\#g}$ and $\mathbf{v} \in \dot{\mathbf{H}}^{-s}_{\#\sigma}$.

(c) The spaces $\dot{\mathbf{H}}^s_{\#g}$ and $\dot{\mathbf{H}}^{-s}_{\#\sigma}$ are orthogonal in the sense of dual product, i.e., $\langle \mathbf{w}, \mathbf{v}\rangle = 0$ for any $\mathbf{w} \in \dot{\mathbf{H}}^s_{\#g}$ and $\mathbf{v} \in \dot{\mathbf{H}}^{-s}_{\#\sigma}$.

(d) There exist the bounded orthogonal projector operators $\mathbb{P}_g : \dot{\mathbf{H}}^s_\# \to \dot{\mathbf{H}}^s_{\#g}$ and $\mathbb{P}_\sigma : \dot{\mathbf{H}}^s_\# \to \dot{\mathbf{H}}^s_{\#\sigma}$ (the Leray projector), while $\mathbf{F} = \mathbb{P}_g\mathbf{F} + \mathbb{P}_\sigma\mathbf{F}$ for any $\mathbf{F} \in \dot{\mathbf{H}}^s_\#$.

For the evolution problems we will systematically use the spaces $L_q(0, T; H^s_\#)$, $s \in \mathbb{R}$, $1 \leq q \leq \infty$, $0 < T < \infty$, which consists of functions that map $t \in (0, T)$ to a function or distributions from $H^s_\#$. For $1 \leq q < \infty$, the space $L_q(0, T; H^s_\#)$ is endowed with the norm

$$\|h\|_{L_q(0,T;H^s_\#)} = \left(\int_0^T \|h(\cdot,t)\|^q_{H^s_\#} dt\right)^{1/q} = \left(\int_0^T \left[\sum_{\xi\in\mathbb{Z}^n} \varrho(\boldsymbol{\xi})^{2s}|\widehat{h}(\boldsymbol{\xi},t)|^2\right]^{q/2} dt\right)^{1/q} < \infty,$$

and for $q = \infty$ with the norm

$$\|h\|_{L_\infty(0,T;H^s_\#)} = \operatorname{ess\,sup}_{t\in(0,T)} \|h(\cdot,t)\|_{H^s_\#} = \operatorname{ess\,sup}_{t\in(0,T)} \left[\sum_{\xi\in\mathbb{Z}^n} \varrho(\boldsymbol{\xi})^{2s}|\widehat{h}(\boldsymbol{\xi},t)|^2\right]^{1/2} < \infty.$$

For a function (or distribution) $h(\mathbf{x},t)$, we will use the notation $h'(\mathbf{x},t) := \partial_t h(\mathbf{x},t) := \dfrac{\partial}{\partial t}h(\mathbf{x},t)$ for the partial derivative in the time variable t. Let X and Y be some Hilbert spaces. We will further need the space

$$W^1(0,T;X,Y) := \{u \in L_2(0,T;X) : u' \in L_2(0,T;Y)\}$$

endowed with the norm

$$\|u\|_{W^1(0,T;X,Y)} = (\|u\|^2_{L_2(0,T;X)} + \|u'\|^2_{L_2(0,T;Y)})^{1/2}.$$

Spaces of such type are considered in [15, Chapter 1, Section 2.2]. We will particularly need the spaces $W^1(0,T;H^s_\#,H^{s'}_\#)$ and their vector counterparts.

Unless stated otherwise, we will assume in this paper that all vector and scalar variables are real-valued (however with complex-valued Fourier coefficients).

3 Weak formulation of the evolution spatially-periodic anisotropic Navier-Stokes problem

Let us consider the following Navier-Stokes problem for the real-valued unknowns $(\mathbf{u},p)$,

$$\partial_t\mathbf{u} - \boldsymbol{\mathfrak{L}}\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} \quad \text{in } \mathbb{T}\times(0,T), \tag{3.1}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \mathbb{T}\times(0,T), \tag{3.2}$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}^0 \text{ in } \mathbb{T}, \tag{3.3}$$

with given data $\mathbf{f}\in L_2(0,T;\dot{\mathbf{H}}^{-1}_\#)$, $\mathbf{u}^0\in\dot{\mathbf{H}}^0_{\#\sigma}$. Note that the time-trace $\mathbf{u}(\cdot,0)$ for $\mathbf{u}$ solving the weak form of (3.1)–(3.2) is well defined, see Definition 3.1 and Remark 3.3.

Let us introduce the bilinear form

$$a_{\mathbb{T}}(\mathbf{u},\mathbf{v}) = a_{\mathbb{T}}(t;\mathbf{u},\mathbf{v}) := \left\langle a^{\alpha\beta}_{ij}(\cdot,t)E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v})\right\rangle_{\mathbb{T}} \ \forall\, \mathbf{u},\mathbf{v}\in\dot{\mathbf{H}}^1_\#. \tag{3.4}$$

By the boundedness condition (1.4) and inequality (2.13) we have

$$\begin{aligned}|a_{\mathbb{T}}(t;\mathbf{u},\mathbf{v})| &\le \|\mathbb{A}\|\|\mathbb{E}(\mathbf{u})\|_{L^{n\times n}_{2\#}}\|\mathbb{E}(\mathbf{v})\|_{L^{n\times n}_{2\#}} \le \|\mathbb{A}\|\|\nabla\mathbf{u}\|_{L^{n\times n}_{2\#}}\|\nabla\mathbf{v}\|_{L^{n\times n}_{2\#}}\\ &\le \|\mathbb{A}\|\|\mathbf{u}\|_{\dot{\mathbf{H}}^1_\#}\|\mathbf{v}\|_{\dot{\mathbf{H}}^1_\#} \quad \forall\,\mathbf{u},\mathbf{v}\in\dot{\mathbf{H}}^1_\#.\end{aligned} \tag{3.5}$$

If the relaxed ellipticity condition (1.3) holds, taking into account the relation $\sum_{i=1}^n E_{ii}(\mathbf{w}) = \operatorname{div}\mathbf{w} = 0$ for $\mathbf{w}\in\dot{\mathbf{H}}^1_{\#\sigma}$, equivalence of the norm $\|\nabla(\cdot)\|_{L^{n\times n}_{2\#}}$ to the norm $\|\cdot\|_{\dot{\mathbf{H}}^1_{\#\sigma}}$ in $\dot{\mathbf{H}}^1_{\#\sigma}$, see (2.13), and the first Korn inequality (2.14), we obtain

$$\begin{aligned}a_{\mathbb{T}}(t;\mathbf{w},\mathbf{w}) = \left\langle a^{\alpha\beta}_{ij}(\cdot,t)E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{w})\right\rangle_{\mathbb{T}} &\ge C_{\mathbb{A}}^{-1}\|\mathbb{E}(\mathbf{w})\|^2_{L^{n\times n}_{2\#}}\\ &\ge \frac12 C_{\mathbb{A}}^{-1}\|\nabla\mathbf{w}\|^2_{L^{n\times n}_{2\#}} \ge \frac14 C_{\mathbb{A}}^{-1}\|\mathbf{w}\|^2_{\dot{\mathbf{H}}^1_{\#\sigma}} \quad \forall\,\mathbf{w}\in\dot{\mathbf{H}}^1_{\#\sigma}.\end{aligned} \tag{3.6}$$

Then (3.5) and (3.6) give

$$\frac14 C_{\mathbb{A}}^{-1}\|\mathbf{w}\|^2_{\dot{\mathbf{H}}^1_{\#\sigma}} \le a_{\mathbb{T}}(t;\mathbf{w},\mathbf{w}) \le \|\mathbb{A}\|\|\mathbf{w}\|^2_{\dot{\mathbf{H}}^1_{\#\sigma}} \quad \forall\,\mathbf{w}\in\dot{\mathbf{H}}^1_{\#\sigma}. \tag{3.7}$$

This inequality implies that $\sqrt{a_{\mathbb{T}}(t;\mathbf{w},\mathbf{w})}$ is an equivalent norm in $\dot{\mathbf{H}}^1_{\#\sigma}$ for almost any t and, moreover,

$$\|\mathbf{w}\|_{L_2(0,T;\dot{\mathbf{H}}^1_\#)} := \left(\int_0^T a_{\mathbb{T}}(t;\mathbf{w}(\cdot,t),\mathbf{w}(\cdot,t))dt\right)^{1/2} \tag{3.8}$$

is an equivalent norm in $L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$.

We use the following definition of weak solution, that for $n\in\{2,3,4\}$ reduces to the weak formulations employed, e.g., in [14, Chapter 1, Problem 6.2], [5, Definition 8.5], [29, Problem 2.1], [30, Chapter 3, Problem 3.1]. However the definition that we use is applicable also to higher dimensions (and allows for those dimensions more general test functions than in [14, Chapter 1, Problem 6.2]).

DEFINITION 3.1. *Let $n\ge 2$, $T>0$, $\mathbf{f}\in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#})$ and $\mathbf{u}^0\in\dot{\mathbf{H}}^0_{\#\sigma}$. A function $\mathbf{u}\in L_\infty(0,T;\dot{\mathbf{H}}^0_{\#\sigma})\cap L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$ is called a weak solution of the evolution space-periodic anisotropic Navier-Stokes initial value problem (3.1)–(3.3) if it solves the initial-variational problem*

$$\left\langle \mathbf{u}'(\cdot,t)+\mathbb{P}_\sigma[(\mathbf{u}(\cdot,t)\cdot\nabla)\mathbf{u}(\cdot,t)],\mathbf{w}\right\rangle_{\mathbb{T}}+a_{\mathbb{T}}(\mathbf{u}(\cdot,t),\mathbf{w})=\langle\mathbf{f}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}},\ \textit{for a.e. } t\in(0,T),\ \forall\,\mathbf{w}\in\dot{\mathbf{H}}^1_{\#\sigma},\quad(3.9)$$

$$\langle\mathbf{u}(\cdot,0),\mathbf{w}\rangle_{\mathbb{T}}=\langle\mathbf{u}^0,\mathbf{w}\rangle_{\mathbb{T}},\ \forall\,\mathbf{w}\in\dot{\mathbf{H}}^0_{\#\sigma}.\quad(3.10)$$

The associated pressure p is a distribution on $\mathbb{T}\times(0,T)$ satisfying (3.1) in the sense of distributions, i.e.,

$$\left\langle \mathbf{u}'(\cdot,t)+(\mathbf{u}(\cdot,t)\cdot\nabla)\mathbf{u}(\cdot,t),\mathbf{w}\right\rangle_{\mathbb{T}}+a_{\mathbb{T}}(\mathbf{u}(\cdot,t),\mathbf{w})+\langle\nabla p(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}$$
$$=\langle\mathbf{f}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}},\ \textit{for a.e. } t\in(0,T),\ \forall\,\mathbf{w}\in\mathbf{C}^\infty_{\#}.\quad(3.11)$$

To justify the weak formulation (3.9), let us act on (3.1) by the Leray projector $\mathbb{P}_\sigma$ and taking into account that $\mathbb{P}_\sigma\partial_t\mathbf{u}=\partial_t\mathbf{u}$ and $\mathbb{P}_\sigma\nabla p=\mathbf{0}$, we obtain

$$\partial_t\mathbf{u}+\mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}]-\mathbb{P}_\sigma\mathfrak{L}\mathbf{u}=\mathbb{P}_\sigma\mathbf{f}\quad\text{in } \mathbb{T}\times(0,T).\quad(3.12)$$

Assuming that $\mathbf{u}\in L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$, $a^{\alpha\beta}_{ij}\in L_\infty(0,T;L_{\infty\#})$, by (1.5) we obtain that $\mathfrak{L}\mathbf{u}\in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#})$ and due to the symmetry conditions (1.2), we get for any $\mathbf{w}\in\dot{\mathbf{H}}^1_{\#\sigma}$ and for a.e. $t\in(0,T)$,

$$-\langle\mathbb{P}_\sigma\mathfrak{L}\mathbf{u},\mathbf{w}\rangle_{\mathbb{T}}=-\langle\mathfrak{L}\mathbf{u},\mathbf{w}\rangle_{\mathbb{T}}=\langle a^{\alpha\beta}_{kj}(\cdot,t)E_{j\beta}(\mathbf{u}),\partial_\alpha\mathbf{w}_k\rangle_{\mathbb{T}}=\langle a^{\alpha\beta}_{kj}(\cdot,t)E_{j\beta}(\mathbf{u}),E_{k,\alpha}(\mathbf{w})\rangle_{\mathbb{T}}=a_{\mathbb{T}}(\mathbf{u},\mathbf{w}).$$

For $\mathbf{f}\in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#})$, we also have $\langle\mathbb{P}_\sigma\mathbf{f}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}=\langle\mathbf{f}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}$. Hence taking the dual product of equation (3.12) with $\mathbf{w}$, we arrive at equation (3.9). The boundedness of the first dual product in (3.9) and the weak initial condition (3.10) are justified in Lemma 3.2 and Remark 3.3 below. Equation (3.11) is deduced in a similar way.

LEMMA 3.2. *Let $n\ge 2$, $T>0$, $a^{\alpha\beta}_{ij}\in L_\infty(0,T;L_{\infty\#})$, $\mathbf{f}\in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#})$ and $\mathbf{u}^0\in\dot{\mathbf{H}}^0_{\#\sigma}$. Let $\mathbf{u}\in L_\infty(0,T;\dot{\mathbf{H}}^0_{\#\sigma})\cap L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$ solve equation (3.9).*

(i) Then

$$\mathbf{D}\mathbf{u}:=\mathbf{u}'+\mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}]\in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#\sigma})\quad\textit{and}\quad\mathbf{D}\mathbf{u}(\cdot,t)\in\dot{\mathbf{H}}^{-1}_{\#\sigma}\ \textit{for a.e. } t\in[0,T],\quad(3.13)$$

while

$$(\mathbf{u}\cdot\nabla)\mathbf{u}\in L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#})\quad\textit{and}\quad(\mathbf{u}\cdot\nabla)\mathbf{u}(\cdot,t)\in\dot{\mathbf{H}}^{-n/2}_{\#}\quad\textit{for a.e. } t\in[0,T],\quad(3.14)$$

$$\mathbf{u}'\in L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})\quad\textit{and}\quad\mathbf{u}'(\cdot,t)\in\dot{\mathbf{H}}^{-n/2}_{\#\sigma}\quad\textit{for a.e. } t\in[0,T],\quad(3.15)$$

and hence $\mathbf{u}\in W^1(\dot{\mathbf{H}}^1_{\#\sigma},\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$.

In addition,

$$\partial_t\|\mathbf{u}\|^2_{\dot{\mathbf{H}}^{-(n-2)/4}_{\#\sigma}}=2\langle\Lambda^{-n/2}_{\#}\mathbf{u}',\Lambda_{\#}\mathbf{u}\rangle_{\mathbb{T}}=2\langle\mathbf{u}',\Lambda^{1-n/2}_{\#}\mathbf{u}\rangle_{\mathbb{T}}=2\langle\Lambda^{1-n/2}_{\#}\mathbf{u}',\mathbf{u}\rangle_{\mathbb{T}}\quad(3.16)$$

for a.e. $t\in(0,T)$ and also in the distribution sense on $(0,T)$.

(ii) Moreover, $\mathbf{u}$ is almost everywhere on $[0,T]$ equal to a function $\widetilde{\mathbf{u}}\in\mathcal{C}^0([0,T];\dot{\mathbf{H}}^{-(n-2)/4}_{\#\sigma})$, and $\widetilde{\mathbf{u}}$ is also $\dot{\mathbf{H}}^0_{\#\sigma}$-weakly continuous in time on $[0,T]$, that is,

$$\lim_{t\to t_0}\langle\widetilde{\mathbf{u}}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}=\langle\widetilde{\mathbf{u}}(\cdot,t_0),\mathbf{w}\rangle_{\mathbb{T}}\quad\forall\,\mathbf{w}\in\mathbf{H}^0_{\#},\ \forall\,t_0\in[0,T].$$

(iii) There exists the associated pressure $p\in L_2(0,T;\dot{H}^{-n/2+1}_{\#})$ that for the given $\mathbf{u}$ is the unique solution of equation (3.1) in this space.

Proof. (i) By (3.9) we obtain

$$|\langle \mathbf{D}\mathbf{u}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}| \le |a_{\mathbb{T}}(t;\mathbf{u},\mathbf{w})| + |\langle \mathbf{f}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}|$$
$$\le \|\mathbb{A}\|\|\mathbf{u}(\cdot,t)\|_{\mathbf{H}^1_\#}\|\mathbf{w}\|_{\mathbf{H}^1_\#} + \|\mathbf{f}(\cdot,t)\|_{\mathbf{H}^{-1}_\#}\|\mathbf{w}\|_{\mathbf{H}^1_\#},\ \text{for a.e. } t\in(0,T),\ \forall\,\mathbf{w}\in\dot{\mathbf{H}}^1_{\#\sigma}.$$

In addition, $\operatorname{div}\mathbf{D}\mathbf{u} := \operatorname{div}\mathbf{u}' + \operatorname{div}\mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] = 0$. Hence $\|\mathbf{D}\mathbf{u}(\cdot,t)\|_{\dot{\mathbf{H}}^{-1}_{\#\sigma}} \le \|\mathbb{A}\|\|\mathbf{u}(\cdot,t)\|_{\mathbf{H}^1_\#} + \|\mathbf{f}(\cdot,t)\|_{\mathbf{H}^{-1}_\#}$ for a.e. $t\in(0,T)$ and thus

$$\|\mathbf{D}\mathbf{u}\|_{L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#\sigma})} \le \|\mathbb{A}\|\|\mathbf{u}\|_{L_2(0,T;\mathbf{H}^1_\#)} + \|\mathbf{f}\|_{L_2(0,T;\mathbf{H}^{-1}_\#)}$$

which implies inclusions (3.13).

By the multiplication Theorem 5.1 and the Sobolev interpolation inequality, we obtain

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\dot{\mathbf{H}}^{-n/2}_\#} = \|\operatorname{div}(\mathbf{u}\otimes\mathbf{u})\|_{\mathbf{H}^{-n/2}_\#} \le \|\mathbf{u}\otimes\mathbf{u}\|_{(H^{1-n/2}_\#)^{n\times n}}$$
$$\le C_*(1/2,1/2,n)\|\mathbf{u}\|^2_{\mathbf{H}^{1/2}_\#} \le C_*(1/2,1/2,n)\|\mathbf{u}\|_{\mathbf{H}^0_\#}\|\mathbf{u}\|_{\mathbf{H}^1_\#}. \tag{3.17}$$

Thus

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L_2(0,T;\dot{\mathbf{H}}^{-n/2}_\#)} \le C_*(1/2,1/2,n)\|\mathbf{u}\|_{L_\infty(0,T;\mathbf{H}^0_\#)}\|\mathbf{u}\|_{L_2(0,T;\mathbf{H}^1_\#)},$$

which implies inclusions (3.14). Further,

$$\|\mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}]\|_{L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})} \le \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L_2(0,T;\dot{\mathbf{H}}^{-n/2}_\#)} \le C_*(1/2,1/2,n)\|\mathbf{u}\|_{L_\infty(0,T;\mathbf{H}^0_\#)}\|\mathbf{u}\|_{L_2(0,T;\mathbf{H}^1_\#)},$$

implying that $\mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] \in L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$. Then the first inclusion in (3.13) leads to the inclusion $\mathbf{u}'\in L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$ and hence to inclusions (3.15).

(ii) Since $\mathbf{u}\in L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$ and $\mathbf{u}'\in L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$, relations (3.16) are implied by Lemma 5.8(i). Moreover, Theorem 5.7 implies that $\mathbf{u}$ is almost everywhere on $[0,T]$ equal to a function $\widetilde{\mathbf{u}}\in\mathcal{C}^0([0,T];\dot{\mathbf{H}}^{-(n-2)/4}_{\#\sigma})$.

We have that $\widetilde{\mathbf{u}}\in L_\infty(0,T;\dot{\mathbf{H}}^0_{\#\sigma})$, $\widetilde{\mathbf{u}}\in\mathcal{C}^0([0,T];\dot{\mathbf{H}}^{-(n-2)/4}_{\#\sigma})$ and $\dot{\mathbf{H}}^0_{\#\sigma}\subset\dot{\mathbf{H}}^{-(n-2)/4}_{\#\sigma}$ with continuous injection. Then Lemma 5.6 (taken from [30, Chapter 3, Lemma 1.4]) implies that $\widetilde{\mathbf{u}}$ is $\dot{\mathbf{H}}^0_{\#\sigma}$-weakly continuous in time.

(iii) The associated pressure p satisfies (3.1) that after applying the projector $\mathbb{P}_g$ can be re-written as

$$\nabla p = \mathbf{F}, \tag{3.18}$$

where

$$\mathbf{F} := \mathbb{P}_g[\mathbf{f} - \partial_t\mathbf{u} + \mathfrak{L}\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u}] = \mathbb{P}_g\mathbf{f} + \mathbb{P}_g\mathfrak{L}\mathbf{u} - \mathbb{P}_g[(\mathbf{u}\cdot\nabla)\mathbf{u}] \in L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#g}) \tag{3.19}$$

due to the first inclusion in (3.14). By Lemma 5.4 for gradient, with $s = 1-n/2$, equation (3.18) has a unique solution p in $L_2(0,T;\dot{H}^{-n/2+1}_\#)$. □

Note that inclusions (3.13) do not generally imply that $\mathbf{u}'(\cdot,t)$ and $\mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}](\cdot,t)$ belong to $\dot{\mathbf{H}}^{-1}_{\#\sigma}$ for a.e. $t\in[0,T]$, but only that their sum does. This is why the first dual product in (3.9) is not written as the sum of the two respective dual products.

REMARK 3.3. *The initial condition (3.10) should be understood for the function* $\mathbf{u}$ *re-defined as the function* $\widetilde{\mathbf{u}}$ *that was introduced in Lemma 3.2(ii) and is* $\mathbf{H}^0_\#$*-weakly continuous in time.*

4 Existence for evolution spatially-periodic anisotropic Navier-Stokes problem

In this section, we prove solution existence for the evolution anisotropic incompressible Navier-Stokes systems, accommodating to anisotropy, variable coefficients and arbitrary $n \geq 2$ the approaches presented, e.g., in [14, Chapter 1, Section 6.5], [5, Chapter 8], [29, Chapter 3], [30, Chapter 3, Section 3], [23, Section 4] for the constant-coefficient isotropic Navier-Stokes equations.

THEOREM 4.1. *Let $n \geq 2$ and $T > 0$. Let $a_{ij}^{\alpha\beta} \in L_\infty(0,T;L_{\infty\#})$ and the relaxed ellipticity condition (1.3) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{-1})$, $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^0$.*

(i) Then there exists a weak solution $\mathbf{u} \in L_\infty(0,T;\dot{\mathbf{H}}_{\#\sigma}^0) \cap L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^1)$ of the anisotropic Navier-Stokes initial value problem (3.1)–(3.3) in the sense of Definition 3.1. Particularly, $\lim_{t\to 0}\langle \mathbf{u}(\cdot,t),\mathbf{v}\rangle_{\mathbb{T}} = \langle \mathbf{u}^0,\mathbf{v}\rangle_{\mathbb{T}} \quad \forall\, \mathbf{v} \in \dot{\mathbf{H}}_{\#\sigma}^0$. There exists also the unique pressure $p \in L_2(0,T;\dot{H}_\#^{-n/2+1})$ associated with the obtained $\mathbf{u}$, that is the solution of equation (3.1) in $L_2(0,T;\dot{H}_\#^{-n/2+1})$.

(ii) Moreover, $\mathbf{u}$ satisfies the following (strong) energy inequality,

$$\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + 2\int_{t_0}^t a_{\mathbb{T}}(\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau \leq \|\mathbf{u}(\cdot,t_0)\|^2_{\mathbf{L}_{2\#}} + 2\int_{t_0}^t \langle \mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}} d\tau \tag{4.1}$$

for any $[t_0,t] \subset [0,T]$. It particularly implies the standard energy inequality,

$$\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + 2\int_0^t a_{\mathbb{T}}(\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau \leq \|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + 2\int_0^t \langle \mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\, d\tau \quad \forall\, t \in [0,T]. \tag{4.2}$$

Proof. We prove the solution existence using the Faedo-Galerkin algorithm, cf., e.g., [13, Chapt. 6, Sections 3, 6], [14, Chapter 1, Section 6.4], [29, Chapter 3, Section 3.3], [30, Chapter 3, Section 3], [23, Section 4].

(a) Let $\{\mathbf{w}_l\} = \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_l, \dots$ be the sequence of real orthonormal eigenfunctions of the Bessel potential operator $\Lambda_\#$ in $\dot{\mathbf{H}}_{\#\sigma}^0$, see Appendix 5.3. This sequence constitutes an orthonormal basis in $\dot{\mathbf{H}}_{\#\sigma}^0$ and is similar to a periodic version of the special basis employed in [14, Chapter 1, Corollary 6.1]. It belongs to $\dot{\mathbf{C}}_{\#\sigma}^\infty$ and can be explicitly expressed in terms of the Fourier harmonics, see Remark 5.3. Such choice of the linear independent functions particularly facilitates the proof of existence for arbitrary dimension $n \geq 2$. Another possible choice is given by the eigenfunctions of the isotropic Stokes operator in $\dot{\mathbf{H}}_{\#\sigma}^0$, cf. [29, Section 2.2], [23, Theorem 2.24].

For each integer $m \geq 1$, let us look for a solution

$$\mathbf{u}_m(\mathbf{x},t) = \sum_{l=1}^m \eta_{l,m}(t)\mathbf{w}_l, \quad \eta_{l,m}(t) \in \mathbb{R}, \tag{4.3}$$

of the following discrete analogue of the initial-variational problem (3.9)–(3.10),

$$\langle \mathbf{u}'_m, \mathbf{w}_k\rangle_{\mathbb{T}} + a_{\mathbb{T}}(\mathbf{u}_m,\mathbf{w}_k) + \langle (\mathbf{u}_m\cdot\nabla)\mathbf{u}_m, \mathbf{w}_k\rangle_{\mathbb{T}} = \langle \mathbf{f},\mathbf{w}_k\rangle_{\mathbb{T}}, \text{ a.e. } t \in (0,T),\ \forall\, k \in \{1,\dots,m\}, \tag{4.4}$$

$$\langle \mathbf{u}_m, \mathbf{w}_k\rangle_{\mathbb{T}}(\cdot,0) = \langle \mathbf{u}^0,\mathbf{w}_k\rangle_{\mathbb{T}}, \quad \forall\, k \in \{1,\dots,m\}. \tag{4.5}$$

For a fixed m, equations (4.4)–(4.5) constitute an initial value problem for the nonlinear system of ordinary differential equations for unknowns $\eta_{l,m}(t)$, $\ell \in \{1,\dots,m\}$,

$$\sum_{l=1}^m \langle \mathbf{w}_l,\mathbf{w}_k\rangle_{\mathbb{T}}\, \partial_t\eta_{l,m}(t) + \sum_{l=1}^m a_{\mathbb{T}}(t;\mathbf{w}_l,\mathbf{w}_k)\,\eta_{l,m}(t) + \sum_{l,j=1}^m \langle (\mathbf{w}_l\cdot\nabla)\mathbf{w}_j,\mathbf{w}_k\rangle_{\mathbb{T}}\eta_{l,m}(t)\eta_{j,m}(t)$$
$$= \langle \mathbf{f},\mathbf{w}_k\rangle_{\mathbb{T}}\,, \text{ a.e. } t \in (0,T), \quad \forall\, k \in \{1,\dots,m\}, \tag{4.6}$$

$$\sum_{l=1}^m \langle \mathbf{w}_l,\mathbf{w}_k\rangle_{\mathbb{T}}\,\eta_{l,m}(0) = \langle \mathbf{u}^0,\mathbf{w}_k\rangle_{\mathbb{T}}, \quad \forall\, k \in \{1,\dots,m\}. \tag{4.7}$$

We have $\langle \mathbf{f}, \mathbf{w}_k\rangle_{\mathbb{T}} \in L_2(0,T)$ and due to the orthonormality of the functions $\mathbf{w}_l$, we have $\langle \mathbf{w}_l, \mathbf{w}_k\rangle_{\mathbb{T}} = \delta_{\ell k}$. Then by the Carathéodory existence theorem, see, e.g. [8, Theorem 5.1], the ODE initial value problem (4.6)–(4.7) has an absolutely continuous solution $\eta_{l,m}(t)$, $l = 1,\dots,m$, on an interval $[0,T_m]$, $0 < T_m \le T$.

Multiplying equations (4.6) by $\eta_{k,m}$ and summing them up over $k \in \{1,\dots,m\}$, and also doing the same with equations (4.7), we obtain

$$\langle \partial_t \mathbf{u}_m, \mathbf{u}_m\rangle_{\mathbb{T}} + a_{\mathbb{T}}(\mathbf{u}_m, \mathbf{u}_m) + \langle (\mathbf{u}_m\cdot\nabla)\mathbf{u}_m, \mathbf{u}_m\rangle_{\mathbb{T}} = \langle \mathbf{f}, \mathbf{u}_m\rangle_{\mathbb{T}},\ \text{a.e. } t\in(0,T_m), \tag{4.8}$$

$$\langle \mathbf{u}_m(\cdot,0), \mathbf{u}_m(\cdot,0)\rangle_{\mathbb{T}} = \langle \mathbf{u}^0, \mathbf{u}_m(\cdot,0)\rangle_{\mathbb{T}}. \tag{4.9}$$

By equality (5.4) for the trilinear term, equation (4.8) is reduced to

$$\frac{1}{2}\partial_t\|\mathbf{u}_m\|^2_{\mathbf{L}_{2\#}} + a_{\mathbb{T}}(\mathbf{u}_m, \mathbf{u}_m) = \langle \mathbf{f}, \mathbf{u}_m\rangle_{\mathbb{T}},\ \text{a.e. } t\in(0,T_m), \tag{4.10}$$

Inequality (3.7) for the quadratic form $a_{\mathbb{T}}$ and Yong's inequality for the right hand side of (4.10) imply

$$\begin{aligned}\partial_t\|\mathbf{u}_m\|^2_{\mathbf{L}_{2\#}} + \frac{1}{2}C_{\mathbb{A}}^{-1}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}^1_\#} &\le \partial_t\|\mathbf{u}_m\|^2_{\mathbf{L}_{2\#}} + 2a_{\mathbb{T}}(\mathbf{u}_m, \mathbf{u}_m) = 2\langle \mathbf{f}, \mathbf{u}_m\rangle_{\mathbb{T}} \le 2\|\mathbf{f}\|_{\dot{\mathbf{H}}^{-1}_\#}\|\mathbf{u}_m\|_{\dot{\mathbf{H}}^1_\#}\\ &\le \frac{1}{4}C_{\mathbb{A}}^{-1}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}^1_\#} + 4C_{\mathbb{A}}\|\mathbf{f}\|^2_{\dot{\mathbf{H}}^{-1}_\#},\ \text{a.e. } t\in(0,T_m).\end{aligned} \tag{4.11}$$

Equation (4.9) implies

$$\|\mathbf{u}_m(\cdot,0)\|^2_{\mathbf{L}_{2\#}} = \langle \mathbf{u}^0, \mathbf{u}_m(\cdot,0)\rangle_{\mathbb{T}} \le \|\mathbf{u}^0\|_{\mathbf{L}_{2\#}}\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{L}_{2\#}}. \tag{4.12}$$

Hence (4.11) and (4.12) lead to

$$\partial_t\|\mathbf{u}_m\|^2_{\mathbf{L}_{2\#}} + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}^1_\#} \le 4C_{\mathbb{A}}\|\mathbf{f}\|^2_{\dot{\mathbf{H}}^{-1}_\#},\ \text{a.e. } t\in(0,T_m), \tag{4.13}$$

$$\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{L}_{2\#}} \le \|\mathbf{u}^0\|_{\mathbf{L}_{2\#}}. \tag{4.14}$$

Integrating (4.13), we get

$$\begin{aligned}\|\mathbf{u}_m(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + \frac{1}{4}C_{\mathbb{A}}^{-1}\int_0^t\|\mathbf{u}_m(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^1_\#}d\tau &\le \|\mathbf{u}_m(\cdot,0)\|^2_{\mathbf{L}_{2\#}} + 4C_{\mathbb{A}}\int_0^t\|\mathbf{f}(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^{-1}_\#}d\tau\\ &\le \|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + 4C_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-1}_\#)},\quad t\in[0,T_m].\end{aligned} \tag{4.15}$$

Estimate (4.15) particularly implies that the ODE initial value problem (4.6)–(4.7) has an absolutely continuous solution $\eta_{l,m}(t)$, $l = 1,\dots,m$, on the whole interval $[0,T]$, where the right hand side $\mathbf{f}$ is prescribed, i.e., we can take $T_m = T$.

Hence from (4.15) we conclude that

$$\|\mathbf{u}_m\|^2_{L_\infty(0,T;\mathbf{L}_{2\#})} = \sup_{t\in[0,T]}\|\mathbf{u}_m(\cdot,t)\|^2_{\mathbf{L}_{2\#}} \le \|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + 4C_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-1}_\#)}, \tag{4.16}$$

$$\|\mathbf{u}_m\|^2_{L_2(0,T;\dot{\mathbf{H}}^1_\#)} \le 4C_{\mathbb{A}}\left(\|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + 4C_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-1}_\#)}\right). \tag{4.17}$$

Recall that $\|\mathbf{u}\|_{\mathbf{L}_{2\#}} = \|\mathbf{u}\|_{\mathbf{H}^0_\#}$, while $\|\mathbf{u}\|_{\mathbf{H}^s_\#} = \|\mathbf{u}\|_{\dot{\mathbf{H}}^s_{\#\sigma}}$ for $\mathbf{u}\in\dot{\mathbf{H}}^s_{\#\sigma}$. Estimates (4.16) and (4.17) mean that the sequence $\{\mathbf{u}_m\}$ is bounded in $L_\infty(0,T;\dot{\mathbf{H}}^0_{\#\sigma}))$ and in $L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$, implying that the sequence has a subsequence still denoted as $\{\mathbf{u}_m\}$ that converges weakly in $L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$ and weakly-star in $L_\infty(0,T;\dot{\mathbf{H}}^0_{\#\sigma})$ to a function $\mathbf{u}\in L_\infty(0,T;\dot{\mathbf{H}}^0_{\#\sigma})\cap L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$. Note also that inequality (4.16) implies also that

$$\|\mathbf{u}(\cdot,t)\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} \le \|\mathbf{u}^0\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} + 4C_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-1}_\#)},\ \text{a.e. } t\in(0,T).$$

(b) Let us also prove that the sequence $\{\mathbf{u}'_m\}$ is bounded in $L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$, cf. [14, Chapter 1, Section 6.4]. To this end, we multiply equations (4.4) by $\mathbf{w}_k$ and sum them up over $k\in\{1,\dots,m\}$, to obtain

$$\mathbf{u}'_m - P_m\mathfrak{L}\mathbf{u}_m + P_m[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m] = P_m\mathbf{f}\,,\ \text{a.e. } t\in(0,T), \tag{4.18}$$

where P_m is the projector operator from $\mathbf{H}^{-n/2}_{\#\sigma}$ to $\mathrm{Span}\{\mathbf{w}_1,\dots,\mathbf{w}_m\}$ defined in (5.10) and we took into account that

$$P_m\mathbf{u}'_m=\sum_{k=1}^m\langle\mathbf{u}'_m,\mathbf{w}_k\rangle_{\mathbb{T}}\mathbf{w}_k=\sum_{k=1}^m\sum_{l=1}^m\eta'_{l,m}(t)\langle\mathbf{w}_l,\mathbf{w}_k\rangle_{\mathbb{T}}\mathbf{w}_k=\sum_{l=1}^m\eta'_{l,m}(t)\mathbf{w}_l=\mathbf{u}'_m.$$

Further, due to Theorem 5.2(iii), for any $\mathbf{h}\in\mathbf{H}^r_{\#}$, $r\in\mathbb{R}$, we have

$$\|P_m\mathbf{h}\|^2_{\dot{\mathbf{H}}^r_{\#\sigma}}\le\|\mathbf{h}\|^2_{\mathbf{H}^r_{\#}}\,. \tag{4.19}$$

By (4.19), (1.1) and (1.4) we have

$$\|P_m\mathfrak{L}\mathbf{u}_m\|^2_{\dot{\mathbf{H}}^{-n/2}_{\#\sigma}}\le\|\mathfrak{L}\mathbf{u}_m\|^2_{\dot{\mathbf{H}}^{-n/2}_{\#\sigma}}\le\|\mathfrak{L}\mathbf{u}_m\|^2_{\mathbf{H}^{-1}_{\#}}\le\|\mathbb{A}\|^2\|\mathbf{u}_m\|^2_{\mathbf{H}^1_{\#}}$$

and then by (4.17),

$$\begin{aligned}\|P_m\mathfrak{L}\mathbf{u}_m\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})}&\le\|\mathfrak{L}\mathbf{u}_m\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#\sigma})}\le\|\mathbb{A}\|^2\|\mathbf{u}_m\|^2_{L_2(0,T;\mathbf{H}^1_{\#})}\\&\le4\|\mathbb{A}\|^2C_{\mathbb{A}}\left(\|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}}+4C_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#})}\right).\end{aligned} \tag{4.20}$$

Next, by (4.19) we obtain

$$\|P_m\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})}\le\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})}\le\|\mathbf{f}\|^2_{L_2(0,T;\mathbf{H}^{-1}_{\#})}\,. \tag{4.21}$$

For any $\mathbf{v}_1\in\mathbf{H}^1_{\#\sigma}$, $\mathbf{v}_2\in\mathbf{H}^1_{\#}$, by Theorem 5.1(b) and the Sobolev interpolation inequality, we obtain

$$\begin{aligned}\|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\|^2_{\dot{\mathbf{H}}^{-n/2}_{\#}}&=\|\mathrm{div}(\mathbf{v}_1\otimes\mathbf{v}_2)\|^2_{\mathbf{H}^{-n/2}_{\#}}\le\|\mathbf{v}_1\otimes\mathbf{v}_2\|^2_{(H^{1-n/2}_{\#})^{n\times n}}\\&\le C^2_*(1/2,1/2,n)\|\mathbf{v}_1\|^2_{\mathbf{H}^{1/2}_{\#}}\|\mathbf{v}_2\|^2_{\mathbf{H}^{1/2}_{\#}}\le C^2_*(1/2,1/2,n)\|\mathbf{v}_1\|_{\mathbf{H}^0_{\#}}\|\mathbf{v}_1\|_{\mathbf{H}^1_{\#}}\|\mathbf{v}_2\|_{\mathbf{H}^0_{\#}}\|\mathbf{v}_2\|_{\mathbf{H}^1_{\#}}.\end{aligned}$$

Thus

$$\|P_m[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m]\|^2_{\dot{\mathbf{H}}^{-n/2}_{\#\sigma}}\le\|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\|^2_{\mathbf{H}^{-n/2}_{\#}}\le C^2_*(1/2,1/2,n)\|\mathbf{u}\|^2_{\mathbf{H}^0_{\#}}\|\mathbf{u}\|^2_{\mathbf{H}^1_{\#}}$$

and then by (4.16) and (4.17),

$$\begin{aligned}\|P_m[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m]\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})}&\le C^2_*(1/2,1/2,n)\|\mathbf{u}_m\|^2_{L_\infty(0,T;\mathbf{H}^0_{\#})}\|\mathbf{u}_m\|^2_{L_2(0,T;\mathbf{H}^1_{\#})}\\&\le4C^2_*(1/2,1/2,n)C_{\mathbb{A}}\left(\|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}}+4C_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#})}\right)^2.\end{aligned} \tag{4.22}$$

Equation (4.18) and inequalities (4.20), (4.21) and (4.22) imply that the sequence $\{\mathbf{u}'_m\}$ is bounded in $L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$ and hence it has a subsequence converging to a function $\mathbf{u}^\dagger\in L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$ weakly in this space.

Let us prove that $\mathbf{u}'=\mathbf{u}^\dagger$. Indeed, for any $\phi\in\mathcal{C}^\infty_c(0,T)$ and $\mathbf{w}\in\dot{\mathbf{H}}^{n/2}_{\#\sigma}$, evidently, $\mathbf{v}:=\mathbf{w}\phi\in L_2(0,T;\dot{\mathbf{H}}^{n/2}_{\#\sigma})=\left(L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})\right)^*$ and we have

$$\int_0^T \langle \mathbf{u}^\dagger(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}\phi(t)dt = \int_0^T \langle \mathbf{u}^\dagger(\cdot,t),\mathbf{v}(\cdot,t)\rangle_{\mathbb{T}}dt$$
$$= \int_0^T \langle \mathbf{u}^\dagger(\cdot,t)-\mathbf{u}'_m(\cdot,t),\mathbf{v}(\cdot,t)\rangle_{\mathbb{T}}dt + \int_0^T \langle \mathbf{u}'_m(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}\phi(t)dt. \quad (4.23)$$

The first integral in the right hand side of (4.23) tends to zero as $m\to\infty$ due to the weak convergence of $\mathbf{u}'_m$ to $\mathbf{u}^\dagger$ in $L_2(0,T;\dot{\mathbf{H}}^{n/2}_{\#\sigma})$. For the second integral in the right hand side of (4.23) we obtain,

$$\int_0^T \langle \mathbf{u}'_m(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}\phi(t)dt = -\int_0^T \langle \mathbf{u}_m(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}\phi'(t)dt$$
$$= \int_0^T \langle \mathbf{u}(\cdot,t)-\mathbf{u}_m(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}\phi'(t)dt - \int_0^T \langle \mathbf{u}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}\phi'(t)dt \quad (4.24)$$

The first integral in the right hand side of (4.24) tends to zero as $m\to\infty$ due to the weak convergence of $\mathbf{u}_m$ to $\mathbf{u}$ in $L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$. Hence, taking the limits of (4.23) and (4.24) as $m\to\infty$, we obtain,

$$\int_0^T \langle \mathbf{u}^\dagger(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}\phi(t)dt = -\int_0^T \langle \mathbf{u}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}\phi'(t)dt = \int_0^T \partial_t\langle \mathbf{u}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}\phi(t)dt,$$

which implies that $\langle \mathbf{u}^\dagger(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}$ is the distributional derivative in time of $\langle \mathbf{u}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}$ and thus as in the proof of Lemma (5.8)(ii) the time derivative commutates with the dual product over $\mathbb{T}$, leading to

$$\langle \mathbf{u}'(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}} = \partial_t\langle \mathbf{u}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}} = \langle \mathbf{u}^\dagger(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}$$

in the sense of distributions on $(0,T)$, for any $\mathbf{w}\in\dot{\mathbf{H}}^{n/2}_{\#\sigma}$. Since $\mathbf{w}$ is an arbitrary test function in $\dot{\mathbf{H}}^{n/2}_{\#\sigma}$, this implies that $\mathbf{u}'=\mathbf{u}^\dagger$ and hence $\mathbf{u}'\in L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$.

Applying now Theorem 5.5 (the Aubin-Lions lemma) with $G=\dot{\mathbf{H}}^1_{\#\sigma}$, $H=\dot{\mathbf{H}}^0_{\#\sigma}$, $K=\dot{\mathbf{H}}^{-n/2}_{\#\sigma}$ and $p=q=2$, we conclude that the subsequence $\{\mathbf{u}_m\}$ can be chosen in such a way that it converges to $\mathbf{u}\in L_\infty(0,T;\dot{\mathbf{H}}^0_{\#\sigma})\cap L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$ also strongly in $L_2(0,T;\dot{\mathbf{H}}^0_{\#\sigma})$.

Since $\mathbf{u}\in L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$ and $\mathbf{u}'\in L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$, Theorem 5.7 implies that $\mathbf{u}$ is almost everywhere on $[0,T]$ equal to a function belonging to $C^0([0,T];\dot{\mathbf{H}}^{-(n-2)/4}_{\#\sigma})$. Further on, under $\mathbf{u}$ we will understand the redefined (on a zero-measure set in $[0,T]$) function belonging to $C^0([0,T];\dot{\mathbf{H}}^{-(n-2)/4}_{\#\sigma})$, which also means that $\|\mathbf{u}(\cdot,t)-\mathbf{u}(\cdot,0)\|_{\dot{\mathbf{H}}^{-(n-2)/4}_{\#\sigma}}\to 0$ as $t\to 0$. Since $\mathbf{u}\in L_\infty(0,T;\dot{\mathbf{H}}^0_{\#\sigma})$ as well, Lemma 5.6 implies that $\mathbf{u}$ is $\dot{\mathbf{H}}^0_{\#\sigma}$-weakly continuous in time on $[0,T]$ and hence $\lim_{t\to0}\langle \mathbf{u}(\cdot,t),\mathbf{v}\rangle_{\mathbb{T}} = \langle \mathbf{u}(\cdot,0),\mathbf{v}\rangle_{\mathbb{T}} \quad \forall\, \mathbf{v}\in\dot{\mathbf{H}}^0_{\#\sigma}$.

(c) Let us prove that the limit function $\mathbf{u}$ solves the initial-variational problem (3.9)–(3.10). First of all, equality (4.18) and inequality (4.20) imply that

$$\|\mathbf{u}'_m + P_m[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m]\|^2_{L_2(0,T;\mathbf{H}^{-1}_{\#\sigma})} \le 2\|P_m\mathbf{f}\|^2_{L_2(0,T;\mathbf{H}^{-1}_{\#})} + 2\|P_m\mathfrak{L}\mathbf{u}_m\|^2_{L_2(0,T;\mathbf{H}^{-1}_{\#})}$$
$$\le 2\|\mathbf{f}\|^2_{L_2(0,T;\mathbf{H}^{-1}_{\#})} + 8\|\mathbb{A}\|^2 C_{\mathbb{A}}\left(\|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + 4C_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#})}\right).$$

Thus the sequence $P_m\mathbf{D}\mathbf{u}_m := \mathbf{u}'_m + P_m[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m]$ is bounded in $L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#\sigma})$ and hence there exists a subsequence of the sequence $\mathbf{u}_m$ such that the corresponding subsequence of the sequence $P_m\mathbf{D}\mathbf{u}_m$ weakly converges in this space to a distribution $\mathbf{U}\in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#\sigma})$. Let us prove that $\mathbf{U}=\mathbf{D}\mathbf{u}:=\mathbf{u}'+\mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}]$. Indeed, for any $\phi\in L_2(0,T)$ and $\mathbf{w}\in\dot{\mathbf{H}}^{n/2}_{\#\sigma}$, evidently,

$$\mathbf{v}:=\mathbf{w}\phi\in L_2(0,T;\dot{\mathbf{H}}^{n/2}_{\#\sigma}) = \left(L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})\right)^* \subset L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma}) = \left(L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#\sigma})\right)^*,$$

and we have

$$\int_0^T \langle \mathbf{Du}-\mathbf{U},\mathbf{v}\rangle_{\mathbb{T}} dt = \int_0^T \langle \mathbf{Du}-P_m\mathbf{Du}_m,\mathbf{v}\rangle_{\mathbb{T}} dt + \int_0^T \langle P_m\mathbf{Du}_m-\mathbf{U},\mathbf{v}\rangle_{\mathbb{T}} dt$$
$$= \int_0^T \langle \mathbf{u}'-\mathbf{u}'_m,\mathbf{v}\rangle_{\mathbb{T}} dt + \int_0^T \langle \mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] - P_m[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m],\mathbf{v}\rangle_{\mathbb{T}} dt + \int_0^T \langle P_m\mathbf{Du}_m-\mathbf{U},\mathbf{v}\rangle_{\mathbb{T}} dt. \quad (4.25)$$

The first and the last integrals in the right hand side of (4.25) tends to zero as $m\to\infty$ due to the weak convergence of $\mathbf{u}'_m$ to $\mathbf{u}'$ in $L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$ and of $P_m\mathbf{Du}_m$ to $\mathbf{U}$ in $L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#\sigma})$. For the middle integral in the right hand side of (4.25) we obtain, as in [14, Section 6.4.4], for any function $\mathbf{w}_k\in\dot{\mathbf{C}}^\infty_{\#\sigma}$ from our basis in $\dot{\mathbf{H}}^{n/2}_{\#\sigma}$, for $m\geq k$,

$$\int_0^T \langle P_m[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m],\mathbf{w}_k\rangle_{\mathbb{T}}\phi(t)dt = \int_0^T \langle (\mathbf{u}_m\cdot\nabla)\mathbf{u}_m,\mathbf{w}_k\rangle_{\mathbb{T}}\phi(t)dt$$
$$= -\int_0^T \langle \mathbf{u}_m\cdot\nabla\mathbf{w}_k,\mathbf{u}_m\rangle_{\mathbb{T}}\phi(t)dt = -\int_0^T \langle \mathbb{P}_\sigma[\mathbf{u}_m\cdot\nabla\mathbf{w}_k],\mathbf{u}_m\rangle_{\mathbb{T}}\phi(t)dt$$
$$\to -\int_0^T \langle \mathbb{P}_\sigma[\mathbf{u}\cdot\nabla\mathbf{w}_k],\mathbf{u}\rangle_{\mathbb{T}}\phi(t)dt = \int_0^T \langle \mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}],\mathbf{w}_k\rangle_{\mathbb{T}}\phi(t)dt, \quad m\to\infty \quad (4.26)$$

by the strong convergence of $\{\mathbf{u}_m\}$ to $\mathbf{u}$ in in $L_2(0,T;\dot{\mathbf{H}}^0_{\#\sigma})$. Since $\mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}]$ belongs to $L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$, $P_m[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m]$ is uniformly bounded in this space and $\{\mathbf{w}_k\}$ is a basis in $\dot{\mathbf{H}}^{n/2}_{\#\sigma}$, we conclude that the convergence in (4.26) implies that

$$\int_0^T \langle P_m[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m],\mathbf{w}\rangle_{\mathbb{T}}\phi(t)dt \to \int_0^T \langle \mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}],\mathbf{w}\rangle_{\mathbb{T}}\phi(t)dt, \quad m\to\infty$$

and thus

$$\int_0^T \langle \mathbf{Du}-\mathbf{U},\mathbf{w}\rangle_{\mathbb{T}}\phi(t)dt = 0 \ \ \forall\,\phi\in L_2(0,T), \quad \forall\,\mathbf{w}\in\dot{\mathbf{H}}^{n/2}_{\#\sigma},$$

implying that $\|\langle \mathbf{Du}-\mathbf{U},\mathbf{w}\rangle_{\mathbb{T}}\|_{L_2(0,T)}=0$ for any $\mathbf{w}\in\dot{\mathbf{H}}^{n/2}_{\#\sigma}$ and thus $\langle \mathbf{Du}(\cdot,t)-\mathbf{U}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}=0$ for a.e. $t\in(0,T)$. Choosing $\mathbf{w}=\Lambda^n_\#(\mathbf{Du}-\mathbf{U})$, we conclude that $\|\mathbf{Du}(\cdot,t)-\mathbf{U}(\cdot,t)\|_{\dot{\mathbf{H}}^{-n/2}_{\#\sigma}}=0$ for a.e. $t\in(0,T)$ and hence $\|\mathbf{Du}-\mathbf{U}\|_{L_2(0,T;\dot{\mathbf{H}}^{-n/2}_{\#\sigma})}=0$, i.e., $\mathbf{Du}=\mathbf{U}\in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#\sigma})$.

Now we continue reasoning as, e.g., in [14, Chapter 1, Section 6.4.4] to conclude that the limit function $\mathbf{u}$ solves the initial-variational problem (3.9)–(3.10). Indeed, let us multiply equation (4.4) by an arbitrary $\phi\in L_2(0,T)$, integrate it in t to obtain.

$$\int_0^T \big[\big\langle \mathbf{u}'_m+\mathbb{P}_\sigma[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m],\mathbf{w}_k\big\rangle_{\mathbb{T}} + a_{\mathbb{T}}(\mathbf{u}_m,\mathbf{w}_k) - \langle \mathbf{f},\mathbf{w}_k\rangle_{\mathbb{T}}\big]\phi(t)dt = 0, \quad \forall\,k\in\{1,2,\dots\}. \quad (4.27)$$

To take the limit of (4.27) as $m\to\infty$, we remark that the terms linearly depending on $\mathbf{u}_m$ tend to the corresponding terms for $\mathbf{u}$ due to the weak convergences discussed before. For the nonlinear term, by (5.4) we have

$$\int_0^T \langle \mathbb{P}_\sigma[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m],\mathbf{w}_k\rangle_{\mathbb{T}}\,\phi(t)dt = \int_0^T \langle (\mathbf{u}_m\cdot\nabla)\mathbf{u}_m,\mathbf{w}_k\rangle_{\mathbb{T}}\,\phi(t)dt = -\int_0^T \langle (\mathbf{u}_m\cdot\nabla)\mathbf{w}_k,\mathbf{u}_m\rangle_{\mathbb{T}}\,\phi(t)dt$$
$$\to -\int_0^T \langle (\mathbf{u}\cdot\nabla)\mathbf{w}_k,\mathbf{u}\rangle_{\mathbb{T}}\,\phi(t)dt = \int_0^T \langle (\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{w}_k\rangle_{\mathbb{T}}\,\phi(t)dt = \int_0^T \langle \mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}],\mathbf{w}_k\rangle_{\mathbb{T}}\,\phi(t)dt,$$

where the limit is due to the strong convergence of $\mathbf{u}_m$ to $\mathbf{u}$ in $L_2(0,T;\dot{\mathbf{H}}^0_{\#\sigma})$ and the smoothness of $\mathbf{w}_k$. Thus, we obtain

$$\int_0^T \left[\left\langle \mathbf{u}' + \mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}], \mathbf{w}_k \right\rangle_{\mathbb{T}} + a_{\mathbb{T}}(\mathbf{u},\mathbf{w}_k) - \langle \mathbf{f},\mathbf{w}_k\rangle_{\mathbb{T}} \right]\phi(t)dt = 0, \quad \forall\, k\in\{1,2,\dots\}. \tag{4.28}$$

Since $\mathbf{Du} = \mathbf{u}' + \mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] \in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#\sigma})$ and $\{\mathbf{w}_k\}$ is a basis in $\dot{\mathbf{H}}^1_{\#\sigma}$, equation (4.28) implies that

$$\int_0^T \left[\left\langle \mathbf{u}' + \mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}], \mathbf{w} \right\rangle_{\mathbb{T}} + a_{\mathbb{T}}(\mathbf{u},\mathbf{w}) - \langle \mathbf{f},\mathbf{w}\rangle_{\mathbb{T}} \right]\phi(t)dt = 0, \quad \forall\, \phi\in L_2(0,T), \quad \forall\, \mathbf{w}\in\dot{\mathbf{H}}^1_{\#\sigma}. \tag{4.29}$$

Equation (4.29) means that

$$\left\| \left\langle \mathbf{u}' + \mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}], \mathbf{w} \right\rangle_{\mathbb{T}} + a_{\mathbb{T}}(\mathbf{u},\mathbf{w}) - \langle \mathbf{f},\mathbf{w}\rangle_{\mathbb{T}} \right\|_{L_2(0,T)} = 0 \quad \forall\, \mathbf{w}\in\dot{\mathbf{H}}^1_{\#\sigma},$$

which implies (3.9).

To prove (3.10), let us employ in (4.27) an arbitrary $\phi\in\mathcal{C}^\infty[0,T]$ such that $\phi(T)=0$, integrate the first term by parts with account of (4.5) and take the limit as $m\to\infty$ to obtain

$$\int_0^T \Big\{ \langle -\mathbf{u}(\cdot,t),\mathbf{w}_k\rangle_{\mathbb{T}}\,\phi'(t) + \langle \mathbb{P}_\sigma[(\mathbf{u}(\cdot,t)\cdot\nabla)\mathbf{u}(\cdot,t)]\phi(t),\mathbf{w}_k\rangle_{\mathbb{T}} + a_{\mathbb{T}}(\mathbf{u}(\cdot,t),\mathbf{w}_k)\phi(t) \\ - \langle \mathbf{f}(\cdot,t),\mathbf{w}_k\rangle_{\mathbb{T}}\phi(t)\Big\}dt = \langle \mathbf{u}^0,\mathbf{w}_k\rangle_{\mathbb{T}}\phi(0), \quad \forall\, k\in\{1,2,\dots\}. \tag{4.30}$$

Replacing in (4.30) $\mathbf{u}$ by its redefined version that is $\dot{\mathbf{H}}^0_{\#\sigma}$-weakly continuous in time (cf. the last paragraph of the step **(b)**) and integrating by parts the first term in (4.30), we get

$$\int_0^T \Big\{ \left\langle \mathbf{u}'(\cdot,t) + \mathbb{P}_\sigma[(\mathbf{u}(\cdot,t)\cdot\nabla)\mathbf{u}(\cdot,t)],\mathbf{w}_k\right\rangle_{\mathbb{T}} + a_{\mathbb{T}}(\mathbf{u}(\cdot,t),\mathbf{w}_k) \\ - \langle \mathbf{f}(\cdot,t),\mathbf{w}_k\rangle_{\mathbb{T}}\Big\}\phi(t)dt = \langle \mathbf{u}^0,\mathbf{w}_k\rangle_{\mathbb{T}}\phi(0) - \langle \mathbf{u}(\cdot,0),\mathbf{w}_k\rangle_{\mathbb{T}}\phi(0), \quad \forall\, k\in\{1,2,\dots\}.$$

Comparing with (4.29) and taking into account that $\phi(0)$ is arbitrary, we obtain that $\langle\mathbf{u}^0 - \langle\mathbf{u}(\cdot,0),\mathbf{w}_k\rangle_{\mathbb{T}}$, and because $\mathbf{w}_k$ is a basis in $\dot{\mathbf{H}}^0_{\#\sigma}$, we conclude that $\mathbf{u}^0=\mathbf{u}(\cdot,0)$ thus proving the initial condition (3.10).

The existence and uniqueness of the associated pressure $p\in L_2(0,T;\dot{H}^{-n/2+1}_{\#})$ follows from Lemma 3.2(iii).

(d) Let us prove the (strong) energy inequality (cf., [30, Chapter 3, Remark 4(ii)], [23, Theorem 4.6] and references therein, for the isotropic constant-coefficient case). Here we will generalise the proof of [23, Theorem 4.6]. To this end, let us consider again the subsequence $\{\mathbf{u}_m\}$ from the previous step, which still satisfies equation (4.10). Let $0\le t_0<t\le T$. Multiplying (4.10) by 2 and integrating it time, we get

$$\|\mathbf{u}_m(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + 2\int_{t_0}^t a_{\mathbb{T}}(\tau;\mathbf{u}_m(\cdot,\tau),\mathbf{u}_m(\cdot,\tau))d\tau = \|\mathbf{u}_m(\cdot,t_0)\|^2_{\mathbf{L}_{2\#}} + 2\int_{t_0}^t \langle\mathbf{f}(\cdot,\tau),\mathbf{u}_m(\cdot,\tau)\rangle_{\mathbb{T}}d\tau. \tag{4.31}$$

We would like to take limits of each term in (4.31) as $m\to\infty$. First of all, since $\mathbf{u}_m$ converges to $\mathbf{u}$ strongly in $L_2(0,T;\dot{\mathbf{H}}^0_{\#\sigma})$, we obtain that

$$\|\mathbf{u}_m(\cdot,\tau)\|^2_{\mathbf{L}_{2\#}} \to \|\mathbf{u}(\cdot,\tau)\|^2_{\mathbf{L}_{2\#}}, \text{ for a.e. } \tau\in[0,T]. \tag{4.32}$$

Further, since $\mathbf{u}_m$ converges to $\mathbf{u}$ weakly in $L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$ and $\mathbf{f}\in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#})\subset(L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma}))^*$, we have

$$\int_{t_0}^t \langle\mathbf{f}(\cdot,\tau),\mathbf{u}_m(\cdot,\tau)\rangle_{\mathbb{T}}d\tau \to \int_{t_0}^t \langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}d\tau, \quad \forall\,[t_0,t]\subset[0,T] \tag{4.33}$$

Finally, $\mathbf{u}_m$ converges to $\mathbf{u}$ weakly in $L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$ and

$$\|\mathbf{w}\|_{L_2(t_0,t;\dot{\mathbf{H}}^1_{\#})} := \left(\int_{t_0}^t a_{\mathbb{T}}(\tau;\mathbf{w}(\cdot,\tau),\mathbf{w}(\cdot,\tau))d\tau\right)^{1/2}$$

is an equivalent norm in $L_2(t_0,t;\dot{\mathbf{H}}^1_{\#\sigma})$, see (3.8). Since $\mathbf{u}$ is a weak limit of $\mathbf{u}_m$ in $L_2(t_0,t;\dot{\mathbf{H}}^1_{\#})$, we have (see, e.g., the Remark in Section 4.43 of [16]) that

$$\|\mathbf{u}\|^2_{L_2(t_0,t;\dot{\mathbf{H}}^1_{\#})} \le \liminf_{m\to\infty} \|\mathbf{u}_m\|^2_{L_2(t_0,t;\dot{\mathbf{H}}^1_{\#})}.$$

Hence

$$\int_{t_0}^t a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau \le \liminf_{m\to\infty}\int_{t_0}^t a_{\mathbb{T}}(\tau;\mathbf{u}_m(\cdot,\tau),\mathbf{u}_m(\cdot,\tau))d\tau, \quad \forall\,[t_0,t]\subset[0,T]. \tag{4.34}$$

Taking $\liminf_{m\to\infty}$ from both sides of (4.31), due to (4.32), (4.33) and (4.34), we obtain (4.1) for a.e. $[t_0,t]\subset[0,T]$.

Similar to the reasoning in the proof of Theorem 4.6 in [23], let us now prove that the (strong) energy inequality (4.1) holds also *for any* $[t_0,t]\subset[0,T]$. Let us take some t_0 for which (4.1) holds for a.e. $t'>t_0$. Let us now choose *any* $t\in(t_0,T]$. Then there exists a sequence $t'_i\to t$ such that

$$\|\mathbf{u}(\cdot,t'_i)\|^2_{\mathbf{L}_{2\#}} + 2\int_{t_0}^{t'_i} a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau \le \|\mathbf{u}(\cdot,t_0)\|^2_{\mathbf{L}_{2\#}} + 2\int_{t_0}^{t'_i}\langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}d\tau.$$

Since $\mathbf{u}\in L_2(0,T;\dot{\mathbf{H}}^1_{\#\sigma})$, we have

$$\int_{t_0}^{t'_i} a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau \to \int_{t_0}^{t} a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau, \qquad \int_{t_0}^{t'_i}\langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}d\tau \to \int_{t_0}^{t}\langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}d\tau.$$

On the other hand, the $\mathbf{L}_{2\#}$-weak continuity of $\mathbf{u}$ implies that

$$\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} \le \liminf_{t'_i\to t}\|\mathbf{u}(\cdot,t'_i)\|^2_{\mathbf{L}_{2\#}}.$$

Thus

$$\begin{aligned}\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + 2\int_{t_0}^t a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau &\le \liminf_{t'_i\to t}\left(\|\mathbf{u}(\cdot,t'_i)\|^2_{\mathbf{L}_{2\#}} + 2\int_{t_0}^{t'_i} a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau\right)\\ &\le \liminf_{t'_i\to t}\left(\|\mathbf{u}(\cdot,t_0)\|^2_{\mathbf{L}_{2\#}} + 2\int_{t_0}^{t'_i}\langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}d\tau\right) = \|\mathbf{u}(\cdot,t_0)\|^2_{\mathbf{L}_{2\#}} + 2\int_{t_0}^{t}\langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}d\tau\end{aligned}$$

By a similar argument, we can take any t_0. □

5 Auxiliary results

5.1 Advection term properties

The divergence theorem and periodicity imply the following identity for any $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\in\mathbf{C}^\infty_\#$.

$$\begin{aligned}\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_{\mathbb{T}} &= \int_{\mathbb{T}}\nabla\cdot(\mathbf{v}_1(\mathbf{v}_2\cdot\mathbf{v}_3))\,d\mathbf{x} - \langle(\nabla\cdot\mathbf{v}_1)\mathbf{v}_3 + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_3,\mathbf{v}_2\rangle_{\mathbb{T}}\\ &= -\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3,\mathbf{v}_2\rangle_{\mathbb{T}} - \langle(\nabla\cdot\mathbf{v}_1)\mathbf{v}_3,\mathbf{v}_2\rangle_{\mathbb{T}}\end{aligned} \tag{5.1}$$

Hence for any $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{C}^\infty_\#$,

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2, \mathbf{v}_2\rangle_{\mathbb{T}} = -\frac{1}{2}\langle(\nabla\cdot\mathbf{v}_1)\mathbf{v}_2, \mathbf{v}_2\rangle_{\mathbb{T}} = -\frac{1}{2}\left\langle \operatorname{div}\mathbf{v}_1, |\mathbf{v}_2|^2\right\rangle_{\mathbb{T}}. \tag{5.2}$$

In view of (5.1) we obtain the identity

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_{\mathbb{T}} = -\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_{\mathbb{T}} \quad \forall\ \mathbf{v}_1 \in \mathbf{C}^\infty_{\#\sigma},\ \mathbf{v}_2,\, \mathbf{v}_3 \in \mathbf{C}^\infty_\#, \tag{5.3}$$

and hence the following well known formula for any $\mathbf{v}_1 \in \mathbf{C}^\infty_{\#\sigma}$, $\mathbf{v}_2 \in \mathbf{C}^\infty_\#$,

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2, \mathbf{v}_2\rangle_{\mathbb{T}} = 0. \tag{5.4}$$

Equation (5.4) evidently holds also for $\mathbf{v}_1$ and $\mathbf{v}_2$ from the more general spaces, for which the left hand side in (5.4) is bounded and to which $\mathbf{C}^\infty_{\#\sigma}$ and $\mathbf{C}^\infty_\#$, respectively, are densely embedded.

5.2 Some point-wise multiplication results

Let us accommodated to the periodic function spaces in $\mathbb{R}^n$, $n \geq 1$, a particular case of a much more general Theorem 1 in Section 4.6.1 of [24] about point-wise products of functions/distributions.

THEOREM 5.1. *Assume $n \geq 1$, $s_1 \leq s_2$ and $s_1 + s_2 > 0$. Then there exists a constant $C_*(s_1, s_2, n) > 0$ such that for any $f_1 \in H^{s_1}_\#$ and $f_2 \in H^{s_1}_\#$,*

(a) $f_1 \cdot f_2 \in H^{s_1}_\#$ *and* $\|f_1\cdot f_2\|_{H^{s_1}_\#} \leq C_*(s_1, s_2, n)\|f_1\|_{H^{s_1}_\#}\|f_2\|_{H^{s_2}_\#}$ *if* $s_2 > n/2$;

(b) $f_1 \cdot f_2 \in H^{s_1+s_2-n/2}_\#$ *and* $\|f_1\cdot f_2\|_{H^{s_1+s_2-n/2}_\#} \leq C_*(s_1, s_2, n)\|f_1\|_{H^{s_1}_\#}\|f_2\|_{H^{s_2}_\#}$ *if* $s_2 < n/2$.

Proof. Items (a) and (b) follow, respectively, from items (i) and (iii) of [24, Theorem 1 in Section 4.6.1] when we take into account the norm equivalence in the standard and periodic Sobolev spaces. □

5.3 Spectrum of the periodic Bessel potential operator

In this section we assume that vector-functions/distributions $\mathbf{u}$ are generally complex-valued and the Sobolev spaces $\dot{\mathbf{H}}^s_{\#\sigma}$ are complex. Let us recall the definition

$$\left(\Lambda^r_\#\,\mathbf{u}\right)(\mathbf{x}) := \sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \varrho(\boldsymbol{\xi})^r\widehat{\mathbf{u}}(\boldsymbol{\xi})e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}} \quad \forall\,\mathbf{u} \in \dot{\mathbf{H}}^s_{\#\sigma},\ s\,rin\mathbb{R}. \tag{5.5}$$

of the continuous periodic Bessel potential operator $\Lambda^r_\# : \dot{\mathbf{H}}^s_{\#\sigma} \to \dot{\mathbf{H}}^{s-r}_{\#\sigma}$, $r \in \mathbb{R}$, see (2.10), (2.11), (2.16).

THEOREM 5.2. *Let $r \in \mathbb{R}$, $r \neq 0$.*

(i) Then the operator $\Lambda^r_\#$ in $\dot{\mathbf{H}}^0_{\#\sigma}$ possesses a (non-strictly) monotone sequence of real eigenvalues $\lambda^{(r)}_j$ and a real orthonormal sequence of associated eigenfunctions $\mathbf{w}_j$ such that

$$\Lambda^r_\#\mathbf{w}_j = \lambda^{(r)}_j\mathbf{w}_j,\ j \geq 1,\ \lambda^{(r)}_j > 0, \tag{5.6}$$

$$\lambda^{(r)}_j \to +\infty,\ j \to +\infty \text{ if } r > 0; \quad \lambda^{(r)}_j \to 0,\ j \to +\infty \text{ if } r < 0; \tag{5.7}$$

$$\mathbf{w}_j \in \dot{\mathbf{C}}^\infty_{\#\sigma}, \quad (\mathbf{w}_j, \mathbf{w}_k)_{\dot{\mathbf{H}}^0_{\#\sigma}} = \delta_{jk} \quad \forall\, j,k > 0. \tag{5.8}$$

(ii) Moreover, the sequence $\{\mathbf{w}_j\}$ is an orthonormal basis in $\dot{\mathbf{H}}^0_{\#\sigma}$, that is

$$\mathbf{u} = \sum_{i=1}^{\infty}\langle\mathbf{u}, \mathbf{w}_j\rangle_{\mathbb{T}}\mathbf{w}_j \tag{5.9}$$

where the series converges in $\dot{\mathbf{H}}^0_{\#\sigma}$ for any $\mathbf{u} \in \dot{\mathbf{H}}^0_{\#\sigma}$.

(iii) In addition, the sequence $\{\mathbf{w}_j\}$ *is also an orthogonal basis in* $\dot{\mathbf{H}}^r_{\#\sigma}$ *with*

$$(\mathbf{w}_j, \mathbf{w}_k)_{\dot{\mathbf{H}}^r_{\#\sigma}} = \lambda_j^{(r)}\lambda_k^{(r)}\delta_{jk} \quad \forall\, j,k>0.$$

and for any $\mathbf{u}\in\dot{\mathbf{H}}^r_{\#\sigma}$ *series* (5.9) *converges also in* $\dot{\mathbf{H}}^r_{\#\sigma}$*, that is, the sequence of partial sums*

$$P_m\mathbf{u} := \sum_{j=1}^m \langle \mathbf{u}, \mathbf{w}_j\rangle_{\mathbb{T}}\mathbf{w}_j \tag{5.10}$$

converges to $\mathbf{u}$ *in* $\dot{\mathbf{H}}^r_{\#\sigma}$ *as* $m\to\infty$*. The operator* P_m *defined by* (5.10) *is for any* $r\in\mathbb{R}$ *the orthogonal projector operator from* $\mathbf{H}^r_{\#}$ *to* $\mathrm{Span}\{\mathbf{w}_1,\dots,\mathbf{w}_m\}$*.*

Proof. Let first $r>0$ and let us consider the continuous periodic Bessel potential operator $\Lambda_{\#}^{-r}:\dot{\mathbf{H}}^0_{\#\sigma}\to\dot{\mathbf{H}}^r_{\#\sigma}$. Hence by the Rellich-Kondrachov theorem the operator $\Lambda_{\#}^{-r}:\dot{\mathbf{H}}^0_{\#\sigma}\to\dot{\mathbf{H}}^0_{\#\sigma}$ is compact. It is also self-adjoint since for any $\mathbf{u},\mathbf{v}\in\dot{\mathbf{H}}^0_{\#\sigma}$ we have,

$$(\Lambda_{\#}^{-r}\mathbf{u},\mathbf{v})_{\dot{\mathbf{H}}^0_{\#\sigma}} = \langle\Lambda_{\#}^{-r}\mathbf{u},\bar{\mathbf{v}}\rangle_{\mathbb{T}} = \langle\mathbf{u},\Lambda_{\#}^{-r}\bar{\mathbf{v}}\rangle_{\mathbb{T}} = (\mathbf{u},\Lambda_{\#}^{-r}\mathbf{v})_{\dot{\mathbf{H}}^0_{\#\sigma}}.$$

Then the Hilbert-Schmidt theorem (see, e.g., [22, Theorem 8.94]) implies that there is a sequence of nonzero real eigenvalues $\left\{\lambda_j^{(-r)}\right\}_{i=1}^\infty$ of the operator $\Lambda_{\#}^{-r}:\dot{\mathbf{H}}^0_{\#\sigma}\to\dot{\mathbf{H}}^0_{\#\sigma}$, such that the sequence $\left|\lambda_j^{(-r)}\right|$ is monotone non-increasing and $\lim_{i\to\infty}\lambda_j^{(-r)}=0$. Furthermore, if each eigenvalue of $\Lambda_{\#}^{-r}$ is repeated in the sequence according to its multiplicity, then there exists an orthonormal (in $\dot{\mathbf{H}}^0_{\#\sigma}$) set $\{\mathbf{w}_j\}_{i=1}^\infty$ of the corresponding eigenfunctions, i.e.,

$$\Lambda_{\#}^{-r}\mathbf{w}_j = \lambda_j^{(-r)}\mathbf{w}_j. \tag{5.11}$$

Moreover, the sequence $\{\mathbf{w}_j\}_{i=1}^\infty$ is an orthonormal basis in $\dot{\mathbf{H}}^0_{\#\sigma}$ for $\dot{\mathbf{H}}^r_{\#\sigma}$ as a subset of $\dot{\mathbf{H}}^0_{\#\sigma}$.

In addition, since the eigenvalues are real, (5.11) implies that the eigenfunctions are either real or appear for the same eigenvalue in complex-conjugate pairs and hence their real and imaginary parts are also eigenfunctions. This means that we can choose the orthonormal basis consisting of real eigenfunctions only.

Since $\dot{\mathbf{H}}^r_{\#\sigma}$ is dense in $\dot{\mathbf{H}}^0_{\#\sigma}$, the sequence $\{\mathbf{w}_j\}_{i=1}^\infty$ is an orthonormal basis for the entire space $\dot{\mathbf{H}}^0_{\#\sigma}$. The operator $\Lambda_{\#}^{-r}$ can be represented as

$$\Lambda_{\#}^{-r}\mathbf{v} = \sum_{i=1}^\infty \lambda_j^{(-r)}\langle\mathbf{v},\mathbf{w}_j\rangle_{\mathbb{T}}\mathbf{w}_j \quad \forall\,\mathbf{v}\in\dot{\mathbf{H}}^0_{\#\sigma}, \tag{5.12}$$

where the series converges in $\dot{\mathbf{H}}^0_{\#\sigma}$.

Let us remark that for any $\mathbf{v}\in\dot{\mathbf{H}}^0_{\#\sigma}$

$$(\Lambda_{\#}^{-r}\mathbf{v},\mathbf{v})_{\dot{\mathbf{H}}^0_{\#\sigma}} = \langle\Lambda_{\#}^{-r}\mathbf{v},\bar{\mathbf{v}}\rangle_{\mathbb{T}} = \langle\Lambda_{\#}^{-r/2}\mathbf{v},\overline{\Lambda_{\#}^{-r/2}\mathbf{v}}\rangle_{\mathbb{T}} = \|\Lambda_{\#}^{-r/2}\mathbf{v}\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} = \|\mathbf{v}\|^2_{\dot{\mathbf{H}}^{r/2}_{\#\sigma}} \ge \|\mathbf{v}\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}},$$

that is, $\Lambda_{\#}^{-r}$ is a positive-definite operator. To conclude that all λ_j are positive, we observe that for the unit real eigenfunctions $\mathbf{w}_j$, (5.11) implies

$$\lambda_j^{(-r)} = \lambda_j^{(-r)}\langle\mathbf{w}_j,\mathbf{w}_j\rangle_{\mathbb{T}} = \langle\Lambda_{\#}^{-r}\mathbf{w}_j,\mathbf{w}_j\rangle_{\mathbb{T}} = \langle\Lambda_{\#}^{-r/2}\mathbf{w}_j,\Lambda_{\#}^{-r/2}\mathbf{w}_j\rangle_{\mathbb{T}} > 0.$$

Applying Λ^r to (5.11), we obtain

$$\Lambda_{\#}^{r}\mathbf{w}_j = \lambda_j^{(r)}\mathbf{w}_j,\ \text{where } \lambda_j^{(r)} = 1/\lambda_j^{(-r)} \tag{5.13}$$

implying (5.6) with $\lambda_j^{(r)} = 1/\lambda_j^{(-r)}$ and the coinciding eigenfunctions for the operators $\Lambda_\#^r$ and $\Lambda_\#^{-r}$.

Since $\mathbf{w}_j \in \dot{\mathbf{H}}^0_{\#\sigma}$ and $\lambda_j \neq 0$, equation (5.11) implies $\mathbf{w}_j \in \dot{\mathbf{H}}^r_{\#\sigma}$. Moreover, applying to (5.11) operator $\Lambda^{-r(k-1)}$, with any integer k, and employing consecutively (5.13) or (5.11), we obtain

$$\Lambda_\#^{-rk}\mathbf{w}_j = (\lambda_j^{(-r)})^k\mathbf{w}_j \quad \forall\, k \in \mathbb{Z}. \tag{5.14}$$

and taking into account the continuity of the operator $\Lambda_\#^{-rk} : \dot{\mathbf{H}}^0_{\#\sigma} \to \dot{\mathbf{H}}^{kr}_{\#\sigma}$ for any integer k, we conclude that $\mathbf{w}_j \in \dot{\mathbf{C}}^\infty_{\#\sigma}$.

Finally, let us prove that the sequence $\{\mathbf{w}_j\}$ is an orthogonal basis also in $\dot{\mathbf{H}}^r_{\#\sigma}$. To this end, let $\mathbf{u} \in \dot{\mathbf{H}}^r_{\#\sigma}$. We know that the series (5.9) converges in $\dot{\mathbf{H}}^0_{\#\sigma}$. Let us prove that it converges also in $\dot{\mathbf{H}}^r_{\#\sigma}$, that is, the sequence of its partial sums converges in this space. Indeed,

$$\sum_{j=1}\langle\mathbf{u},\mathbf{w}_j\rangle_{\mathbb{T}}\mathbf{w}_j = \sum_{j=1}\langle\mathbf{u},\lambda_j^{(r)}\mathbf{w}_j\rangle_{\mathbb{T}}\lambda_j^{(-r)}\mathbf{w}_j = \sum_{j=1}\langle\mathbf{u},\Lambda_\#^r\mathbf{w}_j\rangle_{\mathbb{T}}\Lambda_\#^{-r}\mathbf{w}_j = \Lambda_\#^{-r}\sum_{j=1}\langle\Lambda_\#^r\mathbf{u},\mathbf{w}_j\rangle_{\mathbb{T}}\mathbf{w}_j \tag{5.15}$$

Since $\mathbf{u} \in \dot{\mathbf{H}}^r_{\#\sigma}$ we have that $\Lambda_\#^r\mathbf{u} \in \dot{\mathbf{H}}^0_{\#\sigma}$ implying that the sequence $\sum_{i=1}\langle\Lambda_\#^r\mathbf{u},\mathbf{w}_j\rangle_{\mathbb{T}}\mathbf{w}_j$ converges in $\dot{\mathbf{H}}^0_{\#\sigma}$ to $\Lambda_\#^r\mathbf{u}$ as $m \to \infty$. The continuity of the operator $\Lambda_\#^{-r} : \dot{\mathbf{H}}^0_{\#\sigma} \to \dot{\mathbf{H}}^r_{\#\sigma}$ then implies that the right hand side of (5.15) converges in $\dot{\mathbf{H}}^r_{\#\sigma}$ to $\mathbf{u}$ together with the sequence of the partial sums in the left hand side. This means that series (5.9) converges in $\dot{\mathbf{H}}^r_{\#\sigma}$ to $\mathbf{u}$ as well. Thus the set $\{\mathbf{w}_j\}$ is complete in $\dot{\mathbf{H}}^r_{\#\sigma}$.

The orthogonality of the set $\{\mathbf{w}_j\}$ in $\dot{\mathbf{H}}^r_{\#\sigma}$ is implied by the relations

$$(\mathbf{w}_j,\mathbf{w}_k)_{\dot{\mathbf{H}}^r_{\#\sigma}} = (\Lambda_\#^r\mathbf{w}_j,\Lambda_\#^r\mathbf{w}_k)_{\dot{\mathbf{H}}^0_{\#\sigma}} = (\lambda_j^{(r)}\mathbf{w}_j,\lambda_k^{(r)}\mathbf{w}_k)_{\dot{\mathbf{H}}^0_{\#\sigma}} = \lambda_j^{(r)}\lambda_k^{(r)}\langle\mathbf{w}_j,\mathbf{w}_k\rangle_{\mathbb{T}} = \lambda_j^{(r)}\lambda_k^{(r)}\delta_{jk}.$$

Hence the set $\{\mathbf{w}_j\}$ is an orthogonal basis in $\dot{\mathbf{H}}^r_{\#\sigma}$.

Although we started from $r > 0$, in the proof we covered the cases of both positive and negative r. □

Similar to the reasoning at the end of Section 2.2 in [29], for the eigenvalues and eigenfunctions of the isotropic Stokes operator in periodic setting, let us provide an explicit representation of the eigenvalues and eigenfunctions of the operator $\Lambda_\#^r : \dot{\mathbf{H}}^0_{\#\sigma} \to \dot{\mathbf{H}}^0_{\#\sigma}$, $r \in \mathbb{R}$, $r \neq 0$.

Employing representations (2.6) and (5.5) in (5.6), we obtain for a fixed j,

$$\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n}\varrho(\boldsymbol{\xi})^r\widehat{\mathbf{w}}_j(\boldsymbol{\xi})e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}} = \lambda_j^{(r)}\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n}\widehat{\mathbf{w}}_j(\boldsymbol{\xi})e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}, \tag{5.16}$$

that is,

$$\left(\varrho(\boldsymbol{\xi})^r - \lambda_j^{(r)}\right)\widehat{\mathbf{w}}_j(\boldsymbol{\xi}) = 0 \quad \forall\,\boldsymbol{\xi} \in \dot{\mathbb{Z}}^n. \tag{5.17}$$

This implies that the eigenvalues and the corresponding eigenfunctions of the operator Λ^r can be explicitly represented as $\{\lambda_j^{(r)}\} = \{\lambda_{\boldsymbol{\eta},\beta}^{(r)}\}$, $\{\mathbf{w}_j\} = \{\mathbf{w}_{\boldsymbol{\eta},\beta}\}$, where $\boldsymbol{\eta} \in \dot{\mathbb{Z}}^n$, $\beta = \{1,\dots,n-1\}$,

$$\lambda_{\boldsymbol{\eta},\beta}^{(r)} = \varrho(\boldsymbol{\eta})^r = (1+|\boldsymbol{\eta}|^2)^{r/2}, \quad \mathbf{w}_{\boldsymbol{\eta},\beta}(\mathbf{x}) = \mathring{\mathbf{w}}_{\boldsymbol{\eta},\beta}e^{2\pi i\mathbf{x}\cdot\boldsymbol{\eta}}. \tag{5.18}$$

For a fixed $\boldsymbol{\eta}$, the $n-1$ orthonormal constant real vectors $\mathring{\mathbf{w}}_{\boldsymbol{\eta},\beta}$, $\beta = \{1,\dots,n-1\}$ are obtained by the orthogonalisation in $\mathbb{R}^n$ of the real vector set

$$\widetilde{\mathbf{w}}_{\boldsymbol{\eta},\alpha} = \mathbf{e}_\alpha - \frac{\eta_\alpha\boldsymbol{\eta}}{|\boldsymbol{\eta}|^2}, \quad \alpha = \{1,\dots,n\},$$

where $\mathbf{e}_\alpha$ are canonical (coordinate) vectors in $\mathbb{R}^n$. Note that $(\widetilde{\mathbf{w}}_{\boldsymbol{\eta},\alpha}\cdot\boldsymbol{\eta}) = 0$.

REMARK 5.3. *Relations (5.18) particularly imply that $\lambda_{\boldsymbol{\eta},\beta}^{(r)} = \lambda_{\boldsymbol{\eta},\beta}^r$, where $\lambda_{\boldsymbol{\eta},\beta} := \lambda_{\boldsymbol{\eta},\beta}^{(1)} = \varrho(\boldsymbol{\eta}) = (1+|\boldsymbol{\eta}|^2)^{1/2}$, i.e., $\lambda_j^{(r)} = \lambda_j^r$ and the corresponding eigenfunctions coincide for any $r \in \mathbb{R}$, $r \neq 0$. Since the sequence of eigenfunctions $\{\mathbf{w}_j\}$ corresponding to $\lambda_j^{(r)}$ is the same for any $r \in \mathbb{R}$, $r \neq 0$, Theorem 5.2 implies that the sequence constitutes a real orthogonal basis in any space $\dot{\mathbf{H}}^r_{\#\sigma}$, $r \in \mathbb{R}$.*

5.4 Isomorphism of divergence and gradient operators in periodic spaces

In the following assertion we provide for arbitrary $s \in \mathbb{R}$ and dimension $n \geq 2$ the periodic version of Bogovskii/deRham–type results well known for non-periodic domains and particular values of s, see, e.g., [4], [3] and references therein.

LEMMA 5.4. *Let $s \in \mathbb{R}$ and $n \geq 2$. The operators*

$$\mathrm{div}\,: \dot{\mathbf{H}}^{s+1}_{\#g} \to \dot{H}^{s}_{\#}, \tag{5.19}$$

$$\mathrm{grad}\,: \dot{H}^{s}_{\#} \to \dot{\mathbf{H}}^{s-1}_{\#g} \tag{5.20}$$

are isomorphisms.

Proof. (i) Since $\dot{\mathbf{H}}^{s+1}_{\#g} \subset \dot{\mathbf{H}}^{s+1}_{\#}$, operator (5.19) is continuous. Let $f \in \dot{H}^{s}_{\#}$ and let us consider the equation

$$\mathrm{div}\,\mathbf{F} = f \tag{5.21}$$

for $\mathbf{F} \in \dot{\mathbf{H}}^{s+1}_{\#g}$. Calculating the Fourier coefficients of both sides of the equation, we obtain

$$2\pi i \boldsymbol{\xi} \cdot \widehat{\mathbf{F}}(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \dot{\mathbb{Z}}^n.$$

By inspection one can see that this equation has a solution in the form

$$\widehat{\mathbf{F}}(\boldsymbol{\xi}) = \frac{\boldsymbol{\xi}\widehat{f}(\boldsymbol{\xi})}{2\pi i|\boldsymbol{\xi}|^2}, \quad \boldsymbol{\xi} \in \dot{\mathbb{Z}}^n, \tag{5.22}$$

that is,

$$\widehat{\mathbf{F}}(\boldsymbol{\xi}) = 2\pi i \boldsymbol{\xi}\widehat{q} = \widehat{\nabla q}, \text{ where } \widehat{q} = -\frac{\widehat{f}(\boldsymbol{\xi})}{(2\pi)^2|\boldsymbol{\xi}|^2}, \quad \boldsymbol{\xi} \in \dot{\mathbb{Z}}^n.$$

By (5.22), (2.2) and (2.3), we obtain

$$\|\mathbf{F}\|^2_{\dot{\mathbf{H}}^{s+1}_{\#}} = \sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \varrho(\boldsymbol{\xi})^{2(s+1)}|\widehat{\mathbf{F}}(\boldsymbol{\xi})|^2 = \sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \varrho(\boldsymbol{\xi})^{2s}\frac{\varrho(\boldsymbol{\xi})^2}{(2\pi)^2|\boldsymbol{\xi}|^2}|\widehat{f}(\boldsymbol{\xi})|^2 \leq 2\sum_{\boldsymbol{\xi}\in\dot{\mathbb{Z}}^n} \varrho(\boldsymbol{\xi})^{2s}|\widehat{f}(\boldsymbol{\xi})|^2 = 2\|f\|^2_{\dot{H}^s_{\#}}.$$

Hence the solution $\mathbf{F}$ given by (5.22) belongs to $\dot{\mathbf{H}}^{s+1}_{\#g}$ and satisfies the estimate $\|\mathbf{F}\|_{\dot{\mathbf{H}}^{s+1}_{\#}} \leq \sqrt{2}\|f\|_{\dot{H}^s_{\#}}$. There are no other solutions in $\dot{\mathbf{H}}^{s+1}_{\#g}$ since otherwise the difference, $\widetilde{\mathbf{F}}$, of two solutions of equation (5.21) would satisfy equation $\mathrm{div}\,\widetilde{\mathbf{F}} = 0$, and hence belong to $\dot{\mathbf{H}}^{s+1}_{\#g} \cap \dot{\mathbf{H}}^{s+1}_{\#\sigma} = \{\mathbf{0}\}$. Thus operator (5.19) is an isomorphism.

(ii) By the definition of the space $\dot{\mathbf{H}}^{s-1}_{\#g}$, operator (5.20) is continuous. Let $\mathbf{F} \in \dot{\mathbf{H}}^{s-1}_{\#g}$ and let us consider the equation

$$\nabla f = \mathbf{F} \tag{5.23}$$

for $f \in \dot{H}^s_{\#}$. Equation (5.23) has at most one solution since otherwise the difference of any two solutions, $\widetilde{f}$, would satisfy the equation $\nabla\widetilde{f} = \mathbf{0}$ implying that $\widetilde{f} = const = 0$ because $f \in \dot{H}^s_{\#}$. Taking into account that $\mathbf{F} = \nabla q$ for some $q \in \dot{H}^s_{\#}$, we conclude that there exists a solution of equation (5.23), namely $f = q$.

Let us calculate the norm estimate for this solution. Calculating the Fourier coefficients of both sides of equation (5.23), we obtain

$$2\pi i \boldsymbol{\xi}\widehat{f}(\boldsymbol{\xi}) = \widehat{\mathbf{F}}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \dot{\mathbb{Z}}^n. \tag{5.24}$$

Then

$$\widehat{f}(\boldsymbol{\xi}) = \frac{\boldsymbol{\xi}\cdot\widehat{\mathbf{F}}(\boldsymbol{\xi})}{2\pi i|\boldsymbol{\xi}|^2}, \quad \boldsymbol{\xi} \in \dot{\mathbb{Z}}^n. \tag{5.25}$$

By (5.25), (2.2) and (2.3), we obtain

$$\|f\|^2_{\dot H^s_\#}=\sum_{\boldsymbol\xi\in\dot{\mathbb Z}^n}\varrho(\boldsymbol\xi)^{2s}|\widehat f(\boldsymbol\xi)|^2=\sum_{\boldsymbol\xi\in\dot{\mathbb Z}^n}\frac{\varrho(\boldsymbol\xi)^{2s}}{(2\pi)^2|\boldsymbol\xi|^4}|\boldsymbol\xi\cdot\widehat{\mathbf F}(\boldsymbol\xi)|^2\le 2\sum_{\boldsymbol\xi\in\dot{\mathbb Z}^n}\varrho(\boldsymbol\xi)^{2(s-1)}|\widehat{\mathbf F}(\boldsymbol\xi)|^2=2\|\mathbf F\|^2_{\dot{\mathbf H}^{s-1}_\#}.$$

Hence the solution f given by (5.25) belongs to $\dot H^s_\#$ and satisfies the estimate $\|f\|_{\dot H^s_\#}\le\sqrt2\|\mathbf F\|_{\dot{\mathbf H}^{s-1}_\#}$. Thus operator (5.20) is an isomorphism. □

5.5 Some functional analysis results

The Aubin–Lions Lemma, see [14, Chapter 1, Theorem 5.1], has been generalised in [27]. We provide it in the form of Theorem 4.12 in [23].

THEOREM 5.5 (Aubin–Lions Lemma). *Suppose that $G\subset H\subset K$ where G,H and K are reflexive Banach spaces and the embedding $G\subset H$ is compact. Let $1\le p\le\infty$ and $1\le q<\infty$. If the sequence u_n is bounded in $L_q(0,T;G)$ and $\partial_t u_n$ is bounded in $L_p(0,T;K)$, then there exists a subsequence of u_n that is strongly convergent in $L_q(0,T;H)$.*

The following assertion is available in [30, Chapter 3, Lemma 1.4]

LEMMA 5.6. *Let X and Y be two Banach spaces, such that $X\subset Y$ with a continuous injection. If a function v belongs to $L_\infty(0,T;X)$ and is weakly continuous with values in Y, then v is weakly continuous with values in X.*

Theorem 3.1 and Remark 3.2 in Chapter 1 of [15] imply the following assertion.

THEOREM 5.7. *Let X and Y be separable Hilbert spaces and $X\subset Y$ with continuous injection. Let $u\in W^1(0,T;X,Y)$. Than u almost everywhere on $[0,T]$ equals to a function $\tilde u\in\mathcal C^0([0,T];Z)$, where $Z=[X,Y]_{1/2}$ is the intermediate space, Then the trace $u(0)\in Z$ is defined as the corresponding value of $\tilde u\in\mathcal C^0([0,T];Z)$ at $t=0$.*

Let us prove the following assertion inspired by Lemmas 1.2 and 1.3 in Chapter 3 of [30].

LEMMA 5.8. *Let $s,s'\in\mathbb R$, $s'\le s$ and $u\in W^1(0,T;H^s_\#,H^{s'}_\#)$ be real-valued.*
(i) Then

$$\partial_t\|u\|^2_{H^{(s+s')/2}_\#}=2\langle\Lambda^{s'}_\# u',\Lambda^s_\# u\rangle_{\mathbb T}=2\langle\Lambda^{s'+s}_\# u',u\rangle_{\mathbb T}\tag{5.26}$$

for a.e. $t\in(0,T)$ and also in the distribution sense on $t\in(0,T)$.
(ii) Moreover, for any real $v\in W^1(0,T;H^{-s'}_\#,H^{-s}_\#)$ and $t\in(0,T]$,

$$\int_0^t\left[\langle u'(\tau),v(\tau)\rangle_{\mathbb T}+\langle u(\tau),v'(\tau)\rangle_{\mathbb T}\right]d\tau=\langle u(t),v(t)\rangle_{\mathbb T}-\langle u(0),v(0)\rangle_{\mathbb T}.\tag{5.27}$$

Proof. (i) Since $u\in W^1(0,T;H^s_\#,H^{s'}_\#)$, there exists a sequence of infinitely differentiable functions $\{u_m\}$ from $[0,T]$ onto $H^s_\#$, such that

$$u_m\to u\text{ in }W^1(0,T;H^s_\#,H^{s'}_\#)\quad\text{as }m\to\infty.\tag{5.28}$$

For each u_m, we have

$$\begin{aligned}\partial_t\|u_m(t)\|^2_{H^{(s+s')/2}_\#}&=\partial_t\|\Lambda^{(s+s')/2}_\# u_m(t)\|^2_{H^0_\#}=\partial_t\left\langle\Lambda^{(s+s')/2}_\# u_m(t),\Lambda^{(s+s')/2}_\# u_m(t)\right\rangle_{\mathbb T}\\&=2\mathrm{Re}\left\langle\Lambda^{(s+s')/2}_\# u'_m(t),\Lambda^{(s+s')/2}_\# u_m(t)\right\rangle_{\mathbb T}=2\mathrm{Re}\left\langle\Lambda^{s'}_\# u'_m(t),\Lambda^s_\# u_m(t)\right\rangle_{\mathbb T}.\end{aligned}\tag{5.29}$$

By (5.28),

$$\|u_m\|^2_{H^s_\#} = \|\Lambda^s_\# u_m\|^2_{L_{2\#}} \to \|\Lambda^s_\# u\|^2_{L_{2\#}} = \|u\|^2_{H^s_\#} \text{ in } L_{1\#}(0,T),$$
$$\|u'_m\|^2_{H^{s'}_\#} = \|\Lambda^{s'}_\# u'_m\|^2_{L_{2\#}} \to \|\Lambda^{s'}_\# u'\|^2_{L_{2\#}} = \|u'\|^2_{H^{s'}_\#} \text{ in } L_{1\#}(0,T).$$

Hence

$$\left\langle \Lambda^{s'}_\# u'_m, \Lambda^s_\# u_m \right\rangle_{\mathbb{T}} \to \left\langle \Lambda^{s'}_\# u', \Lambda^s_\# u \right\rangle_{\mathbb{T}} \quad \text{ in } L_{1\#}(0,T).$$

These convergences also hold for a.e. $t \in (0,T)$ and in the distribution sense; therefore we are allowed to pass to the limit in (5.29) in the distribution sense, arriving at (5.26) in the limit.

(ii) Since $u \in W^1(0,T; H^s_\#, H^{s'}_\#)$ and $v \in W^1(0,T; H^{-s'}_\#, H^{-s}_\#)$, the dual products under the integral in (5.27) are bounded in $L_1(0,T)$ and hence the integral is well defined. On the other hand, Theorem 5.7 implies that u and v almost everywhere on $[0,T]$ equal to, respectively, functions $\tilde{u} \in \mathcal{C}^0([0,T]; H^{(s+s')/2}_\#)$ and $\tilde{v} \in \mathcal{C}^0([0,T]; H^{-(s+s')/2}_\#)$. Then the traces $u(t), v(t), u(0), v(0)$ are defined as the corresponding values of $\tilde{u}$ and $\tilde{v}$, implying that the dual products in the last two terms in (5.27) are well defined. Further in the proof we redefine u and v on a set of measure zero in $[0,T]$ as the functions $\tilde{u}$ and $\tilde{v}$, respectively.

There exists a sequence of infinitely differentiable functions $\{v_k\}$ from $[0,T]$ onto $H^{-s'}_\#$, such that $v_k \to v$ in $W^1(0,T; H^{-s'}_\#, H^{-s}_\#)$, $k \to \infty$. For each u_m and v_k, we have

$$\left\langle u'_m(t), v_k(t) \right\rangle_{\mathbb{T}} + \left\langle u_m(t), v'_k(t) \right\rangle_{\mathbb{T}} = \partial_t \langle u_m(t), v_k(t) \rangle_{\mathbb{T}},$$

which after the integration in t leads to

$$\int_0^t \left[\langle u'_m(\tau), v_k(\tau) \rangle_{\mathbb{T}} + \langle u_m(\tau), v'_k(\tau) \rangle_{\mathbb{T}} \right] d\tau = \langle u_m(t), v_k(t) \rangle_{\mathbb{T}} - \langle u_m(0), v_k(0) \rangle_{\mathbb{T}}.$$

Taking the limits as $m \to \infty$ and $k \to \infty$, we get (5.27). □

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