On Solonnikov Parabolicity of the Evolution Anisotropic Stokes and Oseen PDE Systems

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Abstract In this note it is shown that the evolution (non-stationary) anisotropic Stokes and generalised Oseen systems with variable viscosity coefficients in a compressible framework can be classified as parabolic PDE systems in the sense of Solonnikov.

Keywords Partial differential equations · Evolution anisotropic Stokes and Oseen equations · Parabolicity

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1 Introduction: Solonnikov Parabolicity

In this section, we present a generalised notion of parabolicity for PDE systems introduced by Solonnikov in [7, Section 1] (see also [5, Chapter VII, Section 8, Definition 4], [2, Definition I.4]). It can be considered as a parabolic counterpart of the Agmon-Douglis-Nirenberg ellipticity for PDE systems.

Following [5, Chapter VII, Section 8], we will begin with the Petrovskii definition of a scalar parabolic equation. Let $L(x, t, \partial/\partial\mathbf{x}, \partial/\partial t)$ be a scalar linear partial differential operator of arbitrary order with complex coefficients depending on $\mathbf{x}$ and t. Let $\boldsymbol{\xi} \in \mathbb{R}^n$ be an n-dimensional vector and $\varrho \in \mathbb{C}$ be a scalar. It is clear that at any point $(\mathbf{x}, t)$, the function $L(\mathbf{x}, t, i\boldsymbol{\xi}, \varrho)$ is a polynomial in ξ_j and ϱ. Let b be some positive integer and let the degree of the polynomial $L(\mathbf{x}, t, i\boldsymbol{\xi}\lambda, \varrho\lambda^{2b})$ in λ be equal to $2br$, where r is a positive integer. We denote by L_0 the principal part of the polynomial L, i.e., the sum of those terms of L, for which

$$L_0(\mathbf{x}, t, i\boldsymbol{\xi}\lambda, \varrho\lambda^{2b}) = \lambda^{2br} L_0(\mathbf{x}, t, i\boldsymbol{\xi}, \varrho). \tag{1.1}$$

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Definition 1.1 *A scalar partial differential operator L is said to be parabolic ($2b$-parabolic) at a point $(\mathbf{x}, t)$ if for any real vector $\boldsymbol{\xi}$ the roots, ϱ_s, of the principal part, $L_0(\mathbf{x}, t, i\boldsymbol{\xi}, \varrho)$, of the polynomial L in variable ϱ satisfy the condition*

$$\operatorname{Re}\varrho_s \le -\delta|\boldsymbol{\xi}|^{2b} \tag{1.2}$$

for some constant $\delta > 0$.

Definition 1.2 *Let $m \ge 2$ be a positive integer. A matrix differential operator $\mathbb{L}(\mathbf{x}, t, \partial_{\mathbf{x}}, \partial_t)$ with elements $\mathbb{L}_{kj}(\mathbf{x}, t, \partial_{\mathbf{x}}, \partial_t)$, $(k, j = 1, \dots, m)$ will be called parabolic at point $(\mathbf{x}, t)$ in the sense of Solonnikov if:*

(i) the operator

$$L(x, t, \partial/\partial\mathbf{x}, \partial/\partial t) = \det\mathbb{L}(\mathbf{x}, t, \partial_{\mathbf{x}}, \partial_t)$$

is $2b$-parabolic in the sense of Definition 1.1 for some positive integer b;

(ii) there exist integers s_ℓ and t_ℓ, $\boldsymbol{\ell} = 1, \dots, m$ such that the degree of the polynomial $\mathbb{L}_{\boldsymbol{\ell}j}(\mathbf{x}, t, i\boldsymbol{\xi}\lambda, \varrho\lambda^{2b})$ in λ does not exceed $s_\ell + t_j$ (if $s_\ell + t_j < 0$ then $\mathbb{L}_{\boldsymbol{\ell}j} = 0$) and, in addition,

$$\sum_{\ell=1}^{m}(s_\ell + t_\ell) = 2br,$$

where r is the degree of the polynomial $L(\mathbf{x}, t, i\boldsymbol{\xi}, \varrho)$ in the variable ϱ.

2 Anisotropic Stokes and Oseen PDE Systems

Let $n \ge 2$ be an integer, $\mathbf{x} \in \mathbb{R}^n$ denote the space coordinate vector, and $t \in \mathbb{R}$ be time variable. Let $\mathfrak{L}$ denote the second-order differential operator represented in the component-wise divergence form as

$$(\mathfrak{L}\mathbf{u})_k := \partial_\alpha\big(a_{kj}^{\alpha\beta} E_{j\beta}(\mathbf{u})\big), \quad k = 1, \dots, n, \tag{2.1}$$

where $\mathbf{u} = (u_1, \dots, u_n)^\top$, $E_{j\beta}(\mathbf{u}) := \frac{1}{2}(\partial_j u_\beta + \partial_\beta u_j)$ are the entries of the symmetric part $\mathbb{E}(\mathbf{u})$ of the gradient, $\nabla\mathbf{u}$, of $\mathbf{u}$ in space coordinates, and $a_{kj}^{\alpha\beta}(\mathbf{x}, t)$ are variable components of the tensor viscosity coefficient, cf. [1], $\mathbb{A}(\mathbf{x}, t) = \left\{a_{kj}^{\alpha\beta}(\mathbf{x}, t)\right\}_{i,j,\alpha,\beta=1}^{n}$ depending on the space coordinate vector $\mathbf{x}$ and time t. We also denoted $\partial_j = \dfrac{\partial}{\partial x_j}$. Here and further on, the Einstein convention on summation in repeated indices from 1 to n is used unless stated otherwise.

The following symmetry conditions are assumed (cf. [6, Eq. (3.1), (3.3)]),

$$a_{kj}^{\alpha\beta}(\mathbf{x},t) = a_{\alpha j}^{k\beta}(\mathbf{x},t) = a_{k\beta}^{\alpha j}(\mathbf{x},t). \tag{2.2}$$

In addition, we require that the tensor $\mathbb{A}$ satisfies *the relaxed ellipticity condition* in terms of all *symmetric* matrices in $\mathbb{R}^{n\times n}$ with *zero matrix trace*, see [3, 4]. Thus, we assume that there exists a constant $C_{\mathbb{A}} > 0$ such that,

$$a_{kj}^{\alpha\beta}(\mathbf{x},t)\zeta_{k\alpha}\zeta_{j\beta} \geq C_{\mathbb{A}}^{-1}|\boldsymbol{\zeta}|^2, \quad \text{for a.e. } \mathbf{x}, t, \tag{2.3}$$

$$\forall\, \boldsymbol{\zeta} = \{\zeta_{k\alpha}\}_{k,\alpha=1,\dots,n} \in \mathbb{R}^{n\times n} \text{ such that } \boldsymbol{\zeta} = \boldsymbol{\zeta}^\top \text{ and } \sum_{k=1}^{n}\zeta_{kk} = 0,$$

where $|\boldsymbol{\zeta}|^2 = \zeta_{k\alpha}\zeta_{k\alpha}$, and the superscript $^\top$ denotes the transpose of a matrix. Note that in the more common, strong ellipticity condition, inequality (2.3) should be satisfied for all matrices (not only symmetric with zero trace), which makes it much more restrictive.

We assume that $a_{ij}^{\alpha\beta} \in L_\infty(\mathbb{R}^n \times [0,T])$, where $[0,T]$ is some time interval, and the tensor $\mathbb{A}$ is endowed with the norm

$$\|\mathbb{A}\| := \max\left[\|a_{ij}^{\alpha\beta}\|_{L_\infty(\mathbb{R}^n\times[0,T])} : k, j, \alpha, \beta \in \{1\dots,n\}\right] < \infty. \tag{2.4}$$

Symmetry conditions (2.2) lead to the following equivalent form of the operator $\boldsymbol{\mathfrak{L}}$

$$(\boldsymbol{\mathfrak{L}}\mathbf{u})_k = \partial_\alpha\big(a_{kj}^{\alpha\beta}\partial_\beta u_j\big), \quad k = 1,\dots,n. \tag{2.5}$$

Let $\mathbf{u}(\mathbf{x},t)$ be an unknown vector velocity field, $p(\mathbf{x},t)$ be an unknown (scalar) pressure field, $\mathbf{f}(\mathbf{x},t)$ be a given vector field and $g(\mathbf{x},t)$ be a given scalar field defined in $\mathbb{R}^n$, where $t \in \mathbb{R}$ is the time variable. Then the linear PDE system

$$\partial_t\mathbf{u}(\mathbf{x},t) - \boldsymbol{\mathfrak{L}}\mathbf{u}(\mathbf{x},t) + \nabla p(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t), \tag{2.6}$$

$$\operatorname{div}\mathbf{u}(\mathbf{x},t) = g(\mathbf{x},t). \tag{2.7}$$

determines the *anisotropic evolution Stokes system in a compressible framework.*

A more general linear system

$$\partial_t\mathbf{u}(\mathbf{x},t) - \boldsymbol{\mathfrak{L}}\mathbf{u}(\mathbf{x},t) + \nabla p(\mathbf{x},t) + (\mathbf{U}(\mathbf{x},t)\cdot\nabla)\mathbf{u}(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t), \tag{2.8}$$

$$\operatorname{div}\mathbf{u}(\mathbf{x},t) = g(\mathbf{x},t), \tag{2.9}$$

where $\mathbf{U}(\mathbf{x},t)$ is a given vector field defined at a.e. $(\mathbf{x},t)$, is the (generalised) evolution Oseen system.

If $g = 0$ in (2.7) and (2.9), then systems (2.6)–(2.7) and (2.8)–(2.9) are reduced to the evolution *incompressible* anisotropic Stokes and Oseen systems, respectively.

In the *isotropic case*, the tensor $\mathbb{A}$ reduces to

$$a_{kj}^{\alpha\beta}(\mathbf{x},t) = \lambda(\mathbf{x},t)\delta_{k\alpha}\delta_{j\beta} + \mu(\mathbf{x},t)\left(\delta_{\alpha j}\delta_{\beta k} + \delta_{\alpha\beta}\delta_{kj}\right), \ 1 \le k, j, \alpha, \beta \le n, \tag{2.10}$$

where $\lambda, \mu \in L_\infty(\mathbb{R}^n \times [0,T])$, and $c_\mu^{-1} \le \mu(\mathbf{x},t) \le c_\mu$ for a.e. $\mathbf{x}$ and t, with some constant $c_\mu > 0$ (cf., e.g., Appendix III, Part I, Section 1 in [8]). Then it is immediate that condition (2.3) is fulfilled with $C_{\mathbb{A}} = c_\mu/2$ and thus our results apply also to the Stokes and Oseen systems in the *isotropic case*. Moreover, (2.1) becomes

$$\mathfrak{L}\mathbf{u} = (\lambda+\mu)\nabla \operatorname{div}\mathbf{u} + \mu\Delta\mathbf{u} + (\nabla\lambda)\operatorname{div}\mathbf{u} + 2(\nabla\mu)\cdot\mathbb{E}(\mathbf{u}) \tag{2.11}$$

Assuming $\lambda = 0$ and $\mu = 1$ we arrive at the classical mathematical formulations of isotropic, constant-coefficient Stokes and Oseen systems.

3 Classification of the Evolution Anisotropic Stokes and Oseen Systems

Let us consider the Oseen system (2.8)–(2.9). It can be re-written as

$$\mathbf{\Lambda}_{\mathrm{U}}\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix} = \begin{pmatrix}\mathbf{f}\\ g\end{pmatrix}, \tag{3.1}$$

where the operator $\mathbf{\Lambda}_{\mathrm{U}} = \mathbf{\Lambda}_{\mathrm{U}}(\mathbf{x}, t, \partial_{\mathbf{x}}, \partial_t)$ is defined as

$$\mathbf{\Lambda}_{\mathrm{U}}\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix} := \begin{pmatrix}\partial_t\mathbf{u} - \mathfrak{L}\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p\\ \operatorname{div}\mathbf{u}\end{pmatrix}. \tag{3.2}$$

If $\mathbf{U} \equiv \mathbf{0}$, the evolution Oseen system (2.8)–(2.9) reduces to the evolution Stokes system (2.6)–(2.7) that can be re-written as

$$\mathbf{\Lambda}\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix} = \begin{pmatrix}\mathbf{f}\\ g\end{pmatrix}, \tag{3.3}$$

where the operator $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathbf{x}, t, \partial_{\mathbf{x}}, \partial_t)$ is defined as

$$\mathbf{\Lambda}\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix} = \begin{pmatrix}\partial_t\mathbf{u} - \mathfrak{L}\mathbf{u} + \nabla p\\ \operatorname{div}\mathbf{u}\end{pmatrix}. \tag{3.4}$$

Let us classify the operator $\mathbf{\Lambda}$ and the PDE system (2.6)–(2.7). Since Eq. (2.7) does not include the time derivative, ∂_t, PDE system (2.6)–(2.7) and the operator

$\mathbf{\Lambda}$ are not parabolic in the sense of Petrovskii or in the sense of Shirota (see the corresponding definition, e.g., in [5, Chapter VII, Section 8, Definitions 2 and 3], [2, Definitions I.2 and I.5]). However we will prove that under the condition (2.3) of relaxed ellipticity for the viscosity tensor, the operator $\mathbf{\Lambda}$ is parabolic in the sense of Solonnikov.

To this end, we first need to prove several auxiliary results. Replacing in the operator $\mathbf{\Lambda}$ arguments ∂_j by $i\xi_j$ and ∂_t by ρ we express its matrix entries as polynomials in $\rho \in \mathbb{C}$ and $\boldsymbol{\xi} \in \mathbb{R}^n$, i.e.,

$$\Lambda_{\ell j}(\mathbf{x}, t, i\boldsymbol{\xi}, \rho) = \begin{cases} \rho\delta_{\ell j} + \xi_\alpha a_{\ell j}^{\alpha\beta}\xi_\beta, & \ell, j = 1, \dots, n; \\ -i\xi_\ell, & \ell = 1, \dots, n, \;\; j = n+1; \\ -i\xi_j, & \ell = n+1, \;\; j = 1, \dots, n; \\ 0, & \ell = j = n+1. \end{cases} \tag{3.5}$$

Here $a_{\ell j}^{\alpha\beta} = a_{\ell j}^{\alpha\beta}(\mathbf{x}, t)$.

Lemma 3.1 *Let $n \geq 2$ and the relaxed ellipticity condition (2.3) hold at a point $(\mathbf{x}, t)$. Let $\rho \in \mathbb{C}$ and $\boldsymbol{\xi} \in \mathbb{R}^n$. If* $\det \mathbf{\Lambda}(\mathbf{x}, t, i\boldsymbol{\xi}, \rho) = 0$, *then*

$$\operatorname{Re} \rho \leq -\frac{1}{2} C_{\mathbb{A}}^{-1} |\boldsymbol{\xi}|^2. \tag{3.6}$$

Proof Let us assume that $\det \mathbf{\Lambda}(\mathbf{x}, t, i\boldsymbol{\xi}, \rho) = 0$. If $\boldsymbol{\xi} = \mathbf{0}$, then from (3.5) we have $\rho = 0$ implying that inequality (3.6) is satisfied.

Let us now consider the case $\boldsymbol{\xi} \neq \mathbf{0}$. Then there should exist a non-trivial vector $(\mathbf{w}, q) \in \mathbb{C}^n \times \mathbb{C}$ solving the homogeneous algebraic system

$$\mathbf{\Lambda}(\mathbf{x}, t, i\boldsymbol{\xi}, \rho)\begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}. \tag{3.7}$$

If we assume $\mathbf{w} = \mathbf{0}$, then the first n equations of (3.7) imply that $\boldsymbol{\xi} q = \mathbf{0}$, and since $\boldsymbol{\xi} \neq \mathbf{0}$, this means $q = 0$, that is, the vector $(\mathbf{w}, q)$ is trivial. Then we further assume that $\mathbf{w} \neq \mathbf{0}$. The last equation of the system implies

$$\mathbf{w} \cdot \boldsymbol{\xi} = w_j \xi_j = 0. \tag{3.8}$$

After the scalar multiplication of the system (3.7) by the vector $(\overline{\mathbf{w}}, 0)$, where the bar means complex conjugate, we have,

$$\overline{w}_\ell[\Lambda_{\ell j}(\mathbf{x}, t, i\xi, \rho) w_j - i\xi_\ell q] = \rho|\mathbf{w}|^2 + a_0(\mathbf{w}, \mathbf{w}) = 0, \tag{3.9}$$

where

$$a_0(\mathbf{w},\mathbf{w}) := \overline{w}_\ell \xi_\alpha a_{\ell j}^{\alpha\beta} \xi_\beta w_j = a_{\ell j}^{\alpha\beta}(\overline{\mathbf{w}}\otimes\boldsymbol{\xi})_{\ell\alpha}(\mathbf{w}\otimes\boldsymbol{\xi})_{\beta j}$$
$$= a_{\ell j}^{\alpha\beta}(\overline{\mathbf{w}}\otimes\boldsymbol{\xi})^s_{\ell\alpha}(\mathbf{w}\otimes\boldsymbol{\xi})^s_{\beta j}. \quad (3.10)$$

due to the symmetry conditions (2.2). Here $(\mathbf{w}\otimes\boldsymbol{\xi})^s$ is the symmetric part of the matrix $\mathbf{w}\otimes\boldsymbol{\xi}$, i.e.,

$$(\mathbf{w}\otimes\boldsymbol{\xi})^s_{\ell\alpha} := \frac{1}{2}(w_\ell\xi_\alpha + w_\alpha\xi_\ell), \quad \ell,\alpha = 1,\dots,n. \quad (3.11)$$

By (3.8) and the ellipticity condition (2.3), Eq. (3.9) implies

$$\begin{aligned}
-\operatorname{Re}\rho|\mathbf{w}|^2 &= \operatorname{Re} a_0(\mathbf{w},\mathbf{w}) \\
&= a_{\ell j}^{\alpha\beta}(\operatorname{Re}\mathbf{w}\otimes\boldsymbol{\xi})^s_{\ell\alpha}(\operatorname{Re}\mathbf{w}\otimes\boldsymbol{\xi})^s_{\beta j} + a_{\ell j}^{\alpha\beta}(\operatorname{Im}\mathbf{w}\otimes\boldsymbol{\xi})^s_{\ell\alpha}(\operatorname{Im}\mathbf{w}\otimes\boldsymbol{\xi})^s_{\beta j} \\
&\geq C_{\mathbb{A}}^{-1}(|(\operatorname{Re}\mathbf{w}\otimes\boldsymbol{\xi})^s|^2 + |(\operatorname{Im}\mathbf{w}\otimes\boldsymbol{\xi})^s|^2) \\
&= \frac{1}{2}C_{\mathbb{A}}^{-1}(|\operatorname{Re}\mathbf{w}|^2|\boldsymbol{\xi}|^2 + |\operatorname{Im}\mathbf{w}|^2|\boldsymbol{\xi}|^2) = \frac{1}{2}C_{\mathbb{A}}^{-1}|\mathbf{w}|^2|\boldsymbol{\xi}|^2,
\end{aligned}$$

which leads to (3.6). □

Let us introduce the differential operator $L(\mathbf{x},t,\partial_\mathbf{x},\partial_t) = \det\boldsymbol{\Lambda}(\mathbf{x},t,\partial_\mathbf{x},\partial_t)$. Then $L(\mathbf{x},t,i\boldsymbol{\xi},\rho) = \det\boldsymbol{\Lambda}(\mathbf{x},t,i\boldsymbol{\xi},\rho)$ is a polynomial in $\rho\in\mathbb{C}$ and $\boldsymbol{\xi}\in\mathbb{R}^n$ and moreover,

$$L(\mathbf{x},t,i\boldsymbol{\xi}\lambda,\rho\lambda^2) = \lambda^{2n}L(\mathbf{x},t,i\boldsymbol{\xi},\rho). \quad (3.12)$$

Theorem 3.2 *Let $n\geq 2$ and the relaxed ellipticity condition* (2.3) *hold at a point* $(\mathbf{x},t)$. *Then at this point the evolution anisotropic Stokes operator* $\boldsymbol{\Lambda}$ *defined by* (3.4) *is parabolic in the sense of Solonnikov, i.e., satisfies Definition 1.2.*

Proof We can choose $m = n+1$ in Definition 1.2. By (3.12), the scalar operator $L(\mathbf{x},t,\partial_\mathbf{x},\partial_t) = \det\boldsymbol{\Lambda}(\mathbf{x},t,\partial_\mathbf{x},\partial_t)$ satisfies (1.1) with $L_0 = L$, $b=1$, and $r=n$. Due to Lemma 3.1, then the operator $\det\boldsymbol{\Lambda}(\mathbf{x},t,\partial_\mathbf{x},\partial_t)$ satisfies Definition 1.1 with $\delta = \frac{1}{2}C_{\mathbb{A}}^{-1}$. Hence it is 2-parabolic and item (i) in Definition 1.2 is satisfied.

Due to (3.5), item (ii) in Definition 1.2 is satisfied for $s_\ell = t_\ell = 1$, $\ell = 1,\dots,n$ and $s_{n+1} = t_{n+1} = 0$. Note that another choice, e.g., $s_\ell = 0$, $t_\ell = 2$, $\ell = 1,\dots,n$ and $s_{n+1} = -1$, $t_{n+1} = 1$ is also possible. □

It is easy to see that the principal part of the Oseen operator (3.2) is obtained by taking $\mathbf{U}\equiv\mathbf{0}$, that is, it is given by the corresponding Stokes operator, which leads to the following assertion implied by Theorem 3.2.

Corollary 3.3 *Let $n \geq 2$, the relaxed ellipticity condition* (2.3) *hold at a point* $(\mathbf{x}, t)$ *and the function* $\mathbf{U}(\mathbf{x}, t)$ *be defined at* $(\mathbf{x}, t)$. *Then at this point the evolution anisotropic Oseen operator* $\mathbf{\Lambda}_{\mathrm{U}}$ *defined by* (3.2) *is parabolic in the sense of Solonnikov, i.e., satisfies Definition 1.2.*

The classification of the evolution anisotropic Stokes and Oseen operators as parabolic in the sense of Solonnikov allows to apply to them some well-posedness results available for initial-boundary value problems for the Solonnikov parabolic systems, see, e.g., [7], [5, Chapter VII, Section 10], [2, Section VI.3.6], [9].

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