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34

BDIE System to the Mixed BVP for the Stokes Equations with Variable Viscosity

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34.1 Introduction

The mixed (Dirichlet-Neumann) boundary value problem for the steady-state Stokes system of PDEs for an incompressible viscous fluid with variable viscosity coefficient is reduced to a system of direct segregated Boundary-Domain Integral Equations (BDIEs). Mapping properties of the potential type integral operators appearing in these equations are presented in appropriate Sobolev spaces. We also prove the equivalence between the original BVP and the corresponding BDIE system.

Let $\Omega = \Omega^+ \subset \mathbb{R}^3$ be a bounded connected domain with boundary $\partial\Omega = S$, which is a closed and simply connected infinitely differentiable manifold of dimension 2, and $\overline{\Omega} = \Omega \cup S$. The exterior of the domain Ω is denoted as $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$. Moreover, let $S = \overline{S_D} \cup \overline{S_N}$ where both S_N and S_D are non-empty disjointed and simply connected open manifolds of S.

Let $\boldsymbol{v}$ be the velocity vector field; p the pressure scalar field and $\mu \in \mathcal{C}^\infty(\Omega)$ be the variable kinematic viscosity of the fluid such that $\mu(\boldsymbol{x}) > c > 0$.

For a compressible fluid the stress tensor operator, σ_{ij}, for an arbitrary couple $(\boldsymbol{v}, p)$ is defined as

$$\sigma_{ji}(\boldsymbol{v}, p)(\boldsymbol{x}) := -\delta_i^j p(\boldsymbol{x}) + \mu(\boldsymbol{x}) \left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} - \frac{2}{3}\delta_{ij} \operatorname{div} \boldsymbol{v} \right),$$

and the Stokes operator is defined as

$$\begin{aligned}\mathcal{A}_j(\boldsymbol{v}, p)(\boldsymbol{x}) : &= \frac{\partial}{\partial x_i} \sigma_{ji}(\boldsymbol{v}, p)(\boldsymbol{x}) \\ &= \frac{\partial}{\partial x_i} \left(\mu(\boldsymbol{x}) \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta_{ij} \operatorname{div} \boldsymbol{v} \right) \right) - \frac{\partial p}{\partial x_j},\ j, i \in \{1, 2, 3\},\end{aligned}$$

where δ_i^j is Kronecker symbol. Here and henceforth we assume the Einstein summation in repeated indices from 1 to 3. We also denote the Stokes operator as $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_j\}_{j=1}^3$.

For an incompressible fluid $\operatorname{div} \boldsymbol{v} = 0$, which reduces the stress tensor operator and the Stokes operator, respectively, to

$$\sigma_{ij}(\boldsymbol{v},p)(\boldsymbol{x}) = -\delta_i^j p(\boldsymbol{x}) + \mu(\boldsymbol{x})\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right),$$
$$\mathcal{A}_j(\boldsymbol{v},p)(\boldsymbol{x}) = \frac{\partial}{\partial x_i}\left(\mu(\boldsymbol{x})\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\right) - \frac{\partial p}{\partial x_j}.$$

In what follows $H^s(\Omega) = H_2^s(\Omega)$, $H^s(\partial\Omega) = H_2^s(\partial\Omega)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ is an arbitrary real number (see, e.g., [LiMa73], [McL00]). We recall that H^s coincide with the Sobolev–Slobodetski spaces W_2^s for any non-negative s. We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^3)$, $\widetilde{H}^s(\Omega) := \{g : \ g \in H^s(\mathbb{R}^3),\ \operatorname{supp} g \subset \overline{\Omega}\}$; similarly, $\widetilde{H}^s(S_1) = \{g \in H^s(S),\ \operatorname{supp}\ g \subset \overline{S}_1\}$ is the Sobolev space of functions having support in $S_1 \subset S = \partial\Omega$. We will also use the notation like $\boldsymbol{H}^s(\Omega) = [H^s(\Omega)]^n$ for the $n-$dimensional counterparts of all the aforementioned spaces. Let $\boldsymbol{H}^s_{\operatorname{div}}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^s(\Omega) : \operatorname{div} \boldsymbol{v} = 0\}$ be the divergence-free Sobolev space.

We will also make use of the following spaces, (cf. e.g. [Co88] [CMN09])

$$\mathbb{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) := \{(\boldsymbol{v},p) \in \boldsymbol{H}^1(\Omega) \times L_2(\Omega) : \boldsymbol{\mathcal{A}}(\boldsymbol{v},p) \in \boldsymbol{L}_2(\Omega)\},$$
$$\mathbb{H}^{1,0}_{\operatorname{div}}(\Omega;\boldsymbol{\mathcal{A}}) := \{(\boldsymbol{v},p) \in \boldsymbol{H}^1_{\operatorname{div}}(\Omega) \times L_2(\Omega) : \boldsymbol{\mathcal{A}}(\boldsymbol{v},p) \in \boldsymbol{L}_2(\Omega)\},$$

endowed with the same norm, $\|(\boldsymbol{v},p)\|_{\mathbb{H}^{1,0}_{\operatorname{div}}(\Omega;L)} = \|(\boldsymbol{v},p)\|_{\mathbb{H}^{1,0}(\Omega;L)}$, where

$$\|(\boldsymbol{v},p)\|_{\mathbb{H}^{1,0}(\Omega;L)} := \left(\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} + \|p\|^2_{L_2(\Omega)} + \|\boldsymbol{\mathcal{A}}(\boldsymbol{v},p)\|^2_{\boldsymbol{L}_2(\Omega)}\right)^{\frac{1}{2}}.$$

For sufficiently smooth functions $\boldsymbol{v}$ and p in $\Omega^\pm$, we can write the classical traction operators on the boundary S as

$$T_i^\pm(\boldsymbol{v},p)(\boldsymbol{x}) := \gamma^\pm \sigma_{ij}(\boldsymbol{v},p)(\boldsymbol{x})\, n_j(\boldsymbol{x}), \tag{34.1}$$

where $n_j(\boldsymbol{x})$ denote components of the unit outward normal vector $\boldsymbol{n}(\boldsymbol{x})$ to the boundary S of the domain Ω and $\gamma^\pm$ are the trace operators from inside and outside Ω.

Traction operators (34.1) can be continuously extended to the *canonical* traction operators $\boldsymbol{T}^\pm : \mathbb{H}^{1,0}(\Omega^\pm, \boldsymbol{\mathcal{A}}) \to \boldsymbol{H}^{-\frac{1}{2}}(S)$ defined in the weak form similar to [Co88, Mi11, CMN09] as

$$\langle \boldsymbol{T}^\pm(\boldsymbol{v},p), \boldsymbol{w}\rangle_S := \pm \int_{\Omega^\pm} \left[\boldsymbol{\mathcal{A}}(\boldsymbol{v},p)\boldsymbol{\gamma}^{-1}\boldsymbol{w} + \mathcal{E}\left((\boldsymbol{v},p), \boldsymbol{\gamma}^{-1}\boldsymbol{w}\right)\right] d\boldsymbol{x},$$

$$\forall\,(\boldsymbol{v},p)\in\mathbb{H}^{1,0}(\Omega^{\pm},\boldsymbol{\mathcal{A}}),\ \forall\,\boldsymbol{w}\in\boldsymbol{H}^{\frac{1}{2}}(S).$$

Here the operator $\boldsymbol{\gamma}^{-1}:\boldsymbol{H}^{\frac{1}{2}}(S)\to\boldsymbol{H}^{1}(\mathbb{R}^3)$ denotes a continuous right inverse of the trace operator $\boldsymbol{\gamma}:\boldsymbol{H}^{1}(\mathbb{R}^3)\to\boldsymbol{H}^{\frac{1}{2}}(S)$, and the bilinear form $\mathcal{E}$ is defined as

$$\begin{aligned}\mathcal{E}\left((\boldsymbol{v},p),\boldsymbol{u}\right)(\boldsymbol{x}):=\ &\frac{1}{2}\mu(x)\left(\frac{\partial u_i(\boldsymbol{x})}{\partial x_j}+\frac{\partial u_j(\boldsymbol{x})}{\partial x_i}\right)\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j}+\frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right)\\&-\frac{2}{3}\mu(x)\mathrm{div}\,\boldsymbol{v}(\boldsymbol{x})\,\mathrm{div}\,\boldsymbol{u}(\boldsymbol{x})-p(\boldsymbol{x})\mathrm{div}\,\boldsymbol{u}(\boldsymbol{x}).\end{aligned}$$

Furthermore, if $(\boldsymbol{v},p)\in\mathbb{H}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$ and $\boldsymbol{u}\in\boldsymbol{H}^{1}(\Omega)$, the following first Green identity holds, cf. [Co88, Mi11, CMN09],

$$\langle\boldsymbol{T}^{+}(\boldsymbol{v},p),\boldsymbol{\gamma}^{+}\boldsymbol{u}\rangle_S=\int_{\Omega}[\boldsymbol{\mathcal{A}}(\boldsymbol{v},p)\boldsymbol{u}+\mathcal{E}\left((\boldsymbol{v},p),\boldsymbol{u}\right)(\boldsymbol{x})]d\boldsymbol{x}.\tag{34.2}$$

For $(\boldsymbol{v},p)\in\mathbb{H}^{1,0}_{\mathrm{div}}(\Omega^{\pm},\boldsymbol{\mathcal{A}})$ the *canonical* traction operators can be reduced to $\boldsymbol{T}^{\pm}:\mathbb{H}^{1,0}_{\mathrm{div}}(\Omega^{\pm},\boldsymbol{\mathcal{A}})\to\boldsymbol{H}^{-\frac{1}{2}}(S)$ defined as

$$\begin{aligned}\langle\boldsymbol{T}^{\pm}(\boldsymbol{v},p),\boldsymbol{w}\rangle_S:=\pm\int_{\Omega^{\pm}}&\left[\boldsymbol{\mathcal{A}}(\boldsymbol{v},p)\boldsymbol{\gamma}^{-1}_{\mathrm{div}}\boldsymbol{w}+\mathcal{E}(\boldsymbol{v},\boldsymbol{\gamma}^{-1}_{\mathrm{div}}\boldsymbol{w})\right]d\boldsymbol{x}\\&\forall\,(\boldsymbol{v},p)\in\mathbb{H}^{1,0}_{\mathrm{div}}(\Omega^{\pm},\boldsymbol{\mathcal{A}}),\ \forall\,\boldsymbol{w}\in\boldsymbol{H}^{\frac{1}{2}}(S).\end{aligned}$$

Here the operator $\boldsymbol{\gamma}^{-1}_{\mathrm{div}}:\boldsymbol{H}^{\frac{1}{2}}(S)\to\boldsymbol{H}^{1}_{\mathrm{div}}(\mathbb{R}^3)$ denotes a continuous right inverse of the trace operator $\boldsymbol{\gamma}:\boldsymbol{H}^{1}_{\mathrm{div}}(\mathbb{R}^3)\to\boldsymbol{H}^{\frac{1}{2}}(S)$, and the bilinear form $\mathcal{E}$ reduces to

$$\mathcal{E}(\boldsymbol{v},\boldsymbol{u})(\boldsymbol{x}):=\frac{\mu(\boldsymbol{x})}{2}\left(\frac{\partial u_i(\boldsymbol{x})}{\partial x_j}+\frac{\partial u_j(\boldsymbol{x})}{\partial x_i}\right)\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j}+\frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right).$$

For $(\boldsymbol{v},p)\in\mathbb{H}^{1,0}_{\mathrm{div}}(\Omega,\boldsymbol{\mathcal{A}})$ and $\boldsymbol{u}\in\boldsymbol{H}^{1}_{\mathrm{div}}(\Omega)$, the first Green identity takes the same form (34.2), where $\mathcal{E}\left((\boldsymbol{v},p),\boldsymbol{u}\right)(\boldsymbol{x})$ reduces to $\mathcal{E}\left(v,\boldsymbol{u}\right)(\boldsymbol{x})$.

Applying the identity (34.2) to the pairs $(\boldsymbol{v},p)\in\mathbb{H}^{1,0}_{\mathrm{div}}(\Omega,\boldsymbol{\mathcal{A}})$ and $(\boldsymbol{u},q)\in\mathbb{H}^{1,0}_{\mathrm{div}}(\Omega,\boldsymbol{\mathcal{A}})$ with exchanged roles and subtracting the one from the other, we arrive at the second Green identity, cf. [McL00, Mi11],

$$\int_{\Omega}\left[\mathcal{A}_j(\boldsymbol{v},p)u_j-\mathcal{A}_j(\boldsymbol{u},q)v_j\right]d\boldsymbol{x}=\int_S\left[T_j(\boldsymbol{v},p)u_j-T_j(\boldsymbol{u},q)v_j\right]dS.\tag{34.3}$$

Now we are ready to define the mixed boundary value problem for which we aim to derive equivalent boundary-domain integral equation systems (BDIEs) and investigate the existence and uniqueness of their solutions.

For $\boldsymbol{f}\in\boldsymbol{L}_2(\Omega)$, $\boldsymbol{\varphi}_0\in\boldsymbol{H}^{\frac{1}{2}}(S_D)$ *and* $\boldsymbol{\psi}_0\in\boldsymbol{H}^{-\frac{1}{2}}(S_N)$, *find* $(\boldsymbol{v},p)\in\mathbb{H}^{1,0}_{\mathrm{div}}(\Omega,\boldsymbol{\mathcal{A}})$ *such that:*

$$\boldsymbol{\mathcal{A}}(\boldsymbol{v},p)(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x}),\qquad\boldsymbol{x}\in\Omega,\tag{34.4a}$$

$$r_{S_D}\boldsymbol{\gamma}^+\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\varphi}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in S_D, \tag{34.4b}$$

$$r_{S_N}\boldsymbol{T}^+(\boldsymbol{v}, p)(\boldsymbol{x}) = \boldsymbol{\psi}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in S_N. \tag{34.4c}$$

The following assertion can be easily proved by the Lax-Milgram lemma.

Theorem 1 *Mixed boundary value problem* (34.4) *is uniquely solvable.*

34.2 Parametrix and Parametrix-Based Hydrodynamic Potentials

When $\mu(\boldsymbol{x}) = 1$, the operator $\boldsymbol{\mathcal{A}}$ becomes the constant-coefficient Stokes operator $\mathring{\boldsymbol{\mathcal{A}}}$, for which we know an explicit fundamental solution defined by the pair of distributions $(\mathring{\boldsymbol{u}}^k, \mathring{q}^k)$ where $\mathring{u}_j^k$ represent components of the incompressible velocity fundamental solution and $\mathring{q}^k$ represent the components of the pressure fundamental solution (see e.g. [La69], [KoWe06], [HsWe08]).

$$\mathring{u}_j^k(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{8\pi}\left\{\frac{\delta_j^k}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right\},$$

$$\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}) = \frac{x_k - y_k}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}, \qquad j, k \in \{1, 2, 3\}.$$

Therefore $(\mathring{\boldsymbol{u}}^k, \mathring{q}^k)$ satisfy

$$\mathring{\mathcal{A}}_j(\mathring{\boldsymbol{u}}^k, \mathring{q}^k)(\boldsymbol{x}) = \sum_{i=1}^{3}\frac{\partial^2 \mathring{u}_j^k}{\partial x_i^2} - \frac{\partial \mathring{q}^k}{\partial x_j} = \delta_j^k \delta(\boldsymbol{x}-\boldsymbol{y})$$

Let us denote $\mathring{\sigma}_{ij}(\boldsymbol{v}, p) := \sigma_{ij}(\boldsymbol{v}, p)|_{\mu=1}$. Then in the particular case, for $\mu = 1$ and the fundamental solution $(\mathring{\boldsymbol{u}}^k, \mathring{q}^k)_{k=1,2,3}$ of the operator $\mathring{\boldsymbol{\mathcal{A}}}$, the stress tensor $\mathring{\sigma}_{ij}(\mathring{\boldsymbol{u}}^k, \mathring{q}^k)(\boldsymbol{x}-\boldsymbol{y})$ reads

$$\mathring{\sigma}_{ij}(\mathring{\boldsymbol{u}}^k, \mathring{q}^k)(\boldsymbol{x}-\boldsymbol{y}) = \frac{3}{4\pi}\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5},$$

and the boundary traction becomes

$$\begin{aligned}\mathring{T}_i(\boldsymbol{x}; \mathring{\boldsymbol{u}}^k, \mathring{q}^k)(\boldsymbol{x}, \boldsymbol{y}) :&= \mathring{\sigma}_{ij}(\mathring{\boldsymbol{u}}^k, \mathring{q}^k)(\boldsymbol{x}-\boldsymbol{y})\, n_j(\boldsymbol{x}) \\ &= \frac{3}{4\pi}\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5}\, n_j(\boldsymbol{x}).\end{aligned}$$

Let us define a pair of functions $(\boldsymbol{u}^k, q^k)_{k=1,2,3}$ as

$$u_j^k(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{u}_j^k(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{8\pi\mu(\boldsymbol{y})}\left\{\frac{\delta_j^k}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right\}, \tag{34.5}$$

$$q^k(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\frac{x_k - y_k}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}, \qquad j,k \in \{1,2,3\}. \tag{34.6}$$

Then

$$\sigma_{ij}(\boldsymbol{x};\boldsymbol{u}^k,q^k)(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{\sigma}_{ij}(\mathring{\boldsymbol{u}}^k,\mathring{q}^k)(\boldsymbol{x}-\boldsymbol{y}),$$
$$T_i(\boldsymbol{x};\boldsymbol{u}^k,q^k)(\boldsymbol{x},\boldsymbol{y}) := \sigma_{ij}(\boldsymbol{x};\boldsymbol{u}^k,q^k)(\boldsymbol{x},\boldsymbol{y})\,n_j(\boldsymbol{x}) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{T}_i(\boldsymbol{x};\mathring{\boldsymbol{u}}^k,\mathring{q}^k)(\boldsymbol{x},\boldsymbol{y}).$$

Substituting (34.5)-(34.6) to the Stokes system gives

$$\mathcal{A}_j(\boldsymbol{x};\boldsymbol{u}^k,q^k)(\boldsymbol{x},\boldsymbol{y}) = \delta_j^k\delta(\boldsymbol{x}-\boldsymbol{y}) + R_{kj}(\boldsymbol{x},\boldsymbol{y}), \tag{34.7}$$

where

$$R_{kj}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\frac{\partial\mu(\boldsymbol{x})}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{\boldsymbol{u}}^k,\mathring{q}^k)(\boldsymbol{x}-\boldsymbol{y}) = \mathcal{O}(|\boldsymbol{x}-\boldsymbol{y}|)^{-2})$$

is a weakly singular remainder. This implies that $(\boldsymbol{u}^k, q^k)$ is a parametrix of the operator $\boldsymbol{\mathcal{A}}$.

Let us define the parametrix-based Newton-type and remainder vector potentials

$$\mathcal{U}_k\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{U}_{kj}\rho_j(\boldsymbol{y}) := \int_\Omega u_j^k(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,d\boldsymbol{x},$$
$$\mathcal{R}_k\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{R}_{kj}\rho_j(\boldsymbol{y}) := \int_\Omega R_{kj}(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,d\boldsymbol{x}, \quad \boldsymbol{y}\in\mathbb{R}^3,$$

for the velocity, and the scalar Newton-type and remainder potentials

$$\mathcal{Q}\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{Q}_j\rho_j(\boldsymbol{y}) := \int_\Omega \mathring{q}^j(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})d\boldsymbol{x}, \tag{34.8}$$
$$\mathcal{R}^\bullet\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{R}_j^\bullet\rho_j(\boldsymbol{y}) := 2\,\text{v.p.}\int_\Omega \frac{\partial\mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial x_i}\frac{\partial\mu(\boldsymbol{x})}{\partial x_i}\rho_j(\boldsymbol{x})d\boldsymbol{x} - \frac{4}{3}\boldsymbol{v}(\boldsymbol{y})\cdot\nabla\mu(\boldsymbol{y}), \quad \boldsymbol{y}\in\Omega, \tag{34.9}$$

for the pressure. The integral in (34.9) is understood as a 3D strongly singular integral in the Cauchy sense.

For the velocity, let us also define the parametrix-based single layer potential, double layer potential and their respective direct values on the boundary, as follows,

$$V_k\boldsymbol{\rho}(\boldsymbol{y}) = V_{kj}\rho_j(\boldsymbol{y}) := -\int_S u_j^k(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,dS_x, \quad \boldsymbol{y}\notin S,$$
$$W_k\boldsymbol{\rho}(\boldsymbol{y}) = W_{kj}\rho_j(\boldsymbol{y}) := -\int_S T_j(\boldsymbol{x};\boldsymbol{u}^k,q^k)(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,dS_x, \quad \boldsymbol{y}\notin S,$$

$$\mathcal{V}_k\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{V}_{kj}\rho_j(\boldsymbol{y}) := -\int_S u_j^k(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,dS_x, \quad \boldsymbol{y}\in S,$$
$$\mathcal{W}_k\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{W}_{kj}\rho_j(\boldsymbol{y}) := -\int_S T_j(\boldsymbol{x};\boldsymbol{u}^k,q^k)(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,dS_x, \quad \boldsymbol{y}\in S.$$

Let us also denote

$$\mathcal{W}'_k\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{W}'_{kj}\rho_j(\boldsymbol{y}) := -\int_S T_j(\boldsymbol{y};\boldsymbol{u}^k,\mathring{q}^k)(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,dS_x, \quad \boldsymbol{y}\in S.$$

For pressure in the variable coefficient Stokes system, we will need the following the single-layer and double layer potentials,

$$\mathcal{P}\boldsymbol{\rho}(\boldsymbol{y}) = \mathcal{P}_j\rho_j(\boldsymbol{y}) := -\int_S \mathring{q}^j(\boldsymbol{x},\boldsymbol{y})\rho_j(\boldsymbol{x})\,dS_x,$$
$$\Pi\boldsymbol{\rho}(\boldsymbol{y}) = \Pi_j\rho_j(\boldsymbol{y}) := -2\int_S \frac{\partial\mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{x})}\mu(\boldsymbol{x})\rho_j(\boldsymbol{x})\,dS_x, \quad \boldsymbol{y}\notin S.$$

The parametrix-based integral operators, depending on the variable coefficient $\mu(\boldsymbol{x})$, can be expressed in terms of the corresponding integral operators for the constant coefficient case, $\mu=1$,

$$\mathcal{U}_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{\mathcal{U}}_k\boldsymbol{\rho}(\boldsymbol{y}), \tag{34.10}$$
$$\mathcal{R}_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{-1}{\mu(\boldsymbol{y})}\Big[2\frac{\partial}{\partial y_j}\mathring{\mathcal{U}}_{ki}(\rho_j\partial_i\mu)(\boldsymbol{y}) + 2\frac{\partial}{\partial y_i}\mathring{\mathcal{U}}_{kj}(\rho_j\partial_i\mu)(\boldsymbol{y}) + \mathring{\mathcal{Q}}_k(\rho_j\partial_j\mu)(\boldsymbol{y})\Big], \tag{34.11}$$
$$\mathcal{Q}_j\rho_j(\boldsymbol{y}) = \mathring{\mathcal{Q}}_j\rho_j(\boldsymbol{y}), \qquad \mathcal{R}^\bullet_j\rho_j(\boldsymbol{y}) = -2\frac{\partial}{\partial y_i}\mathring{\mathcal{Q}}_j(\rho_j\partial_i\mu)(\boldsymbol{y}) -2\boldsymbol{v}(\boldsymbol{y})\cdot\nabla\mu(\boldsymbol{y}), \tag{34.12}$$
$$V_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{V}_k\boldsymbol{\rho}(\boldsymbol{y}), \qquad W_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{W}_k(\mu\boldsymbol{\rho})(\boldsymbol{y}), \tag{34.13}$$
$$\mathcal{V}_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{\mathcal{V}}_k\boldsymbol{\rho}(\boldsymbol{y}), \qquad \mathcal{W}_k\boldsymbol{\rho}(\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\mathring{\mathcal{W}}_k(\mu\boldsymbol{\rho})(\boldsymbol{y}), \tag{34.14}$$
$$\mathcal{P}_j\rho_j(\boldsymbol{y}) = \mathring{\mathcal{P}}_j\rho_j(\boldsymbol{y}), \qquad \Pi_j\rho_j(\boldsymbol{y}) = \mathring{\Pi}_j(\mu\rho_j)(\boldsymbol{y}), \tag{34.15}$$
$$\mathcal{W}'_k\boldsymbol{\rho} = \mathring{\mathcal{W}}'_k\boldsymbol{\rho} - \left(\frac{\partial_i\mu}{\mu}\mathring{\mathcal{V}}_k\boldsymbol{\rho} + \frac{\partial_k\mu}{\mu}\mathring{\mathcal{V}}_i\boldsymbol{\rho} - \frac{2}{3}\delta_i^k\frac{\partial_j\mu}{\mu}\mathring{\mathcal{V}}_j\boldsymbol{\rho}\right)n_i. \tag{34.16}$$

Note that the velocity potentials defined above are *not incompressible for the variable coefficient* $\mu(y)$.

The following assertions of this section are well known for the constant coefficient case, see e.g. [KoWe06, HsWe08]. Then by relations (34.10)-(34.16) we obtain their counterparts for the variable-coefficient case.

Theorem 2 *The following operators are continuous.*

$$\mathcal{U}_{ik} : \widetilde{H}^s(\Omega) \to H^{s+2}(\Omega), \quad s \in \mathbb{R}, \tag{34.17}$$

$$\mathcal{U}_{ik} : H^s(\Omega) \to H^{s+2}(\Omega), \quad s > -\frac{1}{2}, \tag{34.18}$$

$$\mathcal{R}_{ik} : \widetilde{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{34.19}$$

$$\mathcal{R}_{ik} : H^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{34.20}$$

$$\mathcal{P}_k : H^{s-\frac{3}{2}}(S) \to H^{s-1}(\Omega), \quad s \in \mathbb{R}, \tag{34.21}$$

$$\Pi_k : H^{s-\frac{1}{2}}(S) \to H^{s-1}(\Omega), \quad s \in \mathbb{R}, \tag{34.22}$$

$$\mathcal{Q}_k : \widetilde{H}^{s-2}(\Omega) \to H^{s-1}(\Omega), \quad s \in \mathbb{R}, \tag{34.23}$$

$$\mathcal{R}^{\bullet}_k : H^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2}. \tag{34.24}$$

Let us also denote

$$\mathcal{L}^{\pm}_k \boldsymbol{\rho}(\boldsymbol{y}) := T^{\pm}_k(\boldsymbol{W}\boldsymbol{\rho}, \Pi\boldsymbol{\rho})(\boldsymbol{y}), \quad \boldsymbol{y} \in S,$$

where $T^{\pm}_k$ are the traction operators for the *compressible* fluid.

Theorem 3 *Let $s \in \mathbb{R}$. Let S_1 and S_2 be two non empty manifolds on S with smooth boundary ∂S_1 and ∂S_2, respectively. Then the following operators are continuous,*

$$V_{ik} : H^s(S) \to H^{s+\frac{3}{2}}(\Omega), \qquad W_{ik} : H^s(S) \to H^{s+\frac{1}{2}}(\Omega),$$
$$\mathcal{V}_{ik} : H^s(S) \to H^{s+1}(S), \qquad \mathcal{W}_{ik} : H^s(S) \to H^{s+1}(S),$$
$$r_{S_2}\mathcal{V}_{ik} : \widetilde{H}^s(S_1) \to H^{s+1}(S_2), \qquad r_{S_2}\mathcal{W}_{ik} : \widetilde{H}^s(S_1) \to H^{s+1}(S_2),$$
$$\mathcal{L}^{\pm}_{ik} : H^s(S) \to H^{s-1}(S), \qquad \mathcal{W}'_{ik} : H^s(S) \to H^{s+1}(S).$$

Theorem 4 *If $\boldsymbol{\tau} \in \boldsymbol{H}^{1/2}(S)$, $\boldsymbol{\rho} \in \boldsymbol{H}^{-1/2}(S)$, then the following jump relations hold,*

$$\gamma^{\pm} V_k \boldsymbol{\rho} = \mathcal{V}_k \boldsymbol{\rho}, \qquad \gamma^{\pm} W_k \boldsymbol{\tau} = \mp\frac{1}{2}\tau_k + \mathcal{W}_k \boldsymbol{\tau}$$

$$T^{\pm}_k(\boldsymbol{V}\boldsymbol{\rho}, \mathcal{P}\boldsymbol{\rho}) = \pm\frac{1}{2}\rho_k + \mathcal{W}'_k \boldsymbol{\rho},$$

$$(\mathcal{L}^{\pm}_k - \hat{\mathcal{L}}_k)\boldsymbol{\tau} = -\gamma^{\pm}\left[(\partial_i\mu)\, W_k(\boldsymbol{\tau}) + (\partial_k\mu)\, W_i(\boldsymbol{\tau}) - \frac{2}{3}\delta^k_i(\partial_j\mu)\, W_j\boldsymbol{\tau}\right] n_i,$$

$$\hat{\mathcal{L}}_k(\boldsymbol{\tau}) = \mathring{\mathcal{L}}_k(\mu\boldsymbol{\tau}).$$

Proposition 1 *The following operators are compact,*

$$\mathcal{R}_{ik} : H^s(\Omega) \to H^s(\Omega), \qquad \mathcal{R}^{\bullet}_k : H^s(\Omega) \to H^{s-1}(\Omega), \quad s \in \mathbb{R},$$

$$\gamma^{+}\mathcal{R}_{ik} : H^s(\Omega) \to H^{s-\frac{1}{2}}(S), \qquad T^{\pm}_{ik}(\mathcal{R}, \mathcal{R}^{\bullet}) : H^s(\Omega) \to H^{s-\frac{3}{2}}(S), \quad s > \frac{1}{2}.$$

Proposition 2 *Let $s \in \mathbb{R}$ and S_1 be a non empty submanifold of S with smooth boundary. Then the following operators are compact,*

$$(\mathcal{L}^{\pm}_{ik} - \hat{\mathcal{L}}_{ik}) : \widetilde{H}^s(S_1) \to H^{s-1}(S).$$

34.3 The Third Green Identities

Let $B(\boldsymbol{y}, \epsilon) \subset \Omega$ be a ball of a radius ϵ around a point $\boldsymbol{y} \in \Omega$. Applying the second Green identity (34.3) in the domain $\Omega \setminus B(\boldsymbol{y}, \epsilon)$ to any $(\boldsymbol{v}, p) \in \mathbb{H}^{1,0}_{\text{div}}(\Omega; \boldsymbol{\mathcal{A}})$ and to the fundamental solution $(\boldsymbol{u}^k, q^k)$ and taking the limit as $\epsilon \to 0$, we obtain the following third Green identity

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{T}^{+}(\boldsymbol{v}, p) + \boldsymbol{W}\boldsymbol{\gamma}^{+}\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{A}}(\boldsymbol{v}, p) \quad \text{in } \Omega. \tag{34.25}$$

Similarly, applying the first Green identity (34.2) in the domain $\Omega \setminus B(\boldsymbol{y}, \epsilon)$ to any $(\boldsymbol{v}, p) \in \mathbb{H}^{1,0}_{\text{div}}(\Omega; \boldsymbol{\mathcal{A}})$ and to the pressure part of the constant-coefficient fundamental solution $\mathring{q}^k$, for u_k, and taking the limit as $\epsilon \to 0$, we obtain the following parametrix-based third Green identity for pressure,

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \mathcal{P}\boldsymbol{T}^{+}(\boldsymbol{v}, p) + \Pi\boldsymbol{\gamma}^{+}\boldsymbol{v} = \mathcal{Q}\boldsymbol{\mathcal{A}}(\boldsymbol{v}, p) \quad \text{in } \Omega. \tag{34.26}$$

If the couple $(\boldsymbol{v}, p) \in \mathbb{H}^{1,0}_{\text{div}}(\Omega; \boldsymbol{\mathcal{A}})$ is a solution of the Stokes PDE (34.4a) with variable coefficient, then (34.25) and (34.26) give

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{T}^{+}(\boldsymbol{v}, p) + \boldsymbol{W}\boldsymbol{\gamma}^{+}\boldsymbol{v} = \boldsymbol{\mathcal{U}}\boldsymbol{f}, \tag{34.27}$$

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \mathcal{P}\boldsymbol{T}^{+}(\boldsymbol{v}, p) + \Pi\boldsymbol{\gamma}^{+}\boldsymbol{v} = \mathcal{Q}\boldsymbol{f} \quad \text{in } \Omega. \tag{34.28}$$

We will also need the trace and traction of the third Green identities for $(\boldsymbol{v}, p) \in \mathbb{H}^{1,0}_{\text{div}}(\Omega; \boldsymbol{\mathcal{A}})$ on S:

$$\frac{1}{2}\gamma^{+}\boldsymbol{v} + \boldsymbol{\mathcal{R}}^{+}\boldsymbol{v} - \boldsymbol{\mathcal{V}}\boldsymbol{T}^{+}(\boldsymbol{v}, p) + \boldsymbol{\mathcal{W}}\gamma^{+}\boldsymbol{v} = \gamma^{+}\boldsymbol{\mathcal{U}}\boldsymbol{f}, \tag{34.29}$$

$$\frac{1}{2}\boldsymbol{T}^{+}(\boldsymbol{v}, p) + \boldsymbol{T}^{+}(\boldsymbol{\mathcal{R}}, \mathcal{R}^{\bullet})\boldsymbol{v} - \boldsymbol{\mathcal{W}}'\boldsymbol{T}^{+}(\boldsymbol{v}, p) + \boldsymbol{\mathcal{L}}^{+}\gamma^{+}\boldsymbol{v} = \boldsymbol{T}^{+}(\boldsymbol{\mathcal{U}}, \mathcal{Q})\boldsymbol{f}. \tag{34.30}$$

One can prove the following two assertions that are instrumental for proof of equivalence of the BDIEs and the mixed PDE.

Lemma 1. *Let $\boldsymbol{v} \in \boldsymbol{H}^1_{\text{div}}(\Omega)$, $p \in L_2(\Omega)$, $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$, $\boldsymbol{\Psi} \in \boldsymbol{H}^{-\frac{1}{2}}(S)$ and $\boldsymbol{\Phi} \in \boldsymbol{H}^{\frac{1}{2}}(S)$ satisfy the equations*

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \mathcal{P}\boldsymbol{\Psi} + \Pi\boldsymbol{\Phi} = \mathcal{Q}\boldsymbol{f} \quad \text{in } \Omega,$$
$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\Psi} + \boldsymbol{W}\boldsymbol{\Phi} = \boldsymbol{\mathcal{U}}\boldsymbol{f} \quad \text{in } \Omega.$$

Then $(\boldsymbol{v}, p) \in \mathbb{H}^{1,0}_{\text{div}}(\Omega, \boldsymbol{\mathcal{A}})$ and solve the equation $\boldsymbol{\mathcal{A}}(\boldsymbol{y}; \boldsymbol{v}, p) = \boldsymbol{f}$. Moreover, the following relations hold true:

$$\boldsymbol{V}(\boldsymbol{\Psi} - \boldsymbol{T}^{+}(\boldsymbol{v}, p))(\boldsymbol{y}) - \boldsymbol{W}(\boldsymbol{\Phi} - \gamma^{+}\boldsymbol{v})(\boldsymbol{y}) = \boldsymbol{0}, \qquad \boldsymbol{y} \in \Omega,$$
$$\mathcal{P}(\boldsymbol{\Psi} - \boldsymbol{T}^{+}(\boldsymbol{v}, p))(\boldsymbol{y}) - \Pi(\boldsymbol{\Phi} - \gamma^{+}\boldsymbol{v})(\boldsymbol{y}) = 0, \qquad \boldsymbol{y} \in \Omega.$$

Lemma 2. *Let $S = \overline{S}_1 \cup \overline{S}_2$, where S_1 and S_2 are open non-empty non-intersecting simply connected submanifolds of S with infinitely smooth boundaries. Let $\boldsymbol{\Psi}^* \in \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(S_1)$, $\boldsymbol{\Phi}^* \in \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(S_2)$. If*

$$\boldsymbol{V}\boldsymbol{\Psi}^*(\boldsymbol{x}) - \boldsymbol{W}\boldsymbol{\Phi}^*(\boldsymbol{x}) = \boldsymbol{0} \qquad \mathcal{P}(\boldsymbol{\Psi}^*) - \Pi(\boldsymbol{\Phi}^*) = \boldsymbol{0} \quad \textit{in } \Omega,$$

then $\boldsymbol{\Psi}^* = \boldsymbol{0}$ and $\boldsymbol{\Phi}^* = \boldsymbol{0}$ on S.

34.4 Boundary-domain integral equation system for the mixed problem

We aim to obtain a segregated boundary-domain integral equation system for mixed BVP (34.4). To this end, let the functions $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{\frac{1}{2}}(S)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-\frac{1}{2}}(S)$ be respective continuations of the boundary functions $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{\frac{1}{2}}(S_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-\frac{1}{2}}(S_N)$ from (34.4b) and (34.4c). Let us now represent

$$\gamma^+\boldsymbol{v} = \boldsymbol{\Phi}_0 + \boldsymbol{\varphi}, \quad \boldsymbol{T}^+(\boldsymbol{v}, p) = \boldsymbol{\Psi}_0 + \boldsymbol{\psi} \text{ on } S, \tag{34.31}$$

where $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(S_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(S_D)$ are unknown boundary functions.

Let us now take equations (34.27) and (34.28) in the domain Ω and restrictions of equations (34.29) and (34.30) to the boundary parts S_D and S_N, respectively. Substituting there representations (34.31) and considering further the unknown boundary functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ as formally independent of (segregated from) the unknown domain functions $\boldsymbol{v}$ and p, we obtain the following system of four boundary-domain integral equations for four unknowns, $(\boldsymbol{v}, p) \in \mathbb{H}^{1,0}_{\mathrm{div}}(\Omega, \boldsymbol{\mathcal{A}})$, $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(S_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(S_D)$:

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \mathcal{P}\boldsymbol{\psi} + \Pi\boldsymbol{\varphi} = F_0 \quad \text{in } \Omega, \tag{34.32a}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\psi} + \boldsymbol{W}\boldsymbol{\varphi} = \boldsymbol{F} \quad \text{in } \Omega, \tag{34.32b}$$

$$r_{S_D}\gamma^+\boldsymbol{\mathcal{R}}\boldsymbol{v} - r_{S_D}\boldsymbol{\mathcal{V}}\boldsymbol{\psi} + r_{S_D}\boldsymbol{\mathcal{W}}\boldsymbol{\varphi} = r_{S_D}\gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0 \quad \text{on } S_D, \tag{34.32c}$$

$$r_{S_N}\boldsymbol{T}^+(\boldsymbol{\mathcal{R}}, \mathcal{R}^{\bullet})\boldsymbol{v} - r_{S_N}\boldsymbol{\mathcal{W}}'\boldsymbol{\psi} + r_{S_N}\boldsymbol{\mathcal{L}}^+\boldsymbol{\varphi} = r_{S_N}\boldsymbol{T}^+(\boldsymbol{F}, F_0) - \boldsymbol{\psi}_0 \quad \text{on } S_N, \tag{34.32d}$$

where

$$F_0 = \mathcal{Q}\boldsymbol{f} + \mathcal{P}\boldsymbol{\Psi}_0 - \Pi\boldsymbol{\Phi}_0, \quad \boldsymbol{F} = \boldsymbol{\mathcal{U}}\boldsymbol{f} + \boldsymbol{V}\boldsymbol{\Psi}_0 - \boldsymbol{W}\boldsymbol{\Phi}_0. \tag{34.33}$$

Applying Lemma 1 to (34.33) and taking into account the continuity of operators (34.20) and (34.24), one can prove that $(F_0, \boldsymbol{F}) \in \mathbb{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$.

We denote the right hand side of BDIE system (34.32) as

$$\mathcal{F}^{11} := [F_0, \boldsymbol{F}, r_{S_D}\gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0, r_{S_N}\boldsymbol{T}^+_{\boldsymbol{F},F} - \boldsymbol{\psi}_0]^\top, \tag{34.34}$$

which implies $\mathcal{F}^{11} \in \mathbb{H}^{1,0}(\Omega, \boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{\frac{1}{2}}(S_D) \times \boldsymbol{H}^{-\frac{1}{2}}(S_N)$.

Note that BDIE system (34.32) can be split into the BDIE system of 3 vector equations (34.32b), (34.32c), (34.32d) for 3 vector unknowns, $\boldsymbol{v}$, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, and the separate equation (34.32a) that can be used, after solving the system, to obtain the pressure, p. However since the couple $(\boldsymbol{v}, p)$ shares the space $\mathbb{H}^{1,0}_{\mathrm{div}}(\Omega, \boldsymbol{\mathcal{A}})$, equations (34.32b), (34.32c), (34.32d) are not completely separate from equation (34.32a).

Theorem 5 (Equivalence Theorem) *Let $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$ and let $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{-\frac{1}{2}}(S)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-\frac{1}{2}}(S)$ be some fixed extensions of $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{\frac{1}{2}}(S_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-\frac{1}{2}}(S_N)$ respectively.*

(i) If some $(\boldsymbol{v}, p) \in \mathbb{H}^{1,0}_{\mathrm{div}}(\Omega; \boldsymbol{\mathcal{A}})$ solve mixed BVP (34.4), then $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \mathbb{H}^{1,0}_{\mathrm{div}}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(S_D) \times \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(S_N)$, where

$$\boldsymbol{\varphi} = \gamma^+ \boldsymbol{v} - \boldsymbol{\Phi}_0, \quad \boldsymbol{\psi} = \boldsymbol{T}^+(\boldsymbol{v}, p) - \boldsymbol{\Psi}_0 \quad \text{on } S, \tag{34.35}$$

solve BDIE system (34.32).

(ii) If $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \mathbb{H}^{1,0}_{\mathrm{div}}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(S_D) \times \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(S_N)$ solve the BDIE system (34.32) then $(\boldsymbol{v}, p)$ solve mixed BVP (34.4) and the functions $\boldsymbol{\psi}, \boldsymbol{\varphi}$ satisfy (34.35).

(iii) BDIE system (34.32) is uniquely solvable in $\mathbb{H}^{1,0}_{\mathrm{div}}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(S_D) \times \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(S_N)$.

Proof. (i) Let $(\boldsymbol{v}, p) \in \mathbb{H}^{1,0}_{\mathrm{div}}(\Omega; \boldsymbol{\mathcal{A}})$ be a solution of the BVP. Let us define the functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ by (34.35). By the BVP boundary conditions, $\gamma^+ \boldsymbol{v} = \boldsymbol{\varphi}_0 = \boldsymbol{\Phi}_0$ on S_D and $\boldsymbol{T}^+(\boldsymbol{v}, p) = \boldsymbol{\psi}_0 = \boldsymbol{\Psi}_0$ on S_N. This implies that $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(S_D) \times \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(S_N)$. Taking into account the Green identities (34.26)-(34.30), we immediately obtain that $(p, \boldsymbol{v}, \boldsymbol{\varphi}, \boldsymbol{\psi})$ solve system (34.32).

ii) Conversely, let $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \mathbb{H}^{1,0}_{\mathrm{div}}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(S_D) \times \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(S_N)$ solve BDIE system (34.32). If we take the trace of (34.32b) restricted to S_D, use the jump relations for the trace of W, see Theorem 5, and subtract it from (34.32c), we arrive at $r_{S_D} \gamma^+ \boldsymbol{v} - \dfrac{1}{2} r_{S_D} \boldsymbol{\varphi} = \boldsymbol{\varphi}_0$ on S_D. As $\boldsymbol{\varphi}$ vanishes on S_D, therefore the Dirichlet condition of the BVP is satisfied.

Repeating the same procedure but taking the traction of (34.32a) and (34.32b), restricted to S_N, using the jump relations for the traction of V and subtracting it from (34.32d), we arrive at $r_{S_N} \boldsymbol{T}(\boldsymbol{v}, p) - \dfrac{1}{2} r_{S_N} \boldsymbol{\psi} = \boldsymbol{\psi}_0$ on S_N. As $\boldsymbol{\psi}$ vanishes on S_N, therefore the Neumann condition of the BVP is satisfied. Since $\boldsymbol{\varphi}_0 = \boldsymbol{\Phi}_0$ on S_D and $\boldsymbol{\psi}_0 = \boldsymbol{\Psi}_0$ on S_N, the conditions (34.35) are satisfied, respectively, on S_D and S_N.

Also we have that $\boldsymbol{\Psi} \in \boldsymbol{H}^{-\frac{1}{2}}$ and $\boldsymbol{\Phi} \in \boldsymbol{H}^{-\frac{1}{2}}$. We note that if $(\boldsymbol{v}, p) \in L_2(\Omega) \times \boldsymbol{H}^1_{\mathrm{div}}(\Omega)$ then $\boldsymbol{\mathcal{A}}(\boldsymbol{v}, p) = \boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$. Due to relations (34.32a) and (34.32b) the hypotheses of the Lemma 1 are satisfied with $\boldsymbol{\Psi} = \boldsymbol{\psi} + \boldsymbol{\Psi}_0$ and

$\boldsymbol{\Phi} = \boldsymbol{\varphi} + \boldsymbol{\Phi}_0$. As a result we obtain that $(\boldsymbol{v}, p)$ is a solution of $\boldsymbol{\mathcal{A}}(\boldsymbol{v}, p) = \boldsymbol{f}$ satisfying

$$\boldsymbol{V}(\boldsymbol{\Psi}^*) - \boldsymbol{W}(\boldsymbol{\Phi}^*) = \boldsymbol{0}, \quad \mathcal{P}(\boldsymbol{\Psi}^*) - \Pi(\boldsymbol{\Phi}^*) = 0 \quad \text{in } \Omega, \tag{34.36}$$

where

$$\Psi^* = \boldsymbol{\psi} + \boldsymbol{\Psi}_0 - \boldsymbol{T}^+(\boldsymbol{v}, p) \qquad\qquad \Phi^* = \boldsymbol{\varphi} + \boldsymbol{\Phi}_0 - \gamma^+\boldsymbol{v}$$

Since $\boldsymbol{\Psi}^* \in \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(S_D)$ and $\boldsymbol{\Phi}^* \in \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(S_N)$, and (34.36) hold true, applying Lemma 2 for $S_1 = S_D$ and $S_2 = S_N$ we obtain $\boldsymbol{\Psi}^* = \boldsymbol{\Phi}^* = \boldsymbol{0}$ on S. This implies conditions (34.35).

(iii) The uniqueness of the BDIEs (34.32) follows from from the uniqueness of the BVP, see Theorem 1, and items (i) and (ii). □

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