Theoretical Backgrounds of Durability Analysis by Normalized Equivalent Stress Functionals

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(Received 14 November 2001; accepted 23 January 2002)

Abstract: Generalized durability diagrams and their properties are considered for a material under a multiaxial loading given by an arbitrary function of time. Material strength and durability under such loading are described in terms of durability, safety factor and normalized equivalent stress. Relations between these functionals are analysed. We discuss some material properties including time and load stability, self-degradation (ageing), and monotonic damaging. Phenomenological strength conditions are presented in terms of the normalized equivalent stress. It is shown that the damage based durability analysis is reduced to a particular case of such strength conditions. Examples of the reduction are presented for some known durability models. The approach is applicable to the strength and durability description at creep and impact loading and their combination.

Key Words: Durability, strength conditions, endurance limit, dynamic failure

1. INTRODUCTION

Different forms of durability description are commonly used for time- or history-dependent materials possessing plasticity, creep and/or serving under fatigue or impact loadings. A usual auxiliary means for this is the introduction of a damage measure and an evolution law for this measure, see, for example, [1, 2, 3, 4, 5, 6, 7] and also some remarks in Appendix A. Together with the limiting damage value, which when reached means rupture, this gives a strength condition. Such damage measure is often associated with a geometrical change, that is, with the defect cross-section fraction or the defect volume fraction in a representative volume element or with the stiffness change of the damaged material or it is taken as an abstract internal material parameter. The geometrical damage measures involve difficulties in their experimental evaluation, the stiffness damage measure is not always representative, e.g. for high cyclic fatigue. For abstract damage measures, a value of the measure below the critical one delivers no direct information about the safety or the residual life durability. Different abstract measures and their evolution laws are not easy to compare. In addition, most evolution laws do not take into account a dependence of the damage measure rate on the process history. We remark that such a dependence is considered in [5].

In this paper, we try to show that the phenomenological durability description and analysis can be done completely without such additional means as a damage measure if the loading process $\sigma_{ij}(\tau)$ is known. Note, however, that a damage analysis can be useful for

Mathematics and Mechanics of Solids, **8:** 105–142, 2003 DOI: 10.1177/108128603029768

prediction of the loading process $\sigma_{ij}(\tau, x)$ at each point x, particularly for softening materials. Some damage measures can also be quite helpful for obtaining a phenomenological durability description from micromechanical considerations.

We discuss here a material under a uniform multiaxial stress state. Using some ideas of [8], durability, safety factor and normalized equivalent stress (load factor) are presented in this paper, which are mechanically meaningful and experimentally measurable on the one hand and accumulate process history on the other hand.

Generally, durability analysis includes the following main items: (i) determination of the durability $t^*(\sigma)$ for a prescribed loading process $\sigma_{ij}(\tau)$; (ii) determination of the safety factor $\underline{\lambda}(\sigma; t)$ at an instant t for a prescribed process $\sigma_{ij}(\tau)$; (iii) determination of a damage $\omega(\sigma; t)$ at an instant t for a prescribed process $\sigma_{ij}(\tau)$; (iv) interpolation of the functionals $t^*(\sigma)$, $\underline{\lambda}(\sigma; t)$ and $\omega(\sigma; t)$ along their values for some processes $\sigma^r_{ij}(\tau),\ r = 1...R$. We call $t^*(\sigma)$, $\underline{\lambda}(\sigma; t)$ and $\omega(\sigma; t)$ functionals since each of these maps a function $\sigma_{ij}(\tau)$ into a number. Note that, although a damage measure is mentioned between the main items, it can be considered as an auxiliary parameter, helping in some models to determine the practically interesting parameters $t^*(\sigma)$ and $\underline{\lambda}(\sigma; t)$. This paper is devoted chiefly to a discussion of definitions, properties and mutual connections of the functionals $t^*(\sigma)$, $\underline{\lambda}(\sigma; t)$, and of the normalized equivalent stress functional $\underline{\Lambda}(\sigma; t) = 1/\underline{\lambda}(\sigma; t)$. It develops the results of [9].

2. GENERALIZED DURABILITY DIAGRAM

2.1. Durability and strength stability in time

Let a material undergo a loading program (process) $\sigma_{ij}(\tau)$. We discuss here rupture without specifying the rupture type and we only assume the following: (i) one can unambiguously detect at any time instant whether the body is ruptured or not; and (ii) if the body is ruptured at an instant t, it will be ruptured also at any instant $t_1 > t$ (no repairing mechanism). We say that strength is stable in time (or t-stable) at an instant t under a loading process $\sigma_{ij}(\tau)$ if there is no rupture at t and there exists an instant $t'(\sigma; t) > t$ where no rupture occurs also. (This means that the time interval, where strength is t-stable, is open.) From this definition, strength is unstable in time (or t-unstable) at an instant t under a loading process $\sigma_{ij}(\tau)$ if there is no rupture at t but the body is ruptured at any instant $t'(\sigma; t) > t$ (an example is given by the discontinuous loading process presented in Figure 5(a) at $t = \tau^*$ if $\sigma_1 < \sigma_r \leq \sigma_2$, see below).

If the strength is t-stable at all instants where no rupture appears, then the time $t^*(\sigma)$, at which a rupture for the material appears is called durability or life time. If there exists an instant where the strength is t-unstable, we implement a more general definition of the durability as an instant $t^*(\sigma)$ such that there is no rupture at any $t < t^*(\sigma)$ and the body is ruptured at any $t > t^*(\sigma)$; if there is no rupture at any time $t < \infty$, we say the durability is infinite, $t^*(\sigma) = \infty$.

Thus, $t < t^*(\sigma)$ is the condition of t-stable strength at the instant t. On the other hand, the equality $t = t^*(\sigma) < \infty$ means rupture or strength t-instability at the instant t.

The life time seems to be the main relevant measurable parameter in the durability analysis and all other parameters are derived from it. For different loading processes $\sigma^1_{ij}(\tau)$ and $\sigma^2_{ij}(\tau)$, the durability has different values $t^*(\sigma^1)$ and $t^*(\sigma^2)$ (see Figure 1).

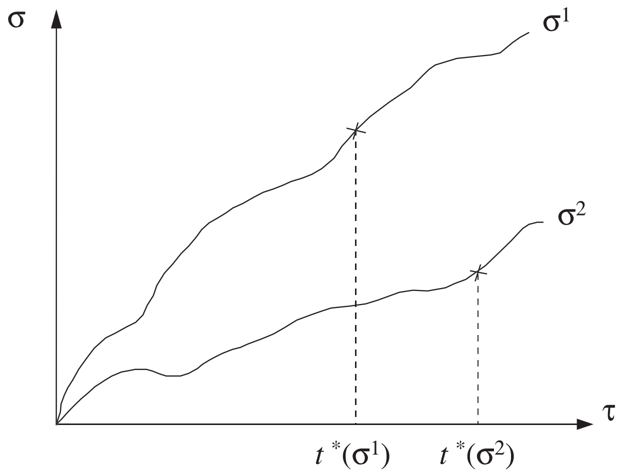

Figure 1. Loading processes and durabilities.

2.2. Durability diagrams

Let

$$H(\tau) = \left\{ \begin{array}{ll} 0, & \tau \leq 0 \\ 1, & \tau > 0 \end{array} \right\}$$

be the Heaviside step-like function. Under a uniaxial step-like loading $\sigma(\tau) = H(\tau)\sigma^0$, where σ^0 is a constant, it is usual to determine experimentally the durability diagram in the axes $s^0 \mapsto t^*(s^0)$. Its counterpart in fatigue under a constant stress range oscillation $\Delta\sigma^0 = \sigma^0_{\max} - \sigma^0_{\min}$ is the Wöhler diagram $\Delta\sigma^0 \mapsto n^*(\Delta\sigma^0)$, where $n^*(\Delta\sigma^0)$ is the number of cycles before rupture.

An example of a simple durability diagram given by a power law (a straight line in the double logarithmic coordinates) can be written as

$$t^*(\sigma^0) = A|\sigma^0|^{-b}.$$

where A and b are positive constants depending on the stress state type (tension, compression or shear). A similar power dependence for a constant in time multiaxial loading $\sigma^0_{ij} =$ constant can be written in the form

$$t^*(\sigma^0) = |\sigma^0|^{-b\left(\tilde{\sigma}^0\right)} A\left(\tilde{\sigma}^0\right). \tag{1}$$

Here $|\sigma^0|$ is a matrix norm of the tensor σ^0_{ij}, for example, $|\sigma^0| = \sqrt{\sum_{i,j=1}^{3} \sigma^0_{ij}\,\sigma^0_{ij}}$; $\tilde{\sigma}^0_{ij} = \sigma^0_{ij}/|\sigma^0|$ is the normalized stress tensor, presenting the tensor σ^0_{ij} shape; $A(\tilde{\sigma}^0)$ and $b(\tilde{\sigma}^0)$ are positive parameters depending on the tensor σ^0_{ij} shape but not on the tensor norm.

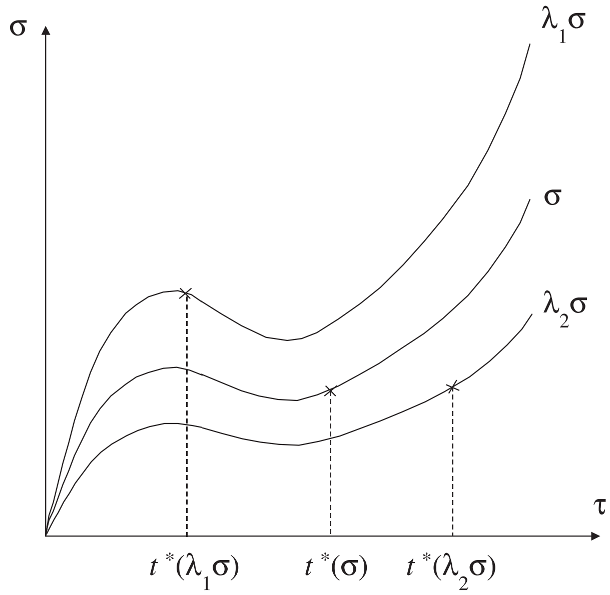

Figure 2. Proportional loading processes and durabilities, $0 < \lambda_2 < 1 < \lambda_1$.

To present a generalized diagram for a multiaxial process described by an arbitrary tensor function $\sigma_{ij}(\tau)$, let us consider a family of proportional processes $\lambda\sigma_{ij}(\tau)$, obtained from the original process $\sigma_{ij}(\tau)$ after its multiplication by a non-negative constant number λ, see Figure 2.

The generalized durability diagram for a process $\sigma_{ij}(\tau)$ is the dependence of the durability $t^*(\lambda\sigma)$ on a parameter $\lambda \geq 0$.

The concept propounded in this paper concerns mainly time- and history-dependent materials but should work also in the particular case of materials independent of time and history. We use the latter extensively for illustrations.

Let us consider, for example, a material independent of time and history, uniaxially loaded by a step-like process $\sigma(\tau) = H(\tau)\sigma^0$, where σ^0 is a constant, and obeying the strength condition $\sigma<\sigma_r$, where σ_r is constant. Then the durability diagram is given by the line $\lambda = \sigma_r/\sigma^0$, that is

$$t^*(\lambda\sigma) = \left\{ \begin{array}{ll} \infty, & \lambda < \sigma_r/\sigma^0 \\ 0, & \lambda \geq \sigma_r/\sigma^0 \end{array} \right\}.$$

If the loading process for the same material is $\sigma(\tau) = a\tau$, where a is a constant, then the durability diagram is a hyperbola $t^*(\lambda\sigma) = \sigma_r/(a\lambda)$.

Let us consider an arbitrary material. Suppose $\sigma_{ij}(\tau)$ is a multiaxial step-like process $\sigma_{ij}(\tau) = H(\tau)\sigma^0_{ij}$, where σ^0_{ij} is a constant tensor, $|\sigma^0| = 1$. It is evident, that the generalized durability diagram $\lambda \mapsto t^*(\lambda\sigma)$ coincides with the classical durability diagram $|\sigma| \mapsto t^*(\sigma)$ for the step-like processes $\sigma_{ij}(\tau) = |\sigma|H(\tau)\sigma^0_{ij}$.

Similarly, the generalized durability diagram $\lambda \mapsto t^*(\lambda\sigma)$ for a periodic loading process $\{\sigma_{ij}(\tau) = H(\tau)f(\tau)\sigma^0_{ij}$, where $\sigma^0_{ij} =$ constant, $|\sigma^0| = 1$, $f(\tau)$ is a t_0-periodic function with the unit range, $\Delta f = f_{max} - f_{min} = 1\}$, coincides with the classical Wöhler diagram $\Delta\sigma \mapsto n^*(\Delta\sigma)$ for the oscillating processes $\sigma_{ij}(\tau) = \Delta\sigma H(\tau)f(\tau)\sigma^0_{ij}$, where $n^* = t^*/t_0$.

Let us consider the general properties of the generalized durability diagram $t^*(\lambda\sigma)$ for an arbitrary material under a given process $\sigma_{ij}(\tau)$. This function is defined on the half axis $\lambda \in [0,\infty)$ and is non-negative. When λ varies, different situations can arise. We plot schematically a durability diagram in Figure 3(a). Although we consider $t^*(\lambda\sigma)$ as a function of λ, the choice of the axes directions on the plot is traditional for the durability analysis. The curves a, b, c at large λ and curves d, e, f at small λ present different possible cases of the diagram behaviour. That is, one of the curves a, b or c continues by one of the curves d, e or f for a particular material under a particular loading $\sigma_{ij}(\tau)$.

We analyse first small durabilities t^*, that is, large λ.

(A): The rupture can occur at $t = t^*(\lambda^0\sigma) = 0$ for a finite but sufficiently large λ^0, curve a in Figure 3(a). It can happen, for example, for some materials under the step-like loading $\sigma_{ij}(\tau) = H(\tau)\sigma^0_{ij}$ where σ^0_{ij} is a constant tensor. Particularly, as mentioned above, this diagram is the horizontal line $\lambda = \lambda^0 = \sigma_r/\sigma^0$ for a time- and history-independent material under such loading.

(B): The durability $t^*(\lambda\sigma_{ij})$ can be non-zero at any finite λ but tends to zero as λ tends to infinity, curve b in Figure 3(a); then $\lambda^0 = \infty$. Particularly, this is the case for a loading process growing continuously from zero, e.g. for $\sigma_{ij}(\tau) = \tau\sigma^0_{ij}$ where σ^0_{ij} is a constant tensor. This is the case also under the step-like loading $\sigma_{ij}(\tau) = H(\tau)\sigma^0_{ij}$ for materials obeying some dynamic strength conditions, see for example Sections 6.2 and 6.3.

(C): There exist loadings for some materials (or material models), that do not cause rupture however large these loadings are. An example is the uniform three-axes compression, $\sigma_{ij} = \delta_{ij}$. Suppose a loading process $\sigma_{ij}(\tau)$ is represented by such a loading on a beginning time interval $0 \le \tau \le t_+$ followed by a loading able to cause rupture at some time. Then there is no rupture on $0 \le \tau \le t_+$ for any non-negative λ, curve c in Figure 3(a), and we can put $\lambda = \lambda^0 = \infty$ on this segment.

Let us consider the durability behaviour at large t^*, that is, at small λ.

Let $\lambda = 0$. The durability $t^*(0)$, when no loading is applied, is either finite or infinite.

(0): The first case means that rupture at $t = t^*(0) < \infty$ is caused not by a mechanical load $\sigma_{ij}(\tau)$, $\tau \ge 0$ but for another reason, for example, by a previous loading history at $\tau < 0$. Other possible reasons for such behaviour can be radiation, corrosion or other chemical reactions, dissolution, etc, which we can refer to as natural or artificial ageing leading to the complete degradation of the material at the time $t^*(0)$. We call the material self-degrading if $t^*(0) < \infty$.

Note that ageing does not necessarily lead to complete degradation. Generally, a material is said to be ageing in strength if $t^*(\sigma^\Delta) \ne t^*(\sigma) + \Delta$, where $\sigma^\Delta_{ij}(\tau) = \sigma_{ij}(\tau + \Delta)$, for some $\sigma_{ij}(\tau)$ and Δ. This means that a shift of a loading process in time does not cause the same shift in durability for an ageing material.

Let us return to the description of Figure 3(a).

(D): The durability $t^*(\lambda\sigma)$ tends to a finite value $t^{*0}(\sigma) \le t^*(0)$ as λ tends to 0, curve d in Figure 3(a). Usually one can expect continuity, i.e. $t^{*0}(\sigma) = t^*(0)$ but it is not always the case since $t^{*0}(\sigma)$, unlike $t^*(0)$, is determined not only by the material properties but also by loading. For example, $t^{*0}(\sigma) < t^*(0)$ for a singular stress $\sigma_{ij}(\tau)$ infinitely growing as

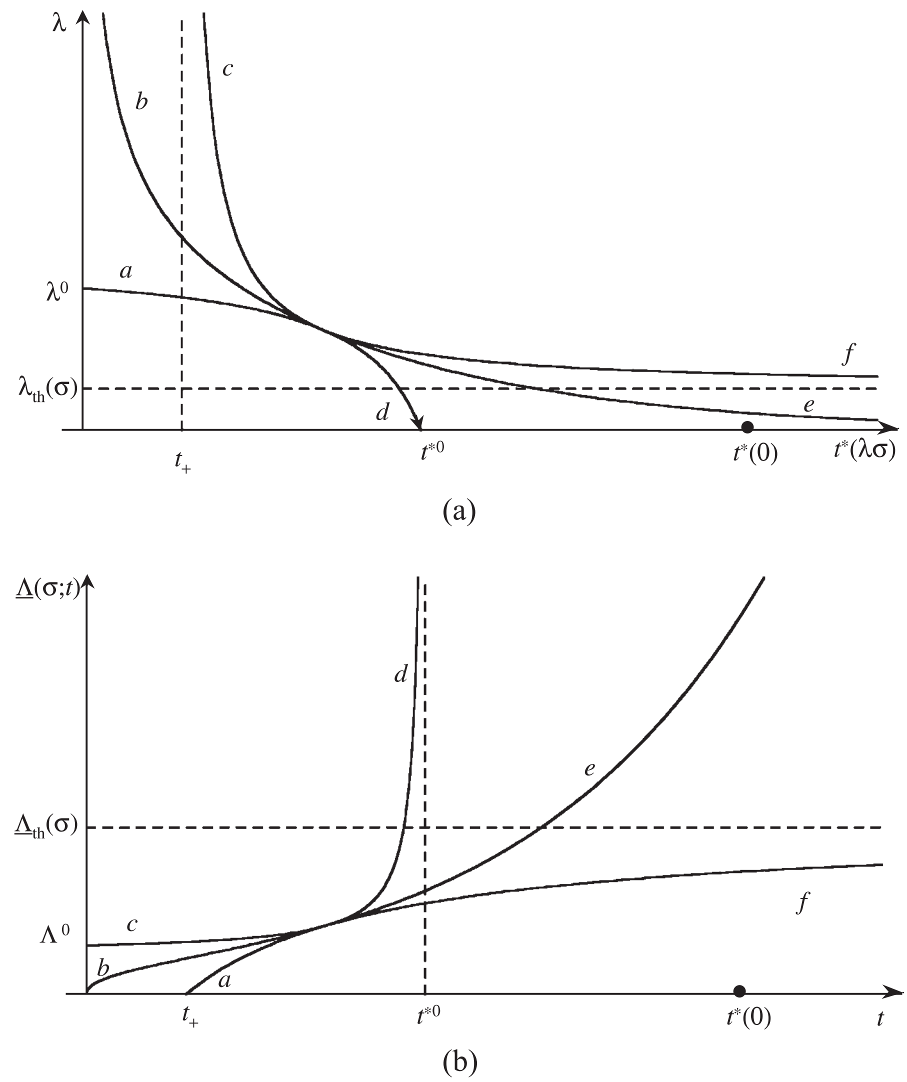

Figure 3. (a) Durability diagram for a process $\sigma_{ij}(\tau)$. (b) Normalized equivalent stress versus t for the process.

τ tends to t^{*0}, i.e. for $\sigma_{ij}(\tau) = \sigma_{ij}^0/(t^{*0} - \tau)$. Obviously, $t^{*0}(\sigma)$ can be finite also for non-self-degrading materials, i.e. for $t^*(0) = \infty$.

If $t^*(0) = \infty$, i.e. the material is not self-degrading, we can have three possible situations.

(E): $t^*(\lambda\sigma) \to t^{*0}(\sigma) = \infty$ as $\lambda \to 0$ and there exists no non-zero threshold; that is, the durability $t^*(\lambda\sigma)$ monotonically grows up to infinity with diminishing λ but is always finite at $\lambda > 0$, curve *e* in Figure 3(a).

(F): $t^*(\lambda\sigma) \to t^{*0}(\sigma) = \infty$ as $\lambda \to 0$ and there exists a threshold value $\lambda_{th}(\sigma) > 0$ such that $t^*(\lambda\sigma) = \infty$ for all λ such as $0 \leq \lambda \leq \lambda_{th}(\sigma)$ and $t^*(\lambda\sigma) < \infty$ for all $\lambda > \lambda_{th}(\sigma)$, curve f in Figure 3(a).

(G): $t^*(\lambda\sigma)$ has no definite limit $t^{*0}(\sigma)$; this means it is not monotonic as $\lambda \to 0$. This can happen for materials and processes that are not MD (see below).

Cases E and F seem to be most usual in the durability analysis.

Let us analyse the durability diagram for intermediate λ.

Firstly, the dependence $t^*(\lambda\sigma)$ on λ can be either monotonically non-increasing or not. In the former case, i.e. if $t^*(\lambda_1\sigma) \geq t^*(\lambda_2\sigma)$ for any numbers $\lambda_2 > \lambda_1 \geq 0$, the process will be called monotonically damaging (MD). A material is MD if all processes are MD for it.

Note that there exist materials that are not MD. For example, strength and durability of solidifying or cemented materials can be essentially increased, if the contracting loading is increased during the solidification or cementation phases, see Figure 4.

Secondly, the durability diagrams can have finite jumps along λ as well as along $t^*(\lambda\sigma)$ axes. Figures 5, 6 and 7 give some examples of such loading processes for a material independent of time and history, in which rupture appears at $\sigma = \sigma_r$.

2.3. Strength stability in proportional load perturbations

Strength is said to be stable with respect to proportional load perturbations (λ-stable) under a process $\sigma_{ij}(\tau)$ at an instant $t < \infty$, if there is no rupture at and before the instant t under $\sigma_{ij}(\tau)$ and under slightly higher or lower loading. More precisely, there exists $\epsilon > 0$ such that there is no rupture at and before the instant t under the process $\lambda\sigma_{ij}(\tau)$ for any $\lambda \in (1-\epsilon, 1+\epsilon)$.

This implies that if the strength in a body is λ-unstable at an instant t_1, it cannot become λ-stable at any instant $t_2 > t_1$.

We denote by $t_{st}^*(\sigma)$ the critical time, that is such that strength is λ-stable at all instants $t < t_{st}^*(\sigma)$ but either rupture or strength λ-instability exists at all $t > t_{st}^*(\sigma)$. If strength is λ-stable at all instants $t < \infty$, we take $t_{st}^*(\sigma) = \infty$.

It is evident that the critical time $t_{st}^*(\sigma)$ is not greater than the durability $t^*(\sigma)$ and the strength is not only λ-stable but also *t*-stable at $t < t_{st}^*(\sigma)$. If $t_{st}^*(\sigma) = t^*(\sigma)$, then either rupture exists or strength is *t*-unstable and λ-(stable or unstable), at time $t = t_{st}^*(\sigma)$. If $t_{st}^*(\sigma) < t^*(\sigma)$, then strength is *t*-stable and λ-(stable or unstable) at time $t = t_{st}^*(\sigma)$ but *t*-stable and λ-unstable at $t \in (t_{st}^*(\sigma), t^*(\sigma))$. This means that the durability diagram has at $\lambda = 1$ a horizontal jump on the half-interval $[t_{st}^*(\sigma), t^*(\sigma))$, where the diagram has no values (see Figure 7(b) at $\sigma_m = \sigma_r$).

Strength is said to be absolutely stable ($t\lambda$-stable) under a process $\sigma_{ij}(\tau)$ at an instant t, if strength is *t*-stable at and before the instant t under $\sigma_{ij}(\tau)$ and under a slightly higher or lower loading. More precisely, there exists $\epsilon > 0$ such that $t < t^*(\lambda\sigma)$ for any $\lambda \in (1-\epsilon, 1+\epsilon)$.

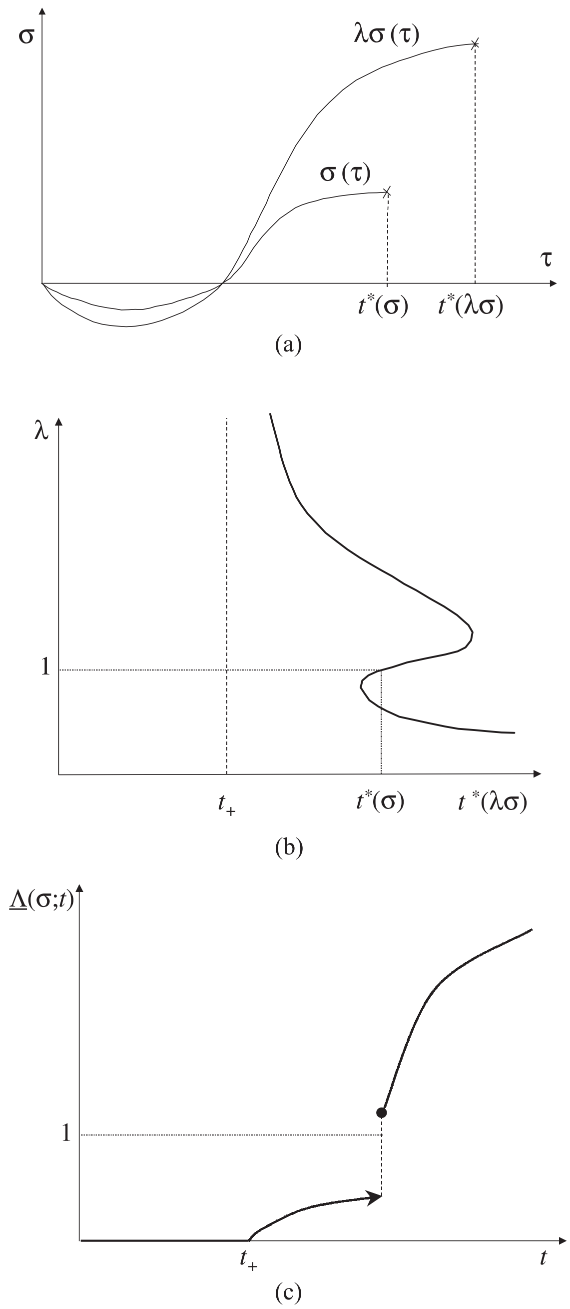

Figure 4. (a) Proportional non-monotonically damaging loading processes for $\lambda = 1$ and $\lambda > 1$. (b) Durability diagram for the process. (c) The normalized equivalent stress for the process.

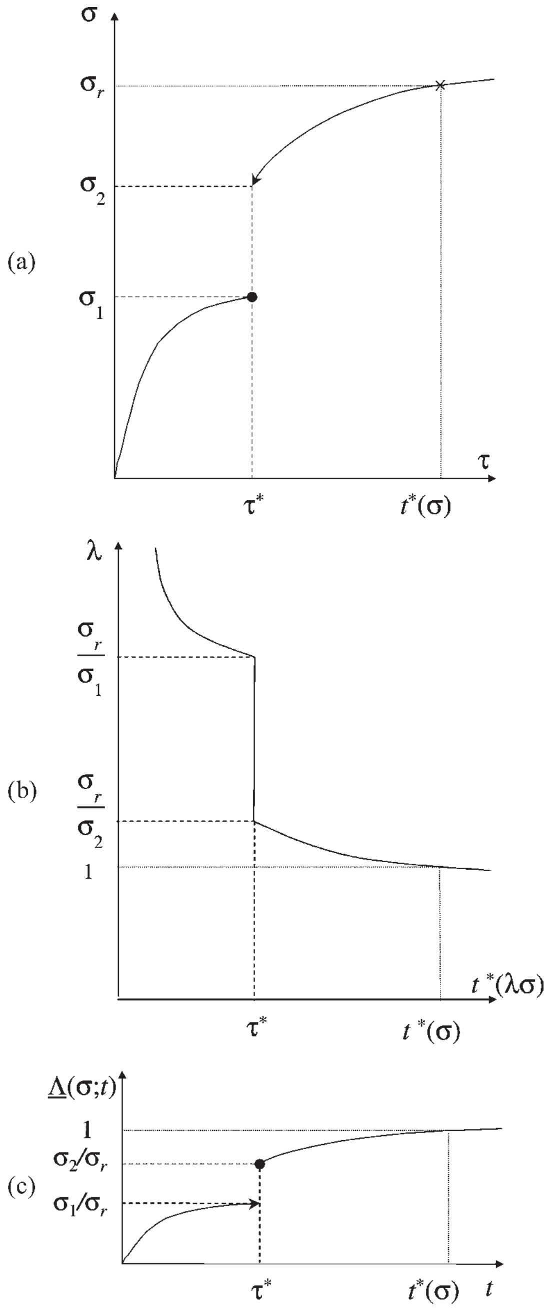

Figure 5. (a) Monotonic piecewise continuous loading process. (b) Piecewise continuous durability diagram generated by the process. (c) The normalized equivalent stress for the process.

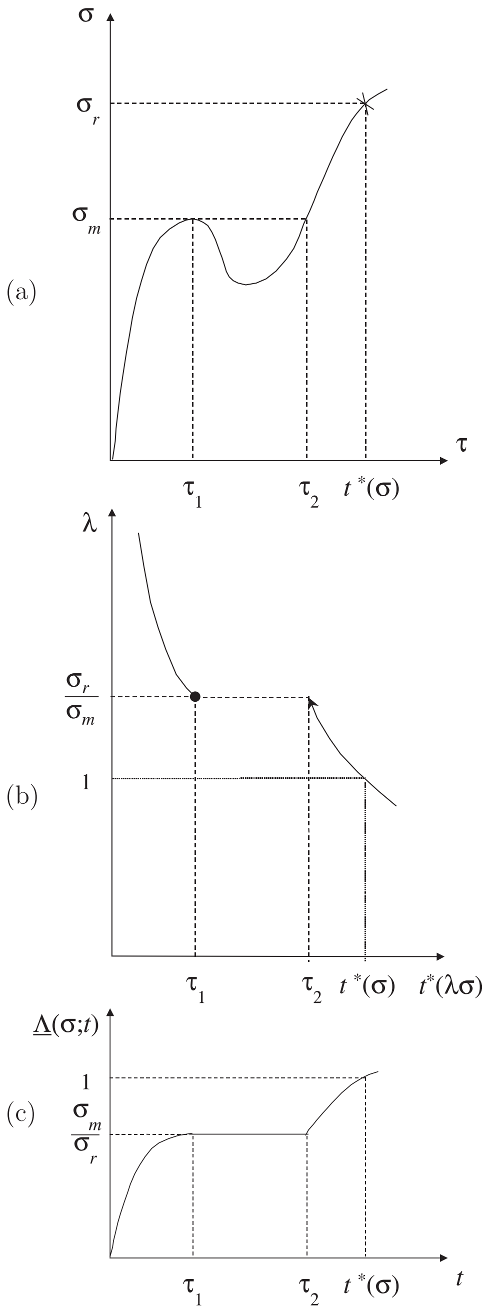

Figure 6. (a) Non-monotonic continuous loading process. (b) Piecewise continuous durability diagram generated by the process. (c) The normalized equivalent stress for the process.

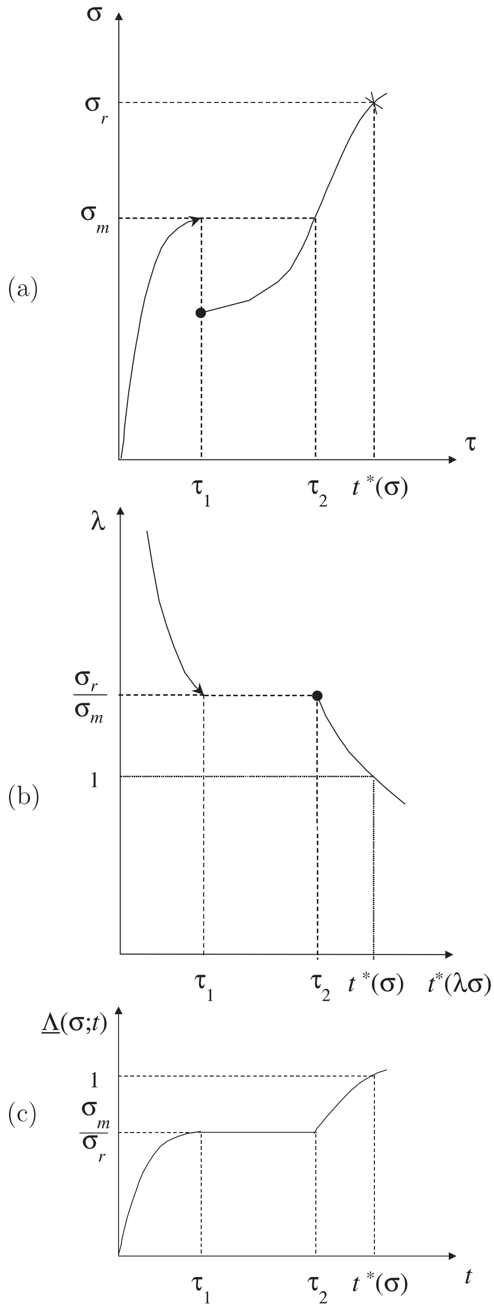

Figure 7. (a) Non-monotonic right-continuous loading process. (b) Piecewise continuous durability diagram generated by the process. (c) The normalized equivalent stress for the process.

Evidently, if strength is absolutely stable at an instant t, it is also t-stable and λ-stable at the same instant t. On the other hand, strength is absolutely stable under a process $\sigma_{ij}(\tau)$ at any instant $t < t^*_{st}(\sigma)$. However, strength can be t-stable and λ-stable but not absolutely stable at $t = t^*_{st}(\sigma) < t^*(\sigma)$.

For $t = \infty$, the above reasoning can be modified by the following way. Endurance is said to be stable with respect to proportional load perturbations (λ-stable) under a process $\sigma_{ij}(\tau)$, if there is no rupture under $\sigma_{ij}(\tau)$ and under a slightly higher or lower loading at any time. More precisely, there exists $\epsilon > 1$ such that there is no rupture at all time instants $t < \infty$ under the process $\sigma_{ij}(\tau)$ for any $\lambda \in (1-\epsilon, 1+\epsilon)$.

Evidently, λ-stable endurance under a process $\sigma_{ij}(\tau)$ implies $t^*_{st}(\sigma) = \infty$. However, the equality $t^*_{st}(\sigma) = \infty$ does not generally imply λ-stable endurance.

Returning to the example description, we note that the discontinuous monotonic process in Figure 5(a) generates a continuous durability diagram $\lambda \mapsto t^*(\lambda\sigma)$ (see Figure 5(b)) with a finite jump along the λ axis. The strength under the process is λ-stable but t-unstable at $\tau = \tau^*$ if $\sigma_1 < \sigma_r \leq \sigma_2$; $t^*_{st}(\sigma) = t^*(\sigma) = \tau^*$ for this case.

The continuous non-monotonic process in Figure 6(a) generates a discontinuous (right-continuous) durability diagram $\lambda \mapsto t^*(\lambda\sigma)$, see Figure 6(b). If there exists strength at an instant t, the strength is absolutely stable. Rupture appears at $t = \tau_1$ if $\sigma_m = \sigma_r$.

The discontinuous (right-continuous) non-monotonic process in Figure 7(a) generates a discontinuous (left-continuous) durability diagram $\lambda \mapsto t^*(\lambda\sigma)$, see Figure 7(b). The strength under the process is t-stable but λ-unstable at $t \in [\tau_1^*, \tau_2^*)$ if $\sigma_m = \sigma_r$; $t^*_{st}(\sigma) = \tau_1 < t^*(\sigma) = \tau_2^*$ for this case. (Recall that one should turn the diagrams in Figures 6(b) and 7(b) appropriately making the axis λ horizontal, to interpret the diagrams right-(left-)continuity literally).

Some relations between strength λ-stability and continuity of the durability diagram are given in Appendix B.

3. SAFETY FACTOR AND NORMALIZED EQUIVALENT STRESS

For a given process $\sigma_{ij}(\tau)$, we can determine (experimentally) a unique finite, infinite or zero value of durability $t^*(\lambda\sigma)$ for any number $\lambda \geq 0$. Consider the inverse task: for any $t \geq 0$, to determine a number $\lambda^*(\sigma; t)$ such that $t^*(\lambda^*(\sigma; t)\sigma) = t$. This is equivalent interpreting the durability diagram $\lambda \mapsto t^*(\lambda\sigma)$ as the dependence $t \mapsto \lambda^*(\sigma; t)$. Examples of the diagrams in Figures 3(a), 4(b), 5(b), 6(b) and 7(b) show that this is not always uniquely possible since the dependence is either not defined or not unique for some instants t. The following definition concerns the cases when this is possible.

Definition 1CM. If the durability $t^*(\lambda\sigma)$ is a continuous and monotonically decreasing function of λ, the temporal safety factor $\underline{\lambda}^T(\sigma; t)$ is the non-negative number, by which the loading process $\sigma_{ij}(\tau)$ must be multiplied to obtain the durability t, i.e. $t^*(\underline{\lambda}^T(\sigma; t)\,\sigma) = t$.

This simple definition of the safety factor is however not applicable if the durability diagram $t^*(\lambda\sigma)$ is not a monotonically decreasing function of λ as in Figures 4(b) and 5(b) at $\tau = \tau^*$, since $\lambda^*(\sigma; t)$ then appears to be multiply defined. It is also not applicable if

$t^*(\lambda\sigma)$ has a horizontal jump as in Figure 6(b) since $\lambda^*(\sigma;t)$ appears to be not defined for $\tau_1 < t < \tau_2$.

If $\sigma_{ij}(\tau)$ is an MD process, i.e. the durability $t^*(\lambda\sigma)$ is a monotonically non-increasing although generally discontinuous function of λ, we generalize the definition in the following way:

Definition 1MD. The temporal safety factor $\underline{\lambda}^T(\sigma;t)$ for a MD process $\sigma_{ij}(\tau)$ is supremum of non-negative numbers λ such that the durability $t^*(\lambda\sigma)$ is greater than t; if there is no such λ, we take $\underline{\lambda}^T(\sigma;t) = 0$.

To overcome the difficulties with non-monotonically damaging processes, we introduce the following general definition of the safety factor $\underline{\lambda}(\sigma;t)$ embracing also the previous particular cases.

Definition 1. The temporal safety factor $\underline{\lambda}^T(\sigma;t)$ is supremum of $\lambda \geq 0$ such that $t^*(\lambda''\sigma) > t$ for any $\lambda'' \in [0,\lambda]$; if there is no such λ, we take $\underline{\lambda}^T(\sigma;t) = 0$. The mapping $(\sigma;t) \mapsto \underline{\lambda}^T(\sigma;t)$ defined on a set of processes $\sigma_{ij}(\tau)$ and time instants t is called the (strength) safety factor functional $\underline{\lambda}^T$.

Note that we can equivalently define $\underline{\lambda}^T(\sigma;t)$ as a non-negative number such that $t^*(\lambda\sigma) > t$ for any $\lambda \in [0,\underline{\lambda}^T(\sigma;t))$ but for any $\lambda > \underline{\lambda}^T(\sigma;t)$ there exists a number $\lambda'' \in [\underline{\lambda}^T(\sigma;t),\lambda]$ such that $t^*(\lambda''\sigma) \leq t$; if there is no such $\underline{\lambda}^T(\sigma;t)$, one should take $\underline{\lambda}^T(\sigma;t) = 0$.

Definition 2. The temporal normalized equivalent stress $\underline{\Lambda}^T(\sigma;t)$ is defined as $1/\underline{\lambda}^T(\sigma;t)$; if $\underline{\lambda}^T(\sigma;t) = 0$, we take $\underline{\Lambda}^T(\sigma;t) = \infty$.

The mapping $(\sigma;t) \mapsto \underline{\Lambda}^T(\sigma;t)$ defined on a set of processes $\sigma_{ij}(\tau)$ and time instants t is called the temporal normalized equivalent stress functional (TNESF) $\underline{\Lambda}^T$.

From the definition, if the durability $t^*(\sigma)$ is known, the value of the TNESF $\underline{\Lambda}^T(\sigma;t)$ is a solution of the scalar equation

$$t^*(\sigma/\Lambda) = t$$

for each instant t and loading process $\sigma(\tau)$ such that the dependence of the durability $t^*(\lambda\,\sigma)$ on λ is continuous and monotonic; if $t^*(\lambda\,\sigma)$ is not continuous or not monotonic, $\underline{\Lambda}^T(\sigma;t)$ is given by Definitions 1–2.

As follows from Definitions 1 and 2, the functionals $\underline{\lambda}^T$ and $\underline{\Lambda}^T$ do exist for any material with unambiguous detection of strength/rupture status and without repairing mechanism, and are unique; that is, they are material characteristics for a prescribed environment (temperature, pre-history, etc).

Remark 1. We can see from Definitions 1MD and 2 (see Appendix C) that it is possible to replace the durability $t^*(\lambda\sigma)$ by the critical time $t^*_{st}(\lambda\sigma)$ in the definitions to arrive at exactly the same functionals $\underline{\lambda}$ and $\underline{\Lambda}$ for MD processes $\sigma(\tau)$.

The temporal safety factor $\underline{\lambda}^T(\sigma;t)$ and the temporal normalized equivalent stress $\underline{\Lambda}^T(\sigma;t)$ are counterparts of the non-local safety factor $\underline{\lambda}(\sigma;x)$ and the non-local normalized equivalent stress (load factor) $\underline{\Lambda}(\sigma;x)$ defined in [8].

The safety factor and the TNESF introduced by Definitions 1–3 are durability-based. One can introduce also the corresponding strength-based functionals, coinciding with durability-based ones everywhere except for the points of their discontinuity in t; we will describe these elsewhere.

For brevity, we sometimes drop the superscript T in the rest of the paper.

To justify the name normalized equivalent stress for $\underline{\Lambda}$, we consider a constant in time process $\sigma_{ij}(\tau)$ =constant. For example, let the material strength be associated with the von Mises equivalent stress $\sigma^e(\sigma) = \sqrt{[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]/2}$, i.e. the strength condition has the form $\sigma^e(\sigma) < \sigma_r(t)$, where the function $\sigma_r(t)$ is a material characteristic (classical durability diagram under the uniaxial tension) and σ_1, σ_2, σ_3 are the principal stresses. Then $\underline{\Lambda}(\sigma; t)$ is defined from the equation $\sigma^e(\sigma/\Lambda) = \sigma_r(t)$, i.e.

$$\underline{\Lambda}(\sigma; t) = \sigma^e(\sigma)/\sigma_r(t). \tag{2}$$

Formula (2) holds true not only for the von Mises equivalent stress but also for the Tresca and any other equivalent stress representations $\sigma^e(\sigma)$ that are functions positively-homogeneous of the order of $+1$.

One can see from Definitions 1 and 2 that the safety factor is a non-increasing and the normalized equivalent stress is a non-decreasing function of time, i.e.

$$\underline{\lambda}(\sigma; t_2) \leq \underline{\lambda}(\sigma; t_1), \ \underline{\Lambda}(\sigma; t_2) \geq \underline{\Lambda}(\sigma; t_1) \quad \text{if} \quad t_2 > t_1. \tag{3}$$

Since, we suppose the opposite: there exists $t_2 > t_1$ such that $\underline{\lambda}(\sigma; t_2) > \underline{\lambda}(\sigma; t_1)$. From the $\underline{\lambda}$ definition then there exists λ such that $\underline{\lambda}(\sigma; t_2) > \lambda > \underline{\lambda}(\sigma; t_1)$ and $t_2 < t^*(\lambda\sigma) \leq t_1$. This contradicts the condition $t_2 > t_1$.

As follows from Definitions 1 and 2 (see Appendix D), for any t, the safety factor functional and the TNESF are non-negative positively-homogeneous functionals of the orders -1 and $+1$, respectively, i.e.

$$\underline{\lambda}(k\sigma; t) = \frac{1}{k}\underline{\lambda}(\sigma; t) \geq 0, \quad \underline{\Lambda}(k\sigma; t) = k\underline{\Lambda}(\sigma; t) \geq 0, \quad \text{for any } k > 0. \tag{4}$$

For infinite time t we get from here the corresponding definition of the endurance safety factor and temporal endurance normalize equivalent stress:

Definition 3. The temporal endurance (threshold) safety factor $\underline{\lambda}^T_{th}(\sigma)$ is the supremum of $\lambda \geq 0$ such that there is no rupture for all $t < \infty$ under the process $\lambda''\sigma$ for any $\lambda'' \in [0, \lambda]$; if there is no such λ, we take $\underline{\lambda}^T_{th}(\sigma) = 0$.

The temporal endurance (threshold) normalized equivalent stress is defined as $\underline{\Lambda}^T_{th}(\sigma) = 1/\underline{\lambda}^T_{th}(\sigma)$; if $\underline{\lambda}^T_{th}(\sigma) = 0$, we take $\underline{\Lambda}^T_{th}(\sigma) = \infty$.

The mappings $\sigma \mapsto \underline{\lambda}^T_{th}(\sigma)$, $\sigma \mapsto \underline{\Lambda}^T_{th}(\sigma)$ defined on a set of processes $\sigma_{ij}(\tau)$ are called the temporal endurance (threshold) safety factor functional $\underline{\lambda}_{th}$ and the temporal endurance (threshold) normalized equivalent stress functional $\underline{\Lambda}_{th}$, respectively.

Owing to monotonicity (3), we can define the endurance functionals also as

$$\underline{\lambda}^T_{th}(\sigma) = \underline{\lambda}^T(\sigma;\infty) := \inf_{t<\infty} \underline{\lambda}^T(\sigma;t), \quad \underline{\Lambda}^T_{th}(\sigma) = \underline{\Lambda}^T(\sigma,\infty) := \sup_{t<\infty} \underline{\Lambda}^T(\sigma;t). \qquad (5)$$

We can point out the cases, described in the previous section, for which $\underline{\lambda}_{th}(\sigma) = 0$: case (0) when the material is self-degrading, i.e. $t^*(0) < \infty$; case (D), i.e. $t^*(\lambda\sigma) \to t^{*0}(\sigma) \neq \infty$ as $\lambda \to 0$; case (E); case (G) since the absence of a limit of the function $t^*(\lambda\sigma)$ as $\lambda \to 0$ implies that there exists $t < \infty$ such that $\underline{\lambda}(\sigma;t) = 0$.

Evidently, the endurance safety factor and the endurance normalized equivalent stress make sense as material characteristics only for non-self-degrading materials. As follows from the self-degradation definition above, a material is self-degrading, if and only if there exists an instant $t^*(0)$ such that $\underline{\lambda}(0;t) = \infty$ for $t < t^*(0)$ and $\underline{\lambda}(0;t) = 0$ for $t \geq t^*(0)$. This statement gives an equivalent definition of self-degradation in terms of the safety factor $\underline{\lambda}$ behaviour.

The safety factor $\underline{\lambda}(\sigma;t)$ as a function of t at a given process $\sigma(\tau)$, can also be considered as a generalized durability diagram $t \mapsto \underline{\lambda}(\sigma;t)$. It coincides with the monotonic continuous parts of the corresponding diagram $\lambda \mapsto t^*(\lambda\sigma) = t$ giving there $\underline{\lambda}(\sigma;t^*(\lambda\sigma)) = \lambda$. It cuts off the non-monotonic (multi-valued) parts of the diagram $\lambda^*(\sigma;t)$ (taking the branch with the lowest λ^* and making a corresponding finite jump in $\underline{\lambda}(\sigma;t)$ in the branch beginning, see Figure 4) and continues the diagram on to the jump segment $[t^*((\lambda-0)\sigma), t^*((\lambda+0)\sigma)]$ where $\lambda^*(\sigma;t)$ does not exist, see Figures 6 and 7. As a result, the diagram looks like a curve in Figure 3(a) consisting of corresponding branches with, in addition, possible vertical jumps but without complications such as in Figures 4(b), 6(b) or 7(b). As shown already in this section, the diagram is monotonically non-increasing in time. The collection of such diagrams for all possible processes in fact defines the functional $\underline{\lambda}$.

From the generalized durability diagram $t \mapsto \underline{\lambda}(\sigma;t)$ for a given process $\sigma_{ij}(\tau)$, presented, for example, in Figure 3(a), we can obtain the corresponding diagram $t \mapsto \underline{\Lambda}(\sigma;t) = 1/\underline{\lambda}(\sigma;t)$ for the normalized equivalent stress $\underline{\Lambda}(\sigma;t)$, see Figure 3(b). Different curves correspond to different possible cases of its behaviour described in points (A)–(F) of Section 2. Generally, the $t \mapsto \underline{\Lambda}(\sigma;t)$ diagram can have vertical jumps and is a non-decreasing function of time (see (3)). Some examples are given in Figures 4(c), 5(c), 6(c) and 7(c).

The diagram $t \mapsto \underline{\Lambda}(\sigma;t)$ can be used in two ways. Firstly, it shows a number $\underline{\Lambda}(\sigma;t)$ such that there is no rupture up to time t for any process $\sigma_{ij}(\tau)/\Lambda'$ with $\Lambda' > \underline{\Lambda}(\sigma;t)$. For example, if the diagram includes the curve f (Figure 3), then the process $\sigma_{ij}(\tau)/\Lambda'$ with $\Lambda' > \underline{\Lambda}_{th}(\sigma)$ causes no rupture for any t. Another way is to use the diagram together with the stable strength condition (6) below for given $\sigma_{ij}(\tau)$ and t. For example, if the diagram includes the curve f, then the process $\sigma_{ij}(\tau)$ causes no rupture for any t if $\underline{\Lambda}_{th}(\sigma) < 1$.

4. STRENGTH AND ENDURANCE CONDITIONS

Let $\sigma_{ij}(\tau)$ be a process and t be a time instant. The following conclusions can be drawn from Definitions 1 and 2 for TNESF:

(i) The inequality

$$\underline{\Lambda}(\sigma; t) < 1 \tag{6}$$

implies absolutely stable strength under the process $\sigma_{ij}(\tau)$ at any instant $\tau \leq t$.

(ii) The equality

$$\underline{\Lambda}(\sigma; t) = 1 \tag{7}$$

implies

(a) either strength t-stable at any $\tau \leq t$ but not absolutely stable at an instant $\tau \leq t$ under the process $\sigma_{ij}(\tau)$, i.e. $t^*(\sigma) > t$ but for any $\lambda > 1$ there exists $\lambda'' \in (1, \lambda]$ such that $t^*(\lambda''\sigma) \leq t$;

(b) or rupture (or t-unstable strength) under the process $\sigma_{ij}(\tau)$ at an instant $\tau \leq t$, i.e. $t^*(\sigma) \leq t$.

(iii) If $\sigma_{ij}(\tau)$ is an MD process, the inequality

$$\underline{\Lambda}(\sigma; t) > 1 \tag{8}$$

implies rupture (or t-unstable strength) under the process $\sigma_{ij}(\tau)$ at an instant $\tau \leq t$, i.e. $t^*(\sigma) \leq t$.

Let us show that, inversely, if strength is absolutely stable for an MD process $\sigma_{ij}(\tau)$ at all $\tau \leq t$ then Equation (6) is satisfied. The strength absolute stability means that there exists $\lambda > 1$ such that $t^*(\lambda''\sigma) > t$ for any $\lambda'' \in [1, \lambda]$. In addition, $t^*(\lambda''\sigma) > t$ also for all $\lambda'' \in [0, 1]$ since the process $\sigma_{ij}(\tau)$ is MD. The application of Definitions 1 and 2 completes the proof of the following statement.

Statement 1. Inequality (6) gives a sufficient (and necessary, if $\sigma_{ij}(\tau)$ is an MD process) condition of absolutely stable strength at all $\tau \leq t$ under the process $\sigma_{ij}(\tau)$.

For the endurance functionals, we similarly have from Definition 3 the following conclusions:

(i) The inequality

$$\underline{\Lambda}_{th}(\sigma) < 1 \tag{9}$$

implies λ-stable endurance under the process $\sigma_{ij}(\tau)$.

(ii) The equality

$$\underline{\Lambda}_{th}(\sigma) = 1 \tag{10}$$

implies

(a) either λ-unstable endurance, that is, there is no rupture under the process $\sigma_{ij}(\tau)$ at any time but for any $\lambda > 1$ there exists $\lambda'' \in (1, \lambda]$ such that $t^*(\lambda''\sigma) < \infty$;

(b) or rupture at an instant $t < \infty$, i.e. $t^*(\sigma) < \infty$.

(iii) If $\sigma_{ij}(\tau)$ is an MD process, the inequality

$$\underline{\Lambda}_{th}(\sigma) > 1 \tag{11}$$

implies rupture at an instant $t < \infty$, i.e. $t^*(\sigma) < \infty$.

Then we have the following statement:

Statement 2. Inequality (9) gives a sufficient (and necessary, if $\sigma_{ij}(\tau)$ is an MD process) condition of λ-stable endurance under the process $\sigma_{ij}(\tau)$.

Conditions (6)–(11) together with the homogeneity of $\underline{\Lambda}$ and $\underline{\Lambda}_{th}$ also show that the functionals do really play the role of normalized equivalent stresses.

It follows from the TNESF definition that if the durability diagram $t^*(\lambda\sigma)$ is known for a process $\sigma_{ij}(\tau)$ for all $\lambda \geq 0$, then the normalized equivalent stress $\underline{\Lambda}(\sigma; t)$ can be obtained for $\sigma_{ij}(\tau)$ for any $t \geq 0$. Let us consider an inverse task. Suppose the values of the TNESF $\underline{\Lambda}(\sigma; t)$ are known for a process $\sigma_{ij}(\tau)$ for any $t \geq 0$. Is it possible to obtain values of the durability diagram $t^*(\lambda\sigma)$ for any $\lambda \geq 0$ for the process $\sigma_{ij}(\tau)$?

It is evident that this is not possible if $\sigma_{ij}(\tau)$ is not an MD process, since the information about the non-monotonic behaviour of $t^*(\lambda\sigma)$ as a function of λ is lost in $\underline{\Lambda}(\sigma; t)$. On the other hand, if not only inequalities (3) holds but $\underline{\Lambda}(\sigma; t)$ is a monotonically increasing and continuous function of t, then it is evident that $t^*(\lambda\sigma)$ is a solution of the following scalar equation

$$\underline{\Lambda}(\sigma; t^*) = 1/\lambda \tag{12}$$

and this solution exists and is unique if $\underline{\Lambda}(\sigma; 0) \leq 1/\lambda \leq \underline{\Lambda}(\sigma; \infty)$.

Note that generally equality (12) cannot be satisfied even for arbitrary MD processes but the following inequality holds for any process

$$\underline{\Lambda}(\sigma; t^*(\lambda\sigma)) \geq 1/\lambda \quad \text{for all } \lambda > 0. \tag{13}$$

Since, using Definition 1 for $\underline{\lambda}(\sigma; t^*(\lambda\sigma))$, we have $t^*(\tilde{\lambda}\sigma) > t^*(\lambda\sigma)$ for all $\tilde{\lambda} \in [0, \underline{\lambda}(\sigma; t^*(\lambda\sigma)))$, then $\underline{\lambda}(\sigma; t^*(\lambda\sigma)) \leq \lambda$ since otherwise $t^*(\lambda\sigma) > t^*(\lambda\sigma)$ which is absurd.

The above discussion shows that, in addition to the non-sensitivity to non-monotonic behaviour of the durability diagram, the TNESF also does not distinguish rupture from not absolutely stable strength. For this reason, it is not the durability $t^*(\sigma)$ but the critical time $t^*_{st}(\sigma) \leq t^*(\sigma)$ which can be obtained from $\underline{\Lambda}(\sigma; t)$ generally. The following statement is proved in Appendix E:

Statement 3. Let $\sigma_{ij}(\tau)$ be an MD process. The critical time $t^*_{st}(\sigma)$ equals the supremum of t such that

$$\underline{\Lambda}(\sigma; t) < 1. \tag{14}$$

Taking into account the homogeneity of $\underline{\Lambda}(\sigma; t)$, one can obtain from Statement 3 the following slightly more general proposition:

Corollary 1. Let $\sigma_{ij}(\tau)$ be an MD process. For any $\lambda > 0$, the critical time $t^*_{st}(\lambda\sigma)$ equals the supremum of t such that $\underline{\Lambda}(\sigma; t) < 1/\lambda$.

The following corollary is proved in Appendix F:

Corollary 2. A time t^{**} is critical, i.e. $t^{**} = t^*_{st}(\sigma)$, for an MD process $\sigma_{ij}(\tau)$ if and only if

$$\underline{\Lambda}(\sigma; t) < 1 \leq \underline{\Lambda}(\sigma; t^{**}) \; \forall \, t < t^{**}. \tag{15}$$

From inequality (15) we also have the following Corollary:

Corollary 3. *If* $\underline{\Lambda}(\sigma; t)$ is left-continuous in time at $t = t^*_{st}(\sigma)$ for an MD process $\sigma_{ij}(\tau)$, then $\underline{\Lambda}(\sigma; t^*_{st}(\sigma)) = 1$.

As remarked before, one can replace the durability $t^*(\lambda\sigma)$ by the critical time $t^*_{st}(\lambda\sigma)$ in Definitions 1 and 2 to arrive at exactly the same functionals $\underline{\lambda}$ and $\underline{\Lambda}$ for an MD process $\sigma(\tau)$. Thus, if the critical time $t^*_{st}(\lambda\sigma)$ is known for an MD process $\sigma_{ij}(\tau)$ at all $\lambda \geq 0$, then the values of the TNESF $\underline{\Lambda}(\sigma; t)$ are uniquely determined for the process $\sigma_{ij}(\tau)$ at any t. Conversely, if the values of the TNESF $\underline{\Lambda}(\sigma; t)$ are known for an MD process $\sigma_{ij}(\tau)$ at all t, then the values of the critical time $t^*_{st}(\lambda\sigma)$ are uniquely determined for the process $\sigma_{ij}(\tau)$ at any $\lambda \geq 0$ and particularly at $\lambda = 1$.

Note that namely the critical time $t^*_{st}(\sigma)$ is necessary for practical design since, as mentioned above, for cases when $t^*_{st}(\sigma) \neq t^*(\sigma)$, the material strength is λ-unstable for $t \in (t^*_{st}(\sigma), t^*(\sigma))$.

5. EXISTENCE AND UNIQUENESS OF THE TNESF

Suppose the material strength under a process $\sigma_{ij}(\tau)$ at an instant t is described by a (necessary and sufficient) strength condition

$$\underline{F}(\sigma; t) < 1 \tag{16}$$

where $\underline{F}$ is a non-linear functional non-decreasing in time, known from experimental data approximation or from a durability theory on the processes $\lambda\sigma_{ij}(\tau)$ for all $\lambda \geq 0$ and for all instants $t'' < t'$ for some $t' > t$. Non-decreasing in time for F means the absence of a repairing mechanism. Then necessity and sufficiency of strength condition (16) implies unambiguous detection of the rupture/strength state at any instant $t < t'$ and consequently of the durability $t^*(\lambda\sigma)$ for all $\lambda \geq 0$ if $t^*(\lambda\sigma) < t'$. Thus, Definitions 1 and 2 are applicable to uniquely determine TNESF $\underline{\Lambda}(\sigma; t)$ on $\sigma_{ij}(\tau)$ at instant t, although this does not always lead to an analytical expression. Owing to the TNESF homogeneity, we have then its values $\underline{\Lambda}(\lambda\sigma; t) = \lambda\underline{\Lambda}(\sigma; t)$ for any $\lambda \geq 0$.

Statement 4. (i) If $\underline{F}(\sigma; t)$ is a non-negative positively-homogeneous functional of the order of $+1$ on σ and non-decreasing in t, then generally $\underline{\Lambda}(\sigma; t) \geq \underline{F}(\sigma; t)$.
(ii) If, in addition $\underline{F}(\sigma; t)$ is right-continuous in the second argument at the considered time t, then simply $\underline{\Lambda}(\sigma; t) = \underline{F}(\sigma; t)$.

The proof is given in Appendix G.

The statement is used in Section 6 to obtain TNESFs from known strength conditions of some durability theories.

6. EXAMPLES OF NORMALIZED EQUIVALENT STRESS FUNCTIONALS

Let us consider examples of calculation of the TNESF $\underline{\Lambda}$ for several known durability theories. It is supposed for all the examples that $\sigma_{ij}(\tau) = 0$ if $\tau \leq 0$.

6.0. TNESF for constant loading

6.0.1. Uniaxial constant loading

For a uniaxial constant loading $\sigma(\tau) = \sigma$ = constant at $\tau > 0$, the temporal strength condition can be written in the form

$$|\sigma| < \sigma^*(\mathrm{sign}(\sigma); t), \tag{17}$$

where the temporal strength $\sigma^*(\mathrm{sign}(\sigma); t)$ is a non-increasing function of time t depending also on the sign of the applied stress σ. Inversely, the durability condition can be written in the form

$$t < t^{*0}(\sigma),$$

where $|\sigma| \mapsto t^{*0}(\sigma) = t^{*0}(\mathrm{sign}(\sigma)|\sigma|)$ is the classical durability diagram for constant loading.

From Equation (17) and Definition 1, we have the strength condition in terms of the TNESF

$$\underline{\Lambda}_0(\sigma, t) = \frac{|\sigma|}{\sigma^*(\mathrm{sign}(\sigma); t)} < 1. \tag{18}$$

6.0.2. Multiaxial constant loading

For a multiaxial constant loading $\sigma_{ij}(\tau) = \sigma_{ij}$ = constant, $\tau > 0$, the temporal strength condition can be written in the form

$$|\sigma| < \sigma^*(\tilde{\sigma}; t), \tag{19}$$

where $|\sigma|$ is a matrix norm of the tensor σ_{ij} and the temporal strength $\sigma^*(\tilde{\sigma}; t)$ is a non-increasing function of time t depending also on the shape $\tilde{\sigma}_{ij} = \sigma_{ij} / |\sigma|$ of the applied stress tensor σ_{ij}. Inversely, the durability condition can be written in the form

$$t < t^{*0}(\sigma),$$

where $|\sigma| \mapsto t^{*0}(\sigma) = t^{*0}(\tilde{\sigma}|\sigma|)$ is the classical durability diagram for constant multiaxial loading.

From Equation (19) and Definition 1, we have the multiaxial strength condition in terms of the TNESF

$$\underline{\Lambda}_0(\sigma, t) = \frac{|\sigma|}{\sigma^*(\tilde{\sigma}; t)} < 1. \tag{20}$$

6.1. Time and history independent material

Let the material strength under an arbitrary (right-continuous) multiaxial loading process $\sigma_{ij}(\tau)$ be determined only by its instant stress tensor value. Then it can be described by the strength condition

$$\Lambda^I(\sigma(\tau)) < 1 \tag{21}$$

where $\Lambda^I(\sigma)$ is a known non-negative positively-homogeneous function of the order of $+1$. For example, Λ^I can be the von Mises normalized equivalent stress $\Lambda^I(\sigma) = \sqrt{[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]/(2\sigma_r^2)}$ or the Tresca normalized equivalent stress $\Lambda^I(\sigma) = \max_{k,m} |\sigma_k - \sigma_m|/\sigma_r$, where, as above, $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses and σ_r is a known uniaxial tensile strength. Evidently, the strength condition on an interval $0 \le \tau \le t$ can be rewritten in the form

$$\underline{\Lambda}_1(\sigma; t) < 1, \qquad \underline{\Lambda}_1(\sigma; t) = \sup_{0 \le \tau \le t} \Lambda^I(\sigma(\tau)) \tag{22}$$

where $\underline{\Lambda}_1$ is the TNESF (for instants when it is right-continuous in t, cf Statement 4).

6.2. Temporal strength condition

Let the dynamic strength of a material under uniaxial loading be described by the Nikiforovsky–Shemyakin temporal strength condition [10, 11]

$$\int_0^t \sigma(\tau) \mathrm{d}\tau < J_r, \tag{23}$$

where J_r is a material parameter. Then the durability $t^*(\lambda\sigma)$ under the process $\lambda\sigma(\tau)$ for $\sigma(\tau) > 0$ is determined from

$$\int_0^{t^*} \lambda\sigma(\tau) \mathrm{d}\tau = J_r$$

and

$$\lambda = J_r \left[\int_0^{t^*} \sigma(\tau) \mathrm{d}\tau \right]^{-1}. \tag{24}$$

If $\sigma(\tau) > 0$ at $\tau > 0$, the right-hand side of (24) is a continuous monotonically decreasing function of t^*, and we have from Definitions 1CM and 2,

$$\underline{\Lambda}_2(\sigma; t) = \frac{1}{J_r} \int_0^t \sigma(\tau)\, \mathrm{d}\tau. \tag{25}$$

For arbitrary processes with not necessarily positive $\sigma(\tau)$ at $\tau > 0$, we have from Definitions 1 and 2 a more general formula for the TNESF:

$$\underline{\Lambda}_2(\sigma; t) = \max \left[0, \frac{1}{J_r} \sup_{0 \le t' \le t} \int_0^{t'} \sigma(\tau)\, \mathrm{d}\tau \right]. \tag{26}$$

6.3. Finitely-temporal strength condition

Let the dynamic strength of a material under uniaxial loading be described by the finitely-temporal (structural-temporal) strength condition (see [12]) at an instant t:

$$\sup_{0 \le t' \le t} \frac{1}{t_r} \int_{t'-t_r}^{t'} \sigma(\tau)\, \mathrm{d}\tau < \sigma_r, \tag{27}$$

where $t_r > 0$ and $\sigma_r > 0$ are material parameters. Then the durability $t^*(\lambda\sigma)$ under the process $\lambda\sigma(\tau)$ is determined as a minimal non-negative solution of

$$\sup_{0 \le t' \le t^*} \frac{1}{t_r} \int_{t'-t_r}^{t'} \lambda\sigma(\tau)\, \mathrm{d}\tau = \sigma_r$$

and

$$\lambda = \sigma_r t_r \left[\sup_{0 \le t' \le t^*} \int_{t'-t_r}^{t'} \sigma(\tau)\, \mathrm{d}\tau \right]^{-1}. \tag{28}$$

From Definitions 1 and 2, then the TNESF is

$$\underline{\Lambda}_3(\sigma; t) = \max \left[0, \frac{1}{\sigma_r t_r} \sup_{0 \le t' \le t} \int_{t'-t_r}^{t'} \sigma(\tau)\, \mathrm{d}\tau \right]. \tag{29}$$

It is evident that $\underline{\Lambda}_2(\sigma; t) = \underline{\Lambda}_3(\sigma; t)$ for $t \le t_r$ if $J_r = \sigma_r t_r$.

6.4. Campbell strength condition

Campbell [13] introduced the following yielding criterion under uniaxial dynamic loading $\sigma(\tau) \geq 0$

$$\int_0^{t^*} \left[\frac{\sigma(\tau)}{\sigma_r}\right]^b \mathrm{d}\tau = t_r, \tag{30}$$

where $t_r > 0$, $\sigma_r > 0$ and b are material parameters. We can treat this also as a dynamic rupture criterion. Then the corresponding strength condition is

$$\underline{\Lambda}_4(\sigma; t) < 1, \tag{31}$$

where

$$\underline{\Lambda}_4(\sigma; t) = \left[\frac{1}{t_r}\int_0^{t} \left[\frac{\sigma(\tau)}{\sigma_r}\right]^b \mathrm{d}\tau\right]^{1/b} \tag{32}$$

is the TNESF, Statement 4.

If $b = 1$, then evidently, $\underline{\Lambda}_4(\sigma; t)$ degenerates into $\underline{\Lambda}_2(\sigma; t)$ for $J_r = \sigma_r t_r$. For $b = 1$ and $t \leq t_r$ it coincides also with $\underline{\Lambda}_3(\sigma; t)$.

6.5. Modified Campbell strength condition

A finitely temporal modification of dynamic yield (rupture) criterion (30) presented in [12] (see also references therein)

$$\frac{1}{t_r}\int_{t^*-t_r}^{t^*} \left[\frac{\sigma(\tau)}{\sigma_r}\right]^b \mathrm{d}\tau = 1, \tag{33}$$

leads to the following strength condition

$$\underline{\Lambda}_5(\sigma; t) < 1, \tag{34}$$

where

$$\underline{\Lambda}_5(\sigma; t) = \sup_{0\leq t'\leq t}\left[\frac{1}{t_r}\int_{t'-t_r}^{t'} \left[\frac{\sigma(\tau)}{\sigma_r}\right]^b \mathrm{d}\tau\right]^{1/b} \tag{35}$$

is the TNESF and $t_r > 0$, $\sigma_r > 0$ and b are material parameters. If $b = 1$, then evidently, $\underline{\Lambda}_5(\sigma; t)$ degenerates into $\underline{\Lambda}_3(\sigma; t)$. For $t \leq t_r$ it coincides also with $\underline{\Lambda}_4(\sigma; t)$ and if, additionally, $b = 1$, then also with $\underline{\Lambda}_2(\sigma; t)$ for $J_r = \sigma_r t_r$.

6.6. Il'ushin durability theory

6.6.1. Linear theory

It was supposed in [5] that there exists an abstract damage tensor $\omega_{ij}(\sigma; t)$ which is a functional defined on load processes $\sigma_{ij}(\tau)$ (for simplicity, here we neglect the dependence of ω also on stress moments considered in [5]). It was supposed that there is no rupture if a set of $\tilde{m}$ strength conditions written in terms of $\omega(\sigma; t)$ is satisfied:

$$M_m(\omega) < c_m, \quad m = 1, ..., \tilde{m}. \tag{36}$$

Here functions M_m and constants c_m are material characteristics associated with an mth rupture mode.

It was supposed in the linear version of the Il'ushin theory [5] that the tensor damage functional $\omega(\sigma; t)$ can be taken in the form

$$\omega_{ij}(\sigma; t) = \int_0^t \varphi_{ijkl}(t - \tau)\, \mathrm{d}\sigma_{kl}(\tau) \tag{37}$$

where functions $\varphi_{ijkl}(\tau)$ are material characteristics independent of $\sigma_{ij}(\tau)$.

Denoting $M(\omega_{ij}) = \max_{m=1,...\tilde{m}}(M_m(\omega_{ij})/c_m)$, we can rewrite Equations (36)–(37) in the form

$$M[\int_0^t \varphi_{ijkl}(t - \tau)\, \mathrm{d}\sigma_{kl}(\tau)] < 1. \tag{38}$$

Since the left-hand side of Equation (38) can be non-monotonic in t at least for non-monotonic processes $\sigma_{kl}(\tau)$, it should be corrected so as not to predict a life after rupture, e.g. in the following way,

$$\sup_{0 \le t' \le t} M[\int_0^{t'} \varphi_{ijkl}(t' - \tau)\mathrm{d}\sigma_{kl}(\tau)] < 1. \tag{39}$$

The corresponding strength condition for a process $\lambda\sigma_{ij}(\tau)$ takes the form

$$\sup_{0 \le t' \le t} M[\lambda \int_0^{t'} \varphi_{ijkl}(t' - \tau)\mathrm{d}\sigma_{kl}(\tau)] < 1. \tag{40}$$

Suppose first that the function M is non-negative and positively homogeneous of the order of $+1$, i.e. $M(\lambda\omega_{ij}) = \lambda M(\omega_{ij})$. Then according to Definitions 1CM and 2, we have the following expression for the TNESF:

$$\underline{\Lambda}_6(\sigma; t) = \sup_{0 \le t' \le t} M[\int_0^{t'} \varphi_{ijkl}(t' - \tau)\, \mathrm{d}\sigma_{kl}(\tau)]. \tag{41}$$

If the function M is not positively homogeneous of the order of $+1$, one should reduce Equation (38) to an equivalent form with a new function M which is already homogeneous.

Another way is to apply the more general Definition 1, which probably demands some numerical calculations. In this case $\underline{\lambda}_6(\sigma;t)$ is the supremum of numbers λ^* such that inequality (40) is satisfied for all $\lambda\in[0,\lambda^*]$. If $M(\lambda\omega_{ij})$ is continuously monotonically growing with λ, then one can obtain $\underline{\lambda}_6(\sigma;t)$ more simply as a solution of the equation obtained from (40) (after replacing the sign '$<$' by the sign '$=$'), instead of finding the supremum. Then $\underline{\Lambda}_6(\sigma;t)=1/\underline{\lambda}_6(\sigma;t)$ according to Definition 2.

Suppose $\sigma_{11}(\tau)$ is a uniaxial process. If we take $\varphi_{ijkl}(t-\tau)=(t-\tau)\delta_{ik}\delta_{jl}$ and $M(\omega_{11})=\omega_{11}/J_r$, then the TNESF $\underline{\Lambda}_6(\sigma;t)$ from the Il'ushin linear durability theory coincides with its counterpart $\underline{\Lambda}_2(\sigma;t)$ given by the Nikiforovsky–Shemyakin temporal strength condition. On the other hand, if we take $\varphi_{ijkl}(t-\tau)=[t-\tau-(t-\tau-t_r)H(t-\tau-t_r)]\delta_{ik}\delta_{jl}$ and $M(\omega_{11})=\omega_{11}/(\sigma_r t_r)$, then the TNESF $\underline{\Lambda}_6(\sigma;t)$ from the Il'ushin linear durability theory coincides with its counterpart $\underline{\Lambda}_3(\sigma;t)$ given by the finitely-temporal strength condition (27).

6.6.2. Non-linear theory

In the non-linear version of the Il'ushin durability theory [5], representation (37) is replaced by a more general non-linear form for the damage tensor functional:

$$\begin{aligned}\omega_{ij}(\sigma;t) &= \sum_{n=1}^{\infty}\int_0^t \ldots \int_0^t \Phi^{(n)}_{iji_1j_1\ldots i_nj_n}(t-\tau_1,\ldots,t-\tau_n)\\ &\times\ \sigma_{i_1j_1}(\tau_1)\ldots\sigma_{i_nj_n}(\tau_n)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n\,. \end{aligned}\tag{42}$$

Then as above, the corresponding strength condition for a process $\lambda\sigma_{ij}(\tau)$ takes the form

$$\begin{aligned}&\sup_{0\le t'\le t} M\left[\sum_{n=1}^{\infty}\lambda^n\int_0^{t'}\ldots\int_0^{t'}\Phi^{(n)}_{iji_1j_1\ldots i_nj_n}(t'-\tau_1,\ldots,t'-\tau_n)\right.\\ &\times\ \left.\sigma_{i_1j_1}(\tau_1)\ldots\sigma_{i_nj_n}(\tau_n)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n\right] < 1.\end{aligned}\tag{43}$$

If the left-hand side of Equation (43) is a monotonically and continuously growing function of λ, then according to Definition 1CM one can obtain $\underline{\lambda}_6(\sigma;t)$ as a solution of the equation obtained from (43) (after replacing the sign '$<$' by the sign '$=$' there). Otherwise, one can apply the more general Definition 1. In this case $\underline{\lambda}_6(\sigma;t)$ is the supremum of the numbers λ^* such that inequality (43) is satisfied for all $\lambda\in[0,\lambda^*]$. Then $\underline{\Lambda}_6(\sigma;t)=1/\underline{\lambda}_6(\sigma;t)$ according to Definition 2.

6.7. Robinson linear rule of damage accumulation

Let a material obey the Robinson hypothesis of creep damage linear accumulation [14, 15] (see also [3, 16]). Then the durability $t^*(\sigma)$ under a multiaxial process $\sigma_{ij}(\tau)$ can be determined from the equation

$$\int_0^{t^*} \frac{\mathrm{d}\tau}{t^{*0}(\sigma(\tau))} = 1. \tag{44}$$

Here $t^{*0}(\sigma(\tau)) = t^{*0}(\sigma_{ij}(\tau)) = t^*(\sigma^0_{ij})|_{\sigma^0_{ij}=\sigma_{ij}(\tau)}$ is a function presenting the classical durability diagram under a multiaxial step-like loading, where σ^0_{ij} = constant. Then $t^*(\lambda\sigma)$ is determined from

$$\int_0^{t^*} \frac{\mathrm{d}\tau}{t^{*0}(\lambda\sigma(\tau))} = 1. \tag{45}$$

Suppose the classical durability diagram $t^*(\sigma^0_{ij})$ is given by the power law (1). Then (45) is reduced to

$$\int_0^{t^*} \frac{(\lambda|\sigma(\tau)|)^{b(\tilde{\sigma}(\tau))}}{A(\tilde{\sigma}(\tau))} d\tau = 1. \tag{46}$$

Suppose additionally that $\sigma_{ij}(\tau)$ is an in-phase (coaxial, proportional) multiaxial process. That is, the shape of the tensor $\sigma_{ij}(\tau)$ does not vary in time: $\sigma_{ij}(\tau) = \sigma^0_{ij}\frac{|\sigma(\tau)|}{|\sigma^0|}$, i.e. $\tilde{\sigma}_{ij}(\tau) = \sigma_{ij}(\tau)/|\sigma(\tau)| = \sigma^0_{ij}/|\sigma^0|$ =constant. Then $A(\tilde{\sigma}(\tau))$ =constant, $b(\tilde{\sigma}(\tau))$ = constant and we have from Equation (46):

$$\lambda = \left[\frac{1}{A(\tilde{\sigma})}\int_0^{t^*} |\sigma(\tau)|^{b(\tilde{\sigma})}\mathrm{d}\tau\right]^{-1/b(\tilde{\sigma})}. \tag{47}$$

Supposing $b > 0$, then the right-hand side of Equation (47) is a continuous monotonically non-increasing function of t^*. From Definitions 1CM and 2, we then have the following representations for the TNESF on in-phase processes:

$$\underline{\Lambda}_7(\sigma; t) = \left[\frac{1}{A(\tilde{\sigma})}\int_0^{t} |\sigma(\tau)|^{b(\tilde{\sigma})}\mathrm{d}\tau\right]^{1/b(\tilde{\sigma})}. \tag{48}$$

Let now $\sigma_{ij}(\tau)$ be an arbitrary multiaxial process but b = constant is a positive material parameter independent on $\tilde{\sigma}_{ij}(\tau)$. Then in a similar way one can obtain for this case the TNESF

$$\underline{\Lambda}_7(\sigma; t) = \left[\int_0^{t} \frac{|\sigma(\tau)|^b}{A(\tilde{\sigma}(\tau))}\mathrm{d}\tau\right]^{1/b}. \tag{49}$$

If $\sigma(\tau) \geq 0$ is a uniaxial process, then the TNESFs (48) and (49) coincide for $A = t_r\sigma_r^b$ with the functional $\underline{\Lambda}_4(\sigma; t)$ associated with the Campbell strength condition.

If the loading is not in-phase and $b(\tilde{\sigma})$ is not constant, or the classical durability diagram is more complicated than (1), then Equation (45) cannot be generally solved with respect to λ analytically, but this can be done numerically. The solution gives $\underline{\lambda}_7(\sigma; t^*)$ and $\underline{\Lambda}_7(\sigma; t^*) = 1/\underline{\lambda}_7(\sigma; t^*)$, if $t^{*0}(\lambda\sigma_{ij}(\tau))$ is a decreasing function of λ. Otherwise one should apply general Definitions 1 and 2 to Equation (45) (where the sign '$=$' must be replaced by the sign '$<$').

6.8. Hoff model for rod creep rupture

Consider an incompressible rod under a nominal stress $\sigma_0(\tau) \geq 0$ at $\tau > 0$. Its creep can be described by the Norton creep law

$$\frac{\mathrm{d}\epsilon(\tau)}{\mathrm{d}\tau} = a\sigma^b(\tau), \tag{50}$$

where ϵ is the creep logarithmic strain, and $a, b > 0$ are material constants. The creep rupture is modelled by Hoff [17] (see also [3, Section 85] and [4, Section 2.2]) taking into account the increase of the actual stress σ, caused by the rod cross-section decrease due to the material incompressibility,

$$\sigma = \sigma_0(\tau)\mathrm{e}^{\epsilon(\tau)}. \tag{51}$$

The strength condition

$$\sup_{0\leq\tau\leq t} \sigma(\tau) < \infty \tag{52}$$

is applied, which can be also rewritten in the form

$$\underline{\Lambda}_8(\sigma; t) < 1, \quad \underline{\Lambda}_8(\sigma; t) = \begin{cases} 0 & \text{if} \quad \sup_{0\leq\tau\leq t} \sigma(\tau) < \infty \\ \infty & \text{if} \quad \sup_{0\leq\tau\leq t} \sigma(\tau) = \infty \end{cases} \tag{53}$$

Substituting Equation (51) into (50) and integrating it with the initial condition $\epsilon(0) = 0$ gives the following relation between actual and nominal stresses

$$\sigma(t) = \sigma_0(t) \left[1 - ab \int_0^t \sigma_0^b(\tau)\,\mathrm{d}\tau\right]^{-1/b}. \tag{54}$$

Using Equation (52), for a process $\lambda\sigma_0(\tau)$ we arrive at the strength condition in terms of σ_0 on a time segment $[0, t]$:

$$\sup_{0\leq\tau\leq t} \lambda\sigma_0(\tau) \left[1 - ab\lambda^b \int_0^\tau \sigma_0^b(\xi)\,\mathrm{d}\xi\right]^{-1/b} < \infty. \tag{55}$$

This means that, in terms of the nominal stress, the TNESF for this model is

$$\underline{\Lambda}_{08}(\sigma_0; t) = \max\left\{\hat{\underline{\Lambda}}_{08}(\sigma_0; t), \underline{\Lambda}_8(\sigma_0; t)\right\}, \tag{56}$$

where

$$\underline{\hat{\Lambda}}_{08}(\sigma_0; t) = \left[ab \int_0^t \sigma_0^b(\tau) \mathrm{d}\tau \right]^{1/b}. \tag{57}$$

Up to the notation $ab = 1/A = 1/(t_r\sigma_r)$, the functional $\underline{\hat{\Lambda}}_{08}$ coincides with the TNESF $\underline{\Lambda}_4(\sigma; t)$ corresponding to the Campbell strength condition and with the functional $\underline{\Lambda}_7(\sigma; t)$ corresponding to the power law of durability and the linear rule of damage accumulation.

6.9. Kachanov damage model

In the Kachanov damage model [1] (see also [3, Section 87] and [4, Section 2.4]) the same problem as in the Hoff model is considered and the same creep law (50) and expression for the actual stress (51) are supposed. However, the strength condition (52) is replaced by the strength condition

$$\sup_{0 \le \tau \le t} \frac{\sigma(\tau)}{1 - \omega(\tau)} < \infty, \tag{58}$$

which is equivalent to Equation (52) supplemented by the strength condition

$$\omega(t) < 1. \tag{59}$$

Here $\omega(t)$ is a damage measure, whose behaviour is described by equation

$$\frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau} = B \left(\frac{\sigma(\tau)}{1 - \omega(\tau)} \right)^k \tag{60}$$

with the initial condition $\omega(0) = 0$; $B, k > 0$ are material constants.

Integrating Equation (60), after some manipulations we can rewrite the strength condition (59) in the form homogeneous with respect to σ:

$$\underline{\hat{\Lambda}}_9(\sigma; t) < 1, \quad \underline{\hat{\Lambda}}_9(\sigma; t) = \left[(k+1)B \int_0^t \sigma^k(\tau)\, \mathrm{d}\tau \right]^{1/k}. \tag{61}$$

Note that $\underline{\hat{\Lambda}}_9$ coincides with $\underline{\Lambda}_4$, $\underline{\Lambda}_7$ and $\underline{\hat{\Lambda}}_{08}$ up to notations.

Recalling the strength condition (53), we finally obtain the TNESF in terms of the actual stress σ:

$$\underline{\Lambda}_9(\sigma; t) = \max(\underline{\Lambda}_8(\sigma; t), \underline{\hat{\Lambda}}_9(\sigma; t)). \tag{62}$$

To obtain the TNESF in terms of the nominal stress, we substitute Equation (54) into (61) and for a process $\lambda\sigma_0(\tau)$ we arrive at the strength condition

$$\lambda^k \int_0^t \sigma_0^k(\tau) \left[1 - ab\lambda^b \int_0^\tau \sigma_0^b(\xi)\, \mathrm{d}\xi \right]^{-k/b} \mathrm{d}\tau < \frac{1}{(k+1)B}. \tag{63}$$

This completes the strength condition (55) on a time segment $[0, t]$.

Taking into account Equation (55) and condition $\sigma_0 \geq 0$, one can see that the left-hand side of Equation (63) is a monotonically increasing function of λ at fixed t and a monotonically non-decreasing function t at fixed λ. This means that, in terms of the nominal stress for this model, the TNESF is

$$\underline{\Lambda}_{09}(\sigma_0; t) = \max(\underline{\Lambda}_{08}(\sigma_0; t), 1/\lambda_{09}(\sigma_0; t)) \tag{64}$$

where $\underline{\Lambda}_{08}(\sigma_0; t)$ is defined in Equation (56) and $\lambda_{09}(\sigma_0; t)$ is a unique non-negative solution of the equation obtained from (63) after the replacement of the inequality by the equality sign (for instants when it is right-continuous in t, see Statement 4). For each t and an arbitrary process $\sigma_0(\tau) \geq 0$, this equation is nonlinear transcendental and can be solved numerically. For $\sigma_0(\tau) = \text{constant}$ at $\tau \geq 0$, this equation is reduced to (cf [1], [3, Section 87], [4, Section 2.4])

$$t = \frac{1}{ab(\lambda\sigma_0)^b} \left\{ 1 - \left[1 - \frac{(b-k)a(\lambda\sigma_0)^{b-k}}{(k+1)B} \right]^{\frac{b}{b-k}} \right\}.$$

6.10. Rabotnov damage model

In the Rabotnov damage model [2] (see also [3, Section 87] and [4, Section 2.4]), we consider the same problem for a rod under creep, as in the Hoff and Kachanov models. However, the influence of the damage on the creep is taken into account in the form

$$\frac{\mathrm{d}\epsilon(\tau)}{\mathrm{d}\tau} = a\sigma^b(\tau)(1 - \omega(\tau))^{-q}, \tag{65}$$

whereas the relation between the nominal σ_0 and actual σ stresses is given by the same formula (51).

An expression for the damage rate more general than Equation (60) is given in the form

$$\frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau} = B\sigma^k(\tau)(1 - \omega(\tau))^{-r}. \tag{66}$$

Here a, b, q, B, k, r are considered to be material constants. It is evident that Equations (65) and (66) degenerate at $q = 0$ and $k = r$ into the relations (50) and (60) used by Kachanov.

The strength condition (59) completed in fact by condition (52) was used by Rabotnov on a time segment $[0, t]$.

Integrating Equation (66), we obtain

$$[1 - \omega(t)]^{r+1} = 1 - (r+1)B \int_0^t \sigma^k(\tau)\,\mathrm{d}\tau. \tag{67}$$

Then we can rewrite the strength condition (59) in the form

$$\underline{\hat{\Lambda}}_{10}(\sigma;t) < 1, \quad \underline{\hat{\Lambda}}_{10}(\sigma;t) = \left[(r+1)B\int_0^t \sigma^k(\tau)\,\mathrm{d}\tau\right]^{1/k}. \tag{68}$$

Recalling the strength condition (53) equivalent to (52), we finally obtain the TNESF for the Rabotnov model in terms of the actual stress σ:

$$\underline{\Lambda}_{10}(\sigma;t) = \max(\underline{\Lambda}_8(\sigma;t), \underline{\hat{\Lambda}}_{10}(\sigma;t)). \tag{69}$$

To obtain the TNESF in terms of the nominal stress, we first substitute Equation (51) into (65), and integrate. Using the resulting expression for ϵ in Equation (51) gives

$$\sigma(\tau) = \sigma_0(\tau)\left\{1 - ab\int_0^\tau \sigma_0^b(\xi)[1-\omega(\xi)]^{-q}\,\mathrm{d}\xi\right\}^{-1/b}. \tag{70}$$

Substituting this into Equation (66), after integrating, we obtain a non-linear equation connecting ω with σ_0

$$[1-\omega(t)]^{r+1} = 1-(r+1)B\int_0^t \sigma_0^k(\tau)\left\{1 - ab\int_0^\tau \sigma_0^b(\xi)[1-\omega(\xi)]^{-q}\,\mathrm{d}\xi\right\}^{-k/b}\mathrm{d}\tau. \tag{71}$$

Let ω'' be a solution of the equation

$$\begin{aligned}[1-\omega''(t)]^{r+1} &= 1-(r+1)B\int_0^t (\lambda''\sigma_0(\tau))^k \\ &\times \left\{1 - ab\int_0^\tau (\lambda''\sigma_0)^b(\xi)[1-\omega''(\xi)]^{-q}\,\mathrm{d}\xi\right\}^{-k/b}\mathrm{d}\tau. \end{aligned} \tag{72}$$

The strength condition (59) for ω'', generated by the process $\lambda''\sigma_0(\tau)$, gives

$$\lambda''^k\int_0^t \sigma_0^k(\tau)\left\{1 - ab\lambda''^b\int_0^\tau \sigma_0^b(\xi)[1-\omega''(\xi)]^{-q}\,\mathrm{d}\xi\right\}^{-k/b}\mathrm{d}\tau < \frac{1}{(r+1)B}. \tag{73}$$

The inequality (73) is reminiscent of Equation (63) but there is the additional multiplier $[1-\omega'']^{-q}$ in (73), which also depends on λ'' through Equation (72).

Let us define a functional $\underline{\hat{\lambda}}_{010}(\sigma_0;t)$, according to Definition 1, as the supremum of non-negative numbers λ such that inequality (73), where ω'' is a solution of (72), is satisfied for any $\lambda'' \in [0,\lambda]$. The functional $\underline{\hat{\lambda}}_{010}$ can also be equivalently defined as a minimal positive solution λ'' of Equation (72) and the corresponding equality obtained from Equation (73), if the solution does exist.

Owing to Equation (70), the strength condition (52) for $\lambda''\sigma_0$ gives

$$\sup_{0\le\tau\le t} \sigma_0(\tau)\left\{1 - ab\lambda''^b \int_0^\tau \sigma_0^b(\xi)[1-\omega''(\xi)]^{-q}\mathrm{d}\xi\right\}^{-1/b} < \infty \tag{74}$$

where ω'' is a solution of Equation (72). Let a functional $\underline{\hat{\lambda}}^0_{010}(\sigma_0; t)$ be the supremum of non-negative numbers λ such that the inequality

$$\lambda'' < \sup_{0\le\tau\le t}\left\{ab\int_0^\tau \sigma_0^b(\xi)[1-\omega''(\xi)]^{-q}\mathrm{d}\xi\right\}^{1/b} \tag{75}$$

where ω'' is a solution of Equation (72), is satisfied for any $\lambda'' \in [0, \lambda]$. The functional $\underline{\hat{\lambda}}^0_{010}$ can be equivalently defined as a minimal positive solution λ'' of Equation (72) and the corresponding equality obtained from Equation (75), if the solution does exist. Finally, the TNESF in terms of the nominal stress for the Rabotnov model has the form

$$\underline{\Lambda}_{010}(\sigma_0; t) = \max(\underline{\Lambda}_{08}(\sigma_0; t), 1/\underline{\hat{\lambda}}^0_{010}(\sigma_0; t), 1/\underline{\hat{\lambda}}_{010}(\sigma_0; t)) \tag{76}$$

(for instants when it is right-continuous in *t*, see Statement 4).

Examples 6.8–6.10 particularly show that the TNESF and the corresponding strength condition for the same material (or model) can look quite different, being presented in terms of the nominal or actual stress. One should always carefully fix the used stress type.

6.11. TNESFs for other damage models

One of the general forms of the continuum damage mechanics (see, for example, [3, 4, 6, 7]) can be written as an expression of effective (micro-)stress tensor $\tilde{\sigma}(\tau)$ in terms of the actual (macro-)stress tensor $\sigma(\tau)$ and a damage (tensor) measure $\omega(\tau)$

$$\tilde{\sigma}(\tau) = f_1(\omega, \sigma), \tag{77}$$

a damage rate equation

$$\frac{\mathrm{d}\omega(\tau)}{\mathrm{d}\tau} = f_2(\omega, \sigma) \tag{78}$$

with the initial condition

$$\omega(0) = 0, \tag{79}$$

and a (necessary and sufficient) strength condition

$$F(\omega; \sigma) < 1. \tag{80}$$

The functions f_1, f_2 and F are considered to be known material characteristics. The models described in Sections 6.9 and 6.10 present particular cases of Equations (77)–(80). To complete the problem, the corresponding constitutive equations of the material and equilibrium equations written in terms of σ or $\tilde{\sigma}$ should be added to Equations (77)–(80). However, we need only Equations (78), (79) and the strength condition (80) to determine the TNESF for such a model in terms of the actual (macro-)stress. Thus, integrating Equation (78) with the initial conditions (79), as in subsections 6.9 and 6.10, we obtain

$$\omega(t) = f_3(\sigma; t), \tag{81}$$

where the functional f_3 is a solution of Equations (78) and (79) for a given process $\sigma(\tau)$. Substituting this into Equation (80), we obtain the following strength condition for a process $\sigma(\tau)$

$$F(f_3(\sigma; t); \sigma) < 1.$$

Assuming the absence of a repairing mechanism, this is equivalent to the condition

$$\sup_{0 \leq t' \leq t} F(f_3(\sigma; t'); \sigma) < 1. \tag{82}$$

Then the left-hand side of Equation (82) is non-decreasing in time and the TNESF can be obtained from Definitions 1–2, see Section 5.

Note that, although the TNESF is determined in this way independently of Equation (77) and of the material constitutive and equilibrium equations, the equations will be necessary to determine the process $\sigma_{ij}(\tau)$ and to calculate a corresponding value of the TNESF.

7. COMPLEX TNESFs FOR COMBINED CREEP, INSTANT AND DYNAMIC LOADING

The TNESF $\underline{\Lambda}^T$ is a material characteristic which is not necessarily connected to a geometrical, stiffness-related or abstract damage measure and can be identified from some durability tests. As shown in the previous sections, any strength condition written in terms of a damage measure can be expressed in terms of a corresponding TNESF (although not always analytically). Let us show some simple ways of constructing TNESFs to include, for example, instant overloading or dynamic effects in addition to creep durability. The Robinson damage accumulation rules mentioned above do not take into account sequence effects, i.e. damage caused by a stress in a particular instant is independent of where it occurs in the load history. We see that this shortcoming can be overcome in a simple way by choosing a proper structure for the TNESF.

Suppose one has a TNESF $\underline{\Lambda}^T(\sigma; t)$ obtained, for example, from a damage measure approach, which does not take into account any influence of instantaneous overloads of material, especially a finite strength under instantaneous loading. Particularly, TNESFs (48) and (49) based on the power-type durability diagrams give such examples. To avoid this shortcoming, one can combine an instant normalized equivalent stress function Λ^I and a temporal TNESF $\underline{\Lambda}^T$ and arrive at a complex strength condition, for example, in the form

$$\underline{\Lambda}^{IT}(\sigma;t) = \sup_{0\leq t'\leq t}\left\{\Lambda^{I}(\sigma(t')) + \underline{\Lambda}^{T}(\sigma;t')\right\} < 1. \tag{83}$$

For example, if $\Lambda^{I}(\sigma) = \sigma_{eq}(\sigma)/\sigma_r$, $\sigma_{eq}(\sigma)$ is, the von Mises, Tresca or other instantaneous equivalent stress, σ_r is an instant uniaxial strength, and $\underline{\Lambda}^{T}(\sigma;t)$ is given by Equation (49), then the TNESF (83) will take the form

$$\underline{\Lambda}^{IT}(\sigma;t) = \sup_{0\leq t'\leq t}\left\{\frac{\sigma_{eq}(\sigma(t'))}{\sigma_r} + \left[\int_0^{t'}\frac{|\sigma(\tau)|^b}{A(\tilde{\sigma}(\tau))}\mathrm{d}\tau\right]^{1/b}\right\} \tag{84}$$

where b is a material parameter and $A(\tilde{\sigma}(\tau))$ is a material function of the normalized stress tensor $\tilde{\sigma}_{ij}(\tau) = \sigma_{ij}(\tau)/|\sigma(\tau)|$ at an instant τ.

If there exist also dynamic effects on the material strength and the instant strength is not well defined, one can replace the instant strength term by a corresponding dynamic TNESF $\underline{\Lambda}^{D}(\sigma(t')$ and arrive, for example, at the following complex TNESF and strength condition:

$$\underline{\Lambda}^{DT}(\sigma;t) = \sup_{0\leq t'\leq t}\left\{\underline{\Lambda}^{D}(\sigma;t') + \underline{\Lambda}^{T}(\sigma;t')\right\} < 1. \tag{85}$$

For example, we can take $\underline{\Lambda}^{T}(\sigma;t')$ in the form (49), and $\underline{\Lambda}^{D}(\sigma;t')$ associated with the Morozov–Petrov–Utkin dynamic strength condition (27) generalized on the multiaxial case in the form

$$\underline{\Lambda}^{D}(\sigma;t') = \frac{1}{\sigma_r}\sigma_{eq}\left(\bar{\sigma}(t';t_r)\right), \quad \bar{\sigma}_{kj}(t';t_r) = \frac{1}{t_r}\int_{t'-t_r}^{t'}\sigma_{kj}(t'')\,\mathrm{d}t'', \tag{86}$$

where $\sigma_{eq}(\sigma)$ is, for example, the von Mises, Tresca or another instantaneous equivalent stress, and σ_r and t_r are material constants. Then TNESF (85) takes the form

$$\underline{\Lambda}^{DT}(\sigma;t) = \sup_{0\leq t'\leq t}\left\{\frac{1}{\sigma_r}\sigma_{eq}\left(\bar{\sigma}(t';t_r)\right) + \left[\int_0^{t'}\frac{|\sigma(\tau)|^b}{A(\tilde{\sigma}(\tau))}\mathrm{d}\tau\right]^{1/b}\right\}. \tag{87}$$

Note that both strength conditions (84) and (87) lead to non-linear summation rules since they include the fading memory terms $\sigma_{eq}(\sigma(t'))/\sigma_r$ or $\sigma_{eq}(\bar{\sigma}(t';t_r)/\sigma_r)$.

Note also that presentations (83) and (85) are not uniquely possible and one can use not only the sum but also other homogeneous combinations of the terms Λ^{I}, $\underline{\Lambda}^{D}$ and $\underline{\Lambda}^{T}$ to get other possible simple forms of the TNESFs describing the interaction of instant, dynamic and long-time effects on the durability. For example, one can take

$$\underline{\Lambda}^{DT}(\sigma;t) = \sup_{0\leq t'\leq t}\left\{[\underline{\Lambda}^{D}(\sigma;t')]^q + [\underline{\Lambda}^{T}(\sigma;t')]^q\right\}^{\frac{1}{q}}, \tag{88}$$

instead of Equation (85), where $q > 0$ can be considered as a material parameter. If $q = 1$, Equation (88) is reduced to (85). The limiting case $q \to \infty$ corresponds to the TNESF

$$\underline{\Lambda}^{DT}(\sigma;t) = \sup_{0\le t'\le t} \max\left\{\underline{\Lambda}^{D}(\sigma;t'), \underline{\Lambda}^{T}(\sigma;t')\right\}. \tag{89}$$

Evidently, we can determine which form fits better to a particular material behaviour from a comparison with experimental data.

8. CONCLUSION AND PERSPECTIVES

The generalized durability diagrams introduced in the paper give an instrument to compare irregular loading processes of different intensities and allow us to introduce the notions of the temporal safety factor and the temporal normalized equivalent stress for such processes. The TNESF is a mechanically meaningful material characteristic, which can be determined from durability macro-experiments without any additional information, such as micro-cracks or micro-void distribution or stiffness change. The concept of normalized equivalent stress forms a basis for the durability and strength description under creep and/or dynamic loadings, which do not need the introduction of any damage measures. On the other hand, the durability analysis, based on damage measures, is reduced to a particular case of the normalized equivalent stress concept. Nevertheless, the continuum damage mechanics remains helpful also when the normalized equivalent stress concept is applied. Particularly, the softening damage measures allow the calculation of a stress redistribution in a structure element and the geometrical damage measures can be used for estimating macroscopic TNESFs from micro-mechanical modelling.

The TNESF concept reduces different durability models to a unique form which facilitates their comparison. Examples of the reduction are presented in this paper. The durability-based TNESF, described in this paper, is deduced from the durability functional $t^*(\sigma)$ but the former seems to be more robust in design applications and corresponding computer codes. Moreover, the TNESF should be more convenient for identification from experimental data owing to its better properties (homogeneity in stress and monotonicity in time). Natural function classes for loading processes σ_{ij} and properties of the TNESFs $\underline{\Lambda}$ on those classes are to be studied. Strength-based TNESFs (coinciding with the durability-based TNESFs at the points of continuity in time) and some methods of their direct interpolation along the durability diagrams under constant loading will be described in a separate paper. Methods for refinement of the TNESF identification (interpolation) from a finite number of experimental data need to be developed for an effective implementation of the concept in engineering practice. The adaptation of the identification ideas of [18] to the TNESFs looks promising. An expansion of approach of [8, 19] to non-local durability analysis for bodies with stress concentration is to be done. This approach can be also extended to fatigue strength analysis under cyclic [20, 21] and non-cyclic loading.

Note also that the similar concept of the temporal normalized equivalent strain functional can be introduced in same way by replacing successively the stress loading process $\sigma_{ij}(\tau)$ by the strain loading process $\varepsilon_{ij}(\tau)$ in the above reasoning.

Acknowledgment. This work was completed under the research grant GR/M24592 "Non-local approach to high cyclic fatigue: Theoretical basis" of the Engineering and Physical Sciences Research Council, UK.

APPENDIX A
REMARKS ON SOME DAMAGE MEASURES

Different damage measures are often introduced to reflect the change in material properties under loading in comparison with a reference state. We present several of the most popular damage measures at an instant t under the loading process $\sigma(\tau)$.

Geometrical damage measures (see [1, 3, 7]):

$$\omega_S(\vec{n};\sigma;t) \ : \ = 1 - S(\sigma;t)/S(0;0) \Longrightarrow S(\sigma;t) = (1-\omega_S(n;\sigma;t))S(0;0);$$

$$\omega_V(\sigma;t) \ : \ = 1 - V(\sigma;t)/V(0;0) \Longrightarrow V(\sigma;t) = (1-\omega_V(\sigma;t))V(0;0).$$

Here $S(\sigma;t)$ is a representative element net cross-section area with a normal vector $\vec{n}$ at an instant t, and $S(0;0)$ is the corresponding area before loading; $V(\sigma;t)$ is a representative element net volume at an instant t, and $V(0;0)$ is the corresponding volume before loading.

Softening damage measure (see [1, 3, 7]):

$$\omega_E(\sigma;t) := I - E(\sigma;t)E^{-1}(0;0) \Longrightarrow E(\sigma;t) = (I-\omega_E(\sigma;t))E(0;0).$$

Here $E(\sigma;t)$ is the (macro-)stiffness tensor at an instant t and $E(0;0)$ is the tensor before loading. The effective (micro-)stress tensor is taken as

$$\tilde{s}(t) = (I-\omega_E(\sigma;t))^{-1}\sigma(t) = E(0;0)E^{-1}(\sigma;t)\sigma(t). \tag{90}$$

Assuming that the damage is isotropic, it is often supposed that the geometric and softening damage measures coincide. General relations between the anisotropic softening and the geometrical damage measures for an elastic medium with cracks can be found in [22].

One of the main ideas of the continuum (softening) damage mechanics is an assumption (see [7]) that all constitutive relations known for an undamaged material hold true also for the damaged material if one replaces there the macro-stresses σ by the effective stresses $\tilde{s}$. There exists also a temptation to use this idea for the strength prediction, that is, to write the strength condition for a damaged material in the form

$$\Lambda_{00}([I-\omega_E(\sigma;t)]^{-1}\sigma(t)) < 1 \tag{91}$$

if we know a strength condition $\Lambda_{00}(\sigma) < 1$ for the virgin material. The problem, however, is that the function $\Lambda_{00}(\sigma)$ is principally unknown since the material at rupture is always damaged (not virgin). An exclusion can be the case when only ageing damage (caused by non-mechanical reasons) is analysed. Another idea, that the softening damage measure is a perfect strength indicator and that one can write the strength condition in the form $F(\omega_E(\sigma;t)) < 1$, does also not always work; a ‘paradoxical’ example when adding damage (crack array) increases strength is presented in [22, Section VII.A].

APPENDIX B
STRENGTH ABSOLUTE STABILITY AND DURABILITY DIAGRAM CONTINUITY

Statement 5. The strength is absolutely stable under a process $\sigma_{ij}(\tau)$ at all $t < t^*(\sigma)$ if and only if the durability $t^*(\lambda\sigma)$ is a lower semi-continuous function of λ at $\lambda = 1$, that is, for any $\delta > 0$ there exists $\epsilon(\delta) > 0$ such that $t^*(\sigma) - t^*(\lambda'\sigma) < \delta$ for any $\lambda' \in (1-\epsilon, 1+\epsilon)$.

Proof. For any $\delta > 0$ we denote $t_\delta = t^*(\sigma) - \delta < t^*(\sigma)$. Suppose strength is absolutely stable at all $t < t^*(\sigma)$ and particularly at the instant t_δ. Then there exists $\epsilon(\delta) > 0$ such that $t^*(\lambda'\sigma) > t_\delta$ for all $\lambda' \in (1-\epsilon, 1+\epsilon)$. Hence, $t^*(\sigma) - t^*(\lambda'\sigma) < \delta$ for all $\lambda' \in (1-\epsilon, 1+\epsilon)$ which proves the lower semi-continuity of $t^*(\lambda\sigma)$.

Conversely, let $t^*(\lambda\sigma)$ be lower semi-continuous in λ at $\lambda = 1$. Then for any $\delta > 0$ there exists $\epsilon(\delta) > 0$ such that $t^*(\sigma) - t^*(\lambda'\sigma) < \delta$ for all $\lambda' \in (1-\epsilon, 1+\epsilon)$. Hence for any $t < t^*(\sigma)$, we take $\delta(t) = t^*(\sigma) - t$ and we have $t < t^*(\lambda'\sigma)$ for all $\lambda' \in (1-\epsilon, 1+\epsilon)$, which proves that the strength is absolutely stable under the process $\sigma_{ij}(\tau)$ at all $t < t^*(\sigma)$.

For MD processes, the lower semi-continuity of $t^*(\lambda\sigma)$ in λ coincides with the right continuity of $t^*(\lambda\sigma)$ in λ and we can reformulate the above statement in the form:

Statement 6. The strength is absolutely stable under an MD process $\sigma_{ij}(\tau)$ at all $t < t^*(\sigma)$ if and only if the durability $t^*(\lambda\sigma)$ is a right-continuous function of λ at $\lambda = 1$; that is, for any $\delta > 0$ there exists $\lambda(\delta) > 1$ such that $|t^*(\sigma) - t^*(\lambda'\sigma)| < \delta$ for any $\lambda' \in [1, \lambda]$.

APPENDIX C
PROOF OF REMARK 1

We define $\underline{\lambda}_{st}(\sigma; t) := \sup\{\lambda : t^*_{st}(\lambda''\sigma) > t \quad \forall \lambda'' \in [0, \lambda]\}$. Since $t^*_{st}(\lambda''\sigma) \le t^*(\lambda''\sigma)$ then $\underline{\lambda}_{st}(\sigma; t) \le \underline{\lambda}(\sigma; t)$.

Suppose $\underline{\lambda}_{st}(\sigma; t) < \underline{\lambda}(\sigma; t)$. Then for any λ_0, such that $\underline{\lambda}_{st}(\sigma; t) < \lambda_0, \lambda_{00} < \underline{\lambda}(\sigma; t)$, we have

$$t^*_{st}(\lambda_0\sigma) \le t < t^*(\lambda_0\sigma). \tag{92}$$

Consequently, strength is λ-unstable in any instant $t' \in (t, t^*(\lambda_0\sigma))$ under the process $\lambda_0\sigma$; that is, rupture appears at or before t' under the process $\lambda_{00}\sigma$ for any $\lambda_{00} > \lambda_0$. Thus, strength is t-unstable under the process $\lambda_0\sigma$ for any $\lambda_0 \in (\underline{\lambda}_{st}(\sigma; t), \underline{\lambda}(\sigma; t))$ and hence $t^*(\lambda_0\sigma) \le t$, and we arrive at a contradiction with the last inequality in inequality (92). Thus $\underline{\lambda}_{st}(\sigma; t) = \underline{\lambda}(\sigma; t)$.

APPENDIX D
PROOF OF POSITIVE HOMOGENEITY FOR STRENGTH FUNCTIONALS

Let $k > 0$. Denoting $\tilde{\lambda} = k\lambda$, $\tilde{\lambda}'' = k\lambda''$, we have from Definition 1

$$\underline{\lambda}(k\sigma; t) \quad : \quad = \sup \lambda : \{t^*(\lambda'' k\sigma) > t \quad \text{for all } \lambda'' \in [0, \lambda]\}$$

$$= \frac{1}{k}\sup(k\lambda) : \{t^*(k\lambda''\sigma) > t \quad \text{for all } k\lambda'' \in [0, k\lambda]\}$$

$$= \frac{1}{k}\sup\tilde{\lambda} : \{t^*(\tilde{\lambda}''\sigma) > t \quad \text{for all } \tilde{\lambda}'' \in [0, \tilde{\lambda}]\} = \frac{1}{k}\underline{\lambda}(\sigma; t).$$

APPENDIX E
PROOF OF STATEMENT 3

Let T be the supremum of t such that Equation (14) is satisfied. Suppose first $T < t^*_{st}(\sigma) \leq \infty$. For any $t > T$, condition (14) is violated, that is $\underline{\lambda} \leq 1$. Consequently, $t^*(\lambda'\sigma) \leq t$ for any $\lambda' > 1$ and any $t > T$ due to the definition of $\underline{\lambda}$ for MD materials; that is, $t^*(\lambda'\sigma) \leq T < t^*_{st}(\sigma)$ for any $\lambda' > 1$. However, this contradicts the definition of the critical time $t^*_{st}(\sigma)$ since the strength appears to be λ-unstable at the instant $T < t^*_{st}(\sigma)$ under the process $\sigma_{ij}(\tau)$. Consequently T cannot be less than $t^*_{st}(\sigma)$.

Suppose now $t^*_{st}(\sigma) < T \leq \infty$. Then we obtain from the definition of T that condition (14) holds for any t such that $t^*_{st}(\sigma) < t < T$. Owing to Statement 1, this implies λ-stable strength at the instant $t > t^*_{st}(\sigma)$ under the process $\sigma_{ij}(\tau)$, which contradicts the definition of the critical time $t^*_{st}(\sigma)$. The contradiction proves that T cannot be greater than $t^*_{st}(\sigma)$.

Hence $T = t^*_{st}(\sigma)$.

APPENDIX F
PROOF OF COROLLARY 2

Suppose first $t^{**} = t^*_{st}(\sigma)$. Then the right-hand side of Equation (15) follows from Statement 1 and the left-hand side follows from Statement 3.

Now suppose Equation (15) is satisfied, then $t^*_{st}(\sigma) \geq t^{**}$ owing to the left-hand side of Equation (15) and to Statement 3. But if $t^*_{st}(\sigma) > t^{**}$ then $\underline{\Lambda}(\sigma; t^{**}) < 1$ owing to Statement 1 which contradicts the right-hand side of Equation (15). Consequently $t^{**} = t^*_{st}(\sigma)$.

APPENDIX G
PROOF OF STATEMENT 4

Let $\underline{F}(\sigma; t) = \infty$, then $\underline{\Lambda}$ cannot be zero since then Statement 1 implies strength and consequently $\underline{F}(\sigma; t) < 1$. On the other hand, $\underline{\Lambda}(\sigma; t)$ cannot be a finite number since then $\underline{\Lambda}(\sigma/C; t) < 1$ for any $C > \underline{\Lambda}(\sigma; t)$, which means strength under the process $\sigma(\tau)/C$ and consequently $\underline{F}(\sigma/C; t) < 1$ and $\underline{F}(\sigma; t) < C$.

Let $\underline{F}(\sigma; t)$ be finite. The homogeneity of $\underline{F}(\sigma; t)$ implies that any process σ is MD. Taking into account that $t < t^*(\lambda\sigma)$ implies $\underline{F}(\lambda\sigma; t) < 1$, we have from Definition 1MD,

$$\underline{\lambda} = \sup\{\lambda > 0 : t < t^*(\lambda\sigma)\} \leq \sup\{\lambda > 0 : \underline{F}(\lambda\sigma; t) < 1\}$$

$$= \sup\{\lambda > 0 : \underline{F}(\sigma; t) < 1/\lambda\} = 1/\underline{F}(\sigma; t).$$

This completes the proof of point (i).

Let now $\underline{F}(\sigma; t)$ be right-continuous in the second argument at the considered time t. If $t^*(\lambda\sigma)$ is a durability under the process $\lambda\sigma$, then $\underline{F}(\lambda\sigma; t^*(\lambda\sigma)) \geq 1$ since otherwise there exists $t > t^*(\lambda\sigma)$ such that $\underline{F}(\lambda\sigma; t) < 1$ due to the right-continuity and non-decreasing of $\underline{F}(\lambda\sigma; t)$; which means $t^*(\lambda\sigma)$ is not the durability. Consequently, $\underline{F}(\lambda\sigma; t) \geq 1$ if $t \geq t^*(\lambda\sigma)$. On the other hand, $t \geq t^*(\lambda\sigma)$ if $\underline{F}(\lambda\sigma; t) \geq 1$ owing to the durability definition. Thus condition $t < t^*(\lambda\sigma)$ is equivalent to condition $\underline{F}(\lambda\sigma; t) < 1$.

Then we have from Definition 1MD

$$\begin{aligned} \underline{\lambda} &= \sup\{\lambda : t < t^*(\lambda\sigma)\} = \sup\{\lambda : \underline{F}(\lambda\sigma; t) < 1\} \\ &= \sup\{\lambda : \underline{F}(\sigma; t) < 1/\lambda\} = 1/\underline{F}(\sigma; t). \end{aligned}$$

Consequently, $\underline{\Lambda} = 1/\underline{\lambda} = \underline{F}(\sigma; t)$. This completes the proof of point (ii).

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