J. Math. Anal. Appl. 400 (2013) 48–67

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Solution regularity and co-normal derivatives for elliptic systems with non-smooth coefficients on Lipschitz domains

Sergey E. Mikhailov

Brunel University London, Department of Mathematics, Uxbridge, UB8 3PH, UK

ARTICLE INFO

Article history:
Received 7 May 2011
Available online 24 October 2012
Submitted by W.L. Wendland

Keywords:
Partial differential equation systems
Non-smooth coefficients
Sobolev spaces
Solution regularity
Classical, generalized and canonical co-normal derivatives
Weak BVP settings

ABSTRACT

Elliptic PDE systems of the second order with coefficients from L_∞ or Hölder–Lipschitz spaces are considered in the paper. Continuity of the operators in corresponding Sobolev spaces is stated and the internal (local) solution regularity theorems are generalized to the non-smooth coefficient case. For functions from the Sobolev space $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, definitions of non-unique generalized and unique canonical co-normal derivatives are considered, which are related to possible extensions of a partial differential operator and the PDE right hand side from the domain Ω to its boundary. It is proved that the canonical co-normal derivatives coincide with the classical ones when both exist. A generalization of the boundary value problem settings, which makes them insensitive to the co-normal derivative inherent non-uniqueness is given.

1. Introduction

It is well known that for a function from the Sobolev space $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, the *strong* co-normal derivative defined on the boundary in the trace sense, does not generally exist. Instead, if the function satisfies a second order partial differential equation (or a system of such equations) with a right-hand side from $H^{s-2}(\Omega)$, a *generalized* co-normal derivative operator can be defined by the first Green's identity; cf. e.g. [10, Lemma 4.3] for $s = 1$. However this definition is related to an extension of the PDE operator and its right hand side from the domain Ω, where they are prescribed, to the domain boundary, where they are not. Since the extensions are non-unique, the generalized co-normal derivative operator appears to be non-unique and non-linear unless a linear relation between the PDE solution and the extension of its right hand side is enforced. This leads to a revision of the boundary value problem settings, to make them insensitive to the co-normal derivative inherent non-uniqueness. For functions u from a subspace of $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, which can be mapped by the (extended) PDE operator into the space $\widetilde{H}^t(\Omega)$, $t \geq -\frac{1}{2}$, one can define a *canonical* co-normal derivative (cf. [6, Theorem 1.5.3.10] and [5, Lemma 3.2] for $s = 1$, $t = 0$), which is unique, linear in u, and coincides with the co-normal derivative in the trace sense if the latter does exist. These notions were developed, to some extent, in [12] for a PDE with an infinitely smooth coefficient on a domain with an infinitely smooth boundary. In [14] the analysis was generalized to the co-normal derivative operators for some elliptic PDE systems with infinitely smooth coefficients and the right hand side from $H^{s-2}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, on a Lipschitz domain.

In this paper, we extend the previous results to solutions of elliptic second order PDE systems on interior or exterior Lipschitz domains with compact boundaries and L_∞ or Hölder–Lipschitz coefficients. To show that the canonical co-normal derivatives coincide with the classical ones, some new facts about solution regularity of PDEs with non-smooth coefficients are also proved in the paper.

E-mail address: sergey.mikhailov@brunel.ac.uk.

0022-247X/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.10.045

The paper is arranged as follows. Section 2 provides a number of auxiliary facts on Sobolev (Bessel potential) spaces. In Section 3, we describe some L_∞-based Sobolev–Slobodetski spaces that essentially coincide with the Hölder–Lipschitz spaces, to use them for PDE coefficients, and prove boundedness of PDE operators with such coefficients in appropriate Sobolev spaces. In Section 4 we generalize the well known result about the local solution regularity of elliptic PDE systems to the case of relaxed smoothness of the PDE coefficients. In addition to the differentiation argument employed usually in the solution regularity analysis, we use for our proof also the Bessel potential operator that appeared to be more suitable for Hölder-smooth coefficients. The solution regularity theorems are implemented then in Section 6. In Section 5 all results of [14] about the generalized co-normal derivatives for PDE systems with smooth coefficients are extended to non-smooth coefficients. Particularly, we introduce and analyse the generalized co-normal derivatives on interior and exterior Lipschitz domains (with compact boundaries), associated with elliptic systems of second order PDEs with the right hand side from $H^{s-2}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$. The weak settings of Dirichlet, Neumann and mixed problems (revised versions for the latter two) are considered and it is shown that they are well posed in spite of the inherent non-uniqueness of the generalized co-normal derivatives. In Section 6 we introduce and analyse the canonical co-normal derivative operator uniquely defined on some subspaces $H^{s,t}(\Omega; A)$ of the usual Sobolev spaces $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, $-\frac{1}{2} \le t$, generalizing the corresponding results of [14] to the case of non-smooth coefficients of the PDE operator. It is proved that for elliptic systems the canonical co-normal derivative coincides with the classical (strong) one for the cases when both exist. Some auxiliary estimates and necessary assertions from [14] are provided in two Appendices.

The present paper updates and complements the preliminary results from [13].

2. Some function spaces

2.1. Sobolev spaces

Unless otherwise stated, we suppose that $\Omega = \Omega^+$ is an interior or exterior open domain of $\mathbb{R}^n$, whose boundary $\partial\Omega$ is a compact, Lipschitz, $(n-1)$-dimensional set. Let $\overline{\Omega}$ denote the closure of Ω and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ its complement. In what follows $\mathcal{D}(\Omega) = C^\infty_{\text{comp}}(\Omega)$ denotes the space of Schwartz test functions, $\mathcal{D}(\overline{\Omega}) := \{\varphi = \phi|_\Omega, \phi \in \mathcal{D}(\mathbb{R}^n)\}$, while $\mathcal{D}^*(\Omega)$ denotes the space of Schwartz distributions; $H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$, $H^s(\partial\Omega) = H^s_2(\partial\Omega)$ are the Sobolev (Bessel potential) spaces, where $s \in \mathbb{R}$ is a number (see, e.g., [9]).

We denote by $\widetilde{H}^s(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^s(\mathbb{R}^n)$, which can be characterized as $\widetilde{H}^s(\Omega) = \{g : g \in H^s(\mathbb{R}^n), \operatorname{supp} g \subset \overline{\Omega}\}$; see e.g. [10, Theorem 3.29]. The space $H^s(\Omega)$ consists of restrictions on Ω of distributions from $H^s(\mathbb{R}^n)$, $H^s(\Omega) := \{g|_\Omega : g \in H^s(\mathbb{R}^n)\}$, and $H^s_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$. We recall that $H^s(\Omega)$ coincide with the Sobolev–Slobodetski spaces $W^s_2(\Omega)$ for any non-negative s. We denote $H^s_{\text{loc}}(\Omega) := \{g : \varphi g \in H^s(\Omega)\ \forall\varphi \in \mathcal{D}(\Omega)\}$. We will use also the notation $H^s_{\text{loc}}(\overline{\Omega}) := \{g : \varphi g \in H^s(\Omega)\ \forall\varphi \in \mathcal{D}(\overline{\Omega})\}$ and note that $H^s_{\text{loc}}(\overline{\Omega}) = H^s(\Omega)$ for interior domains but not for the exterior ones.

Note that distributions from $H^s(\Omega)$ and $H^s_0(\Omega)$ are defined only in Ω, while distributions from $\widetilde{H}^s(\Omega)$ are defined in $\mathbb{R}^n$ and include the distributions supported only on the boundary $\partial\Omega$. For $s \ge 0$, we can identify $\widetilde{H}^s(\Omega)$ with the subset of functions from $H^s(\Omega)$, whose extensions by zero outside Ω belong to $H^s(\mathbb{R}^n)$, cf. [10, Theorem 3.33], i.e., identify functions $u \in \widetilde{H}^s(\Omega)$ with their restrictions, $u|_\Omega \in H^s(\Omega)$. However generally we will distinguish distributions $u \in \widetilde{H}^s(\Omega)$ and $u|_\Omega \in H^s(\Omega)$, especially for $s \le -\frac{1}{2}$.

We denote by $H^s_{\partial\Omega}$ the subspace of $H^s(\mathbb{R}^n)$ (and of $\widetilde{H}^s(\Omega)$), whose elements are supported on $\partial\Omega$, i.e., $H^s_{\partial\Omega} := \{g : g \in H^s(\mathbb{R}^n), \operatorname{supp} g \subset \partial\Omega\}$. A characterization of this space is provided in Theorem B.1 in Appendix B. To simplify notations for vector-valued functions, $u : \Omega \to \mathbb{C}^m$, we will often write $u \in H^s(\Omega)$ instead of $u \in H^s(\Omega)^m = H^s(\Omega; \mathbb{C}^m)$, etc.

As usual (see e.g. [9,10]), for two elements from dual complex Sobolev spaces the bilinear dual product $\langle\cdot,\cdot\rangle_\Omega$ associated with the sesquilinear inner product $(\cdot,\cdot)_\Omega := (\cdot,\cdot)_{L_2(\Omega)}$ in $L_2(\Omega)$ is defined as

$$\langle u, v\rangle_{\mathbb{R}^n} := \int_{\mathbb{R}^n} [\mathcal{F}^{-1}u](\xi)[\mathcal{F}v](\xi)d\xi =: (\mathcal{F}\bar{u}, \mathcal{F}v)_{\mathbb{R}^n} =: (\bar{u}, v)_{\mathbb{R}^n}, \quad u \in H^s(\mathbb{R}^n), v \in H^{-s}(\mathbb{R}^n), \tag{2.1}$$

$$\langle u, v\rangle_\Omega := \langle u, V\rangle_{\mathbb{R}^n} =: (\bar{u}, v)_\Omega \quad \text{if } u \in \widetilde{H}^s(\mathbb{R}^n), v \in H^{-s}(\Omega), v = V|_\Omega \text{ with } V \in H^{-s}(\mathbb{R}^n), \tag{2.2}$$

$$\langle u, v\rangle_\Omega := \langle U, v\rangle_{\mathbb{R}^n} =: (\bar{u}, v)_\Omega \quad \text{if } u \in H^s(\mathbb{R}^n), v \in \widetilde{H}^{-s}(\Omega), u = U|_\Omega \text{ with } U \in H^s(\mathbb{R}^n) \tag{2.3}$$

for $s \in \mathbb{R}$, where $\bar{g}$ is the complex conjugate of g, while $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the distributional Fourier transform operator and its inverse, respectively, that for integrable functions take form

$$\hat{g}(\xi) = [\mathcal{F}g](\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} g(x)dx, \quad g(x) = [\mathcal{F}^{-1}\hat{g}](x) := \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\hat{g}(\xi)d\xi.$$

For vector-valued elements $u \in H^s(\mathbb{R}^n)^m$, $v \in H^{-s}(\mathbb{R}^n)^m$, $s \in \mathbb{R}$, definition (2.1) should be understood as

$$\langle u, v\rangle_{\mathbb{R}^n} := \int_{\mathbb{R}^n} \hat{u}(\xi)\cdot\hat{v}(\xi)d\xi = \int_{\mathbb{R}^n} \hat{u}(\xi)^\top\hat{v}(\xi)d\xi =: (\bar{\hat{u}}, \hat{v})_{\mathbb{R}^n} =: (\bar{u}, v)_{\mathbb{R}^n},$$

where $\hat{u}\cdot\hat{v} = \hat{u}^\top\hat{v} = \sum_{k=1}^m \hat{u}_k\hat{v}_k$ is the product of two m-dimensional vectors.

Let $\mathcal{J}^s$ be the Bessel potential operator defined as

$$[\mathcal{J}^s g](x) = \mathcal{F}^{-1}_{\xi\to x}\{(1+|\xi|^2)^{s/2}\hat{g}(\xi)\}. \tag{2.4}$$

The inner product in $H^s(\Omega)$, $s \in \mathbb{R}$, is defined as follows:

$$(u, v)_{H^s(\mathbb{R}^n)} := (\mathcal{J}^s u, \mathcal{J}^s v)_{\mathbb{R}^n} = \int_{\mathbb{R}^n} (1+\xi^2)^s \overline{\hat{u}(\xi)}\hat{v}(\xi)d\xi = \left\langle \overline{u}, \mathcal{J}^{2s} v\right\rangle_{\mathbb{R}^n}, \quad u, v \in H^s(\mathbb{R}^n),$$

$$(u, v)_{H^s(\Omega)} := ((I-P)U, (I-P)V)_{H^s(\mathbb{R}^n)}, \qquad u = U|_\Omega, \qquad v = V|_\Omega, \quad U, V \in H^s(\mathbb{R}^n).$$

Here $P : H^s(\mathbb{R}^n) \to \widetilde{H}^s(\mathbb{R}^n \setminus \bar{\Omega})$ is the orthogonal projector; see e.g. [10, p. 77].

3. Elliptic PDE systems with non-smooth coefficients

3.1. Some Sobolev–Slobodetski and Hölder–Lipschitz spaces

For an open set Ω let $W^\mu_\infty(\Omega)$, $\mu \geq 0$, be the Sobolev–Slobodetski space equipped with the norm

$$\|g\|_{W^\mu_\infty(\Omega)} := \sum_{0\leq|\alpha|\leq\mu} \|\partial^\alpha g\|_{L_\infty(\Omega)} < \infty$$

for integer μ, and with the norm

$$\|g\|_{W^\mu_\infty(\Omega)} := \|g\|_{W^{\lfloor\mu\rfloor}_\infty(\Omega)} + |g|_{W^\mu_\infty(\Omega)} < \infty, \quad |g|_{W^\mu_\infty(\Omega)} := \sum_{|\alpha|=\lfloor\mu\rfloor} \left\| \frac{\partial^\alpha g(x) - \partial^\alpha g(y)}{|x-y|^{\mu-\lfloor\mu\rfloor}} \right\|_{L_\infty(\Omega\times\Omega)}$$

for non-integer μ, where $\lfloor\mu\rfloor$ is the integer part of μ. Evidently $W^0_\infty(\Omega) = L_\infty(\Omega)$, while (possibly after adjusting functions on zero measure sets, cf. [20, Chapter V, Section 4, Proposition 6]) $W^\mu_\infty(\Omega)$ is the usual Hölder space $C^\mu(\Omega) = C^{0,\mu}(\Omega)$ for $0 < \mu < 1$, $W^\mu_\infty(\Omega) = C^{\lfloor\mu\rfloor,\mu-\lfloor\mu\rfloor}(\Omega)$ for non-integer $\mu > 1$, and $W^\mu_\infty(\Omega)$ is the Lipschitz space $C^{\mu-1,1}(\Omega)$ for integer $\mu \geq 1$.

Let $\mathbb{R}_+(s)$ be the set of all non-negative numbers if s is an integer and of all positive numbers otherwise.

Definition 3.1. For an open set Ω and $\mu \geq 0$ let $\bar{C}^\mu(\overline{\Omega})$ be the set of restrictions on Ω of all functions from $W^\mu_\infty(\mathbb{R}^n)$, equipped with the norm $\|g\|_{\bar{C}^\mu(\overline{\Omega})} = \inf_{G|_\Omega=g} \|G\|_{W^\mu_\infty(\mathbb{R}^n)}$.

The set $\bar{C}^\mu_+(\overline{\Omega})$ is defined as $\bar{C}^\mu(\overline{\Omega})$ for integer non-negative μ and as $\bigcup_{\nu>\mu} \bar{C}^\nu(\overline{\Omega})$ for non-integer nonnegative μ. Evidently $g \in \bar{C}^\mu_+(\overline{\Omega})$ if and only if $g \in \bar{C}^{\mu+\epsilon}(\overline{\Omega})$ for some $\epsilon \in \mathbb{R}_+(\mu)$.

Obviously $\|g\|_{W^0_\infty(\Omega)} = \|g\|_{\bar{C}^0(\overline{\Omega})} = \|g\|_{L_\infty(\Omega)}$ i.e. $\bar{C}^0(\overline{\Omega}) = W^0_\infty(\Omega) = L_\infty(\Omega)$, and $\|g\|_{W^\mu_\infty(\Omega)} \leq \|g\|_{\bar{C}^\mu(\overline{\Omega})}$, $\bar{C}^\mu(\overline{\Omega}) \subset W^\mu_\infty(\Omega)$ for $\mu > 0$. The space $\bar{C}^\mu(\overline{\Omega})$ for $\mu > 0$ is similar to the space $C_c^{\lfloor\mu\rfloor,\mu-\lfloor\mu\rfloor}(\bar{\Omega})$ for non-integer μ and to the space $C_c^{\mu-1,1}(\bar{\Omega})$ for integer μ, used in [6, p. 21], except that functions from $\bar{C}^\mu(\overline{\Omega})$ may not have a compact support in $\mathbb{R}^n$ assumed for functions from $C_c^{k,\alpha}(\bar{\Omega})$.

Theorem 3.2. *Let Ω be an open set, $s \in \mathbb{R}$, $g \in \bar{C}^\mu(\overline{\Omega})$, $\mu - |s| \in \mathbb{R}_+(s)$. Then $gv \in H^s(\Omega)$ for every $v \in H^s(\Omega)$, and $\|gv\|_{H^s(\Omega)} \leq C\|g\|_{\bar{C}^\mu(\overline{\Omega})}\|v\|_{H^s(\Omega)}$, where C is independent of g, v or Ω.*

Proof. Note that the theorem is close to the statement given in [6, Theorem 1.4.1.1] without proof.

Let first $\Omega = \mathbb{R}^n$. The case $s = 0$ is evident. For $s > 0$ the estimate can be obtained from [21, Theorem 2(b)] with parameters $s_1 = \mu$, $s_2 = s$, $p_1 = \infty$, $q_1 = p_2 = q_2 = p = q = 2$ there (see also [6, Theorem 1.4.4.2]). A simpler proof for all $s \in \mathbb{R}$ is available in [3, Section 9, Theorems 11–13].

When $\Omega \neq \mathbb{R}^n$, let $V \in H^s(\mathbb{R}^n)$ and $G \in W^s_\infty(\mathbb{R}^n)$ be such that $v = V|_\Omega$, $\|V\|_{H^s(\mathbb{R}^n)} = \|v\|_{H^s(\Omega)}$, $g = G|_\Omega$, $\|G\|_{W^\mu_\infty(\mathbb{R}^n)} < 2\|g\|_{\bar{C}^\mu(\overline{\Omega})}$. Then $GV \in H^s(\mathbb{R}^n)$ by the previous paragraph and

$$\|gv\|_{H^s(\Omega)} \leq \|GV\|_{H^s(\mathbb{R}^n)} \leq C\|G\|_{W^\mu_\infty(\mathbb{R}^n)}\|V\|_{H^s(\mathbb{R}^n)} < 2C\|g\|_{\bar{C}^\mu(\overline{\Omega})}\|v\|_{H^s(\Omega)}. \quad \square$$

Note that the condition on g in Theorem 3.2 is equivalent to the membership $g \in \bar{C}^{|s|}_+(\overline{\Omega})$.

3.2. PDE systems

Let us consider in an open set Ω a system of m complex linear differential equations of the second order with respect to m unknown functions $\{u_i\}_{i=1}^m = u : \Omega \to \mathbb{C}^m$, which for sufficiently smooth u and f has the following strong form,

$$Au(x) := -\sum_{i,j=1}^n \partial_i[a_{ij}(x)\,\partial_j u(x)] + \sum_{j=1}^n b_j(x)\,\partial_j u(x) + c(x)u(x) = f(x), \quad x \in \Omega, \tag{3.1}$$

where $f : \Omega \to \mathbb{C}^m$, $\partial_j := \partial/\partial x_j$ $(j = 1, 2, \dots, n)$, $a(x) = \{a_{ij}(x)\}_{i,j=1}^n = \{\{a_{ij}^{kl}(x)\}_{k,l=1}^m\}_{i,j=1}^n$, $b(x) = \{\{b_i^{kl}(x)\}_{k,l=1}^m\}_{i=1}^n$ and $c(x) = \{c^{kl}(x)\}_{k,l=1}^m$, i.e., $a_{ij}, b_i, c : \Omega \to \mathbb{C}^{m\times m}$ for fixed indices i, j. If $m = 1$, then (3.1) is a scalar equation. The PDE system formally adjoint to (3.1) is given in the strong form as

$$A^*v(x) := -\sum_{i,j=1}^n \partial_i[\bar{a}_{ji}^\top(x)\,\partial_j v(x)] - \sum_{j=1}^n \partial_j[\,\bar{b}_j^\top(x)v(x)] + \bar{c}^\top(x)v(x) = f(x), \quad x \in \Omega. \tag{3.2}$$

Definition 3.3. For $\sigma \in \mathbb{R}$, we will say that the coefficients of Eq. (3.1) belong to the class $\mathcal{C}_+^\sigma(\overline{\Omega})$, i.e. $\{a, b, c\} \in \mathcal{C}_+^\sigma(\overline{\Omega})$, if $a \in \bar{C}_+^{|\sigma|}(\overline{\Omega})$, $b \in \bar{C}_+^{\mu_b(\sigma)}(\overline{\Omega})$, $\mu_b(\sigma) := \max(0, |\sigma - \frac{1}{2}| - \frac{1}{2})$, $c \in \bar{C}_+^{\mu_c(\sigma)}(\overline{\Omega})$, $\mu_c(\sigma) := \max(0, |\sigma| - 1)$.

For an open set Ω, as usual, $\{a, b, c\} \in \mathcal{C}_{+\mathrm{loc}}^\sigma(\Omega)$ means that $\{a, b, c\} \in \mathcal{C}_+^\sigma(\overline{\Omega'})$ for any $\overline{\Omega'} \subset \Omega$.

Note that if $\sigma_1 \le \sigma \le \sigma_2$, then $\mathcal{C}_+^{\sigma_1}(\overline{\Omega}) \bigcap \mathcal{C}_+^{\sigma_2}(\overline{\Omega}) \subset \mathcal{C}_+^{\sigma}(\overline{\Omega}) \subset \mathcal{C}_+^{\sigma_1}(\overline{\Omega}) \bigcup \mathcal{C}_+^{\sigma_2}(\overline{\Omega})$.

Let $u \in H^s(\Omega)$, $\{a, b, c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega})$, $f \in H^{s-2}(\Omega)$, $s \in \mathbb{R}$. Equation system (3.1) is understood in the distributional sense as

$$\langle Au, v\rangle_\Omega = \langle f, v\rangle_\Omega \quad \forall v \in \mathcal{D}(\Omega),$$

where $v : \Omega \to \mathbb{C}^m$ and

$$\langle Au, v\rangle_\Omega := \mathcal{E}(u, v) \quad \forall v \in \mathcal{D}(\Omega), \tag{3.3}$$

$$\mathcal{E}(u, v) = \mathcal{E}_\Omega(u, v) := \sum_{i,j=1}^n \left\langle a_{ij}\partial_j u, \partial_i v\right\rangle_\Omega + \sum_{j=1}^n \left\langle b_j\partial_j u, v\right\rangle_\Omega + \langle cu, v\rangle_\Omega\,. \tag{3.4}$$

Let us denote

$$s_b(s) = \begin{cases} s-1 & \text{if } s < 1 \\ 0 & \text{if } 1 \le s \le 2, \\ s-2 & \text{if } 2 < s \end{cases} \qquad s_c(s) = \begin{cases} s & \text{if } s < 0 \\ 0 & \text{if } 0 \le s \le 2. \\ s-2 & \text{if } 2 < s \end{cases} \tag{3.5}$$

Taking into account that $\mu_b(s-1) = |s_b(s)|$ and $\mu_c(s-1) = |s_c(s)|$, Theorem 3.2 implies that $a_{ij}\partial_j u \in H^{s-1}(\Omega)$, $b_j\partial_j u \in H^{s_b(s)}(\Omega)$, $cu \in H^{s_c(s)}(\Omega)$. Thus bilinear form (3.4) is well defined for any $v \in \mathcal{D}(\Omega)$ and moreover, the bilinear functional $\mathcal{E} : \{H^s(\Omega), \widetilde{H}^{2-s}(\Omega)\} \to \mathbb{C}$ is bounded for any $s \in \mathbb{R}$. Since the set $\mathcal{D}(\Omega)$ is dense in $\widetilde{H}^{2-s}(\Omega)$, expression (3.3) defines then a bounded linear operator $A : H^s(\Omega) \to H^{s-2}(\Omega) = [\widetilde{H}^{2-s}(\Omega)]^*$, $s \in \mathbb{R}$,

$$\langle Au, v\rangle_\Omega := \mathcal{E}(u, v) \quad \forall v \in \widetilde{H}^{2-s}(\Omega). \tag{3.6}$$

Similar to the operator A, the weak form of the operator A^* for any $v \in H^{2-s}(\Omega)$, $s \in \mathbb{R}$, is

$$\langle A^*v, u\rangle_\Omega := \mathcal{E}^*(v, u) \quad \forall u \in \widetilde{H}^s(\Omega), \tag{3.7}$$

where $\mathcal{E}^*(v, u) = \overline{\mathcal{E}(\bar{u}, \bar{v})}$ is the bilinear form and so defined operator $A^* : H^{2-s}(\Omega) \to H^{-s}(\Omega) = [\widetilde{H}^s(\Omega)]^*$ is bounded for any $s \in \mathbb{R}$.

The above paragraph can be summarized as the following assertion.

Theorem 3.4. *If $s \in \mathbb{R}$ and $\{a, b, c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega})$, then bilinear form (3.4), $\mathcal{E} : \{H^s(\Omega), \widetilde{H}^{2-s}(\Omega)\} \to \mathbb{C}$ is bounded, while expressions (3.6) and (3.7) define bounded linear operators $A : H^s(\Omega) \to H^{s-2}(\Omega)$ and $A^* : H^{2-s}(\Omega) \to H^{-s}(\Omega)$, respectively.*

Note that for the particular important case $s = 1$, the conditions on the coefficients in Theorem 3.4 mean $a, b, c \in L_\infty(\Omega)$.

4. Local regularity of solutions to elliptic systems with Hölder–Lipschitz coefficients

In this section we extend the well known result about the local regularity of elliptic PDE solutions, to the case of relaxed smoothness of the PDE coefficients. This will be used then to prove counterparts of [14, Theorems 3.12 and 3.16] in Section 6.2.

The local regularity of solutions to elliptic PDEs (3.1) and (3.2) for the case of infinitely smooth coefficients is well known (see e.g. [19,1,9]). For non-infinitely smooth coefficients, the case $a, b, c \in C^{k,1}(\bar{\Omega})$, $s_1 = 1$, $s_2 = k$ with integer $k \ge 0$ can be found in [10, Theorem 4.16], and the case $a \in C^{0,1}(\bar{\Omega})$, $b = 0$, $c = const$, $s_2 \in (-3/2, -1/2)$ in [18, Theorem 4], extended in [4] to general elliptic systems with all coefficients from $C^{0,1}(\bar{\Omega})$. In Theorems 4.3 and 4.4 below we prove the local regularity results for arbitrary Hölder coefficients and wider ranges of the Sobolev space indices s_1 and s_2.

Let us define the matrix function $\mathcal{A}(x, \xi) := \sum_{i,j=1} a_{ij}(x)\xi_i\xi_j$ for $\xi \in \mathbb{R}^n$. The partial differential operator A is elliptic in the sense of Petrovsky at a point x, where the coefficients $a_{ij}^{kl}(x)$ are defined, if $\det \mathcal{A}(x, \xi) \ne 0$ for any non-zero $\xi \in \mathbb{R}^n$

(see e.g. [15, Section 55]), evidently implying $|\det \mathcal{A}(x,\xi)| \ge C(x)|\xi|^{2m}$ for all $\xi \in \mathbb{R}^n$ with some positive $C(x)$, which in turn gives the following estimate for the matrix norm $|\cdot|$ of the inverse matrix $\mathcal{A}^{-1}(x,\xi)$,

$$|\mathcal{A}^{-1}(x,\xi)| \le C_0(x)|\xi|^{-2} \quad \forall \xi \in \mathbb{R}^n \tag{4.1}$$

with some $C_0(x) > 0$. We say that the operator A is elliptic in a domain if it is elliptic at each point of the domain.

Note that we will need the ellipticity in this paper only in proving solution regularity in Theorems 4.3 and 4.4, which will be then used only to prove equivalence of the strong and canonical co-normal derivatives in Section 6.2.

Differentiation or Nirenberg difference quotient arguments are employed usually in the solution regularity analysis in [17,19,1,9], but we will also need for our proof some powers of the Bessel potential operator $\mathcal{J}$ to deal with the Hölder-smooth coefficients along with the solution and the right hand side in some range of Sobolev spaces and have to prove first Lemma 4.1 and Corollary 4.2 about commutators.

Lemma 4.1. *Let s be real, k be an integer, $w \in H^s(\mathbb{R}^n)^m$, $g \in W_\infty^{\sigma+\varepsilon}(\mathbb{R}^n)^m$, $\sigma = \left|s-k+\frac{1}{2}\right| + |k| + \frac{1}{2}$ and $\varepsilon \in \mathbb{R}_+(\sigma)$. Then $\mathcal{J}^{2k}(gw) - g\mathcal{J}^{2k}w \in H^{s-2k+1}(\mathbb{R}^n)^m$ and*

$$\|\mathcal{J}^{2k}(gw) - g\mathcal{J}^{2k}w\|_{H^{s-2k+1}(\mathbb{R}^n)^m} \le C|k|\, \|g\|_{W_\infty^{\sigma+\varepsilon}(\mathbb{R}^n)^m} \|w\|_{H^s(\mathbb{R}^n)^m}. \tag{4.2}$$

Proof. The proof below is given for $m = 1$, generalization to the vector case, $m > 1$, is evident. For $k = 0$ the lemma is trivial. Let now $k > 0$. Denoting the Fourier convolution by $*$ we have due to (2.4),

$$\begin{aligned} K(\xi) &:= \mathcal{F}[\mathcal{J}^{2k}(gw) - g\mathcal{J}^{2k}w](\xi) = (1+|\xi|^2)^k(\widehat{g} * \widehat{w})(\xi) - (\widehat{g} * \mathcal{F}[\mathcal{J}^{2k}w])(\xi) \\ &= \int_{\mathbb{R}^n} [(1+|\xi|^2)^k - (1+|\xi-\eta|^2)^k]\widehat{g}(\eta)\widehat{w}(\xi-\eta)d\eta \\ &= \int_{\mathbb{R}^n} [\eta\cdot\xi + \eta\cdot(\xi-\eta)]f_k(\xi,\xi-\eta)\widehat{g}(\eta)\widehat{w}(\xi-\eta)d\eta \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}^n} \widehat{\nabla g}(\eta)\cdot(\xi+\xi-\eta)f_k(\xi,\xi-\eta)\widehat{w}(\xi-\eta)d\eta, \end{aligned}$$

where

$$f_k(\xi,\xi-\eta) := \frac{(1+|\xi|^2)^k - (1+|\xi-\eta|^2)^k}{|\xi|^2 - |\xi-\eta|^2} = \frac{p^{2k}(\xi) - p^{2k}(\xi-\eta)}{p^2(\xi) - p^2(\xi-\eta)} = \sum_{j=1}^k p^{2(k-j)}(\xi)p^{2(j-1)}(\xi-\eta)$$

and $p(\zeta) := (1+|\zeta|^2)^{1/2}$. This implies

$$\begin{aligned} K(\xi) &= \frac{1}{2\pi i}\sum_{j=1}^k p^{2(k-j)}(\xi)\left[\xi\cdot\int_{\mathbb{R}^n}\widehat{\nabla g}(\eta)p^{2(j-1)}(\xi-\eta)\widehat{w}(\xi-\eta)d\eta\right. \\ &\quad \left. + \int_{\mathbb{R}^n}\widehat{\nabla g}(\eta)\cdot(\xi-\eta)p^{2(j-1)}(\xi-\eta)\widehat{w}(\xi-\eta)d\eta\right] \\ &= \frac{-1}{4\pi^2}\sum_{j=1}^k p^{2(k-j)}(\xi)\mathcal{F}\left[\nabla\cdot\left\{(\nabla g)\mathcal{J}^{2(j-1)}w\right\} + \nabla g\cdot\mathcal{J}^{2(j-1)}\nabla w\right](\xi). \end{aligned}$$

Taking into account Theorem 3.2, we obtain,

$$\begin{aligned} \|\mathcal{J}^{2k}(gw) - g\mathcal{J}^{2k}w\|_{H^{s-2k+1}(\mathbb{R}^n)} &= \|p^{s-2k+1}K\|_{L_2(\mathbb{R}^n)} \\ &= \frac{1}{4\pi^2}\left\|\sum_{j=1}^k p^{s+1-2j}\mathcal{F}\left[\nabla\cdot\left\{(\nabla g)\mathcal{J}^{2(j-1)}w\right\} + \nabla g\cdot\mathcal{J}^{2(j-1)}\nabla w\right]\right\|_{L_2(\mathbb{R}^n)} \\ &\le \frac{1}{4\pi^2}\sum_{j=1}^k \left\|\nabla\cdot\left\{(\nabla g)\mathcal{J}^{2(j-1)}w\right\} + \nabla g\cdot\mathcal{J}^{2(j-1)}\nabla w\right\|_{H^{s+1-2j}(\mathbb{R}^n)} \\ &\le C_1\sum_{j=1}^k\left[\|g\|_{W_\infty^{|s+2-2j|+1+\varepsilon_1}(\mathbb{R}^n)} + \|g\|_{W_\infty^{|s+1-2j|+1+\varepsilon_1}(\mathbb{R}^n)}\right]\|w\|_{H^s(\mathbb{R}^n)} \end{aligned}$$

for any $\varepsilon_1 \in \mathbb{R}_+(s)$. That is,

$$\|\mathcal{J}^{2k}(gw) - g\mathcal{J}^{2k}w\|_{H^{s-2k+1}(\mathbb{R}^n)} \le C_1k(\|g\|_{W_\infty^{|s|+1+\varepsilon_1}(\mathbb{R}^n)} + \|g\|_{W_\infty^{|s-2k+1|+1+\varepsilon_1}(\mathbb{R}^n)})\|w\|_{H^s(\mathbb{R}^n)}. \tag{4.3}$$

Let now $k < 0$. If we denote $v = \mathcal{J}^{2k}w \in H^{s-2k}(\mathbb{R}^n)$, then by inequality (4.3), where $2k$ is replaced with $-2k$ and $s-2k$ with s, we obtain,

$$\begin{aligned}\|\mathcal{J}^{2k}(gw) - g\mathcal{J}^{2k}w\|_{H^{s-2k+1}(\mathbb{R}^n)} &= \|\mathcal{J}^{2k}[g\mathcal{J}^{-2k}v - \mathcal{J}^{-2k}(gv)]\|_{H^{s-2k+1}(\mathbb{R}^n)} = \|g\mathcal{J}^{-2k}v - \mathcal{J}^{-2k}(gv)\|_{H^{s+1}(\mathbb{R}^n)}\\ &\le C_1|k|(\|g\|_{W_\infty^{|s-2k|+1+\varepsilon_1}(\mathbb{R}^n)} + \|g\|_{W_\infty^{|s+1|+1+\varepsilon_1}(\mathbb{R}^n)})\|v\|_{H^{s-2k}(\mathbb{R}^n)}\\ &= C_1|k|(\|g\|_{W_\infty^{|s-2k|+1+\varepsilon_1}(\mathbb{R}^n)} + \|g\|_{W_\infty^{|s+1|+1+\varepsilon_1}(\mathbb{R}^n)})\|w\|_{H^{s}(\mathbb{R}^n)}.\end{aligned}$$

Inequality (4.2) follows for both positive and negative k if we remark that

$$\sigma := \left|s - k + \frac{1}{2}\right| + |k| + \frac{1}{2} = \begin{cases}\max(|s|+1, |s-2k+1|+1), & k > 0\\ \max(|s-2k|+1, |s+1|+1) & k < 0.\end{cases} \quad \square$$

Let us denote by A_0 the principal divergence part of the operator A from (3.1), i.e.,

$$A_0u(x) := -\sum_{i,j=1}^{n} \partial_i[a_{ij}(x)\,\partial_j u(x)]. \tag{4.4}$$

Bearing in mind that the Bessel potential operators $\mathcal{J}^{2k}$ commutate with differentiation, Lemma 4.1 implies the following assertion.

Corollary 4.2. *Let s be real, k be an integer, $u \in H^s(\mathbb{R}^n)^m$, $a_{ij} \in W_\infty^{\sigma+\varepsilon}(\mathbb{R}^n)^{m\times m}$, $\sigma = \left|s-k-\frac{1}{2}\right| + |k| + \frac{1}{2}$, $\varepsilon \in \mathbb{R}_+(\sigma)$. Then $\mathcal{J}^{2k}(A_0u) - A_0\mathcal{J}^{2k}u \in H^{s-2k-1}(\mathbb{R}^n)^m$ and*

$$\|\mathcal{J}^{2k}(A_0u) - A_0\mathcal{J}^{2k}u\|_{H^{s-2k-1}(\mathbb{R}^n)^m} \le C|2k|\|a\|_{W_\infty^{\sigma+\varepsilon}(\mathbb{R}^n)^m}\|u\|_{H^s(\mathbb{R}^n)^m}.$$

If Ω is an open set while a set Ω' is such that $\overline{\Omega'} \subset \Omega$, we will denote this as $\Omega' \Subset \Omega$.

Now we are in a position to prove the following local regularity theorem.

Theorem 4.3. *Let Ω be an open set in $\mathbb{R}^n$, $s_1 \in \mathbb{R}$, $m \ge 1$, $u \in H^{s_1}_{\mathrm{loc}}(\Omega)^m$, $f \in H^{s_2}_{\mathrm{loc}}(\Omega)^m$, $s_2 > s_1 - 2$. If u satisfies either*

(a) *elliptic (in the sense of Petrovsky) system (3.1), $Au = f$, in Ω with $\{a, b, c\} \in \mathcal{C}^{s_1-1}_{+\mathrm{loc}}(\Omega) \bigcap \mathcal{C}^{s_2+1}_{+\mathrm{loc}}(\Omega)$ or*

(b) *elliptic (in the sense of Petrovsky) system (3.2), $A^*u = f$, in Ω with $\{a, b, c\} \in \mathcal{C}^{1-s_1}_{+\mathrm{loc}}(\Omega) \bigcap \mathcal{C}^{-s_2-1}_{+\mathrm{loc}}(\Omega)$, then $u \in H^{s_2+2}_{\mathrm{loc}}(\Omega)^m$.*

Proof. Note that the theorem hypothesis $s_2 > s_1 - 2$ implies that either $s_1 \ne 1$ or $s_2 \ne -1$ and thus $a \in \bar{C}^{\mu}_{\mathrm{loc}}(\Omega)$ for some $\mu > 0$ and particularly, $a \in C(\Omega)$ (maybe after adjusting a on a zero measure set, that we will assume to be done). We give a proof only for part (a) of the theorem, organized in several steps, for part (b) it is similar.

Step (0). As usual, cf. e.g. [9, Chapter 2, Theorem 3.1], let us first consider the case $a = const$, $b = 0$, $c = 0$ and $\Omega = \mathbb{R}^n$. Suppose a function U satisfies Eq. (3.1). Application of the Fourier transform reduces this equation to $(2\pi)^2\mathcal{A}(\xi)\widehat{U}(\xi) = \widehat{f}(\xi)$. Resolving it for $\widehat{U}$ and applying ellipticity estimate (4.1), we obtain $(1+|\xi|^2)|\widehat{U}(\xi)| \le C_1|\widehat{f}(\xi)| + |\widehat{U}(\xi)|$ with $C_1 = (2\pi)^{-2}C_0$, implying

$$\|U\|_{H^{s+2}(\mathbb{R}^n)} \le C_1\|f\|_{H^s(\mathbb{R}^n)} + \|U\|_{H^s(\mathbb{R}^n)} \quad \forall s \in \mathbb{R}. \tag{4.5}$$

Step (i). Let now the coefficients $\{a, b, c\} \in \mathcal{C}^{s_1-1}_{+\mathrm{loc}}(\Omega) \bigcap \mathcal{C}^{s_2+1}_{+\mathrm{loc}}(\Omega)$ be not generally constant, Ω be not generally $\mathbb{R}^n$, and $u \in H^{s_1}_{\mathrm{loc}}(\Omega)$. Let $B_\rho = B_{y,\rho} \subset \Omega' \Subset \Omega$ be an open ball of radius ρ centred at a point $y \in \Omega$. Let a, b, c and u be extended outside Ω' to $\{a^e, b^e, c^e\} \in \mathcal{C}^{s_1-1}_{+}(\mathbb{R}^n) \bigcap \mathcal{C}^{s_2+1}_{+}(\mathbb{R}^n)$ and $u^e \in H^{s_1}(\mathbb{R}^n)$, and we will further drop the superscript e for brevity.

Let $\eta \in \mathcal{D}(B_\rho)$ be a cut-off function such that $\eta(x) = 1$ in $B_{\rho/2}$. Then $U_\eta(x) := \eta(x)u(x)$ belongs to $H^{s_1}(\mathbb{R}^n)$, is compactly supported in B_ρ and satisfies equation

$$A_{0y}U_\eta = \eta f + A_\eta u - A_0^- U_\eta \quad \text{in } \mathbb{R}^n. \tag{4.6}$$

Here A_{0y} is the principal part of the operator with the coefficient matrix $a(y)$, thus constant in x, i.e.,

$$A_{0y}U_\eta := -\sum_{i,j=1}^{n} a_{ij}(y)\partial_i\partial_j U_\eta, \tag{4.7}$$

$$A_\eta u := -\sum_{i,j=1}^{n}(\partial_i\eta)a_{ij}\partial_j u - \sum_{i,j=1}^{n}\partial_i[(\partial_j\eta)a_{ij}u] - \sum_{j=1}^{n}\eta b_j\partial_j u - \eta c u, \tag{4.8}$$

$$A_0^- U_\eta := -\sum_{i,j=1}^{n}\partial_i(a_{ij}^-\,\partial_j U_\eta), \tag{4.9}$$

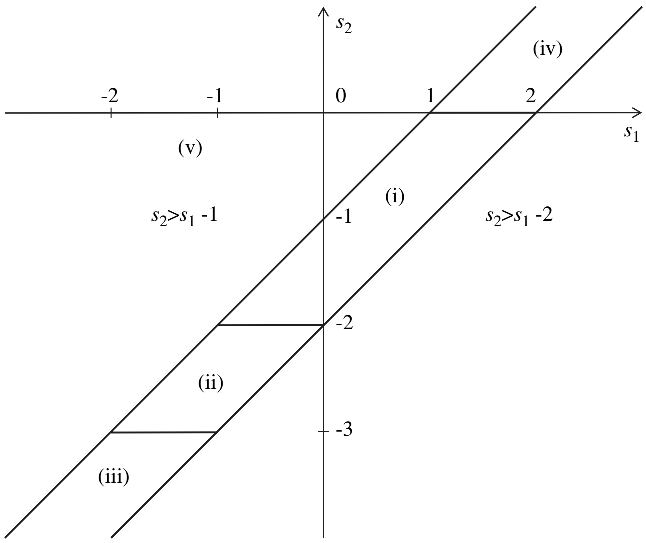

Fig. 1. Regions of parameters s_1, s_2.

and $a^-(x) := a(x) - a(y)$. Let

$$s_2 + 1 \le s_1 < s_2 + 2; \tag{4.10}$$

see Fig. 1. Then by Theorem 3.2,

$$\begin{aligned} \|A_\eta u\|_{H^{s_2}(\mathbb{R}^n)} &\le C_2 \left[\|(\nabla\eta)a\nabla u\|_{H^{s_2}(\mathbb{R}^n)} + \|(\nabla\eta)au\|_{H^{s_2+1}(\mathbb{R}^n)} + \|\eta b\nabla u\|_{H^{s_2}(\mathbb{R}^n)} + \|\eta cu\|_{H^{s_2}(\mathbb{R}^n)}\right] \\ &\le CC_2 \left[\|\nabla\eta\|_{W_\infty^{|s_1-1|+\varepsilon_1}(\mathbb{R}^n)} \|a\nabla u\|_{H^{s_1-1}(\mathbb{R}^n)} + \|\nabla\eta\|_{W_\infty^{|s_2+1|+\varepsilon_2}(\mathbb{R}^n)} \|au\|_{H^{s_2+1}(\mathbb{R}^n)}\right. \\ &\quad \left. + \|\eta\|_{W_\infty^{|s_2|+\varepsilon_2}(\mathbb{R}^n)} \|b\nabla u\|_{H^{s_2}(\mathbb{R}^n)} + \|\eta\|_{W_\infty^{|s_2|+\varepsilon_2}(\mathbb{R}^n)} \|cu\|_{H^{s_2}(\mathbb{R}^n)}\right] \le C_3(\eta)\|u\|_{H^{s_1}(\mathbb{R}^n)}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} C_3(\eta) &:= CC_2C' \left[\|\nabla\eta\|_{W_\infty^{|s_1-1|+\varepsilon_1}(\mathbb{R}^n)} \|a\|_{W_\infty^{|s_1-1|+\varepsilon_1}(\mathbb{R}^n)} + \|\nabla\eta\|_{W_\infty^{|s_2+1|+\varepsilon_2}(\mathbb{R}^n)} \|a\|_{W_\infty^{|s_2+1|+\varepsilon_2}(\mathbb{R}^n)}\right. \\ &\quad \left. + \|\eta\|_{W_\infty^{|s_2|+\varepsilon_2}(\mathbb{R}^n)} \|b\|_{W_\infty^{\mu_b^0+\varepsilon_b^0}(\mathbb{R}^n)} + \|\eta\|_{W_\infty^{|s_2|+\varepsilon_2}(\mathbb{R}^n)} \|c\|_{W_\infty^{\mu_c^0+\varepsilon_c^0}(\mathbb{R}^n)}\right], \end{aligned} \tag{4.12}$$

$$\mu_b^0 := \min\{|s| : s_2 \le s \le s_1 - 1\} = \max\{s_2, 1 - s_1, 0\} = \max\{\mu_b(s_1 - 1), \mu_b(s_2 + 1)\},$$
$$\mu_c^0 := \min\{|s| : s_2 \le s \le s_1\} = \max\{s_2, -s_1, 0\} = \max\{\mu_c(s_1 - 1), \mu_c(s_2 + 1)\},$$

by Definition 3.3 and condition (4.10), while by the theorem hypothesis there exist $\varepsilon_1 \in \mathbb{R}_+(s_1)$, $\varepsilon_2 \in \mathbb{R}_+(s_2)$, $\varepsilon_b^0 \in \mathbb{R}_+(\mu_b^0)$, $\varepsilon_c^0 \in \mathbb{R}_+(\mu_c^0)$ such that the norms of the coefficients a, b, c are bounded in (4.12).

Let us assume the condition

$$|s_2 + 1| < 1 \tag{4.13}$$

in addition to condition (4.10), which correspond to region (i) in Fig. 1.

Let us define $a_0^-(x) := \begin{cases} a^-(x), & x \in \bar{B}_\rho \\ a^-(x\rho/|x|), & x \notin \bar{B}_\rho. \end{cases}$

Then it is easy to see that $\|a_0^-\|_{W_\infty^{|s_2+1|+\varepsilon_2}(\mathbb{R}^n)} = \|a_0^-\|_{C^{|s_2+1|+\varepsilon_2}(\mathbb{R}^n)} = \|a^-\|_{C^{|s_2+1|+\varepsilon_2}(\bar{B}_\rho)}$ for some ε_2 such that

$$\varepsilon_2 \in \mathbb{R}_+(s_2), \quad |s_2 + 1| + \varepsilon_2 < 1. \tag{4.14}$$

Thus, since supp $U_\eta \subset B$, we have from (4.9) by Theorem 3.2,

$$\begin{aligned} \|A_0^- U_\eta\|_{H^{s_2}(\mathbb{R}^n)} &\le C_4 \|a^-\nabla U_\eta\|_{H^{s_2+1}(\mathbb{R}^n)} = C_4 \|a_0^-\nabla U_\eta\|_{H^{s_2+1}(\mathbb{R}^n)} \\ &\le CC_4 \|a_0^-\|_{W_\infty^{|s_2+1|+\varepsilon_2/2}(\mathbb{R}^n)} \|\nabla U_\eta\|_{H^{s_2+1}(\mathbb{R}^n)} \le CC_4 \|a^-\|_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)} C_5 \|U_\eta\|_{H^{s_2+2}(\mathbb{R}^n)}. \end{aligned} \tag{4.15}$$

Applying estimate (4.5) to Eq. (4.6) and taking into account estimates (4.11) and (4.15), we then have under conditions (4.10) and (4.14),

$$C_6(\rho)\|U_\eta\|_{H^{s_2+2}(\mathbb{R}^n)} \le C_7(\eta)\|f\|_{H^{s_2}(B_\rho)} + C_8(\eta)\|u\|_{H^{s_1}(B_\rho)}, \tag{4.16}$$

$$C_6(\rho) := 1 - C_1CC_4C_5\|a^-\|_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)}, \qquad C_7(\eta) := C_1C\|\eta\|_{\bar{C}^{|s_2|+\varepsilon_2}(\bar{B}_\rho)}, \qquad C_8(\eta) := C_1C_3(\eta) + C_7(\eta).$$

The parameter $C_7(\eta)$ and, due to the theorem hypotheses, also $C_3(\eta)$ and thus $C_8(\eta)$ are finite for any $\rho \in (0, \infty)$. We will prove that $C_6(\rho)$ is positive for sufficiently small ρ under conditions (4.10) and (4.13).

Let first $s_2 = -1$, and consider estimate (4.16) with $\varepsilon_2 = 0$. Since $a^-(y) = 0$ and a^- is continuous in $\bar{B}_\rho$, for any sufficiently small $\rho > 0$, the norm $\|a^-\|_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)} = \|a^-\|_{C(\bar{B}_\rho)}$ becomes small enough for $C_6(\rho)$ in (4.16) to be positive.

Let now $0 < |s_2+1| < 1$. Due to the theorem hypothesis, there exists $\varepsilon_2 \in (0, 1-|s_2+1|)$ such that $a^- \in C^{|s_2+1|+\varepsilon_2}(\bar{B}_\rho)$, which implies the following estimate,

$$\|a^-\|_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)} = \|a^-\|_{C(\bar{B}_\rho)} + |a|_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)} \le \|a^-\|_{C(\bar{B}_\rho)} + (2\rho)^{\varepsilon_2/2}|a|_{C^{|s_2+1|+\varepsilon_2}(\bar{B}_\rho)},$$

$$|a|_{C^{|s_2+1|+\varepsilon_2}(\bar{B}_\rho)} := \sup_{x,x'\in\bar{B}_\rho} \frac{|a(x)-a(x')|}{|x-x'|^{|s_2+1|+\varepsilon_2}} \le |a|_{C^{|s_2+1|+\varepsilon_2}(\bar{\Omega}')} < \infty.$$

Thus again for any sufficiently small $\rho > 0$, the norm $\|a^-\|_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)}$ becomes small enough for $C_6(\rho)$ in (4.16) to be positive.

This means $U_\eta \in H^{s_2+2}(\mathbb{R}^n)$ implying $u \in H^{s_2+2}(B_{y,\rho(y)/2})$ for arbitrary point $y \in \Omega$ under conditions (4.10) and (4.13). Thus any compact subdomain $\bar{\Omega}'$ of the open domain Ω has an open cover by the balls $B_{y,\rho(y)/2}$ such that $u \in H^{s_2+2}(B_{y,\rho(y)/2})$. Due to the compactness of $\bar{\Omega}'$, there exists a finite subset of the balls, $B^j := B_{y^j,\rho(y^j)/2}, j = 1, 2, \ldots, J$, still covering $\bar{\Omega}'$. Let $\{\varphi_j(x) \in \mathcal{D}(B^j)\}_{j=1}^J$ be a partition of unity, $\sum_{j=1}^J \varphi_j(x) = 1$ for any $x \in \Omega'$ and $U_j \in H^{s_2+2}(\mathbb{R}^n)$ be such that $U_j = u$ on B^j and $\|U^j\|_{H^{s_2+2}(\mathbb{R}^n)} = \|u\|_{H^{s_2+2}(B^j)}$. Then by Theorem 3.2,

$$\|u\|_{H^{s_2+2}(\Omega')} = \left\|\sum_{j=1}^J \varphi_j u\right\|_{H^{s_2+2}(\Omega')} = \left\|\sum_{j=1}^J \varphi_j U^j\right\|_{H^{s_2+2}(\Omega')} \le \sum_{j=1}^J \|\varphi_j U^j\|_{H^{s_2+2}(\mathbb{R}^n)}$$
$$\le C\sum_{j=1}^J \|\varphi_j\|_{W^\mu_\infty(\mathbb{R}^n)}\|U^j\|_{H^{s_2+2}(\mathbb{R}^n)} = C\sum_{j=1}^J \|\varphi_j\|_{W^\mu_\infty(\mathbb{R}^n)}\|u\|_{H^{s_2+2}(B^j)},$$

for any $\mu > |s_2+2|$. Thus $u \in H^{s_2+2}(\Omega')$ for any compact $\bar{\Omega}' \subset \Omega$, implying $u \in H^{s_2+2}_{\mathrm{loc}}(\Omega)$ under conditions (4.10) and (4.13).

Step (ii). Let us prove the theorem under conditions $s_2+1 \le s_1 < s_2+2$, $-3 < s_2 \le -2$, that are satisfied in region (ii) in Fig. 1. We proceed as in Step (i) but instead of estimate (4.15) for the term $A_0^- U_\eta$ we split it into two parts

$$A_0^- U_\eta = A_{01}^- U_\eta + A_{02}^- U_\eta, \quad A_{01}^- U_\eta := \sum_{i,j=1}^n \partial_i((\partial_j a_{ij}^-)U_\eta),\ A_{02}^- U_\eta := -\sum_{i,j=1}^n \partial_i\partial_j(a_{ij}^- U_\eta)$$

and estimate each of them as follows:

$$\|A_{01}^- U_\eta\|_{H^{s_2}(\mathbb{R}^n)} \le C_4\|(\nabla a^-)U_\eta\|_{H^{s_2+1}(\mathbb{R}^n)} \le C_4\|(\nabla a^-)U_\eta\|_{H^{s_1}(\mathbb{R}^n)}$$
$$\le CC_4\|a^-\|_{W_\infty^{|s_1|+1+\varepsilon_1/2}(\mathbb{R}^n)}\|U_\eta\|_{H^{s_1}(\mathbb{R}^n)} = CC_4\|a^-\|_{W_\infty^{|s_1-1|+\varepsilon_1/2}(\mathbb{R}^n)}\|U_\eta\|_{H^{s_1}(\mathbb{R}^n)}$$

where we have taken into account that $s_1 < 0$ in region (ii), and

$$\|A_{02}^- U_\eta\|_{H^{s_2}(\mathbb{R}^n)} \le C_4\|a^- U_\eta\|_{H^{s_2+2}(\mathbb{R}^n)} \le CC_4\|a_0^- U_\eta\|_{H^{s_2+2}(\mathbb{R}^n)} \le CC_4\|a^-\|_{C^{|s_2+2|+\varepsilon_2/2}(\bar{B}_\rho)}\|U_\eta\|_{H^{s_2+2}(\mathbb{R}^n)}.$$

Taking into account that $\|A_{02}^- U_\eta\|_{H^{s_2}(\mathbb{R}^n)}$ can be made arbitrarily small by choosing sufficiently small ball radius ρ, as in Step (i), since $0 \le |s_2+2| < 1$, this proves the theorem for region (ii).

Step (iii). Let us prove the theorem under conditions

$$s_2+1 \le s_1 < s_2+2, \quad s_2 \le -3, \tag{4.17}$$

that are satisfied in region (iii) in Fig. 1. For arbitrary $\Omega' \Subset \Omega$ let $\eta \in C^\infty(\Omega)$ with supp $\eta \Subset \Omega$ and $\eta = 1$ in Ω'. Then the function $U_\eta = \eta u \in H^{s_1}(\mathbb{R}^n)$ satisfies equation

$$A_0 U_\eta = f_\eta, \quad f_\eta = \eta f + A_\eta u \text{ in } \mathbb{R}^n, \tag{4.18}$$

where A_0 is given by (4.4), A_η by (4.8), while $A_\eta u \in H^{s_2}(\mathbb{R}^n)$ by estimate (4.11). This implies $f_\eta \in H^{s_2}(\mathbb{R}^n)$.

Let $k = -\left\lfloor\frac{-1-s_2}{2}\right\rfloor = \left\lceil\frac{1+s_2}{2}\right\rceil$ and let us denote $v := \mathcal{J}^{2k}U_\eta$. Then $k \le -1$ by the second condition in (4.17), while $v \in H^{s_1-2k}(\mathbb{R}^n)$. Acting by $\mathcal{J}^{2k}$ on (4.18), we arrive at the following equation for v

$$A_0(v) = f_v, \quad \text{in } \mathbb{R}^n, \tag{4.19}$$

where $f_v = \mathcal{J}^{2k} f_\eta - [\mathcal{J}^{2k} A_0 u - A_0 \mathcal{J}^{2k} u]$. To employ Corollary 4.2 with $s = s_1$, we have for its parameter,

$$\sigma = \left| s_1 - k - \frac{1}{2} \right| + |k| + \frac{1}{2} = -s_1 + k + \frac{1}{2} + |k| + \frac{1}{2} = 1 - s_1$$

since $0 < -k \le \frac{-1-s_2}{2} < \frac{1-s_1}{2}$ due to the first condition in (4.17). Then by the theorem hypothesis on the coefficients, the conditions of Corollary 4.2 are satisfied, which implies $[\mathcal{J}^{2k} A_0 u - A_0 \mathcal{J}^{2k} u] \in H^{s_1-2k-1}(\mathbb{R}^n)$. Then taking into account the first condition in (4.17) again, we obtain $f_v \in H^{s_2-2k}(\mathbb{R}^n)$. Denoting $s_1' = s_1 - 2k$, $s_2' = s_2 - 2k$, we arrive at Eq. (4.19) for $v \in H^{s_1'}(\mathbb{R}^n)$ with $f_v \in H^{s_2'}(\mathbb{R}^n)$, where $s_2' + 1 \le s_1' < s_2' + 2, -3 < s_2' \le 1$, and coefficients $a \in \bar{C}_+^{|s_1-1|}(\overline{\Omega}) \bigcap \bar{C}_+^{|s_2+1|}(\overline{\Omega}) \subset \bar{C}_+^{|s_1'-1|}(\overline{\Omega}) \bigcap \bar{C}_+^{|s_2'+1|}(\overline{\Omega})$, which is covered by Steps (i) and (ii) implying $v \in H^{s_2'+2}(\mathbb{R}^n) = H^{s_2+2-2k}(\mathbb{R}^n)$. Thus, $U_\eta := \mathcal{J}^{-2k} v \in H^{s_2+2}(\mathbb{R}^n)$. This gives $u \in H^{s_2+2}(\Omega')$, which implies the theorem claim in region (iii).

Step (iv). Let us prove the theorem under conditions $s_2 + 1 \le s_1 < s_2 + 2$, $s_2 \ge 0$, that are satisfied in region (iv) in Fig. 1. Let α be a multiindex such that $|\alpha| = \lfloor s_2 \rfloor + 1$. Then (4.18) implies

$$A_0 \partial^\alpha U_\eta = \partial^\alpha f_\eta + f_\eta^\alpha, \quad f_\eta^\alpha = A_0 \partial^\alpha U_\eta - \partial^\alpha A_0 U_\eta. \tag{4.20}$$

Since f_η^α is a commutator, we obtain that $f_\eta^\alpha \in H^{s_1-|\alpha|-1}(R^n) \subset H^{s_2-|\alpha|}(R^n)$, where the theorem hypothesis on smoothness of the coefficient matrix a and Theorem 3.2 were taken into account. Then $\partial^\alpha f_\eta + f_\eta^\alpha \in H^{s_2-\alpha}(R^n)$ giving $\partial^\alpha U_\eta \in H_{\mathrm{loc}}^{s_2-|\alpha|+2}(R^n)$ by Step (i), which implies $u \in H_{\mathrm{loc}}^{s_2+2}(\Omega)$, i.e. the theorem claim for region (iv).

Step (v). Now we finally prove the theorem for $s_2 > s_1 - 1$, i.e. for region (v). Since $f \in H^{s_2}(\Omega')$ on any open set $\Omega' \Subset \Omega$, we have also $f \in H^{s_1-1}(\Omega')$, i.e., we arrive at the situation covered by Steps (i)–(iv) with $s_2 = s_1 - 1$, which implies $u \in H^{s_1+1}(\Omega')$. If $s_1 \le s_2$, we iterate this procedure, obtaining at the end $u \in H^{s_2+2}(\Omega')$, i.e. the theorem claim, if $s_2 - s_1$ is an integer, or $u \in H^{s_1+k}(\Omega')$, where $k = \lfloor s_2 - s_1 + 2 \rfloor$, otherwise. Recalling in the latter case that $f \in H^{s_2}(\Omega')$ we can apply the corresponding step from (i) to (iv) again, which finishes the proof for region (v). □

Theorem 4.3 gives solution regularity on any sub-domain Ω' with compact closure $\bar{\Omega}' \subset \Omega$. The following theorem generalizes it to sub-domain Ω' with non-compact closure $\bar{\Omega}' \subset \Omega$ and particularly proves regularity at infinity for exterior (unbounded) domains.

Theorem 4.4. *Let Ω be $\mathbb{R}^n$ or an open exterior or interior domain with a compact boundary in $\mathbb{R}^n$, $s_1 \in \mathbb{R}$, $s_2 > s_1 - 2$, $u \in H^{s_1}(\Omega')^m$ and $f \in H^{s_2}(\Omega')^m$ on any open set $\Omega' \Subset \Omega$, $m \ge 1$. Let u satisfy either*

(a) *elliptic (in the Petrovsky sense) system (3.1), $Au = f$, in Ω with $\{a, b, c\} \in \mathcal{C}_+^{s_1-1}(\mathbb{R}^n) \bigcap \mathcal{C}_+^{s_2+1}(\mathbb{R}^n)$ or*

(b) *elliptic (in the Petrovsky sense) system (3.2), $A^* u = f$, in Ω with $\{a, b, c\} \in \mathcal{C}_+^{1-s_1}(\mathbb{R}^n) \bigcap \mathcal{C}_+^{-s_2-1}(\mathbb{R}^n)$, and in the case of the infinite domain Ω there exist finite matrices $a_{ij}(\infty) := \lim_{x\to\infty} a_{ij}(x)$ satisfying the ellipticity condition $\det \sum_{i,j=1} a_{ij}(\infty)\xi_i\xi_j \neq 0$. Then $u \in H^{s_2+2}(\Omega')^m$ on any $\Omega' \Subset \Omega$.*

Proof. The theorem claim for subdomains $\Omega' \Subset \Omega$ with compact closure is implied by Theorem 4.3. To complete the proof, we have to consider an infinite subdomain $\Omega' \Subset \Omega$ of an infinite domain Ω. Note that the theorem hypothesis $s_2 > s_1 - 2$ implies that either $s_1 \neq 1$ or $s_2 \neq -1$ and thus $a \in \bar{C}^\mu(\Omega)$ for any $\Omega' \Subset \Omega$ for some $\mu > 0$ and particularly, $a \in C(\Omega)$ (maybe after adjusting a on a zero measure set, that we will assume to be done).

The proof follows the pattern of the proof of Theorem 4.3 and we will mostly refer to that proof instead of repeating it whenever possible. We give only a proof for part (a) of the theorem; the proof for part (b) is similar.

Step (i). Let the coefficients a, b, c be not generally constant, Ω be either $\mathbb{R}^n$ or an open exterior domain with a compact boundary in $\mathbb{R}^n$. In the latter case let u be extended outside Ω to $u^e \in H^{s_1}(\mathbb{R}^n)$, and we will further drop the superscript e for brevity. Let $B_\rho = B_{0,\rho}$ be an open ball of radius ρ centred at zero. Let ρ be sufficiently large, so that B_ρ includes the boundary of Ω (if $\Omega \neq \mathbb{R}^n$). Let us chose a cut-off function $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ in $\mathbb{R}^n \setminus B_{2\rho}$ and $\eta(x) = 0$ in B_ρ. Denoting $U_\eta(x) := \eta(x)u(x)$ we obtain that $\operatorname{supp} U_\eta \subset \mathbb{R}^n \setminus B_\rho \subset \Omega$.

Then the function U_η satisfies equation

$$A_{0\infty} U_\eta = \eta f + A_\eta u - A_\infty^- U_\eta \quad \text{in } \mathbb{R}^n. \tag{4.21}$$

Here $A_{0\infty}$ is the principal part of the operator with the constant coefficient matrix $a(\infty)$, i.e.,

$$A_{0\infty} U_\eta := -\sum_{i,j=1}^n a_{ij}(\infty)\partial_i\partial_j U_\eta, \tag{4.22}$$

$$A_\eta u = -\sum_{i,j=1}^n (\partial_i \eta) a_{ij} \partial_j u - \sum_{i,j=1}^n \partial_i[(\partial_j \eta) a_{ij} u] - \sum_{j=1}^n \eta b_j \partial_j u - \eta c u, \tag{4.23}$$

$$A^-_\infty U_\eta = -\sum_{i,j=1}^n \partial_i(a^-_{ij}\partial_j U_\eta), \tag{4.24}$$

where $a^-(x) = a(x) - a(\infty)$.

Let

$$s_2 + 1 \le s_1 < s_2 + 2; \tag{4.25}$$

see Fig. 1. Then by Theorem 3.2, we again, as in the proof of Theorem 4.3 arrive at estimate (4.11), where $C_3(\rho)$ is defined by (4.12).

Let us define $a^-_\infty(x) = \begin{cases} a^-(x), & x \in \mathbb{R}^n \setminus B_\rho \\ \frac{|x|}{\rho} a^-\left(\frac{x\rho}{|x|}\right), & x \in \bar{B}_\rho. \end{cases}$

Then evidently $\|a^-_\infty\|_{C(\mathbb{R}^n)} = \|a^-\|_{C(\mathbb{R}^n\setminus B_\rho)} \to 0$ as $\rho \to \infty$. Moreover, it is easy to check (see Appendix A) that $\|a^-_\infty\|_{C^\mu(\mathbb{R}^n)} \le 3\|a^-\|_{C^\mu(\mathbb{R}^n\setminus B_\rho)}$ for any $\mu \in [0,1]$ and sufficiently large ρ, and $\|a^-_\infty\|_{C^\mu(\mathbb{R}^n)} \to 0$ as $\rho \to \infty$ if $a^- \in C^{\mu+\epsilon}(\Omega)$ for some $\varepsilon > 0$.

Thus, since supp $U_\eta \subset \mathbb{R}^n \setminus B_\rho$, we have from (4.24) by Theorem 3.2,

$$\begin{aligned}\|A^-_\infty U_\eta\|_{H^{s_2}(\mathbb{R}^n)} &\le C_4\|a^-\nabla U_\eta\|_{H^{s_2+1}(\mathbb{R}^n)} = C_4\|a^-_\infty\nabla U_\eta\|_{H^{s_2+1}(\mathbb{R}^n)} \\ &\le CC_4\|a^-_\infty\|_{C^{|s_2+1|+\varepsilon_2/2}(\mathbb{R}^n)}\|\nabla U_\eta\|_{H^{s_2+1}(\mathbb{R}^n)} \le CC_4\|a^-_\infty\|_{C^{|s_2+1|+\varepsilon_2/2}(\mathbb{R}^n)}C_5\|U_\eta\|_{H^{s_2+2}(\mathbb{R}^n)}\end{aligned} \tag{4.26}$$

for any ε_2 such that

$$\varepsilon_2 \in \mathbb{R}_+(s_2), \quad |s_2+1| + \varepsilon_2/2 < 1. \tag{4.27}$$

Applying estimate (4.5) to Eq. (4.21) and taking into account estimates (4.11) and (4.26), we then have under conditions (4.25) and (4.27),

$$\begin{aligned}&C_6(\rho)\|U_\eta\|_{H^{s_2+2}(\mathbb{R}^n)} \le C_7(\rho)\|f\|_{H^{s_2}(\mathbb{R}^n\setminus\bar{B}_\rho)} + C_8(\rho)\|u\|_{H^{s_1}(\mathbb{R}^n\setminus\bar{B}_\rho)}, \\ &C_6(\rho) := 1 - C_1CC_4C_5\|a^-_\infty\|_{C^{|s_2+1|+\varepsilon_2/2}(\mathbb{R}^n)}, \qquad C_7(\rho) := C_1C\|\eta\|_{\tilde{C}^{|s_2|+\varepsilon_2}(\mathbb{R}^n\setminus B_\rho)}, \\ &C_8(\rho) := C_1C_3(\rho) + C_7(\rho).\end{aligned} \tag{4.28}$$

The parameter $C_7(\rho)$ and, due to the theorem hypotheses, also $C_3(\rho)$ and thus $C_8(\rho)$ are finite for any $\rho \in (0,\infty)$.

Further in this step we prove that $C_6(\rho)$ is positive for sufficiently large ρ under conditions $s_2+1 \le s_1 < s_2+2$, $|s_2+1| < 1$, which correspond to region (i) in Fig. 1.

Let first $s_2 = -1$, and consider estimate (4.28) with $s_2 + 1 = \varepsilon_2 = 0$. Since $a^-(\infty) = 0$, the norm $\|a^-_\infty\|_{C^{|s_2+1|+\varepsilon_2/2}(\mathbb{R}^n)} = \|a^-_\infty\|_{C(\mathbb{R}^n)}$ for sufficiently large $\rho < \infty$ becomes small enough for $C_6(\rho)$ in (4.28) to be positive.

Let now $0 < |s_2+1| < 1$. Due to the theorem hypothesis, there exists $\varepsilon_2 \in (0, 1-|s_2+1|)$ such that $a^- \in C^{|s_2+1|+\varepsilon_2}(\mathbb{R}^n \setminus B_\rho)$, which implies $\|a^-_\infty\|_{C^{|s_2+1|+\varepsilon_2/2}(\mathbb{R}^n)} \to 0$ as $\rho \to 0$. Thus again for sufficiently large ρ, the norm $\|a^-_\infty\|_{C^{|s_2+1|+\varepsilon_2/2}(\mathbb{R}^n)}$ becomes small enough for $C_6(\rho)$ in (4.16) to be positive.

This means that in both the cases $U_\eta \in H^{s_2+2}(\mathbb{R}^n)$ implying $u \in H^{s_2+2}(\Omega' \setminus B_{2\rho})$ for sufficiently large ρ. Taking into account that $u \in H^{s_2+2}(\Omega' \bigcap B_{3\rho})$ for any ρ by Theorem 4.3, we arrive at the present theorem claim in region (i).

Step (ii). Let us prove the theorem under conditions $s_2+1 \le s_1 < s_2+2$, $-3 < s_2 \le -2$, that are satisfied in region (ii) in Fig. 1. We proceed as in Step (i) but instead of estimate (4.26) for the term $A^-_\infty U_\eta$ we split it into two parts

$$A^-_\infty U_\eta = A^-_{\infty 1}U_\eta + A^-_{\infty 2}U_\eta, \quad A^-_{\infty 1}U_\eta := \sum_{i,j=1}^n \partial_i((\partial_j a^-_{ij})U_\eta), \; A^-_{\infty 2}U_\eta := -\sum_{i,j=1}^n \partial_i\partial_j(a^-_{ij}U_\eta)$$

and estimate each of them as follows:

$$\begin{aligned}\|A^-_{\infty 1}U_\eta\|_{H^{s_2}(\mathbb{R}^n)} &\le C_4\|(\nabla a^-)U_\eta\|_{H^{s_2+1}(\mathbb{R}^n)} \le C_4\|(\nabla a^-)U_\eta\|_{H^{s_1}(\mathbb{R}^n)} \\ &\le CC_4\|a^-\|_{W_\infty^{|s_1|+1+\varepsilon_1/2}(\mathbb{R}^n)}\|U_\eta\|_{H^{s_1}(\mathbb{R}^n)} = CC_4\|a^-\|_{W_\infty^{|s_1-1|+\varepsilon_1/2}(\mathbb{R}^n)}\|U_\eta\|_{H^{s_1}(\mathbb{R}^n)}\end{aligned}$$

where we took into account that $s_1 < 0$ in region (ii), and

$$\|A^-_{\infty 2}U_\eta\|_{H^{s_2}(\mathbb{R}^n)} \le C_4\|a^-U_\eta\|_{H^{s_2+2}(\mathbb{R}^n)} \le CC_4\|a^-_0U_\eta\|_{H^{s_2+2}(\mathbb{R}^n)} \le CC_4\|a^-\|_{C^{|s_2+2|+\varepsilon_2/2}(\bar{B}_\rho)}\|U_\eta\|_{H^{s_2+2}(\mathbb{R}^n)}.$$

Taking into account that $\|A^-_{\infty 2}U_\eta\|_{H^{s_2}(\mathbb{R}^n)}$ can be made arbitrarily small by choosing sufficiently large ρ, as in Step (i), since $0 \le |s_2+2| < 1$, this proves the theorem for region (ii).

Steps (iii)–(v) The proofs of the theorem under condition $s_2 < -3$ and under condition $s_2 \ge 0$, in addition to condition (4.25) coincide word-for-word with the proof in Steps (iii) and (iv), respectively, of Theorem 4.3, while for $s_2 > s_1 - 1$ with the proof in Step (v) of the same theorem. □

Remark 4.5. Conditions on the PDE coefficients in Theorem 4.4 can be evidently relaxed to the corresponding conditions for all domains $\Omega' \Subset \Omega$ (implying that the coefficients are extendable from such Ω' to the whole $\mathbb{R}^n$ such that the conditions hold) supplemented with the continuity of the coefficient a at infinity for the extensions.

Remark 4.6. Theorem 4.4 proof works also for domains Ω with a non-compact boundary and for an open set Ω' for which there exist another open set Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$ and a cut-off function $\eta' \in C^\infty(\mathbb{R}^n)$ with sufficient number of bounded derivatives in $\mathbb{R}^n$ such that $\eta'(x) = 1$ in Ω' and $\eta'(x) = 0$ in $\mathbb{R}^n \setminus \Omega''$. In the first paragraph of Step (i) we can chose then a cut-off function $\eta_\rho \in C^\infty(\mathbb{R}^n)$ such that $\eta_\rho(x) = 1$ in $\mathbb{R}^n \setminus B_{2\rho}$ and $\eta_\rho(x) = 0$ in B_ρ. Defining $\eta(x) := \eta'(x)\eta_\rho(x)$ we have $\eta(x) = 1$ in $\Omega' \setminus B_{2\rho}$ and $\eta(x) = 0$ in $(\mathbb{R}^n \setminus \Omega'') \bigcup B_\rho$. Then the support of $U_\eta(x) := \eta(x)u(x)$ belongs to $\bar{\Omega}'' \setminus B_\rho \subset \Omega$ and we can follow the proof of Theorem 4.4 as before.

5. Extensions and generalized co-normal derivatives for PDE systems with non-smooth coefficients

5.1. Extension of partial differential operators

Let $\frac{1}{2} < s < \frac{3}{2}$ and $\{a, b, c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega})$ (which for the case $s = 1$ means $a, b, c \in L_\infty(\Omega)$). In addition to the operator A defined by (3.6), let us consider also the *aggregate* partial differential operator $\check{A}$, defined as,

$$\langle \check{A}u, v\rangle_\Omega := \check{\mathcal{E}}(u, v) \quad \forall v \in H^{2-s}(\Omega), \tag{5.1}$$

where

$$\check{\mathcal{E}}(u, v) = \check{\mathcal{E}}_\Omega(u, v) := \sum_{i,j=1}^{n} \left\langle \widetilde{E}^{s-1}(a_{ij}\partial_j u), \partial_i v\right\rangle_\Omega + \sum_{j=1}^{n} \left\langle \widetilde{E}^{s_b(s)}(b_j\partial_j u), v\right\rangle_\Omega + \left\langle \widetilde{E}^{s_c(s)}(cu), v\right\rangle_\Omega , \tag{5.2}$$

$\widetilde{E}^{s-1} : H^{s-1}(\Omega) \to \widetilde{H}^{s-1}(\Omega)$, $\widetilde{E}^{s_b(s)} : H^{s_b(s)}(\Omega) \to \widetilde{H}^{s_b(s)}(\Omega)$, $\widetilde{E}^{s_c(s)} : H^{s_c(s)}(\Omega) \to \widetilde{H}^{s_c(s)}(\Omega)$ are bounded extension operators, which are unique by [14, Theorem 2.16] (Theorem B.3 in Appendix B) since $-\frac{1}{2} < s-1 < \frac{1}{2}$ and $-\frac{1}{2} < s_b(s) \le 0$, $s_c(s) = 0$ by (3.5). Then the bilinear form $\check{\mathcal{E}}(u, v) : H^s(\Omega) \times H^{2-s}(\Omega) \to \mathbb{C}$ is bounded by Theorem 3.2, implying that the operator $\check{A} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega) = [H^{2-s}(\Omega)]^*$ is bounded, for $\frac{1}{2} < s < \frac{3}{2}$.

Note that by (2.2)–(2.3) one can rewrite (5.1) also as $(\check{A}u, v)_\Omega := \Phi(u, v)\ \forall v \in H^{2-s}(\Omega)$, where $\Phi(u, v) = \overline{\check{\mathcal{E}}(u, \bar{v})}$ is the sesquilinear form.

If $s = 1$, i.e. $u, v \in H^1(\Omega)$, then evidently

$$\check{\mathcal{E}}(u, v) = \mathcal{E}(u, v) = \int_\Omega \left[\sum_{i,j=1}^{n}(a_{ij}\partial_j u)\cdot\partial_i v + \sum_{j=1}^{n}(b_j\partial_j u)\cdot v + cu\cdot v\right] dx.$$

For $\frac{1}{2} < s < \frac{3}{2}$ and $\{a, b, c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega})$ let us consider also the *aggregate* operator $\check{A}^* : H^{2-s}(\Omega) \to \widetilde{H}^{-s}(\Omega) = [H^s(\Omega)]^*$, defined as,

$$\langle \check{A}^*v, u\rangle_\Omega := \check{\mathcal{E}}^*(v, u) \quad \forall u \in H^s(\Omega), \tag{5.3}$$

$$\check{\mathcal{E}}^*(v, u) = \overline{\check{\mathcal{E}}(\bar{u}, \bar{v})} = \Phi(\bar{u}, v) = \sum_{i,j=1}^{n} \left\langle \bar{a}_{ij}\partial_j u, \widetilde{E}^{1-s}\partial_i v\right\rangle_\Omega + \sum_{j=1}^{n} \left\langle \bar{b}_j\partial_j u, \widetilde{E}^{-s_b(s)}v\right\rangle_\Omega + \left\langle \bar{c}u, \widetilde{E}^{-s_c(s)}v\right\rangle_\Omega \tag{5.4}$$

by (5.2) since $(\widetilde{E}^p)^* = \widetilde{E}^{-p}$ for $-\frac{1}{2} < p < \frac{1}{2}$ by [14, Theorem 2.16] (Theorem B.3 in Appendix B).

Due to Theorem 3.2 and relations (5.1), (5.3) and (5.4), we have the following statement.

Theorem 5.1. *If $\frac{1}{2} < s < \frac{3}{2}$ and $\{a, b, c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega})$, then bilinear form (5.2), $\check{\mathcal{E}} : \{H^s(\Omega), H^{2-s}(\Omega)\} \to \mathbb{C}$ is bounded and expressions (5.1) and (5.3) define bounded linear operators $\check{A} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$, $\check{A}^* : H^{2-s}(\Omega) \to \widetilde{H}^{-s}(\Omega)$, and the aggregate second Green's identity holds true in the following form,*

$$\langle \check{A}u, \bar{v}\rangle_\Omega = \langle u, \overline{\check{A}^*v}\rangle_\Omega, \quad u \in H^s(\Omega), v \in H^{2-s}(\Omega), \frac{1}{2} < s < \frac{3}{2}. \tag{5.5}$$

For any $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, the functional $\check{A}u$ belongs to $\widetilde{H}^{s-2}(\Omega)$ and is an extension of the functional $Au \in H^{s-2}(\Omega)$ from the domain of definition $\widetilde{H}^{2-s}(\Omega)$ to the domain of definition $H^{2-s}(\Omega)$. Similarly, for any $v \in H^{2-s}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, the functional $\check{A}^*v$ belongs to $\widetilde{H}^{-s}(\Omega)$ and is an extension of the functional $A^*v \in H^{-s}(\Omega)$ from the domain of definition $\widetilde{H}^s(\Omega)$ to the domain of definition $H^s(\Omega)$.

The distribution $\check{A}u$ is not the only possible extension of the functional Au, and any functional of the form

$$\check{A}u + g, \quad g \in H^{s-2}_{\partial\Omega} \tag{5.6}$$

gives another extension. On the other hand, any extension of the domain of definition of the functional Au from $\widetilde{H}^{2-s}(\Omega)$ to $H^{2-s}(\Omega)$ has evidently form (5.6). The existence of such extensions is provided in [14, Theorem 2.16] (Theorem B.3 in Appendix B).

5.2. Generalized co-normal derivatives

Let $\gamma^+ : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ denote the trace operator, which is bounded on Lipschitz domains for $\frac{1}{2} < s < \frac{3}{2}$. For $u \in H^s(\Omega)$, $s > \frac{3}{2}$, and $a \in C(\bar{\Omega})$, the strong (classical) co-normal derivative operator

$$T_c^+ u(x) := \sum_{i,j=1}^{n} a_{ij}(x)\, \gamma^+[\partial_j u(x)] \nu_i(x) \tag{5.7}$$

is well defined on $\partial\Omega$ in the sense of traces. Here $\gamma^+[\partial_j u] \in H^{s-\frac{3}{2}}(\partial\Omega) \subset L_2(\partial\Omega)$ if $\frac{3}{2} < s < \frac{5}{2}$, while the outward (to Ω) unit normal vector $\nu(x)$ at the point $x \in \partial\Omega$ belongs to $L_\infty(\partial\Omega)$ for the Lipschitz boundary $\partial\Omega$, implying $T_c^+ u \in L_2(\partial\Omega)$. Note that for Lipschitz domains, $T_c^+ u$ does not generally belong to $H^s(\partial\Omega)$, $s > 0$, even for infinitely smooth u.

A definition of the generalized co-normal derivative is given in [10, Lemma 4.3] for $s = 1$ (cf. also [8, Lemma 2.2] for the generalized co-normal derivative on a manifold boundary) and in [14] for $\frac{1}{2} < s < \frac{3}{2}$ and infinitely smooth coefficients. We can now extend the definition to the range of Sobolev spaces and non-smooth coefficients.

Definition 5.2. Let Ω be a Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, $u \in H^s(\Omega)$, $\{a, b, c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega})$, and $Au = \tilde{f}|_\Omega \in H^{s-2}(\Omega)$ in Ω for some $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$. Let us define *the generalized co-normal derivative* $T^+(\tilde{f}, u) \in H^{s-\frac{3}{2}}(\partial\Omega)$ as

$$\left\langle T^+(\tilde{f}, u), w \right\rangle_{\partial\Omega} := \check{\mathcal{E}}(u, \gamma_{-1}w) - \langle \tilde{f}, \gamma_{-1}w \rangle_\Omega = \langle \check{A}u - \tilde{f}, \gamma_{-1}w \rangle_\Omega \quad \forall w \in H^{\frac{3}{2}-s}(\partial\Omega),$$

where $\gamma_{-1} : H^{\frac{3}{2}-s}(\partial\Omega) \to H^{2-s}(\Omega)$ is a bounded right inverse to the trace operator.

Theorem 5.3. *Under the hypotheses of Definition 5.2, the generalized co-normal derivative $T^+(\tilde{f}, u)$ is independent of the operator γ_{-1}, the estimate $\|T^+(\tilde{f}, u)\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \le C_1 \|u\|_{H^s(\Omega)} + C_2 \|\tilde{f}\|_{\widetilde{H}^{s-2}(\Omega)}$ takes place, and the first Green's identity holds in the following form,*

$$\left\langle T^+(\tilde{f}, u), \gamma^+ v \right\rangle_{\partial\Omega} = \check{\mathcal{E}}(u, v) - \langle \tilde{f}, v \rangle_\Omega = \langle \check{A}u - \tilde{f}, v \rangle_\Omega \quad \forall v \in H^{2-s}(\Omega). \tag{5.8}$$

Proof. The proof of the theorem coincides word-for-word with the proof of its counterpart for infinitely smooth coefficients; see Theorem 3.2 in [14]. □

Because of the involvement of $\tilde{f}$, the generalized co-normal derivative $T^+(\tilde{f}, u)$ is generally *non-linear* in u. It becomes linear if a linear relation is imposed between u and $\tilde{f}$ (including behaviour of the latter on the boundary $\partial\Omega$), thus fixing an extension of $\tilde{f}|_\Omega = Au$ into $\widetilde{H}^{s-2}(\Omega)$. For example, $\tilde{f}|_\Omega$ can be extended as $\check{f} := \check{A}u$, which generally does not coincide with $\tilde{f}$. Then obviously, $T^+(\check{f}, u) = T^+(\check{A}u, u) = 0$, meaning that the co-normal derivatives associated with any other possible extension $\tilde{f}$ appear to be aggregated in $\check{f}$ as

$$\langle \check{f}, v \rangle_\Omega = \langle \tilde{f}, v \rangle_\Omega + \left\langle T^+(\tilde{f}, u)\,, \gamma^+ v \right\rangle_{\partial\Omega} \quad \forall v \in H^{2-s}(\Omega) \tag{5.9}$$

due to (5.8). This justifies the term *aggregate* for the extension $\check{f}$, and thus for the operator $\check{A}u$.

As follows from Definition 5.2, the generalized co-normal derivative is still linear with respect to the couple $(\tilde{f}, u)$, i.e., $T^+(\alpha_1 \tilde{f}_1, \alpha_1 u_1) + T^+(\alpha_2 \tilde{f}_2, \alpha_2 u_2) = T^+(\alpha_1 \tilde{f}_1 + \alpha_2 \tilde{f}_2, \alpha_1 u_1 + \alpha_2 u_2)$ for any complex numbers α_1, α_2.

In fact, for a given function $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, any distribution $\tau \in H^{s-\frac{3}{2}}(\partial\Omega)$ may be nominated as a co-normal derivative of u, by an appropriate extension $\tilde{f}$ of the distribution $Au \in H^{s-2}(\Omega)$ into $\widetilde{H}^{s-2}(\Omega)$. This extension is again given by the second Green's identity (5.8) re-written as follows (cf. [2, Section 2.2, item 4] for $s = 1$),

$$\langle \tilde{f}, v \rangle_\Omega := \check{\mathcal{E}}(u, v) - \left\langle \tau, \gamma^+ v \right\rangle_{\partial\Omega} = \langle \check{A}u - \gamma^{+*}\tau, v \rangle_\Omega \quad \forall v \in H^{2-s}(\Omega). \tag{5.10}$$

Here the operator $\gamma^{+*} : H^{s-\frac{3}{2}}(\partial\Omega) \to \widetilde{H}^{s-2}(\Omega)$ is adjoint to the trace operator, $\langle \gamma^{+*}\tau, v \rangle_\Omega := \left\langle \tau, \gamma^+ v \right\rangle_{\partial\Omega}$ for all $\tau \in H^{s-\frac{3}{2}}(\partial\Omega)$ and $v \in H^{2-s}(\Omega)$. Evidently, the distribution $\tilde{f}$ defined by (5.10) belongs to $\widetilde{H}^{s-2}(\Omega)$ and is an extension of the distribution Au into $\widetilde{H}^{s-2}(\Omega)$ since $\gamma^+ v = 0$ for $v \in \widetilde{H}^{2-s}(\Omega)$.

For $u \in C^1(\overline{\Omega}) \subset H^1(\Omega)$, one can take τ equal to the strong co-normal derivative, $T_c^+ u \in L_\infty(\partial\Omega)$, and relation (5.10) can be considered as the *classical extension* of $f = Au \in H^{-1}(\Omega)$ to $\tilde{f}_c \in \widetilde{H}^{-1}(\Omega)$, which is evidently linear.

For a sufficiently smooth function v and $a, b \in C(\bar{\Omega})$, let

$$T_{*c}^+ v(x) := \sum_{i,j=1}^{n} \bar{a}_{ji}^\top(x)\, \gamma^+[\partial_j v(x)]\nu_i(x) + \sum_{i=1}^{n} \bar{b}_i^\top(x)\gamma^+ v(x)\nu_i$$

be the strong (classical) modified co-normal derivative (it corresponds to $\widetilde{\mathfrak{B}}_\nu v$ in [10]), associated with the operator A^*.

If $v \in H^{2-s}(\Omega)$, $\{a, b, c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega})$, $\frac{1}{2} < s < \frac{3}{2}$, and $A^* v = \tilde{f}_*|_\Omega$ in Ω for some $\tilde{f}_* \in \widetilde{H}^{-s}(\Omega)$, we define *the generalized modified co-normal derivative* $T_*^+(\tilde{f}_*, v) \in H^{\frac{1}{2}-s}(\partial\Omega)$, associated with the operator A^*, similar to Definition 5.2, as

$$\left\langle T_*^+(\tilde{f}_*, v)\,,\, w\right\rangle_{\partial\Omega} := \check{\mathcal{E}}^*(v, \gamma_{-1} w) - \langle \tilde{f}_*, \gamma_{-1} w\rangle_\Omega \quad \forall w \in H^{s-\frac{1}{2}}(\partial\Omega).$$

As in Theorem 5.3, this leads to the following first Green's identity for the function v,

$$\left\langle T_*^+(\tilde{f}_*, v), \gamma^+ u\right\rangle_{\partial\Omega} = \check{\mathcal{E}}^*(v, u) - \langle \tilde{f}_*, u\rangle_\Omega \quad \forall u \in H^s(\Omega), \tag{5.11}$$

which by (5.4) implies

$$\left\langle \gamma^+ u, \overline{T_*^+(\tilde{f}_*, v)}\right\rangle_{\partial\Omega} = \check{\mathcal{E}}(u, \bar{v}) - \langle u, \bar{\tilde{f}}_*\rangle_\Omega \quad \forall u \in H^s(\Omega). \tag{5.12}$$

If, in addition, $Au = \tilde{f}|_\Omega$ in Ω with some $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, then combining (5.12) and the first Green's identity (5.8) for u, we arrive at the following generalized second Green's identity,

$$\langle \tilde{f}, \bar{v}\rangle_\Omega - \langle u, \bar{\tilde{f}}_*\rangle_\Omega = \left\langle \gamma^+ u, \overline{T_*^+(\tilde{f}_*, v)}\right\rangle_{\partial\Omega} - \left\langle T^+(\tilde{f}, u)\,,\, \overline{\gamma^+ v}\right\rangle_{\partial\Omega}. \tag{5.13}$$

By (5.1), (5.3), (5.8) and (5.11), this, of course, leads to the aggregate second Green's identity (5.5).

5.3. *Generalized weak settings of boundary value problems*

Similar to the case of infinitely smooth coefficients in [14, Section 3.2], let us consider the generalized BVP weak settings for PDE system (3.1) on an interior Lipschitz domain for $\frac{1}{2} < s < \frac{3}{2}$ and $\{a, b, c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega})$.

The Dirichlet problem: for $f \in H^{s-2}(\Omega)$ and $\varphi_0 \in H^{s-\frac{1}{2}}(\partial\Omega)$, find $u \in H^s(\Omega)$ such that

$$\langle Au, v\rangle_\Omega = \langle f, v\rangle_\Omega \quad \forall v \in \widetilde{H}^{2-s}(\Omega), \tag{5.14}$$

$$\gamma^+ u = \varphi_0 \quad \text{on } \partial\Omega, \tag{5.15}$$

where Au is defined by (3.6).

The Neumann problem: for $\check{f} \in \widetilde{H}^{s-2}(\Omega)$, find $u \in H^s(\Omega)$ such that

$$\langle \check{A}u, v\rangle_\Omega = \langle \check{f}, v\rangle_\Omega \quad \forall v \in H^{2-s}(\Omega), \tag{5.16}$$

where $\check{A}u$ is defined by (5.1).

The mixed (Dirichlet–Neumann) problem: for $\check{f}_m \in [H_0^{2-s}(\Omega, \partial_D\Omega)]^*$ and $\varphi_0 \in H^{s-\frac{1}{2}}(\partial_D\Omega)$, find $u \in H^s(\Omega)$ such that

$$\langle \check{A}_{\partial_D\Omega} u, v\rangle_\Omega = \langle \check{f}_m, v\rangle_\Omega \quad \forall v \in H_0^{2-s}(\Omega, \partial_D\Omega), \tag{5.17}$$

$$\gamma^+ u = \varphi_0 \quad \text{on } \partial_D\Omega. \tag{5.18}$$

Here $\check{A}_{\partial_D\Omega} : H^s(\Omega) \to [H_0^{2-s}(\Omega, \partial_D\Omega)]^*$ is the *mixed aggregate* operator defined as

$$\langle \check{A}_{\partial_D\Omega} u, v\rangle_\Omega := \langle \check{A}u, v\rangle_\Omega = \check{\mathcal{E}}(u, v) \quad \forall v \in H_0^{2-s}(\Omega, \partial_D\Omega)$$

where, respectively, the Dirichlet and Neumann parts of the boundary, $\partial_D\Omega$ and $\partial_N\Omega = \partial\Omega \setminus \overline{\partial_D\Omega}$ are nonempty, open submanifolds of $\partial\Omega$, and $H_0^s(\Omega, \partial_D\Omega) = \{w \in H^s(\Omega) : \gamma^+ w = 0 \text{ on } \partial_D\Omega\}$. The operator $\check{A}_{\partial_D\Omega}$ is bounded by the same argument as the aggregate operator $\check{A}$. For any $u \in H^s(\Omega)$, the distribution $\check{A}_{\partial_D\Omega} u$ belongs to $[H_0^{2-s}(\Omega, \partial_D\Omega)]^*$ and is an extension of the functional $Au \in H^{s-2}(\Omega)$ from the domain of definition $\widetilde{H}^{2-s}(\Omega) = H_0^{2-s}(\Omega) \subset H_0^{2-s}(\Omega, \partial_D\Omega)$ to the domain of definition $H_0^{2-s}(\Omega, \partial_D\Omega)$, and a restriction of the functional $\check{A}u \in \widetilde{H}^{s-2}(\Omega)$ from the domain of definition $H^{2-s}(\Omega) \supset H_0^{2-s}(\Omega, \partial_D\Omega)$ to the domain of definition $H_0^{2-s}(\Omega, \partial_D\Omega)$.

Note that one can take $v = \bar{w}$ to make the settings (5.14)–(5.18) in terms of the sesquilinear inner product and look more like the usual variational formulations; cf. e.g. [9].

The Dirichlet problem setting (5.14)–(5.15) coincides with the usual one, c.f. [10], (i.e., does not need a generalization), and the co-normal derivative does not evidently participate in it. The Neumann and mixed problems are formulated in terms of the aggregate right hand sides $\check{f}$ and $\check{f}_m$, respectively, prescribed on their own, i.e., without necessary splitting them into the given right hand side of the PDE inside the domain Ω and the part related with the co-normal derivative prescribed on the boundary. If, however, a PDE right hand side extension $\tilde{f}$ and an associated non-zero generalized co-normal derivative $T^+(\tilde{f}, u) = \tau$ are prescribed instead, then $\check{f}$ can be expressed through it by relation (5.9) and $\check{f}_m$ by relation

$$\langle \check{f}_m, v\rangle_\Omega = \langle \tilde{f}, v\rangle_\Omega + \left\langle \tau, \gamma^+ v\right\rangle_{\partial_N\Omega} = \langle \tilde{f} + \gamma^{+*}\tau, v\rangle_\Omega \quad \forall v \in H_0^{2-s}(\Omega, \partial_D\Omega),$$

also obtained from (5.9), where it is taken into account that the trace $\gamma^+ v$ belongs to $\widetilde{H}^{s-\frac{1}{2}}(\partial_N\Omega)$ for $v \in H_0^{2-s}(\Omega, \partial_D\Omega)$ and $\gamma^{+*} : H^{s-\frac{3}{2}}(\partial_N\Omega) \to [H_0^{2-s}(\Omega, \partial_D\Omega)]^*$ is a continuous operator adjoint to the operator γ^+.

Thus the co-normal derivative does not enter, in fact, the generalized weak settings of the Dirichlet, Neumann or mixed problem, implying that the non-uniqueness of $T^+(\tilde{f}, u)$ for a given function $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, does not influence the BVP weak settings, (cf. [2, Section 2.2, item 4] for $s = 1$). On the other hand, for a given $u \in H^s(\Omega)$ the aggregate right hand sides $\check{f}$ and $\check{f}_m$ are uniquely determined by u from (5.16) and (5.17), as are, of course, f and φ_0 by (5.14) and (5.15)/ (5.18).

Remark also that the formulation of the Neumann and mixed BVPs in terms of the aggregate right hand side can be also illustrated by a physical interpretation. For the Neumann problem, for example, if A is a partial differential operator of the Lamé system of linear elasticity in a body $\Omega \subset \mathbb{R}^3$ for the displacement vector $u \in H^1(\Omega)$, then $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ is the distributed volume force vector acting on the body and $T^+(\tilde{f}, u) = \tau \in H^{-\frac{1}{2}}(\partial\Omega)$ is the prescribed traction vector on the boundary. Then $\tau \in H^{-\frac{1}{2}}(\partial\Omega)$ from the mechanical point of view is indistinguishable from the corresponding volume force $\gamma^{+*}\tau \in \tilde{H}^{-1}(\Omega)$ concentrated on the boundary surface and thus can be summed up with $\tilde{f}$ to produce the aggregate right hand side $\check{f}$.

6. Canonical co-normal derivatives for PDE systems with non-smooth coefficients

6.1. Canonical operator extension and the co-normal derivative

As we have seen above, for an arbitrary $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, the co-normal derivative $T^+(\tilde{f}, u)$ is generally non-uniquely determined by u. An exception is $T^+(\check{A}u, u) \equiv 0$, which was in fact implemented in the generalized weak setting of the boundary value problems in Section 5.3. But such zero co-normal derivative evidently differs from the strong co-normal derivative T_c^+u, given by (5.7) for sufficiently smooth u. Another way of making the generalized co-normal derivative unique for $u \in H^1(\Omega)$ was presented in [7, Lemma 5.1.1] and is in fact associated with an extension of $Au \in H^{-1}(\Omega)$ to $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, such that $\tilde{f}$ is orthogonal in $H^{-1}(\mathbb{R}^n)$ to $H^{-1}_{\partial\Omega} \subset H^{-1}(\mathbb{R}^n)$. However it appears (see [14, Lemma A.1]), that even for infinitely smooth functions f such extension $\tilde{f}$ does not generally belong to $L_2(\mathbb{R}^n)$, which implies that the so-defined co-normal derivative operator from [7, Lemma 5.1.1] is not a bounded extension of the strong co-normal derivative operator.

Nevertheless, we can point out some subspaces of $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, where a unique definition of the co-normal derivative by u is still possible and leads to the strong co-normal derivative for sufficiently smooth u. Following [14], we define below one such sufficiently wide subspace.

Definition 6.1. Let $s \in \mathbb{R}$ and $A_* : H^s(\Omega) \to \mathcal{D}^*(\Omega)$ be a linear operator. For $t \geq -\frac{1}{2}$, we introduce a space $H^{s,t}(\Omega; A_*) := \{g : g \in H^s(\Omega), A_*g|_\Omega = \tilde{f}_g|_\Omega, \tilde{f}_g \in \widetilde{H}^t(\Omega)\}$ equipped with the graphic norm, $\|g\|^2_{H^{s,t}(\Omega;A_*)} := \|g\|^2_{H^s(\Omega)} + \|\tilde{f}_g\|^2_{\widetilde{H}^t(\Omega)}$.

If $s_1 \leq s_2$ and $t_1 \leq t_2$, then we have the embedding, $H^{s_2,t_2}(\Omega; A_*) \subset H^{s_1,t_1}(\Omega; A_*)$. Some other properties of the space $H^{s,t}(\Omega; A_*)$ studied in [14, Section 3.2] are provided in Appendix B.

We will further use the space $H^{s,t}(\Omega; A_*)$ for the case when the operator A_* is the operator A from (3.3) or the formally adjoint operator A^* from (3.7).

Definition 6.2. Let $s \in \mathbb{R}$, $t \geq -\frac{1}{2}$. The operator $\tilde{A}$ mapping functions $u \in H^{s,t}(\Omega; A)$ to the extension of the distribution $Au \in H^t(\Omega)$ to $\widetilde{H}^t(\Omega)$ will be called *the canonical extension* of the operator A.

Remark 6.3. If $s \in \mathbb{R}, t \geq -\frac{1}{2}$, then $\|\tilde{A}u\|_{\widetilde{H}^t(\Omega)} \leq \|u\|_{H^{s,t}(\Omega;A)}$ by the definition of the space $H^{s,t}(\Omega; A)$, i.e., the linear operator $\tilde{A} : H^{s,t}(\Omega; A) \to \widetilde{H}^t(\Omega)$ is continuous. Moreover, if $-\frac{1}{2} < t < \frac{1}{2}$, then by Theorem B.3 and uniqueness of the extension of $H^t(\Omega)$ to $\widetilde{H}^t(\Omega)$, we have the representation $\tilde{A} := \widetilde{E}^t A$.

Remark 6.4. Note that in the case of non-smooth coefficients of the operator A, the inclusion $u \in H^s(\Omega)$, $s > 3/2$, does not generally imply that $u \in H^{s,t}(\Omega; A)$ for some $t \geq -\frac{1}{2}$, unlike the case of infinitely (or at least sufficiently) smooth coefficients. Particularly, even $u \in \mathcal{D}(\bar{\Omega})$ does not generally belong to $H^{1,-\frac{1}{2}}(\Omega; A)$ unless $a \in \bar{C}^\mu(\Omega)$ for some $\mu > 1/2$ (and $b, c \in L_\infty(\Omega)$) by Theorem 3.2, i.e., the usual assumption $a \in L_\infty(\Omega)$ is not generally sufficient for this.

As in [13, Definition 3] for scalar PDEs, let us define the *canonical* co-normal derivative operator. This extends [6, Theorem 1.5.3.10] and [5, Lemma 3.2] where co-normal derivative operators acting on functions from $H^{1,0}_p(\Omega;\Delta)$ and $H^{1,0}(\Omega;A)$, respectively, were defined.

Definition 6.5. For $u\in H^{s,-\frac12}(\Omega;A)$, $\{a,b,c\}\in\mathcal{C}^{s-1}_+(\overline{\Omega})$, $\frac12<s<\frac32$, we define the *canonical co-normal derivative* as $T^+u:=T^+(\tilde{A}u,u)\in H^{s-\frac32}(\partial\Omega)$, i.e.,

$$\left\langle T^+u,w\right\rangle_{\partial\Omega}:=\check{\mathcal{E}}(u,\gamma_{-1}w)-\langle\tilde{A}u,\gamma_{-1}w\rangle_\Omega=\langle\check{A}u-\tilde{A}u,\gamma_{-1}w\rangle_\Omega\quad\forall w\in H^{\frac32-s}(\partial\Omega),$$

where $\gamma_{-1}:H^{s-\frac12}(\partial\Omega)\to H^s(\Omega)$ is a bounded right inverse to the trace operator.

Thus, unlike the generalized co-normal derivative, the canonical co-normal derivative is uniquely defined by the function u and the operator A only, uniquely fixing an extension of the latter on the boundary, and is linear in u.

Theorem 5.3 for the generalized co-normal derivative and Definition 6.1 imply the following assertion.

Theorem 6.6. *Under hypotheses of Definition 6.5, the canonical co-normal derivative T^+u is independent of the operator γ_{-1}, the operator $T^+:H^{s,-\frac12}(\Omega;A)\to H^{s-\frac32}(\partial\Omega)$ is continuous, and the first Green's identity holds in the following form,*

$$\left\langle T^+u,\gamma^+v\right\rangle_{\partial\Omega}=\left\langle T^+(\tilde{A}u,u),\gamma^+v\right\rangle_{\partial\Omega}=\check{\mathcal{E}}(u,v)-\langle\tilde{A}u,v\rangle_\Omega=\langle\check{A}u-\tilde{A}u,v\rangle_\Omega\quad\forall v\in H^{2-s}(\Omega).$$

Definitions 5.2 and 6.5 imply that the generalized co-normal derivative of $u\in H^{s,-\frac12}(\Omega;A)$, $\frac12<s<\frac32$, for any other extension $\tilde{f}\in\widetilde{H}^{s-2}(\Omega)$ of the distribution $Au|_\Omega\in H^{-\frac12}(\Omega)$ can be expressed as

$$\left\langle T^+(\tilde{f},u),w\right\rangle_{\partial\Omega}=\left\langle T^+u,w\right\rangle_{\partial\Omega}+\langle\tilde{A}u-\tilde{f},\gamma_{-1}w\rangle_\Omega\quad\forall w\in H^{\frac32-s}(\partial\Omega).$$

Note that the distributions $\check{A}u-\tilde{f}$, $\check{A}u-\tilde{A}u$ and $\tilde{A}-\tilde{f}$ belong to $H^{2-s}_{\partial\Omega}$ since $\tilde{A}u$, $\check{A}u$, $\tilde{f}$ belong to $\widetilde{H}^{2-s}(\Omega)$, while $\tilde{A}u|_\Omega=\check{A}u|_\Omega=\tilde{f}|_\Omega=Au|_\Omega\in H^{s-2}(\Omega)$.

Since by Theorem 6.6 the canonical co-normal derivative does not depend on the extension operator γ_{-1}, the latter can be always chosen such that $\gamma_{-1}w$ has a support only near the boundary, which means that the co-normal derivative T^+u is determined by the behaviour of u near the boundary. We can formalize this in the following statement.

Theorem 6.7. *Let Ω and $\Omega'\subset\Omega$ be interior or exterior open Lipschitz domains, $\partial\Omega\subset\partial\Omega'$, $u\in H^{s,-\frac12}(\Omega;A)$, $u\in H^{s,-\frac12}(\Omega';A)$, $\{a,b,c\}\in\mathcal{C}^{s-1}_+(\overline{\Omega})$, $\frac12<s<\frac32$, while T^+u and T'^+u be the canonical co-normal derivatives on $\partial\Omega$ and $\partial\Omega'$ respectively. Then $T^+u=r_{\partial\Omega}T'^+u$.*

Proof. The proof is word-for-word the proof of the counterpart for infinitely smooth coefficients; see Theorem 3.10 in [14]. □

Theorem 6.7 can be considered as an alternative definition of the canonical co-normal derivative on $\partial\Omega$, where the domain Ω' can be chosen arbitrarily small, and particularly can be taken interior when Ω is exterior (with compact boundary). Note that a similar reasoning holds also for the generalized co-normal derivative.

If $\frac12<s<\frac32$, $\{a,b,c\}\in\mathcal{C}^{s-1}_+(\overline{\Omega})$ and $v\in H^{2-s,-\frac12}(\Omega;A^*)$, then similar to Definitions 6.2 and 6.5 we can introduce the *canonical extension* $\widetilde{A}^*$ of the operator A^*, and the *canonical modified co-normal derivative* $T^+_*v:=T^+_*(\widetilde{A}^*v,v)\in H^{\frac12-s}(\partial\Omega)$, i.e.,

$$\left\langle T^+_*v,w\right\rangle_{\partial\Omega}:=\check{\mathcal{E}}^*(v,\gamma_{-1}w)-\langle\widetilde{A}^*v,\gamma_{-1}w\rangle_\Omega\quad\forall w\in H^{s-\frac12}(\partial\Omega).$$

Then the first Green's identity (5.12) becomes,

$$\left\langle\gamma^+u,\overline{T^+_*v}\right\rangle_{\partial\Omega}=\check{\mathcal{E}}(u,\bar{v})-\langle u,\overline{\widetilde{A}^*v}\rangle_\Omega\quad\forall u\in H^s(\Omega).$$

For $v\in H^{2-s,-\frac12}(\Omega;A^*)$ and $u\in H^s(\Omega)$, $Au=\tilde{f}|_\Omega$ in Ω, where $\tilde{f}\in\widetilde{H}^{s-2}(\Omega)$, the second Green's identity (5.13) takes form,

$$\langle\tilde{f},\bar{v}\rangle_\Omega-\left\langle u,\overline{\widetilde{A}^*v}\right\rangle_\Omega=\left\langle\gamma^+u,\overline{T^+_*v}\right\rangle_{\partial\Omega}-\left\langle T^+(\tilde{f},u),\overline{\gamma^+v}\right\rangle_{\partial\Omega}.\tag{6.1}$$

This form was a starting point in formulation and analysis of the extended boundary-domain integral equations in [11].

If, moreover, $u\in H^{s,-\frac12}(\Omega;A)$, we obtain from (6.1) the second Green's identity for the canonical extensions and canonical co-normal derivatives,

$$\left\langle\tilde{A}u,\bar{v}\right\rangle_\Omega-\left\langle u,\overline{\widetilde{A}^*v}\right\rangle_\Omega=\left\langle\gamma^+u,\overline{T^+_*v}\right\rangle_{\partial\Omega}-\left\langle T^+u,\overline{\gamma^+v}\right\rangle_{\partial\Omega}.\tag{6.2}$$

Particularly, if $u\in H^{1,0}(\Omega;A)$, $v\in H^{1,0}(\Omega;A^*)$, with $a,b,c\in L_\infty(\Omega)$, then (6.2) takes the familiar form, cf. [5, Lemma 3.4],

$$\int_\Omega[\overline{v(x)}Au(x)-u(x)\overline{A^*v(x)}]dx=\left\langle\gamma^+u,\overline{T^+_*v}\right\rangle_{\partial\Omega}-\left\langle T^+u,\overline{\gamma^+v}\right\rangle_{\partial\Omega}.$$

6.2. Classical versus canonical co-normal derivatives

In this section we generalize to the case when the PDE coefficients are not infinitely smooth, the results of [14] on conditions when the canonical co-normal derivative T^+u coincides with the strong co-normal derivative T_c^+u, if the latter does exist in the trace sense. To do this, we will need higher smoothness of the coefficients than necessary for continuity of the PDEs in Theorems 3.4 and 5.1. First of all, we make the following observation; c.f. Remark 6.4.

Remark 6.8. Theorem 3.2 and Definition 3.3 imply that if $\{a, b, c\} \in \mathcal{C}_+^{t+1}(\overline{\Omega})$, $t \geq -\frac{1}{2}$, then $\mathcal{D}(\overline{\Omega}) \subset H^{s,t}(\Omega; A)$ (and moreover, $\mathcal{D}(\overline{\Omega}) \subset H^{s,t+\epsilon}(\Omega; A)$ for some $\epsilon \in \mathbb{R}_+(t)$) for any $s \in \mathbb{R}$.

Now we are in the position to generalize the density theorem from [14, Theorem 3.12] to non-smooth coefficients and exterior domains.

Theorem 6.9. *Let Ω be an interior or exterior Lipschitz domain and $s \in \mathbb{R}$, $-\frac{1}{2} \leq t < \frac{1}{2}$. Let $\{a, b, c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega}) \bigcap \mathcal{C}_+^{t+1}(\overline{\Omega})$, the operator A be elliptic (in the sense of Petrovsky) on $\overline{\Omega}$ and, if Ω is exterior, there exists a finite $a(\infty) := \lim_{x\to\infty} a(x)$, which also satisfies the ellipticity condition. Then $\mathcal{D}(\overline{\Omega})$ is dense in $H^{s,t}(\Omega; A)$.*

Proof. We adopt here for the non-smooth coefficients and exterior domains the proof from [14, Theorem 3.12].

For every continuous linear functional l on $H^{s,t}(\Omega; A)$ there exist distributions $\tilde{h} \in \widetilde{H}^{-s}(\Omega)$ and $g \in H^{-t}(\Omega)$ such that $l(u) = \langle \tilde{h}, u\rangle_\Omega + \langle g, \tilde{A}u\rangle_\Omega\ \forall u \in H^{s,t}(\Omega; A)$.

Remark 6.8 and the theorem hypothesis on the coefficients imply that $\mathcal{D}(\overline{\Omega}) \subset H^{s,t}(\Omega; A)$. To prove the lemma claim, it suffices to show that any l, which vanishes on $\mathcal{D}(\overline{\Omega})$, will vanish on any $u \in H^{s,t}(\Omega; A)$.

If $l(\phi) = 0$ for any $\phi \in \mathcal{D}(\overline{\Omega})$, then

$$\langle \tilde{h}, \phi\rangle_\Omega + \langle g, \tilde{A}\phi\rangle_\Omega = 0. \tag{6.3}$$

Let us consider the case $-\frac{1}{2} < t < \frac{1}{2}$ first and extend g outside Ω to $\tilde{g} = \widetilde{E}^{-t}g \in \widetilde{H}^{-t}(\Omega)$; cf. Theorem B.3. Let $\Omega' \supset \overline{\Omega}$ be some domain, where the operator A is still elliptic. Such domain exists since the coefficients $a(x)$ are continuous and the ellipticity condition holds in the closed domain $\overline{\Omega}$. Then Eq. (6.3) gives

$$\begin{aligned}\langle \tilde{h}, \phi\rangle_{\Omega'} + \langle \tilde{g}, A\phi\rangle_{\Omega'} &= \langle \tilde{h}, \phi\rangle_\Omega + \langle \tilde{g}, A\phi\rangle_\Omega = \langle \tilde{h}, \phi\rangle_\Omega + \langle \widetilde{E}^{-t}g, A\phi\rangle_\Omega \\ &= \langle \tilde{h}, \phi\rangle_\Omega + \langle g, \widetilde{E}^{t}A\phi\rangle_\Omega = \langle \tilde{h}, \phi\rangle_\Omega + \langle g, \tilde{A}\phi\rangle_\Omega = 0\end{aligned} \tag{6.4}$$

for any $\phi \in \mathcal{D}(\Omega')$. This means

$$A^*\tilde{g} = -\tilde{h} \quad \text{in } \Omega' \tag{6.5}$$

in the sense of distributions, where A^* is the operator formally adjoint to A. If $t \leq s-2$, then evidently $\tilde{g} \in \widetilde{H}^{2-s}(\Omega)$. If $t > s-2$, then (6.5) and the local regularity Theorem 4.4(b) imply $\tilde{g} \in H^{2-s}(\Omega'')$ for any Ω'' such that $\Omega \Subset \Omega'' \Subset \Omega'$ and consequently $\tilde{g} \in \widetilde{H}^{2-s}(\Omega)$.

In the case $t = -\frac{1}{2}$, one can extend $g \in H^{\frac{1}{2}}(\Omega)$ outside $\overline{\Omega}$ by zero to $\tilde{g} \in \widetilde{H}^{\frac{1}{2}-\epsilon}(\Omega)$, $0 < \epsilon$, and prove as in the previous paragraph that $\tilde{g} \in \widetilde{H}^{2-s}(\Omega)$.

If $-\frac{1}{2} < t < \frac{1}{2}$ or $[t = -\frac{1}{2}, s \leq \frac{3}{2}]$ then for any $u \in H^{s,t}(\Omega; A)$, we have,

$$l(u) = \langle -A^*\tilde{g}, u\rangle_\Omega + \langle g, \tilde{A}u\rangle_\Omega = -\langle \tilde{g}, Au\rangle_\Omega + \langle \tilde{g}, Au\rangle_\Omega = 0.$$

Thus l is identically zero.

On the other hand, if $t = -\frac{1}{2}$, $s > \frac{3}{2}$, let $\{\tilde{g}_k\} \in \mathcal{D}(\Omega)$ be a sequence converging, as $k \to \infty$, to g in $H_0^{\frac{1}{2}}(\Omega) = H^{\frac{1}{2}}(\Omega)$, cf. Theorem B.2, and thus to $\tilde{g}$ in $\widetilde{H}^{2-s}(\Omega)$. Then for any $u \in H^{s,\frac{1}{2}}(\Omega; A)$, we have,

$$l(u) = \langle -A^*\tilde{g}, u\rangle_\Omega + \langle g, \tilde{A}u\rangle_\Omega = \lim_{k\to\infty}\left\{\langle -A^*\tilde{g}_k, u\rangle_\Omega + \langle \tilde{g}_k, \tilde{A}u\rangle_\Omega\right\} = \lim_{k\to\infty}\left\{-\langle \tilde{g}_k, Au\rangle_\Omega + \langle \tilde{g}_k, Au\rangle_\Omega\right\} = 0,$$

which completes the proof. □

Let us prove an analogue of Lemma 3.13 from [14].

Lemma 6.10. *Let Ω be a Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, $\{a, b, c\} \in \mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega})$, $u \in H^{s,-\frac{1}{2}}(\Omega; A)$, and $\{u_k\} \in \mathcal{D}(\overline{\Omega})$ be a sequence such that*

$$\|u_k - u\|_{H^{s,-\frac{1}{2}}(\Omega;A)} \to 0 \quad \text{as } k \to \infty. \tag{6.6}$$

Then $\|T_c^+u_k - T^+u\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \to 0$ as $k \to \infty$.

Proof. By the lemma hypothesis on the coefficients, there exists $\epsilon \in (0, s-\frac{1}{2})$ such that $\sum_{j=1}^n a_{ij}\partial_j u_k \in H^{\frac{1}{2}+\epsilon}(\Omega)$. Then for any $\gamma_{-1}w \in H^{2-s}(\Omega)$ and $u_k \in \mathcal{D}(\overline{\Omega})$ there exist sequences $\{W_p\}_{p=1}^\infty, \{U_{qi}\}_{q=1}^\infty \in \mathcal{D}(\overline{\Omega})$ such that

$$\lim_{p\to\infty} \|\gamma_{-1}w - W_p\|_{H^{2-s}(\Omega)} = 0, \qquad \lim_{q\to\infty} \left\| \sum_{j=1}^n a_{ij}\partial_j u_k - U_{qi} \right\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} = 0$$

and we have

$$\begin{aligned}
&\check{\mathcal{E}}(u_k, \gamma_{-1}w) - \sum_{j=1}^n \left\langle \widetilde{E}^{s_b(s)}(b_j\partial_j u_k), \gamma_{-1}w\right\rangle_\Omega - \left\langle \widetilde{E}^{s_c(s)}(cu_k), \gamma_{-1}w\right\rangle = \sum_{i,j=1}^n \left\langle \widetilde{E}^{s-1}(a_{ij}\partial_j u_k), \partial_i \gamma_{-1}w\right\rangle_\Omega \\
&= \lim_{p,q\to\infty} \sum_{i=1}^n \left\langle \widetilde{E}^{s-1}U_{qi}, \partial_i W_p\right\rangle_\Omega = \lim_{p,q\to\infty} \sum_{i=1}^n \left\{ \int_{\partial\Omega} U_{qi}\nu_i W_p\, d\Gamma - \int_\Omega (\partial_i U_{qi}) W_p\, d\Omega \right\} \\
&= \sum_{i,j=1}^n \left\{ \int_{\partial\Omega} (a_{ij}\partial_j u_k)\nu_i w\, d\Gamma - \left\langle \widetilde{E}^{-\frac{1}{2}+\epsilon}\partial_i(a_{ij}\partial_j u_k), \gamma_{-1}w\right\rangle_\Omega \right\} \\
&= \left\langle T_c^+ u_k, w\right\rangle_{\partial\Omega} + \langle \tilde{A}u_k, \gamma_{-1}w\rangle_\Omega - \sum_{j=1}^n \left\langle \widetilde{E}^{s_b(s)}(b_j\partial_j u_k), \gamma_{-1}w\right\rangle_\Omega - \left\langle \widetilde{E}^{s_c(s)}(cu_k), \gamma_{-1}w\right\rangle,
\end{aligned}$$

that is, the first Green's identity holds for the classical co-normal derivative, $\mathcal{E}(u_k, \gamma_{-1}w) = \left\langle T_c^+ u_k, w\right\rangle_{\partial\Omega} + \langle \tilde{A}u_k, \gamma_{-1}w\rangle_\Omega$. Thus we have for any $w \in H^{\frac{3}{2}-s}(\partial\Omega)$,

$$\left|\left\langle T^+u - T_c^+ u_k, w\right\rangle_{\partial\Omega}\right| = \left| \check{\mathcal{E}}(u-u_k, \gamma_{-1}w) - \langle \tilde{A}(u-u_k), \gamma_{-1}w\rangle_\Omega \right| \le C\|u-u_k\|_{H^{s,-\frac{1}{2}}(\Omega;A)} \|w\|_{H^{\frac{3}{2}-s}(\partial\Omega)},$$

which implies $\|T_c^+ u_k - T^+u\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \le C\|u-u_k\|_{H^{s,-\frac{1}{2}}(\Omega;A)} \to 0$ as $k\to\infty$. □

Note that a sequence satisfying (6.6) does always exist for Lipschitz domains by Theorem 6.9 since Definition 3.3 implies $\mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega}) \subset \mathcal{C}_+^{s-1}(\overline{\Omega})$ if $\frac{1}{2} < s < \frac{3}{2}$.

The following statement gives the equivalence of the classical co-normal derivative (in the trace sense) and the canonical co-normal derivative, for functions from $H^s(\Omega)$, $s > \frac{3}{2}$.

Corollary 6.11. *If Ω is an interior or exterior Lipschitz domain, $\{a, b, c\} \in \mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega})$ and $u \in H^s(\Omega)$, $s > \frac{3}{2}$, then $T^+u = T_c^+u$.*

Proof. The proof coincides with the proof of [14, Corollary 3.14] if we remark that $\gamma^+[\partial_j u] \in H^{s-\frac{3}{2}}(\partial\Omega)$ for $\frac{3}{2} < s < \frac{5}{2}$ and the condition $\{a, b, c\} \in \mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega})$ implies that $T_c^+u \in L_2(\partial\Omega)$ and $u \in H^{s,-\frac{1}{2}}(\Omega; A) \subset H^{1,-\frac{1}{2}}(\Omega; A)$. □

Similar to [16, p. 85] we introduce the following definition.

Definition 6.12. Let Ω_k, Ω be Lipschitz domains. We say that $\Omega_k \to \Omega$ as $k \to \infty$ if $\partial\Omega_k$ are represented using the same system of covering charts ω_j as $\partial\Omega$ for all sufficiently large k, and $\lim_{k\to\infty} |\zeta_{jk} - \zeta_j|_{C^{0,1}(\bar{\omega}_j)} = 0$, where ζ_{jk} and ζ_j are the corresponding Lipschitz functions for the boundary representation.

Lemma 6.13. *Let Ω and $\Omega_k \Subset \Omega$ be Lipschitz domains such that $\Omega_k \to \Omega$ as $k \to \infty$ (cf. Definition 6.12). If $u \in H^{s,t}(\Omega; A)$ for some $s \in (\frac{1}{2}, \frac{3}{2})$ and $t \in (-\frac{1}{2}, \frac{1}{2})$ and $\{a, b, c,\} \in \mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega})$, then $\langle T^+u, \gamma^+v\rangle_{\partial\Omega} = \lim_{k\to\infty}\langle T_c^+u, \gamma^+v\rangle_{\partial\Omega_k}$ for any $v \in H^{2-s}(\Omega)$.*

Proof. By Theorem 6.7 it suffices to consider only an interior domain Ω. Let $\Omega_k' := \Omega \setminus \overline{\Omega_k}$ be the layer between $\partial\Omega$ and $\partial\Omega_k$. By the solution regularity Theorem 4.3, $u \in H^{t'+2}(\Omega_k)$ for some $t' > -\frac{1}{2}$. On $\partial\Omega_k$ then $T^+u = T_c^+u \in L_2(\partial\Omega_k)$ by Corollary 6.11. Then

$$\begin{aligned}
\langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T_c^+u, \gamma^+v\rangle_{\partial\Omega_k} &= \langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T^+u, \gamma^+v\rangle_{\partial\Omega_k} = \langle T^+u, \gamma^+v\rangle_{\partial\Omega_k'} \\
&= \check{\mathcal{E}}_{\Omega_k'}(u, v) - \langle \tilde{A}_{\Omega_k'}u, v\rangle_{\Omega_k'} = \check{\mathcal{E}}_{\Omega_k'}(u, v) - \langle Au, \tilde{v}_{\Omega_k'}\rangle_{\Omega_k'}, \qquad (6.7)
\end{aligned}$$

where $\tilde{A}_{\Omega_k'}u = \widetilde{E}^t_{\Omega_k'} r_{\Omega_k'}Au \in \widetilde{H}^t(\Omega_k')$ and $\tilde{v}_{\Omega_k'} = \widetilde{E}^{-t}_{\Omega_k'} r_{\Omega_k'}v \in \widetilde{H}^{-t}(\Omega_k')$ are the unique extensions of $r_{\Omega_k'}Au \in H^t(\Omega_k')$ and $r_{\Omega_k'}v \in H^{2-s}(\Omega_k') \subset H^{-t}(\Omega_k')$, respectively.

By (5.2) and Theorem B.3 we have for the first term on the right hand side of (6.7), for $\frac{1}{2} < s \le 1$ and any $\epsilon \in \mathbb{R}_+(s)$,

$$\begin{aligned} |\check{\mathcal{E}}_{\Omega'_k}(u,v)| &\le C\sum_{i,j=1}^n \|a_{ij}\|_{\bar{C}^{|s-1|+\epsilon}(\bar\Omega)}\|\partial_j u\|_{H^{s-1}(\Omega'_k)}\|\partial_i v\|_{H^{1-s}(\Omega)} \\ &\quad + C\sum_{j=1}^n \|b_j\|_{\bar{C}^{|s-1|+\epsilon}(\bar\Omega)}\|\partial_j u\|_{H^{s-1}(\Omega'_k)}\|v\|_{H^{1-s}(\Omega)} + C\|c\|_{L_\infty(\Omega)}\|u\|_{H^0(\Omega'_k)}\|v\|_{H^0(\Omega)} \\ &\le \{C_1\|\nabla u\|_{H^{s-1}(\Omega'_k)} + C_2\|\nabla u\|_{H^{s-1}(\Omega'_k)}\}\|v\|_{H^{2-s}(\Omega)} + C_3\|u\|_{H^0(\Omega'_k)}\|v\|_{H^0(\Omega)} \to 0, \quad k\to\infty \end{aligned} \tag{6.8}$$

by Lemma B.4 since the Lebesgue measure of Ω'_k tends to zero. For $1 < s < \frac{3}{2}$ similarly,

$$\begin{aligned} |\check{\mathcal{E}}_{\Omega'_k}(u,v)| &\le C\sum_{i,j=1}^n \|a_{ij}\|_{\bar{C}^{|s-1|+\epsilon}(\bar\Omega)}\|\partial_j u\|_{H^{s-1}(\Omega)}\|\partial_i v\|_{H^{1-s}(\Omega'_k)} \\ &\quad + C\sum_{j=1}^n \|b_j\|_{L_\infty(\Omega)}\|\partial_j u\|_{H^{s-1}(\Omega)}\|v\|_{H^{1-s}(\Omega'_k)} + C\|c\|_{L_\infty(\Omega)}\|u\|_{H^{s-1}(\Omega)}\|v\|_{H^0(\Omega'_k)} \\ &\le \{C_4\|\nabla v\|_{H^{1-s}(\Omega'_k)} + C_5\|v\|_{H^0(\Omega'_k)}\}\|u\|_{H^s(\Omega)} + C_6\|u\|_{H^{s-1}(\Omega)}\|v\|_{H^0(\Omega'_k)} \to 0, \quad k\to\infty. \end{aligned} \tag{6.9}$$

The norms of the coefficients a, b, c in (6.8) and (6.9) are bounded due to the lemma hypothesis.

For the last term in (6.7) we have by Lemmas B.4 and B.5,

$$|\langle Au, \tilde v_{\Omega'_k}\rangle_{\Omega'_k}| \le \|Au\|_{H^t(\Omega'_k)}\|\tilde v_{\Omega'_k}\|_{\widetilde{H}^{-t}(\Omega'_k)} \le C\|Au\|_{H^t(\Omega'_k)}\|v\|_{H^{-t}(\Omega)} \le C\|Au\|_{H^t(\Omega'_k)}\|v\|_{H^{2-s}(\Omega)} \to 0, \quad k\to\infty,$$

if $-\frac{1}{2} < t \le 0$. On the other hand, if $0 < t < \frac{1}{2}$, then again by Lemmas B.4 and B.5,

$$|\langle Au, \tilde v_{\Omega'_k}\rangle_{\Omega'_k}| = |\langle \tilde A_{\Omega'_k}u, v\rangle_{\Omega'_k}| \le \|\tilde A_{\Omega'_k}u\|_{\widetilde{H}^t(\Omega'_k)}\|v\|_{H^{-t}(\Omega'_k)} \le C\|Au\|_{H^t(\Omega)}\|v\|_{H^{-t}(\Omega'_k)} \to 0, \quad k\to\infty. \quad \square$$

Lemma 6.13 allows to show that the classical and canonical co-normal derivatives coincide also in another case (apart from the one in Corollary 6.11). First note that $C^1(\overline{\Omega}) \subset H^1(\Omega)$ for any interior domain Ω and $C^1(\overline{\Omega'}) \subset H^1(\Omega')$ for any interior subdomain Ω' of exterior domain Ω, but $C^1(\overline{\Omega})$ is not a subset of $H^{1,-\frac{1}{2}}(\Omega; A)$. For $u \in C^1(\overline{\Omega})$, evidently, $\lim_{k\to\infty}\langle T^+_c u, \gamma^+ v\rangle_{\partial\Omega_k} = \langle T^+_c u, \gamma^+ v\rangle_{\partial\Omega}$ for any $v \in H^{2-s}(\Omega^+)$ if $\Omega_k \to \Omega$ as $k \to \infty$, $\overline{\Omega}_k \subset \Omega$. Then Lemma 6.13 immediately implies the following assertion.

Corollary 6.14. *If Ω is a Lipschitz domain, $\{a, b, c\} \in \mathcal{C}^{\frac{1}{2}}_+(\overline{\Omega})$ and $u \in C^1(\overline{\Omega}) \bigcap H^{1,t}_{\mathrm{loc}}(\overline{\Omega}; A)$ for some $t \in (-\frac{1}{2}, \frac{1}{2})$, then $T^+u = T^+_c u$.*

Acknowledgment

This research was supported by the grant EP/H020497/1: "Mathematical Analysis of Localized Boundary-Domain Integral Equations for Variable-Coefficient Boundary Value Problems" from the EPSRC, UK.

Appendix A. Some estimates

We will prove here some estimates used in Step (i) of the proof of Theorem 4.4. Let $0 \le \mu \le 1$ and

$$|a^-_\infty|_{C^\mu(B_\rho)} := \sup_{\substack{x',x''\in B_\rho\\ x'\neq x''}} \frac{|a^-_\infty(x'') - a^-_\infty(x')|}{|x''-x'|^\mu}.$$

Let $x', x'' \in B_\rho$ and $|x''| \ge |x'|$ for definiteness. Then

$$\frac{|a^-_\infty(x'') - a^-_\infty(x')|}{|x''-x'|^\mu} = \frac{\left|\frac{|x''|}{\rho}a^-\left(\frac{x''\rho}{|x''|}\right) - \frac{|x'|}{\rho}a^-\left(\frac{x'\rho}{|x'|}\right)\right|}{|x''-x'|^\mu} \le A + B,$$

where

$$A := \frac{|x''|}{\rho}\frac{\left|a^-\left(\frac{x''\rho}{|x''|}\right) - a^-\left(\frac{x'\rho}{|x'|}\right)\right|}{|x''-x'|^\mu}, \qquad B := \frac{\big||x''|-|x'|\big|}{\rho|x''-x'|^\mu}\left|a^-\left(\frac{x'\rho}{|x'|}\right)\right|.$$

The term A can be expressed as

$$A=\left(\frac{|x''|}{\rho}\right)^{1-\mu}\frac{\left|a^-\left(\frac{x''\rho}{|x''|}\right)-a^-\left(\frac{x'\rho}{|x'|}\right)\right|}{|\tilde{\Delta}|^{\mu}},\quad \tilde{\Delta}:=\frac{x''\rho}{|x''|}-\frac{x'\rho}{|x'|}\frac{|x'|}{|x''|}.$$

Let $\Delta:=\frac{x''\rho}{|x''|}-\frac{x'\rho}{|x'|}$. Then $|\tilde{\Delta}|\geq\rho\geq|\Delta|/2$ if $x'\cdot x''\leq 0$, while $|\tilde{\Delta}|\geq|\Delta|\,|\sin(\widehat{x',\Delta})|\geq|\Delta|\sin(\widehat{\pi/4})=|\Delta|/\sqrt{2}$ if $x'\cdot x''>0$. Thus in both cases,

$$A\leq 2\left(\frac{|x''|}{\rho}\right)^{1-\mu}\frac{\left|a^-\left(\frac{x''\rho}{|x''|}\right)-a^-\left(\frac{x'\rho}{|x'|}\right)\right|}{|\Delta|^{\mu}}\leq 2|a^-|_{C^{\mu}(\partial B_\rho)}\leq 2|a^-|_{C^{\mu}(\mathbb{R}^n\setminus B_\rho)}\quad\text{if }\mu\in[0,1].$$

On the other hand,

$$B\leq\frac{\left(|x''|-|x'|\right)^{1-\mu}}{\rho}\|a^-\|_{C(\partial B_\rho)}\leq\|a^-\|_{C(\mathbb{R}^n\setminus B_\rho)}$$

for $\mu\in[0,1]$ and $\rho\geq 1$. This implies $\|a^-_\infty\|_{C^{\mu}(B_\rho)}\leq 2\|a^-\|_{C^{\mu}(\mathbb{R}^n\setminus B_\rho)}$ and considering also the case $x'\in\mathbb{R}^n\setminus B_\rho$, $x''\in B_\rho$ and the case $x',x''\in\mathbb{R}^n\setminus B_\rho$, we arrive at the desired estimate $\|a^-_\infty\|_{C^{\mu}(\mathbb{R}^n)}\leq 3\|a^-\|_{C^{\mu}(\mathbb{R}^n\setminus B_\rho)}$.

If $a^-\in C^{\mu_1}(\mathbb{R}^n)$ for some μ_1 such that $0\leq\mu<\mu_1\leq 1$, then

$$\begin{aligned}\frac{1}{3}\|a^-_\infty\|_{C^{\mu}(\mathbb{R}^n)}&\leq\|a^-\|_{C^{\mu}(\mathbb{R}^n\setminus B_\rho)}\leq\|a^-\|_{C(\mathbb{R}^n\setminus B_\rho)}+|a^-|_{C^{\mu}(\mathbb{R}^n\setminus B_\rho)}\\&=\|a^-\|_{C(\mathbb{R}^n\setminus B_\rho)}+\sup_{\substack{|x'-x''|\leq r,x'\neq x''\\x',x''\in\mathbb{R}^n\setminus B_\rho}}\frac{|a^-(x'')-a^-(x')|}{|x''-x'|^{\mu}}+\sup_{\substack{|x'-x''|>r\\x',x''\in\mathbb{R}^n\setminus B_\rho}}\frac{|a^-(x'')-a^-(x')|}{|x''-x'|^{\mu}}\\&\leq\|a^-\|_{C(\mathbb{R}^n\setminus B_\rho)}+r^{\mu_1-\mu}\sup_{\substack{|x'-x''|\leq r,x'\neq x''\\x',x''\in\mathbb{R}^n\setminus B_\rho}}\frac{|a^-(x'')-a^-(x')|}{|x''-x'|^{\mu_1}}+2r^{-\mu}\sup_{x\in\mathbb{R}^n\setminus B_\rho}|a^-(x)|\\&\leq(1+2r^{-\mu})\|a^-\|_{C(\mathbb{R}^n\setminus B_\rho)}+r^{\mu_1-\mu}\|a^-\|_{C^{\mu_1}(\mathbb{R}^n)}.\end{aligned}$$

Thus for any $\varepsilon>0$ we can chose sufficiently small $r>0$ so that the last term on the right hand side is less than $\varepsilon/2$ and then chose ρ sufficiently large so that the first term on the right hand side is less than $\varepsilon/2$ since $a^-(x)\to 0$ as $x\to\infty$. This means $\|a^-_\infty\|_{C^{\mu}(\mathbb{R}^n)}\to 0$ as $\rho\to\infty$.

Appendix B. On Sobolev spaces characterization, traces and extensions

To make this paper more self-contained we provide here some assertions from [14] about Sobolev spaces characterization, traces and extensions.

Theorem B.1 (*[14, Theorem 2.10]*). *Let Ω be a Lipschitz domain in $\mathbb{R}^n$.*
(i) *If $t\geq-\frac{1}{2}$, then $H^t_{\partial\Omega}=\{0\}$.*
(ii) *If $-\frac{3}{2}<t<-\frac{1}{2}$, then $g\in H^t_{\partial\Omega}$ if and only if $g=\gamma^*v$, i.e., $\langle g,W\rangle_{\mathbb{R}^n}=\langle v,\gamma W\rangle_{\partial\Omega}\ \forall W\in H^{-t}(\mathbb{R}^n)$,*

with $v=\gamma^*_{-1}g\in H^{t+\frac{1}{2}}(\partial\Omega)$, i.e., $\langle v,w\rangle_{\partial\Omega}=\langle g,\gamma_{-1}w\rangle_{\mathbb{R}^n}\ \forall w\in H^{-t-\frac{1}{2}}(\partial\Omega)$, where v is independent of the choice of the non-unique operators γ_{-1}, γ^*_{-1}, and the estimate $\|v\|_{H^{t+\frac{1}{2}}(\partial\Omega)}\leq C\|g\|_{H^t(\mathbb{R}^n)}$ holds with C independent of t.

Theorem B.2 (*[14, Theorem 2.12]*). *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and $s\leq\frac{1}{2}$. Then $\mathcal{D}(\Omega)$ is dense in $H^s(\Omega)$, i.e., $H^s(\Omega)=H^s_0(\Omega)$.*

Theorem B.3 (*[14, Theorem 2.16]*). *Let Ω be a Lipschitz domain and $-\frac{3}{2}<s<\frac{1}{2}$, $s\neq-\frac{1}{2}$. There exists a bounded linear extension operator $\widetilde{E}^s:H^s(\Omega)\to\widetilde{H}^s(\Omega)$, such that $\widetilde{E}^s g|_\Omega=g$, $\forall g\in H^s(\Omega)$. For $-\frac{1}{2}<s<\frac{1}{2}$ the extension operator is unique, $(\widetilde{E}^s)^*=\widetilde{E}^{-s}$ and $\|\widetilde{E}^s g\|_{\widetilde{H}^s(\Omega)}\leq C\|g\|_{H^s(\Omega)}$, where C depends only on s and on the maximum of the Lipschitz constants of the representation functions ζ_j for the boundary $\partial\Omega$; see Definition 6.12.*

Lemma B.4 (*[14, Lemma 2.17]*). *Let Ω and $\Omega'\subset\Omega$ be open sets, and $s\leq 0$. If $u\in H^s(\Omega)$, then $\|u\|_{H^s(\Omega')}\to 0$ as the Lebesgue measure of Ω' tends to zero.*

Lemma B.5 (*[14, Lemma 2.18]*). *Let $\Omega_k\subset\Omega$ be a sequence of Lipschitz domains converging to a Lipschitz domain Ω and $-\frac{1}{2}<s<\frac{1}{2}$. If $u\in H^s(\Omega)$ and $\tilde{u}_k=\widetilde{E}^s u|_{\Omega_k}$, then there exists a constant C independent of u and k such that $\|\tilde{u}_k\|_{\widetilde{H}^s(\Omega_k)}\leq C\|u\|_{H^s(\Omega)}$ for all sufficiently large k.*

Remark B.6 (*[14, Remark 3.14]*)**.** If $s \in \mathbb{R}$, $-\frac{1}{2} < t < \frac{1}{2}$, and $A_* : H^s(\Omega) \to H^t(\Omega)$ is a linear continuous operator, then $H^{s,t}(\Omega; A_*) = H^s(\Omega)$ by Theorem B.3.

Lemma B.7 (*[14, Lemma 3.5]*)**.** *Let $s \in \mathbb{R}$. If a linear operator $A_* : H^s(\Omega) \to \mathcal{D}^*(\Omega)$ is continuous, then the space $H^{s,t}(\Omega; A_*)$ is complete for any $t \geq -\frac{1}{2}$.*

References

[1] S. Agmon, Lectures on Elliptic Boundary Value Problems, Van Nostrand, New York, 1965.
[2] M.S. Agranovich, Spectral problems for second-order strongly elliptic systems in smooth and non-smooth domains, Russian Math. Surv. 57 (5) (2003) 847–920.
[3] M.S. Agranovich, Sobolev Spaces, 2006. http://agranovich.nm.ru/NEZAV7.HTM.
[4] M.S. Agranovich, To the theory of the Dirichlet and Neumann problems for strongly elliptic systems in Lipschitz domains, Funct. Anal. Appl. 41 (4) (2007) 247–263.
[5] M. Costabel, Boundary integral operators on Lipschitz domains: elementary results, SIAM J. Math. Anal. 19 (1988) 613–626.
[6] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, Boston–London–Melbourne, 1985.
[7] G. Hsiao, W. Wendland, Boundary Integral Equations, Springer, Berlin–Heidelberg, 2008.
[8] M. Kohr, C. Pintea, W.L. Wendland, Brinkman-type operators on Riemannian manifolds: transmission problems in Lipschitz and C^1 domains, Potential Anal. 32 (2010) 229–273.
[9] J.-L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Vol. 1, Springer, Berlin–Heidelberg–New York, 1972.
[10] W. McLean, Strongly Elliptic Systems and Boundary Integral Equations, Cambridge University Press, Cambridge, UK, 2000.
[11] S.E. Mikhailov, Analysis of extended boundary-domain integral and integro-differential equations of some variable-coefficient BVP, in: K. Chen (Ed.), Advances in Boundary Integral Methods—Proceedings of the 5th UK Conference on Boundary Integral Methods, University of Liverpool Publ, Liverpool, UK, 2005, pp. 106–125.
[12] S.E. Mikhailov, Analysis of united boundary-domain integro-differential and integral equations for a mixed BVP with variable coefficient, Math. Methods Appl. Sci. 29 (2006) 715–739.
[13] S.E. Mikhailov, About traces, extensions and co-normal derivative operators on Lipschitz domains, in: C. Constanda, S. Potapenko (Eds.), Integral Methods in Science and Engineering: Techniques and Applications, Birkhäuser, Boston, 2008, pp. 149–160.
[14] S.E. Mikhailov, Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains, J. Math. Anal. Appl. 378 (2011) 324–342.
[15] C. Miranda, Partial Differential Equations of Elliptic Type, Springer, Berlin, 1970.
[16] J. Nečas, Les Méthodes Directes en Théorie des Équations Elliptique, Masson, Paris, 1967.
[17] L. Nirenberg, Remarks on strongly elliptic partial differential equations, Comm. Pure Appl. Math. 8 (1955) 649–675.
[18] G. Savaré, Regularity results for elliptic equations in Lipschitz domains, J. Functional Anal. 152 (1998) 176–201.
[19] L. Schwartz, Su alcuni problemi della teoria delle equazioni defferenziali lineari di tipo ellittico, Rendiconti del seminario Matematico e Fisico di Milano 27 (1958) 14–249.
[20] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, 1970.
[21] J.L. Zolezio, Multiplication dans les espaces de Besov, Proc. Royal Soc. Edinburgh 78A (1977) 113–117.