COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **20**, Number **3**, March **2021**

doi:10.3934/cpaa.2021009

pp. **1103–1133**

BOUNDARY-DOMAIN INTEGRAL EQUATIONS EQUIVALENT TO AN EXTERIOR MIXED BVP FOR THE VARIABLE-VISCOSITY COMPRESSIBLE STOKES PDES

Sergey E. Mikhailov*

Department of Mathematics, Brunel University London, UK

Carlos F. Portillo

Department of Quantitative Methods, Universidad Loyola Andalucía, Sevilla, Spain

(Communicated by Alain Miranville)

Abstract. Two direct systems of Boundary-Domain Integral Equations, BDIEs, associated with a mixed boundary value problem for the stationary compressible Stokes system with variable viscosity coefficient in an exterior domain of $\mathbb{R}^3$ are derived. This is done by employing the Stokes surface and volume potentials based on an appropriate parametrix (Levi function) in the third Green identities for the velocity and pressure. Mapping properties of the potentials in weighted Sobolev spaces are analysed. Finally, the equivalence between the BDIE systems and the BVP is shown and the isomorphism of operators defined by the BDIE systems is proved.

1. **Introduction.** In this paper we consider Boundary-Domain Integral Equations, BDIEs, for the stationary variable-viscosity Stokes system of partial differential equations (PDEs). The Stokes equations describe viscous fluid flows and approximate the Navier-Stokes system under the small Reynolds number. The Stokes equations model also incompressible linearly elastic materials with variable shear modulus but we will mainly use the terminology related to fluids. Here we will also allow for variable compressibility (for example, due to variation of the fluid temperature).

Boundary integral equations and the hydrodynamic potentials for the Stokes system with constant viscosity, have been extensively studied by numerous authors, see e.g., [19, 20, 14, 32, 33, 17, 36, 37]. This approach normally requires the fundamental solution to be available in an explicit form, especially if the boundary integral equations are then to be solved numerically. In the case of constant viscosity, fundamental solutions for both, velocity and pressure, are available.

2020 *Mathematics Subject Classification.* Primary: 31B10, 35Q30, 5J57, 45F15; Secondary: 76D07.

Key words and phrases. Partial differential equations, Stokes system, compressibility, variable viscosity coefficient, exterior domain, parametrix, Levi function, integral potentials, boundary-domain integral equations, weighted Sobolev spaces, mathematical analysis, equivalence, isomorphism.

This research was supported by the grants EP/H020497/1, EP/M013545/1, and 1636273 from the EPSRC, UK, and also by Brunel University London.

* Corresponding author.

However, explicit fundamental solutions are not available in the variable-coefficient case for which a *parametrix* (Levi function), see, e.g., [23, 5, 25, 26], can be used instead, in order to derive systems of Boundary Domain Integral Equations (BDIEs). Parametrix for a scalar PDE is not unique and neither is it in the case of a PDE system, particularly the Stokes system. Choosing the right parametrix is crucial in order to establish relatively simple relationships of the surface and volume potentials with their counterparts in the constant coefficient case, which is essential in proving the equivalence and invertibility theorems. The boundary-domain integral equations to the mixed BVP in bounded domains for the compressible Stokes system with variable viscosity have been investigated in [9] (see also [25, 27] for the incompressible case).

In this paper, we derive two direct BDIE systems associated with the considered mixed boundary value problem for the stationary compressible Stokes system with variable viscosity, defined in an *exterior domain of* $\mathbb{R}^3$. This is done by employing the Stokes surface and volume potentials based on the parametrix (Levi function) used in [25, 27, 9] in the third Green identities for the velocity and pressure. Then we analyse mapping properties of the potentials in weighted Sobolev spaces. Finally, we show equivalence between the BDIE systems and the BVP and prove the invertibility theorems for the operators defined by the BDIE systems.

2. **Preliminaries.**

Let $\Omega := \Omega^+$ be a unbounded (exterior) simply connected domain in $\mathbb{R}^3$ and let $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$ be the complementary (bounded) subset of Ω. The boundary $\partial\Omega$ is simply connected, closed and infinitely smooth for simplicity. Furthermore, $\partial\Omega := \overline{\partial\Omega_N} \cup \overline{\partial\Omega_D}$ where both $\partial\Omega_N$ and $\partial\Omega_D$ are non-empty, simply-connected, open, disjoint manifolds of $\partial\Omega$. In addition, the border of these two smooth submanifolds is also infinitely smooth.

In what follows, $H^s(\Omega)$, $H^s(\partial\Omega)$ are the Bessel potential spaces, where $s\in\mathbb{R}$ is an arbitrary real number (see, e.g., [20, 21]). We recall that H^s coincide with the Sobolev-Slobodetsky spaces W_2^s for any non-negative s. For an open set Ω', we, as usual, denote $\mathcal{D}(\Omega') = C^\infty_{comp}(\Omega')$, while $\mathcal{D}(\overline{\Omega'})$ is the restriction to $\overline{\Omega'}$ of the space $\mathcal{D}(\mathbb{R}^3)$. In what follows we use the bold notation: $\boldsymbol{H}^s(\Omega) = [H^s(\Omega)]^3$ for 3-dimensional vector spaces. We denote by $\widetilde{\boldsymbol{H}}^s(\Omega)$ the subspace of $\boldsymbol{H}^s(\mathbb{R}^3)$ defined as $\widetilde{\boldsymbol{H}}^s(\Omega) := \{\boldsymbol{g} : \boldsymbol{g} \in \boldsymbol{H}^s(\mathbb{R}^3), \operatorname{supp} \boldsymbol{g} \subset \overline{\Omega}\}$; similarly, $\widetilde{\boldsymbol{H}}^s(S_1) = \{\boldsymbol{g} \in \boldsymbol{H}^s(\partial\Omega),\ \operatorname{supp} \boldsymbol{g} \subset \overline{S}_1\}$ is the Sobolev space of functions having support in $S_1 \subset \partial\Omega$. We will use the following notation for derivative operators: $\partial_j = \partial_{x_j} := \frac{\partial}{\partial x_j}$ with $j = 1, 2, 3$; $\nabla := (\partial_1, \partial_2, \partial_3)$.

Furthermore, to ensure unique solvability of the BVPs in exterior domains, we will need the *weighted Sobolev spaces*, see, e.g., [13, 29, 10, 11, 20, 30, 1, 6, 15]. Let us first introduce the weighted Lebesgue space

$$L_2(\rho^{-1}; \Omega) = \{g : \rho^{-1} g \in L_2(\Omega)\},$$

where

$$\rho(\boldsymbol{x}) = (1 + |\boldsymbol{x}|^2)^{1/2}.$$

Let $\mathcal{H}^1(\Omega)$ denote the following weighted Sobolev (Beppo-Levi) space

$$\mathcal{H}^1(\Omega) := \{g \in L_2(\rho^{-1}; \Omega) : \nabla g \in \boldsymbol{L}_2(\Omega)\}$$

endowed with the corresponding norm

$$\|g\|^2_{\mathcal{H}^1(\Omega)} := \|\rho^{-1} g\|^2_{L_2(\Omega)} + \|\nabla g\|^2_{L_2(\Omega)}.$$

The analogous vector counterpart of $\mathcal{H}^1(\Omega)$ reads

$$\boldsymbol{\mathcal{H}}^1(\Omega) := \{\boldsymbol{g} \in \boldsymbol{L}_2(\rho^{-1};\Omega) : \operatorname{grad}\boldsymbol{g} \in L_2(\Omega)^{3\times 3}\}.$$

It is well known that $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{\mathcal{H}}^1(\Omega)$, see [13] (cf. also [6, Section 2] and more references therein). If Ω is unbounded, then the seminorm

$$|\boldsymbol{g}|_{\boldsymbol{\mathcal{H}}^1(\Omega)} := \|\nabla \boldsymbol{g}\|_{\boldsymbol{L}_2(\Omega)},$$

is equivalent to the norm $\|\boldsymbol{g}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)}$ in $\boldsymbol{\mathcal{H}}^1(\Omega)$ [20, Chapter XI, Part B, §1]. If Ω^- is bounded, then $\boldsymbol{\mathcal{H}}^1(\Omega^-) = \boldsymbol{H}^1(\Omega^-)$. If Ω' is a bounded subdomain of an unbounded domain Ω and $\boldsymbol{g} \in \boldsymbol{\mathcal{H}}^1(\Omega)$, then $\boldsymbol{g} \in \boldsymbol{H}^1(\Omega')$.

Let $\widetilde{\boldsymbol{\mathcal{H}}}^1(\Omega)$ be the completion of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{\mathcal{H}}^1(\mathbb{R}^3)$; it can be also characterised as $\widetilde{\boldsymbol{\mathcal{H}}}^1(\Omega) = \{\boldsymbol{g} : \ \boldsymbol{g} \in \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3), \ \operatorname{supp}\boldsymbol{g} \subset \overline{\Omega}\}$. Let $\widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) := [\boldsymbol{\mathcal{H}}^1(\Omega)]^*$ and $\boldsymbol{\mathcal{H}}^{-1}(\Omega) := [\widetilde{\boldsymbol{\mathcal{H}}}^1(\Omega)]^*$ be the corresponding dual spaces. Evidently, the space $\boldsymbol{L}_2(\rho;\Omega) \subset \boldsymbol{\mathcal{H}}^{-1}(\Omega)$.

For any distribution $\boldsymbol{g}$ in $\widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$, we have the following representation property (see [29, Section 2.5]), $g_j = \sum_{i=1}^3 \partial_i g_{ij} + g_j^0, \quad g_{ij} \in L_2(\mathbb{R}^3)$, $g_j^0 \in L_2(\rho;\mathbb{R}^3)$ and $g_{ij}, g_j^0 = 0$ outside the domain Ω, $i,j \in \{1,2,3\}$. Consequently, $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$ and $\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ is dense in $\boldsymbol{\mathcal{H}}^{-1}(\mathbb{R}^3)$.

Let μ be the viscosity coefficient, p the pressure field and $\boldsymbol{v}$ the velocity field. In this paper, for an arbitrary couple $(p,\boldsymbol{v})$, the stress tensor operator, σ_{ij}, and the Stokes operator, $\mathcal{A}_j$, are defined for a compressible fluid as

$$\sigma_{ji}(p,\boldsymbol{v})(\boldsymbol{x}) := -\,\delta_i^j p(\boldsymbol{x}) + \mu(\boldsymbol{x})\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} - \frac{2}{3}\delta_i^j \operatorname{div}\boldsymbol{v}(\boldsymbol{x})\right), \tag{2.1}$$

$$\begin{aligned} \mathcal{A}_j(p,\boldsymbol{v})(\boldsymbol{x}) :=& \frac{\partial}{\partial x_i}\sigma_{ji}(p,\boldsymbol{v})(\boldsymbol{x}) \qquad (2.2)\\ =& \frac{\partial}{\partial x_i}\left(\mu(\boldsymbol{x})\left(\frac{\partial v_j(\boldsymbol{x})}{\partial x_i} + \frac{\partial v_i(\boldsymbol{x})}{\partial x_j} - \frac{2}{3}\delta_i^j \operatorname{div}\boldsymbol{v}(\boldsymbol{x})\right)\right) - \frac{\partial p(\boldsymbol{x})}{\partial x_j}, j, i \in \{1,2,3\}, \end{aligned}$$

where δ_i^j is the Kronecker symbol. Henceforth we assume the Einstein summation in repeated indices from 1 to 3 if not stated otherwise.

Note that (2.1) is a particular case of a more general relation between stress and strain rate tensors for isotropic compressible Newtonian fluids (cf., e.g., Appendix III, Part I, Section 1, p.339 in [34]),

$$\sigma_{ji}(p,\boldsymbol{v})(\boldsymbol{x}) := -\delta_i^j p(\boldsymbol{x}) + \mu(\boldsymbol{x})\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right) + \lambda(\boldsymbol{x})\delta_i^j \operatorname{div}\boldsymbol{v}(\boldsymbol{x}), \tag{2.3}$$

where $\mu(\boldsymbol{x}) > 0$ and $\lambda(\boldsymbol{x}) \in \mathbb{R}$. In this paper we take $\lambda(\boldsymbol{x}) = -\frac{2}{3}\mu(\boldsymbol{x})$, which corresponds to the assumption that the strain rate tensor does not contribute to the volumetric part (matrix trace) σ_{ii} of the stress tensor σ_{ji}.

Throughout this paper, we will assume the following condition to ensure boundedness properties of the integral operators introduced further on.

Condition 2.1.

$$\mu \in \mathcal{C}^1(\mathbb{R}^3) \cap L_\infty(\mathbb{R}^3) \ : \ \rho\nabla\mu \in \boldsymbol{L}_\infty(\mathbb{R}^3).$$

In addition, there exist constants C_1 and C_2 such that

$$0 < C_1 < \mu(\boldsymbol{x}) < C_2. \tag{2.4}$$

Remark 2.2. *If μ satisfies condition 2.1, then the functions μ and $\dfrac{1}{\mu}$ are multipliers in the space $\mathcal{H}^1(\Omega)$, i.e., there exists positive constants $C_3(\mu)$ and $C_4(\mu)$ independent of h such that*

$$\|\mu h\|_{\mathcal{H}^1(\Omega)} \le C_3(\mu)\|h\|_{\mathcal{H}^1(\Omega)}, \quad \|\frac{h}{\mu}\|_{\mathcal{H}^1(\Omega)} \le C_4(\mu)\|h\|_{\mathcal{H}^1(\Omega)}, \quad \forall\, h \in \mathcal{H}^1(\Omega). \tag{2.5}$$

The operator $\boldsymbol{\mathcal{A}}$ acting on $(p, \boldsymbol{v}) \in L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)$ is well defined in the weak sense as long as the variable coefficient $\mu(\boldsymbol{x})$ is essentially bounded, i.e. $\mu \in L_\infty(\Omega)$. Indeed, in the sense of distributions the operator $\boldsymbol{\mathcal{A}}$ is defined as

$$\langle \boldsymbol{\mathcal{A}}(p, \boldsymbol{v}), \boldsymbol{u}\rangle_\Omega = -\mathcal{E}((p, \boldsymbol{v}), \boldsymbol{u}), \quad \forall \boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\Omega), \tag{2.6}$$

where

$$\mathcal{E}\left((p, \boldsymbol{v}), \boldsymbol{u}\right) := \int_\Omega E\left((p, \boldsymbol{v}), \boldsymbol{u}\right)(\boldsymbol{x})\, dx, \tag{2.7}$$

and the function $E\left((p, \boldsymbol{v}), \boldsymbol{u}\right)$ is defined as

$$\begin{aligned} E\left((p, \boldsymbol{v}), \boldsymbol{u}\right)(\boldsymbol{x}) :=& \frac{1}{2}\mu(\boldsymbol{x})\left(\frac{\partial u_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial u_j(\boldsymbol{x})}{\partial x_i}\right)\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right) \\ & - \frac{2}{3}\mu(\boldsymbol{x})\mathrm{div}\boldsymbol{v}(\boldsymbol{x})\,\mathrm{div}\boldsymbol{u}(\boldsymbol{x}) - p(\boldsymbol{x})\mathrm{div}\boldsymbol{u}(\boldsymbol{x}). \end{aligned} \tag{2.8}$$

The bilinear form $\mathcal{E} : \left[L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)\right] \times \widetilde{\boldsymbol{\mathcal{H}}}^1(\Omega) \to \mathbb{R}$ is evidently bounded. Thus, by the density of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\widetilde{\boldsymbol{\mathcal{H}}}^1(\Omega)$, the operator

$$\boldsymbol{\mathcal{A}} : L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \to \boldsymbol{\mathcal{H}}^{-1}(\Omega)$$

defined by (2.6) for any $\boldsymbol{u} \in \widetilde{\boldsymbol{\mathcal{H}}}^1(\Omega)$ is also bounded and gives the weak form of operator (2.2).

We will also make use of the following space, (cf., e.g., [7, 6]),

$$\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) := \{(p, \boldsymbol{v}) \in L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) : \boldsymbol{\mathcal{A}}(p, \boldsymbol{v}) \in \boldsymbol{L}_2(\rho; \Omega)\},$$

endowed with the norm, $\|\cdot\|_{\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})}$, where

$$\|(p, \boldsymbol{v})\|_{\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})} := \left(\|p\|^2_{L_2(\Omega)} + \|\boldsymbol{v}\|^2_{\boldsymbol{\mathcal{H}}^1(\Omega)} + \|\rho\boldsymbol{\mathcal{A}}(p, \boldsymbol{v})\|^2_{\boldsymbol{L}_2(\Omega)}\right)^{1/2}.$$

Similar to [22, Theorem 3.12], one can prove the following assertion.

Theorem 2.3. *Let μ satisfy condition 2.1. Then the space $\mathcal{D}(\overline{\Omega}) \times \boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$.*

For sufficiently smooth functions $(p, \boldsymbol{v}) \in H^{s-1}(\Omega^\pm) \times \boldsymbol{H}^s(\Omega^\pm)$ with $s > 3/2$, we can define the classical traction (conormal derivative) operators, $\boldsymbol{T}^{c\pm} = \{T_i^{c\pm}\}_{i=1}^3$, on the boundary $\partial\Omega$ as

$$\begin{aligned} T_i^{c\pm}(p, \boldsymbol{v})(\boldsymbol{x}) &:= [\gamma^\pm\sigma_{ij}(p, \boldsymbol{v})(\boldsymbol{x})]n_j(\boldsymbol{x}) \\ &= -n_i(\boldsymbol{x})\gamma^\pm p(\boldsymbol{x}) + n_j(\boldsymbol{x})\mu(\boldsymbol{x})\gamma^\pm\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j} + \frac{\partial v_j(\boldsymbol{x})}{\partial x_i} - \frac{2}{3}\delta_i^j\mathrm{div}\boldsymbol{v}(\boldsymbol{x})\right), \ \boldsymbol{x} \in \partial\Omega, \end{aligned} \tag{2.9}$$

where $n_j(\boldsymbol{x})$ denote the components of the unit normal vector $\boldsymbol{n}(\boldsymbol{x})$ to the boundary $\partial\Omega$ *directed outwards the exterior domain* Ω. Moreover, $\boldsymbol{\gamma}^\pm$ denote the trace operators from inside and outside Ω which according to the trace theorem satisfy the mapping property $\gamma^\pm : \boldsymbol{\mathcal{H}}^1(\Omega) \to \boldsymbol{H}^{1/2}(\partial\Omega)$.

Traction operators (2.9) can be continuously extended to the *canonical* traction operators $\boldsymbol{T}^{\pm}:\boldsymbol{\mathcal{H}}^{1,0}(\Omega^{\pm},\boldsymbol{\mathcal{A}})\to\boldsymbol{H}^{-1/2}(\partial\Omega)$ defined in the weak form (cf. [7, 6, 25, 9]), as

$$\langle\boldsymbol{T}^{+}(p,\boldsymbol{v}),\boldsymbol{w}\rangle_{\partial\Omega}:=\int_{\Omega^{\pm}}\left[\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\boldsymbol{\gamma}^{+}_{-1}\boldsymbol{w}+E\left((p,\boldsymbol{v}),\boldsymbol{\gamma}^{+}_{-1}\boldsymbol{w}\right)\right]dx$$
$$\forall\,(p,\boldsymbol{v})\in\boldsymbol{\mathcal{H}}^{1,0}(\Omega^{\pm},\boldsymbol{\mathcal{A}}),\ \forall\,\boldsymbol{w}\in\boldsymbol{H}^{1/2}(\partial\Omega),$$

where the operator $\boldsymbol{\gamma}^{+}_{-1}:\boldsymbol{H}^{1/2}(\partial\Omega)\to\boldsymbol{\mathcal{H}}^{1}(\Omega)$ denotes a continuous right inverse of the trace operator $\boldsymbol{\gamma}^{+}:\boldsymbol{\mathcal{H}}^{1}(\Omega)\to\boldsymbol{H}^{1/2}(\partial\Omega)$.

Furthermore, if $(p,\boldsymbol{v})\in\boldsymbol{\mathcal{H}}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$ and $\boldsymbol{u}\in\boldsymbol{\mathcal{H}}^{1}(\Omega)$, the following first Green identity holds, cf. [12, Section 2.3],

$$\langle\boldsymbol{T}^{+}(p,\boldsymbol{v}),\boldsymbol{\gamma}^{+}\boldsymbol{u}\rangle_{\partial\Omega}=\int_{\Omega}[\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})\boldsymbol{u}+E\left((p,\boldsymbol{v}),\boldsymbol{u}\right)(\boldsymbol{x})]dx. \tag{2.10}$$

Applying identity (2.10) to the pairs $(p,\boldsymbol{v}),(q,\boldsymbol{u})\in\boldsymbol{\mathcal{H}}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$ with exchanged roles and subtracting the one from the other, we arrive at the second Green identity, cf. [21, 22],

$$\begin{aligned}&\langle\boldsymbol{T}^{+}(p,\boldsymbol{v}),\boldsymbol{\gamma}^{+}\boldsymbol{u}\rangle_{\partial\Omega}-\langle\boldsymbol{T}^{+}(q,\boldsymbol{u}),\boldsymbol{\gamma}^{+}\boldsymbol{v}\rangle_{\partial\Omega}\\&=\int_{\Omega}\Big[\ \mathcal{A}_j(p,\boldsymbol{v})u_j-\mathcal{A}_j(q,\boldsymbol{u})v_j+q\operatorname{div}\boldsymbol{v}-p\operatorname{div}\boldsymbol{u}\Big]\,dx.\end{aligned} \tag{2.11}$$

In this paper, we derive the systems of boundary-domain integral equations, which are equivalent to the following mixed compressible exterior Stokes problem. Mixed problem. For $\boldsymbol{f}\in\boldsymbol{L}_2(\rho,\Omega)$, $\boldsymbol{\varphi}_0\in\boldsymbol{H}^{1/2}(\partial\Omega_D)$, $g\in L_2(\Omega)$ and $\boldsymbol{\psi}_0\in\boldsymbol{H}^{-1/2}(\partial\Omega_N)$, find $(p,\boldsymbol{v})\in\boldsymbol{\mathcal{H}}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$ such that:

$$\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})=\boldsymbol{f}\qquad \textit{in }\Omega, \tag{2.12a}$$
$$\operatorname{div}\boldsymbol{v}(\boldsymbol{x})=g\qquad \textit{in }\ \Omega, \tag{2.12b}$$
$$\boldsymbol{\gamma}^{+}\boldsymbol{v}=\boldsymbol{\varphi}_0,\qquad \textit{on }\partial\Omega_D \tag{2.12c}$$
$$\boldsymbol{T}^{+}(p,\boldsymbol{v})=\boldsymbol{\psi}_0\quad \textit{on }\partial\Omega_N. \tag{2.12d}$$

Theorem 2.4. *Let μ satisfy condition 2.1. The mixed BVP (2.12) has at most one solution in the space $\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$.*

Proof. Let us suppose that there are two possible solutions: $(p_1,\boldsymbol{v}_1)$ and $(p_2,\boldsymbol{v}_2)$ belonging to the space $\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$, that satisfy the BVP (2.12). Then, the pair $(p,\boldsymbol{v}):=(p_2,\boldsymbol{v}_2)-(p_1,\boldsymbol{v}_1)\in\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ satisfies the homogeneous version mixed BVP (2.12). Substituting $(q,\boldsymbol{u})=(p,\boldsymbol{v})$ in the first Green identity (2.10), which holds since $(p,\boldsymbol{v})\in\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$, we obtain,

$$\int_{\Omega}\frac{1}{2}\mu(\boldsymbol{x})\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j}+\frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right)^2dx\ =\ 0.$$

As $\mu(\boldsymbol{x})>0$, this can be satisfied only if $\boldsymbol{v}$ is a rigid motion, i.e., $\boldsymbol{v}(\boldsymbol{x})=\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}$ with some constant vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, [21, Lemma 10.5]. However, taking into account the Dirichlet condition $\boldsymbol{\gamma}^{+}\boldsymbol{v}=\boldsymbol{0}$, we deduce that $\boldsymbol{v}\equiv\boldsymbol{0}$. By the Neumann condition $\boldsymbol{T}^{+}(p,\boldsymbol{v})=\boldsymbol{0}$, it is easy to conclude that $p=0$. Hence, $\boldsymbol{v}_1=\boldsymbol{v}_2$ and $p_1=p_2$. □

BVP (2.12) can be represented by the following operator,

$$\boldsymbol{\mathcal{A}}_M:\boldsymbol{\mathcal{H}}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})\to\boldsymbol{L}_2(\rho;\Omega)\times L_2(\Omega)\times\boldsymbol{H}^{1/2}(\partial\Omega_D)\times\boldsymbol{H}^{-1/2}(\partial\Omega_N). \tag{2.13}$$

which is continuous, and by Theorem 2.4 also injective.

3. **Parametrix and remainder.** When $\mu(\boldsymbol{x}) = 1$, the operator $\boldsymbol{\mathcal{A}}$ becomes the constant-coefficient Stokes operator $\mathring{\boldsymbol{\mathcal{A}}}$, for which we know an explicit fundamental solution defined by the pair of functions $(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)$, where summation in k is not assumed, $\mathring{u}_j^k$ represent components of the incompressible velocity fundamental solution and $\mathring{q}^k$ represent the components of the pressure fundamental solution (see e.g. [19, 17, 14]).

$$\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}) = -\frac{(x_k - y_k)}{4\pi|\boldsymbol{x} - \boldsymbol{y}|^3} = \frac{\partial}{\partial x_k}\left(\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}\right), \tag{3.1}$$

$$\mathring{u}_j^k(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{8\pi}\left\{\frac{\delta_j^k}{|\boldsymbol{x} - \boldsymbol{y}|} + \frac{(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x} - \boldsymbol{y}|^3}\right\}, \quad j, k \in \{1, 2, 3\}. \tag{3.2}$$

Therefore, the couple $(\mathring{q}^k, \mathring{\boldsymbol{u}}^k)$ satisfies

$$\frac{\partial}{\partial x_k}\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_k^2}\left(\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}\right) = -\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{3.3}$$

$$\mathring{\mathcal{A}}_j(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x}, \boldsymbol{y})) = \sum_{i=1}^{3}\frac{\partial^2 \mathring{u}_j^k(\boldsymbol{x}, \boldsymbol{y})}{\partial x_i^2} - \frac{\partial \mathring{q}^k(\boldsymbol{x}, \boldsymbol{y})}{\partial x_j} = \delta_j^k \delta(\boldsymbol{x} - \boldsymbol{y}), \tag{3.4}$$

$$\mathrm{div}_{\boldsymbol{x}} \mathring{\boldsymbol{u}}^k(\boldsymbol{x}, \boldsymbol{y}) = 0. \tag{3.5}$$

Here and henceforth, $\delta(\cdot)$ is Dirac's distribution.

Let us denote $\mathring{\sigma}_{ij}(p, \boldsymbol{v}) := \sigma_{ij}(p, \boldsymbol{v})|_{\mu=1}$, $\mathring{T}_i^c(p, \boldsymbol{v}) := T_i^c(p, \boldsymbol{v})|_{\mu=1}$. Then by (2.1) the stress tensor of the fundamental solution reads as

$$\mathring{\sigma}_{ij}(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x}, \boldsymbol{y})) = \frac{3}{4\pi}\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x} - \boldsymbol{y}|^5},$$

and the classical boundary traction of the fundamental solution becomes

$$\begin{aligned}&\mathring{T}_i^c(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x}, \boldsymbol{y}))\\ :=&\mathring{\sigma}_{ij}(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x}, \boldsymbol{y}))\, n_j(\boldsymbol{x}) = \frac{3}{4\pi}\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x} - \boldsymbol{y}|^5}\, n_j(\boldsymbol{x}).\end{aligned}$$

Let us define a pair of functions $(q^k, \boldsymbol{u}^k)_{k=1}^3$,

$$q^k(\boldsymbol{x}, \boldsymbol{y}) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}) = -\frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\frac{x_k - y_k}{4\pi|\boldsymbol{x} - \boldsymbol{y}|^3}, \quad j, k \in \{1, 2, 3\}, \tag{3.6}$$

$$u_j^k(\boldsymbol{x}, \boldsymbol{y}) = \frac{-1}{\mu(\boldsymbol{y})}\mathring{u}_j^k(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{8\pi\mu(\boldsymbol{y})}\left\{\frac{\delta_j^k}{|\boldsymbol{x} - \boldsymbol{y}|} + \frac{(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x} - \boldsymbol{y}|^3}\right\}. \tag{3.7}$$

Then by (2.1),

$$\sigma_{ij}(\boldsymbol{x})(q^k(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{u}^k(\boldsymbol{x}, \boldsymbol{y})) = \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{\sigma}_{ij}(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x}, \boldsymbol{y})), \tag{3.8}$$

$$\begin{aligned}T_i(\boldsymbol{x})(q^k(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{u}^k(\boldsymbol{x}, \boldsymbol{y})) &:= \sigma_{ij}(\boldsymbol{x})(q^k(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{u}^k(\boldsymbol{x}, \boldsymbol{y}))\, n_j(\boldsymbol{x})\\ &= \frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}\mathring{T}_i(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x}, \boldsymbol{y}), \mathring{\boldsymbol{u}}^k(\boldsymbol{x}, \boldsymbol{y})).\end{aligned} \tag{3.9}$$

No summation in k is assumed in (3.8)-(3.9).

Substituting (3.6)-(3.7) in the Stokes system with variable coefficient, (2.2) gives

$$\mathcal{A}_j(\boldsymbol{x})(q^k(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{u}^k(\boldsymbol{x}, \boldsymbol{y})) = \delta_j^k \delta(\boldsymbol{x} - \boldsymbol{y}) + R_{kj}(\boldsymbol{x}, \boldsymbol{y}), \tag{3.10}$$

where

$$R_{kj}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\mu(\boldsymbol{y})}\frac{\partial\mu(\boldsymbol{x})}{\partial x_i}\mathring{\sigma}_{ij}(\boldsymbol{x})(\mathring{q}^k(\boldsymbol{x},\boldsymbol{y}),\mathring{\boldsymbol{u}}^k(\boldsymbol{x},\boldsymbol{y}))$$
$$= \frac{3}{4\pi\mu(\boldsymbol{y})}\frac{\partial\mu(\boldsymbol{x})}{\partial x_i}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^5} = \mathcal{O}(|\boldsymbol{x}-\boldsymbol{y}|)^{-2}) \tag{3.11}$$

is a weakly singular remainder and no summation in k is assumed in (3.10)-(3.11). This implies that $(q^k, \boldsymbol{u}^k)$ is a parametrix of the operator $\boldsymbol{\mathcal{A}}$. Let us keep in mind that we have not assumed summation on the index k in (3.8)-(3.11).

Note that a parametrix is generally not unique (cf. [26] for BDIEs based on an alternative parametrix for a scalar PDE). The possibility to factor out $\frac{\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}$ in (3.8)-(3.9) and $\frac{\nabla\mu(\boldsymbol{x})}{\mu(\boldsymbol{y})}$ in (3.11) is due to the careful choice of the parametrix in form (3.6)-(3.7) and this essentially simplifies the analysis of parametrix-based potentials and BDIE systems further on.

4. **Hydrodynamic potentials.** Let first h and $\boldsymbol{h}$ be sufficiently smooth scalar and vector functions on $\overline{\Omega}$, e.g., $h \in \mathcal{D}(\overline{\Omega})$, $\boldsymbol{h} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$. Let us define the parametrix-based Newton-type and remainder vector potentials for the velocity,

$$[\boldsymbol{\mathcal{U}}\boldsymbol{h}]_k(\boldsymbol{y}) = \mathcal{U}_{kj}h_j(\boldsymbol{y}) := \int_\Omega u_j^k(\boldsymbol{x},\boldsymbol{y})h_j(\boldsymbol{x})dx,$$
$$[\boldsymbol{\mathcal{R}}\boldsymbol{h}]_k(\boldsymbol{y}) = \mathcal{R}_{kj}h_j(\boldsymbol{y}) := \int_\Omega R_{kj}(\boldsymbol{x},\boldsymbol{y})h_j(\boldsymbol{x})dx,$$

and the scalar Newton-type and remainder potentials for the pressure,

$$[\boldsymbol{\mathcal{Q}}h]_j(\boldsymbol{y}) = \mathcal{Q}_j h(\boldsymbol{y}) := \int_\Omega q^j(\boldsymbol{y},\boldsymbol{x})h(\boldsymbol{x})dx = -\int_\Omega q^j(\boldsymbol{x},\boldsymbol{y})h(\boldsymbol{x})dx, \tag{4.1}$$

$$\mathcal{Q}\boldsymbol{h}(\boldsymbol{y}) := \boldsymbol{\mathcal{Q}}\cdot\boldsymbol{h}(\boldsymbol{y}) = \mathcal{Q}_j h_j(\boldsymbol{y}) = \int_\Omega q^j(\boldsymbol{y},\boldsymbol{x})h_j(\boldsymbol{x})dx = -\int_\Omega q^j(\boldsymbol{x},\boldsymbol{y})h_j(\boldsymbol{x})dx, \tag{4.2}$$

$$\mathcal{R}^\bullet\boldsymbol{h}(\boldsymbol{y}) = \mathcal{R}_j^\bullet h_j(\boldsymbol{y}) := -2\,\text{p.v.}\int_\Omega \frac{\partial\mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial x_i}\frac{\partial\mu(\boldsymbol{x})}{\partial x_i}h_j(\boldsymbol{x})dx - \frac{4}{3}h_j\frac{\partial\mu}{\partial y_j} \tag{4.3}$$

$$= -2\left\langle \partial_i\mathring{q}^j(\cdot,\boldsymbol{y})\,,\,h_i\partial_j\mu\right\rangle_\Omega - 2h_i(\boldsymbol{y})\partial_i\mu(\boldsymbol{y}), \tag{4.4}$$

for $\boldsymbol{y}\in\mathbb{R}^3$. The integral in (4.3) is understood as a 3D strongly singular integral (in the sense of the Cauchy principal value). The bilinear form in (4.4) should be understood in the sense of distributions, and the equality between (4.3) and (4.4) holds since

$$\begin{aligned}\left\langle\partial_i\mathring{q}^j(\cdot,\boldsymbol{y}),h_i\partial_j\mu\right\rangle_\Omega &= -\left\langle\mathring{q}^j(\cdot,\boldsymbol{y}),\partial_i(h_i\partial_j\mu)\right\rangle_\Omega + \left\langle n_i\mathring{q}^j(\cdot,\boldsymbol{y}),h_i\partial_j\mu\right\rangle_{\partial\Omega}\\
&= -\lim_{\epsilon\to0}\left\langle\mathring{q}^j(\cdot,\boldsymbol{y}),\partial_i(h_i\partial_j\mu)\right\rangle_{\Omega_\epsilon} + \left\langle n_i\mathring{q}^j(\cdot,\boldsymbol{y}),h_i\partial_j\mu\right\rangle_{\partial\Omega}\\
&= \lim_{\epsilon\to0}\left\langle\partial_i\mathring{q}^j(\cdot,\boldsymbol{y}),h_i\partial_j\mu\right\rangle_{\Omega_\epsilon} - \lim_{\epsilon\to0}\left\langle n_i\mathring{q}^j(\cdot,\boldsymbol{y}),h_i\partial_j\mu\right\rangle_{\partial\Omega_\epsilon\setminus\partial\Omega}\\
&=\text{v.p.}\int_\Omega\frac{\partial\mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial x_i}\frac{\partial\mu(\boldsymbol{x})}{\partial x_i}h_j(\boldsymbol{x})dx - \frac{1}{3}h_j\frac{\partial\mu}{\partial y_j}\ ,\end{aligned}$$

where $\Omega_\epsilon = \Omega\setminus\bar{B}_\epsilon(\boldsymbol{y})$ and $B_\epsilon(\boldsymbol{y})$ is the ball of radius ϵ centred in $\boldsymbol{y}$, which implies that

$$-2\left\langle\partial_i\mathring{q}^j(\cdot,\boldsymbol{y})\,,\,h_i\partial_j\mu\right\rangle_\Omega - 2h_i(\boldsymbol{y})\partial_i\mu(\boldsymbol{y})$$

$$= -\,2\,\text{v.p.}\int_\Omega \frac{\partial \mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial x_i}\frac{\partial \mu(\boldsymbol{x})}{\partial x_i}h_j(\boldsymbol{x})dx - \frac{4}{3}h_j(\boldsymbol{y})\frac{\partial \mu(\boldsymbol{y})}{\partial y_j} = \mathcal{R}^\bullet \boldsymbol{h}(\boldsymbol{y}).$$

In addition, we will introduce the operators $\mathbf{U}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{R}^\bullet$ whose definitions coincide, respectively, with the definition of the operators $\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{R}}^\bullet$ with the sole difference that $\Omega = \mathbb{R}^3$.

Let us now define the parametrix-based velocity single layer potential and double layer potential as follows:

$$[\boldsymbol{V}\boldsymbol{h}]_k(\boldsymbol{y}) = V_{kj}h_j(\boldsymbol{y}) := -\int_{\partial\Omega} u_j^k(\boldsymbol{x},\boldsymbol{y})h_j(\boldsymbol{x})\,dS(\boldsymbol{x}),\ \ \boldsymbol{y}\notin\partial\Omega,$$
$$[\boldsymbol{W}\boldsymbol{h}]_k(\boldsymbol{y}) = W_{kj}h_j(\boldsymbol{y}) := -\int_{\partial\Omega} T_j^c(\boldsymbol{x}; q^k, \boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y})h_j(\boldsymbol{x})\,dS(\boldsymbol{x}),\ \ \boldsymbol{y}\notin\partial\Omega.$$

For the pressure we will need the following single-layer and double layer potentials:

$$\Pi^s\boldsymbol{h}(\boldsymbol{y}) = \Pi_j^s h_j(\boldsymbol{y}) := \int_{\partial\Omega} \mathring{q}^j(\boldsymbol{x},\boldsymbol{y})h_j(\boldsymbol{x})dS(\boldsymbol{x}),\ \boldsymbol{y}\notin\partial\Omega$$
$$\Pi^d\boldsymbol{h}(\boldsymbol{y}) = \Pi_j^d h_j(\boldsymbol{y}) := 2\int_{\partial\Omega} \frac{\partial \mathring{q}^j(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{x})}\mu(\boldsymbol{x})h_j(\boldsymbol{x})dS(\boldsymbol{x}),\ \boldsymbol{y}\notin\partial\Omega.$$

It is easy to observe that the parametrix-based integral operators, with the variable coefficient μ, can be expressed in terms of the corresponding integral operators for the constant-coefficient case, $\mu = 1$, marked by $\mathring{}$,

$$\boldsymbol{\mathcal{U}}\boldsymbol{h} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{h}, \tag{4.5}$$
$$[\boldsymbol{\mathcal{R}}\boldsymbol{h}]_k = \frac{-1}{\mu}\Big[\partial_j\mathring{\mathcal{U}}_{ki}(h_j\partial_i\mu) + \partial_i\mathring{\mathcal{U}}_{kj}(h_j\partial_i\mu) - \mathring{\mathcal{Q}}_k(h_j\partial_j\mu)\Big], \tag{4.6}$$
$$\boldsymbol{\mathcal{Q}}h = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{Q}}}(\mu h), \tag{4.7}$$
$$\mathcal{R}^\bullet\boldsymbol{h} = -2\partial_i\mathring{\mathcal{Q}}_j(h_j\partial_i\mu) - 2h_j\partial_j\mu, \tag{4.8}$$
$$\boldsymbol{V}\boldsymbol{h} = \frac{1}{\mu}\mathring{\boldsymbol{V}}\boldsymbol{h}, \qquad \boldsymbol{W}\boldsymbol{h} = \frac{1}{\mu}\mathring{\boldsymbol{W}}(\mu\boldsymbol{h}), \tag{4.9}$$
$$\Pi^s\boldsymbol{h} = \mathring{\Pi}^s\boldsymbol{h}, \qquad \Pi^d\boldsymbol{h} = \mathring{\Pi}^d(\mu\boldsymbol{h}). \tag{4.10}$$

We will further use (4.5)-(4.10) as definitions of the potentials in the left-hand sides of these relations, when the densities h and $\boldsymbol{h}$ are more general functions or distributions on Ω or $\partial\Omega$.

Note that although the constant-coefficient velocity potentials $\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{h}$, $\mathring{\boldsymbol{V}}\boldsymbol{h}$ and $\mathring{\boldsymbol{W}}\boldsymbol{h}$ are divergence-free in $\Omega^\pm$, the corresponding potentials $\boldsymbol{\mathcal{U}}\boldsymbol{h}$, $\boldsymbol{V}\boldsymbol{h}$ and $\boldsymbol{W}\boldsymbol{h}$ are *not divergence-free for the variable coefficient* $\mu(\boldsymbol{y})$. Note also that by (3.1) and (4.1),

$$\mathring{\mathcal{Q}}_j h = -\partial_j\mathcal{N}_\Delta h, \tag{4.11}$$

where

$$\mathcal{N}_\Delta h(\boldsymbol{y}) = -\frac{1}{4\pi}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}h(\boldsymbol{x})dx \tag{4.12}$$

is the harmonic Newton potential. Hence

$$\text{div}\mathring{\boldsymbol{\mathcal{Q}}}h = \partial_j\mathring{\mathcal{Q}}_j h = -\Delta\mathcal{N}_\Delta h = -h. \tag{4.13}$$

Moreover, for the constant-coefficient potentials we have the following well-known relations,

$$\mathring{\mathcal{A}}(\mathring{\Pi}^s \boldsymbol{h}, \mathring{\boldsymbol{V}}\boldsymbol{h}) = \boldsymbol{0}, \quad \mathring{\mathcal{A}}(\mathring{\Pi}^d \boldsymbol{h}, \mathring{\boldsymbol{W}}\boldsymbol{h}) = \boldsymbol{0} \quad \text{in } \Omega^\pm, \tag{4.14}$$

$$\mathring{\mathcal{A}}(\mathring{\mathcal{Q}}\boldsymbol{h}, \mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{h}) = \boldsymbol{h}. \tag{4.15}$$

In addition, by (4.11) and (4.13),

$$\begin{aligned}\mathring{\mathcal{A}}_j(\frac{4}{3}h, -\mathring{\boldsymbol{\mathcal{Q}}}h) &= -\partial_i\left(\partial_i \mathring{\mathcal{Q}}_j h + \partial_j \mathring{\mathcal{Q}}_i h - \frac{2}{3}\delta_i^j \operatorname{div} \mathring{\boldsymbol{\mathcal{Q}}} h\right) - \frac{4}{3}\partial_j h \\ &= -(\Delta \mathring{\mathcal{Q}}_j h + \partial_j \operatorname{div} \mathring{\boldsymbol{\mathcal{Q}}} h - \frac{2}{3}\partial_j \operatorname{div} \mathring{\boldsymbol{\mathcal{Q}}} h) - \frac{4}{3}\partial_j h = 0. \end{aligned} \tag{4.16}$$

4.1. **Mapping properties.** The following assertions are well known for the constant coefficient case, see e.g. Lemmas A.3 and A.4 in [15] and references therein. Then by relations (4.5)-(4.10), we obtain their counterparts for the variable-coefficient case. Let us highlight that the operators $\boldsymbol{U}, \boldsymbol{Q}, Q, \boldsymbol{R}, R^\bullet$ are defined in the same way as $\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}, \mathcal{Q}, \boldsymbol{\mathcal{R}}$ and $\mathcal{R}^\bullet$ if we take $\Omega = \mathbb{R}^3$.

Remark 4.1. *For sufficiently smooth h, the Newtonian volume potential over $\mathbb{R}^3$, cf. (4.12), is defined as*

$$N_\Delta h(\boldsymbol{y}) = \int_{\mathbb{R}^3} E_\Delta(\boldsymbol{x}, \boldsymbol{y})\, h(\boldsymbol{x})\, d\boldsymbol{x}, \tag{4.17}$$

where

$$E_\Delta(\boldsymbol{x}, \boldsymbol{y}) = \frac{-1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}$$

is the fundamental solution of the Laplace equation and moreover $\mathcal{N}_\Delta \Delta h = \Delta \mathcal{N}_\Delta h = h$, i.e. the operator N_Δ is inverse to to the Laplace operator Δ. On the other hand, it is well known (see, e.g., [31, Theorem 1.2], [13, Theorem III.2]*) that the Laplace operator $\Delta : \mathcal{H}^1(\mathbb{R}^3) \to \mathcal{H}^{-1}(\mathbb{R}^3)$ has a continuous inverse, $\Delta^{-1} : \mathcal{H}^{-1}(\mathbb{R}^3) \to \mathcal{H}^1(\mathbb{R}^3)$ and thus $N_\Delta h = \Delta^{-1} h$ for any $h \in \mathcal{D}(\mathbb{R}^3)$. As remarked in* [6]*, due to the density of $\mathcal{D}(\mathbb{R}^3)$ in $\mathcal{H}^{-1}(\mathbb{R}^3)$ this provides a continuous extension of the operator N_Δ defined by* (4.17) *to the extended continuous Newtonian potential operator*

$$N_\Delta : \mathcal{H}^{-1}(\mathbb{R}^3) \to \mathcal{H}^1(\mathbb{R}^3). \tag{4.18}$$

Theorem 4.2. *The following operators are continuous under condition 2.1,*

$$\boldsymbol{U} : \boldsymbol{\mathcal{H}}^{-1}(\mathbb{R}^3) \to \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3), \tag{4.19}$$

$$\boldsymbol{\mathcal{U}} : \widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) \to \boldsymbol{\mathcal{H}}^1(\Omega), \tag{4.20}$$

$$\boldsymbol{Q} : L_2(\mathbb{R}^3) \to \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3), \tag{4.21}$$

$$\boldsymbol{\mathcal{Q}} : L_2(\Omega) \to \boldsymbol{\mathcal{H}}^1(\Omega), \tag{4.22}$$

$$Q : \boldsymbol{\mathcal{H}}^{-1}(\mathbb{R}^3) \to L_2(\mathbb{R}^3), \tag{4.23}$$

$$\mathcal{Q} : \widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) \to L_2(\Omega), \tag{4.24}$$

$$\boldsymbol{R} : \boldsymbol{L}_2(\rho^{-1}; \mathbb{R}^3) \to \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3), \tag{4.25}$$

$$\boldsymbol{\mathcal{R}} : \boldsymbol{L}_2(\rho^{-1}; \Omega) \to \boldsymbol{\mathcal{H}}^1(\Omega), \tag{4.26}$$

$$R^\bullet : \boldsymbol{L}_2(\rho^{-1}; \mathbb{R}^3) \to L_2(\mathbb{R}^3), \tag{4.27}$$

$$\mathcal{R}^\bullet : \boldsymbol{L}_2(\rho^{-1}; \Omega) \to L_2(\Omega). \tag{4.28}$$

Proof. Let us consider relations (4.5) and (4.7). The continuity of operators $\boldsymbol{U}, \boldsymbol{\mathcal{U}}$, Q and $\mathcal{Q}$ in (4.19), (4.20), (4.23), and (4.24) then follows from the continuity of the corresponding operators $\mathring{\boldsymbol{\mathcal{U}}}$, $\mathring{\boldsymbol{U}}$, $\mathring{\mathcal{Q}}$ and $\mathring{Q}$ provided in [15, Lemma A.3].

Let us prove now the continuity of operator (4.25), which follows if we prove the continuity of operators in the right hand side of (4.6). Let us note that by condition 2.1, μ and $\frac{1}{\mu}$ are bounded and act as multipliers in the space $\boldsymbol{\mathcal{H}}^1(\Omega)$. In addition, condition 2.1 states that $\rho\partial_i\mu \in L_\infty(\mathbb{R}^3)$. Consequently, for any function $h_j \in L_2(\rho^{-1};\mathbb{R}^3)$, we have that $h_j\partial_i\mu \in L_2(\mathbb{R}^3)$, see the proof of [6, Theorem 4.1]. It is easy to prove that the operator $\nabla : L_2(\mathbb{R}^3) \to \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3)$ is continuous, which implies that $\nabla(h_j\partial_i\mu) \in \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3)$. Let us prove continuity of the first operator in the right hand side of (4.6). First, we assume that $h_j\partial_i\mu$ in $\mathcal{D}(\mathbb{R}^3)$. Then

$$\partial_j\mathring{\mathcal{U}}_{ki}(h_j\partial_i\mu) = -\mathring{\mathcal{U}}_{ki}\partial_j(h_j\partial_i\mu). \tag{4.29}$$

By the density of $\mathcal{D}(\mathbb{R}^3)$ in $L_2(\mathbb{R}^3)$ and the continuity of operator $\mathring{\boldsymbol{U}} : \boldsymbol{\mathcal{H}}^{-1}(\mathbb{R}^3) \to \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3)$, cf. (4.19), we can extend relation (4.29) from $h_j\partial_i\mu \in \mathcal{D}(\mathbb{R}^3)$ to $h_j\partial_i\mu \in L_2(\mathbb{R}^3)$. Then, the continuity of operator $\boldsymbol{h} \mapsto \partial_j\mathring{\mathcal{U}}_{ki}(h_j\partial_i\mu) : \boldsymbol{L}_2(\rho^{-1};\mathbb{R}^3) \to \boldsymbol{\mathcal{H}}^1(\mathbb{R}^3)$ follows. The continuity of other two operators in the right hand side of (4.6) can be proved in a similar way. Consequently, operator (4.25) is continuous. The continuity of operator (4.25) implies the continuity of operator (4.26).

Taking into account (4.8), the continuity of operator (4.27) will follow from the continuity of the first operator in the right hand side of (4.8). Let $h_j \in L_2(\rho^{-1};\mathbb{R}^3)$. Applying a similar density argument, as in the previous paragraph we can deduce $\partial_j\mathring{\mathcal{Q}}(h_j\partial_i\mu) = -\mathring{\mathcal{Q}}\partial_j(h_j\partial_i\mu)$. Since, $\partial_j(h_j\partial_i\mu) \in \mathcal{H}^{-1}(\mathbb{R}^3)$, the we have the inclusion $\partial_j\mathring{\mathcal{Q}}(h_j\partial_i\mu) \in L_2(\mathbb{R}^3)$ for any $h_j \in L_2(\rho^{-1};\mathbb{R}^3)$, with the corresponding norm estimate. This implies the continuity of operator (4.27). Continuity of operator (4.28) is implied by the continuity of operator (4.27).

The mapping properties of operators (4.21) and (4.22) differ from the ones for operators (4.23) and (4.24) and need to be proved separately.

Let us consider $\phi \in \mathcal{D}(\mathbb{R}^3)$. Then by (4.11) and (4.12) we have

$$\begin{aligned}\mathring{\mathcal{Q}}_j\phi &= -\partial_j N_\Delta\phi = -\int_{\mathbb{R}^3}\frac{\partial E_\Delta}{\partial y_j}(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\mathbb{R}^3}\frac{\partial E_\Delta}{\partial x_j}(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{x})\,d\boldsymbol{x}\\ &= -\int_{\mathbb{R}^3}E_\Delta(\boldsymbol{x},\boldsymbol{y})\frac{\partial\phi(\boldsymbol{x})}{\partial x_j}\,d\boldsymbol{x} = -N_\Delta(\partial_j\phi).\end{aligned} \tag{4.30}$$

For any $h \in L_2(\mathbb{R}^3)$,

$$\begin{aligned}\|\partial_j h\|_{\mathcal{H}^{-1}(\mathbb{R}^n)} &= \sup_{\xi\in\mathcal{D}(\mathbb{R}^3),\|\xi\|_{\mathcal{H}^1(\mathbb{R}^n)}=1}|\langle\partial_j h,\xi\rangle_{\mathbb{R}^3}| = \sup_{\xi\in\mathcal{D}(\mathbb{R}^3),\|\xi\|_{\mathcal{H}^1(\mathbb{R}^n)}=1}|\langle h,\partial_j\xi\rangle_{\mathbb{R}^3}|\\ &\le \sup_{\xi\in\mathcal{D}(\mathbb{R}^3),\|\xi\|_{\mathcal{H}^1(\mathbb{R}^n)}=1}\|h\|_{L_2(\mathbb{R}^3)}\|\partial_j\xi\|_{L_2(\mathbb{R}^3)} \le \|h\|_{L_2(\mathbb{R}^3)}.\end{aligned} \tag{4.31}$$

Due to the density of $\mathcal{D}(\mathbb{R}^3)$ in $\mathcal{H}^1(\mathbb{R}^3)$, this implies that $\partial_j h \in \mathcal{H}^{-1}(\mathbb{R}^n)$ and moreover the operator $\partial_j : L_2(\mathbb{R}^3) \to \mathcal{H}^{-1}(\mathbb{R}^3)$ is continuous.

As a result, the density of $\mathcal{D}(\mathbb{R}^3)$ in $L_2(\mathbb{R}^3)$ and the continuity of operator (4.18) in (4.30) imply that $\mathring{\mathcal{Q}}_j\phi = -\mathcal{N}_\Delta(\partial_j\phi) \in \mathcal{H}^1(\mathbb{R}^3)$ for any $\phi \in L_2(\mathbb{R}^3)$ and moreover, the operator $\mathring{\mathcal{Q}}_j : L_2(\mathbb{R}^3) \to \mathcal{H}^1(\mathbb{R}^3)$ is continuous. Then operator (4.21) and thus operator (4.22) are continuous as well. □

Theorem 4.3. *The following operators are continuous under condition 2.1*

$$\boldsymbol{V} : \boldsymbol{H}^{-1/2}(\partial\Omega) \to \boldsymbol{\mathcal{H}}^1(\Omega), \tag{4.32}$$

$$\Pi^s : \boldsymbol{H}^{-1/2}(\partial\Omega) \to L_2(\Omega), \tag{4.33}$$

$$\boldsymbol{W} : \boldsymbol{H}^{1/2}(\partial\Omega) \to \boldsymbol{\mathcal{H}}^1(\Omega), \tag{4.34}$$

$$\Pi^d : \boldsymbol{H}^{1/2}(\partial\Omega) \to L_2(\Omega). \tag{4.35}$$

Proof. Let us consider relations (4.9) and (4.10). The continuity of the operators $\boldsymbol{V}$, Π^s, $\boldsymbol{W}$ and Π^d then follows from the continuity of the operators $\mathring{\boldsymbol{V}}$, $\mathring{\boldsymbol{W}}$, $\mathring{\Pi}^s$ and $\mathring{\Pi}^d$ which has already being proved in [15, Lemma 3.3]. □

In the proofs further further, second order derivatives of the coefficient $\mu(\boldsymbol{x})$ will appear and apart from Condition 2.1, we will sometimes need to assume the following additional condition.

Condition 4.4.

$$\mu \in \mathcal{C}^2(\mathbb{R}^3) \ : \ \rho^2 \partial_j\partial_i\mu \in L_\infty(\mathbb{R}^3). \tag{4.36}$$

Theorem 4.5. *The following operators are continuous under Conditions 2.1 and 4.4,*

$$(\Pi^s, \boldsymbol{V}) : \boldsymbol{H}^{-1/2}(\partial\Omega) \to \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \tag{4.37}$$

$$\left(\Pi^d, \boldsymbol{W}\right) : \boldsymbol{H}^{1/2}(\partial\Omega) \to \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \tag{4.38}$$

$$\left(\mathring{\mathcal{Q}}, \boldsymbol{\mathcal{U}}\right) : \boldsymbol{L}_2(\rho; \Omega) \to \boldsymbol{\mathcal{H}}^{1,0}(\mathbb{R}^3; \boldsymbol{\mathcal{A}}), \tag{4.39}$$

$$(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) : \boldsymbol{\mathcal{H}}^1(\Omega) \to \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \tag{4.40}$$

$$\left(\frac{4}{3}\mu I, -\boldsymbol{\mathcal{Q}}\right) : L_2(\Omega) \to \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}). \tag{4.41}$$

Proof. Let us consider first the single layer potentials $(\Pi^s\boldsymbol{h}, \boldsymbol{V}\boldsymbol{h}) \in \boldsymbol{\mathcal{H}}^1(\Omega) \times L_2(\Omega)$ for $\boldsymbol{h} \in \boldsymbol{H}^{-1/2}(\partial\Omega)$. Let us apply the operator $\boldsymbol{\mathcal{A}}$ taking into consideration (4.9) and (4.10)

$$\begin{aligned}\mathcal{A}_j(\Pi^s\boldsymbol{h}, \boldsymbol{V}\boldsymbol{h}) &= \mathcal{A}_j\left(\mathring{\Pi}^s\boldsymbol{h}, \frac{1}{\mu}\mathring{\boldsymbol{V}}\boldsymbol{h}\right)\\ &= \mathring{\mathcal{A}}_j\left(\mathring{\Pi}^s\boldsymbol{h}, \mathring{V}_k\boldsymbol{h}\right) + \partial_k\left(\mu\left[\partial_j(1/\mu)\mathring{V}_k\boldsymbol{h} + \partial_k(1/\mu)\mathring{V}_j\boldsymbol{h} - \frac{2}{3}\delta_j^k\partial_i(1/\mu)\mathring{V}_i\boldsymbol{h}\right]\right).\end{aligned}$$

Now, the term $\mathring{\mathcal{A}}_j(\mathring{\Pi}^s\boldsymbol{h}, \mathring{V}_k\boldsymbol{h})$ vanishes and due Conditions 2.1 and 4.4, the last term belongs to $\boldsymbol{L}_2(\rho; \Omega)$ since $\mathring{\boldsymbol{V}}\boldsymbol{h} \in \boldsymbol{\mathcal{H}}^1(\Omega)$, which implies the continuity of operator (4.37).

The same argument works for the double layer potential $\left(\boldsymbol{W}, \Pi^d\right)\boldsymbol{h}$ with $\boldsymbol{h} \in \boldsymbol{H}^{1/2}(\partial\Omega)$and implies the continuity of operator (4.38). In addition it works for the Newtonian potentials $(\boldsymbol{\mathcal{U}}, \mathcal{Q})$ with the sole difference that $\mathring{\mathcal{A}}_j\left(\mathring{\mathcal{Q}}\boldsymbol{h}, \mathring{\mathcal{U}}_k\boldsymbol{h}\right) = h_j$ and $\boldsymbol{h} \in \boldsymbol{L}_2(\rho; \Omega)$. This implies the continuity of operator (4.39).

For operator (4.40), $\boldsymbol{h} \in \boldsymbol{\mathcal{H}}^1(\Omega) \subset \boldsymbol{L}_2(\rho^{-1}; \Omega)$ and hence the operator $(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) : \boldsymbol{\mathcal{H}}^1(\Omega) \to L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)$ is continuous due to Theorem 4.2. Let us prove that $\boldsymbol{\mathcal{A}}(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) : \boldsymbol{\mathcal{H}}^1(\Omega) \to \boldsymbol{L}_2(\rho; \Omega)$ is continuous. Indeed, by (2.2),

$$\mathcal{A}_j(\mathcal{R}^\bullet\boldsymbol{h}, \mathcal{R}\boldsymbol{h}) = \mathring{\mathcal{A}}_j(\mathcal{R}^\bullet\boldsymbol{h}, \mu\boldsymbol{\mathcal{R}}\boldsymbol{h}) - 2\,\partial_i\mathbb{M}_{ij}(\boldsymbol{\mathcal{R}}\boldsymbol{h}), \tag{4.42}$$

where

$$\mathbb{M}_{ij}(\mathbf{u}) := \frac{1}{2}(u_j\partial_i\mu + u_i\partial_j\mu) - \frac{1}{3}\delta_{ij}u_l\partial_l\mu.$$

Hence due to Theorem 4.2 and Conditions 2.1 and 4.4, the operator $\partial_i \mathbb{M}_{ij}\boldsymbol{\mathcal{R}}$: $\boldsymbol{\mathcal{H}}^1(\Omega)\to L_2(\rho;\Omega)$ is continuous. Moreover, by (4.6), (4.8) and (4.15), $\mathring{\mathcal{A}}_j(\mathcal{R}^\bullet \boldsymbol{h}, \mu\boldsymbol{\mathcal{R}}\boldsymbol{h}) = -2\,\partial_i\mathbb{M}_{ij}(\boldsymbol{h})$, hence by Conditions 2.1 and 4.4 the operator $\mathring{\mathcal{A}}_j(\mathcal{R}^\bullet, \mu\boldsymbol{\mathcal{R}}) : \boldsymbol{\mathcal{H}}^1(\Omega)\to L_2(\rho;\Omega)$ is continuous. Then (4.42) implies the continuity of operator $\boldsymbol{\mathcal{A}}(\mathcal{R}^\bullet, \mu\boldsymbol{\mathcal{R}})$: $\boldsymbol{\mathcal{H}}^1(\Omega) \to \boldsymbol{L}_2(\rho;\Omega)$ and hence of operator (4.40).

For operator (4.41) we proceed in a similar manner to obtain that

$$\boldsymbol{\mathcal{A}}\left(\frac{4}{3}\mu h, -\boldsymbol{\mathcal{Q}}h\right) = \boldsymbol{\mathcal{A}}\left(\frac{4}{3}\mu h, -\frac{1}{\mu}\mathring{\boldsymbol{\mathcal{Q}}}(\mu h)\right)$$
$$= \mathring{\mathcal{A}}_j\left(\frac{4}{3}\mu h, -\mathring{\boldsymbol{\mathcal{Q}}}(\mu h)\right) + 2\,\partial_i\mathbb{M}_{ij}(\boldsymbol{\mathcal{Q}}h) = 2\,\partial_i\mathbb{M}_{ij}(\boldsymbol{\mathcal{Q}}h)$$

due to (4.16). By the continuity of operator (4.22) in Theorem 4.2 and due to Conditions 2.1 and 4.4, the operator $\partial_i\mathbb{M}_{ij}\boldsymbol{\mathcal{Q}} : L_2(\Omega) \to \boldsymbol{L}_2(\rho;\Omega)$ is continuous, implying the continuity of operator (4.41). $\square$

Let us now define direct values on the boundary of the parametrix-based velocity single layer and double layer potentials and introduce the notations for the conormal derivative of the latter, for sufficiently smooth scalar and vector functions h and $\boldsymbol{h}$ on $\partial\Omega$, e.g., $h \in \mathcal{D}(\partial\Omega)$, $\boldsymbol{h} \in \boldsymbol{\mathcal{D}}(\partial\Omega)$,

$$[\boldsymbol{\mathcal{V}}\boldsymbol{h}]_k(\boldsymbol{y}) = \mathcal{V}_{kj}h_j(\boldsymbol{y}) := -\int_{\partial\Omega} u_j^k(\boldsymbol{x},\boldsymbol{y})h_j(\boldsymbol{x})\,dS(\boldsymbol{x}), \qquad \boldsymbol{y}\in\partial\Omega, \quad (4.43)$$

$$[\boldsymbol{\mathcal{W}}\boldsymbol{h}]_k(\boldsymbol{y}) = \mathcal{W}_{kj}h_j(\boldsymbol{y}) := -\int_{\partial\Omega} T_j^c(\boldsymbol{x}; q^k, \boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y})h_j(\boldsymbol{x})\,dS(\boldsymbol{x}), \qquad \boldsymbol{y}\in\partial\Omega, \quad (4.44)$$

$$[\boldsymbol{\mathcal{W}}'\boldsymbol{h}]_k(\boldsymbol{y}) = \mathcal{W}'_{kj}h_j(\boldsymbol{y}) := -\int_{\partial\Omega} T_j^c(\boldsymbol{y}; q^k, \boldsymbol{u}^k)(\boldsymbol{x},\boldsymbol{y})h_j(\boldsymbol{x})\,dS(\boldsymbol{x}), \qquad \boldsymbol{y}\in\partial\Omega, \quad (4.45)$$

$$\boldsymbol{\mathcal{L}}^\pm\boldsymbol{h}(\boldsymbol{y}) := \boldsymbol{T}^\pm(\Pi^d\boldsymbol{h}, \boldsymbol{W}\boldsymbol{h})(\boldsymbol{y}), \qquad \boldsymbol{y}\in\partial\Omega. \quad (4.46)$$

Here $\boldsymbol{T}^\pm$ are the canonical derivative (traction) operators for the *compressible* fluid that are well defined due to Theorem 4.5.

Similar to the potentials in the domain, we can also express the boundary operators in terms of their counterparts with the constant coefficient $\mu = 1$,

$$\boldsymbol{\mathcal{V}}\boldsymbol{h} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{h}, \qquad \boldsymbol{\mathcal{W}}\boldsymbol{h} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{W}}}(\mu\boldsymbol{h}), \quad (4.47)$$

$$[\boldsymbol{\mathcal{W}}'\boldsymbol{h}]_k = [\mathring{\boldsymbol{\mathcal{W}}}'\boldsymbol{h}]_k - \left(\frac{\partial_i\mu}{\mu}[\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{h}]_k + \frac{\partial_k\mu}{\mu}[\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{h}]_i - \frac{2}{3}\delta_i^k\frac{\partial_j\mu}{\mu}[\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{h}]_j\right)n_i. \quad (4.48)$$

We will further use relations (4.47) and (4.48) as definitions of the potentials $\boldsymbol{\mathcal{V}}\boldsymbol{h}$, $\boldsymbol{\mathcal{W}}\boldsymbol{h}$, and $\boldsymbol{\mathcal{W}}'\boldsymbol{h}$ when their densities h and $\boldsymbol{h}$ are more general functions or distributions on $\partial\Omega$.

Theorem 4.6. *Let $s \in \mathbb{R}$. Let S_1 and S_2 be two non empty manifolds on $\partial\Omega$ with smooth boundaries ∂S_1 and ∂S_2, respectively. Then the following operators are continuous under Conditions 2.1 and 4.4,*

$$\boldsymbol{\mathcal{V}} : \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1}(\partial\Omega), \qquad \boldsymbol{\mathcal{W}} : \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1}(\partial\Omega), \quad (4.49)$$

$$r_{S_2}\boldsymbol{\mathcal{V}} : \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^{s+1}(S_2), \qquad r_{S_2}\boldsymbol{\mathcal{W}} : \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^{s+1}(S_2), \quad (4.50)$$

$$\boldsymbol{\mathcal{L}}^\pm : \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s-1}(\partial\Omega), \qquad \boldsymbol{\mathcal{W}}' : \boldsymbol{H}^s(\partial\Omega) \to \boldsymbol{H}^{s+1}(\partial\Omega). \quad (4.51)$$

Moreover, the following operators are compact,

$$r_{S_2}\boldsymbol{\mathcal{V}} : \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^s(S_2), \tag{4.52}$$

$$r_{S_2}\boldsymbol{\mathcal{W}} : \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^s(S_2), \tag{4.53}$$

$$r_{S_2}\boldsymbol{\mathcal{W}}' : \widetilde{\boldsymbol{H}}^s(S_1) \to \boldsymbol{H}^s(S_2). \tag{4.54}$$

Proof. As in Theorem 4.4 of [9], the continuity of operators in (4.49)-(4.51) follows from relations (4.47)-(4.48) and continuity of the counterpart operators for the constant coefficient case, see e.g. [17, 14]. Then compactness of operators (4.52)-(4.54) is implied by the Rellich compactness embedding theorem. □

Theorem 4.7. *If $\boldsymbol{\tau} \in \boldsymbol{H}^{1/2}(\partial\Omega)$, $\boldsymbol{h} \in \boldsymbol{H}^{-1/2}(\partial\Omega)$, then the following relations hold on $\partial\Omega$ under Conditions 2.1 and 4.4,*

$$\gamma^{\pm}\boldsymbol{V}\boldsymbol{h} = \boldsymbol{\mathcal{V}}\boldsymbol{h}, \qquad \gamma^{\pm}\boldsymbol{W}\boldsymbol{\tau} = \mp\frac{1}{2}\boldsymbol{\tau} + \boldsymbol{\mathcal{W}}\boldsymbol{\tau} \tag{4.55}$$

$$\boldsymbol{T}^{\pm}(\Pi^s\boldsymbol{h}, \boldsymbol{V}\boldsymbol{h}) = \pm\frac{1}{2}\boldsymbol{h} + \boldsymbol{\mathcal{W}}'\boldsymbol{h}. \tag{4.56}$$

Proof. The proof of the theorem directly follows from relations (4.9), (4.47)-(4.48) and the analogous jump properties for the counterparts of the operators for the constant coefficient case of $\mu = 1$, see, e.g., [14, Lemma 5.6.5]. □

Let us introduce the notations

$$\mathring{\boldsymbol{\mathcal{L}}}\boldsymbol{\tau}(\boldsymbol{y}) = \mathring{\boldsymbol{\mathcal{L}}}^{\pm}\boldsymbol{\tau}(\boldsymbol{y}) := \mathring{\boldsymbol{T}}^{\pm}(\mathring{\Pi}^d\boldsymbol{\tau}, \mathring{\boldsymbol{W}}\boldsymbol{\tau})(\boldsymbol{y}), \quad \widehat{\boldsymbol{\mathcal{L}}}\boldsymbol{\tau}(\boldsymbol{y}) := \mathring{\boldsymbol{\mathcal{L}}}(\mu\boldsymbol{\tau})(\boldsymbol{y}), \quad \boldsymbol{y} \in \partial\Omega, \tag{4.57}$$

where the first equality is implied by the Lyapunov-Tauber theorem for the constant-coefficient Stokes potentials.

The following theorem is proved similar to [9, Theorem 4.6] but with different spaces involved.

Theorem 4.8. *Let Conditions 2.1 and 4.4 hold and $\boldsymbol{\tau} \in \boldsymbol{H}^{1/2}(\partial\Omega)$. Then*

$$(\mathcal{L}_k^{\pm} - \widehat{\mathcal{L}}_k)\boldsymbol{\tau} = \gamma^{\pm}\left(\mu\left[\partial_i\left(\frac{1}{\mu}\right)\mathring{W}_k(\mu\boldsymbol{\tau}) + \partial_k\left(\frac{1}{\mu}\right)\mathring{W}_i(\mu\boldsymbol{\tau}) - \frac{2}{3}\delta_i^k\partial_j\left(\frac{1}{\mu}\right)\mathring{W}_j(\mu\boldsymbol{\tau})\right]\right)n_i. \tag{4.58}$$

Proof. By Theorem 4.5, the operator $(\Pi^d, \boldsymbol{W}) : \boldsymbol{H}^{1/2}(\partial\Omega) \to \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ is continuous. By Theorem 2.3, there exists a sequence $(\pi^m, \boldsymbol{w}^m)_{m=1}^{\infty} \subset \mathcal{D}(\overline{\Omega}) \times \boldsymbol{\mathcal{D}}(\overline{\Omega})$ converging to $(\mathring{\Pi}^d(\mu\boldsymbol{\tau}), \mathring{\boldsymbol{W}}(\mu\boldsymbol{\tau}))$ in $\boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$. Then, due to (4.9)-(4.10), the sequence $(\pi^m, \frac{1}{\mu}\boldsymbol{w}^m)_{m=1}^{\infty} \subset \mathcal{D}(\overline{\Omega}) \times \boldsymbol{\mathcal{D}}(\overline{\Omega})$ converges to $(\mathring{\Pi}^d(\mu\boldsymbol{\tau}), \frac{1}{\mu}\mathring{\boldsymbol{W}}(\mu\boldsymbol{\tau})) = (\Pi^d\boldsymbol{\tau}, \boldsymbol{W}\boldsymbol{\tau})$ in $\boldsymbol{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ and by continuity of the canonical traction operators $\boldsymbol{T}^{\pm} : \boldsymbol{H}^{1,0}(\Omega^{\pm}; \boldsymbol{\mathcal{A}}) \to \boldsymbol{H}^{-1/2}(\partial\Omega)$, and definition (4.46) we can establish the following equality

$$\mathcal{L}_k^{\pm}\boldsymbol{\tau} := T_k^{\pm}(\Pi^d\boldsymbol{\tau}, \boldsymbol{W}\boldsymbol{\tau}) = T_k^{\pm}(\Pi^d\boldsymbol{\tau}, \boldsymbol{W}\boldsymbol{\tau}) = \lim_{m\to\infty} T_k^{\pm}(\pi^m, \frac{1}{\mu}\boldsymbol{w}^m). \tag{4.59}$$

On the other hand,

$$T_k^{\pm}(\pi^m, \frac{1}{\mu}\boldsymbol{w}^m) = T_k^{c\pm}(\pi^m, \frac{1}{\mu}\boldsymbol{w}^m) = \gamma^{\pm}\sigma_{ik}(\pi^m, \frac{1}{\mu}\boldsymbol{w}^m)n_i$$
$$= \gamma^{\pm}\mathring{\sigma}_{ik}(\pi^m, \boldsymbol{w}^m)n_i + \gamma^{\pm}\left(\mu\left[\partial_i\left(\frac{1}{\mu}\right)w_k^m + \partial_k\left(\frac{1}{\mu}\right)w_i^m - \frac{2}{3}\delta_i^k\partial_j\left(\frac{1}{\mu}\right)w_j^m\right]\right)n_i$$
$$\to \mathring{\mathcal{L}}_k^{\pm}(\mu\boldsymbol{\tau}) + G^{\pm}(\boldsymbol{\tau}) \quad \text{as } m \to \infty,$$

where

$$G^{\pm}(\boldsymbol{\tau}) := \gamma^{\pm}\left(\mu\left[\partial_i\left(\frac{1}{\mu}\right)\mathring{W}_k(\mu\boldsymbol{\tau}) + \partial_k\left(\frac{1}{\mu}\right)\mathring{W}_i(\mu\boldsymbol{\tau}) - \frac{2}{3}\delta_i^k\partial_j\left(\frac{1}{\mu}\right)\mathring{W}_j(\mu\boldsymbol{\tau})\right]\right)n_i$$

since

$$\gamma^{\pm}\mathring{\sigma}_{ik}(\pi^m,\boldsymbol{w}^m)n_i = \mathring{T}_k^{c\pm}(\pi^m,\boldsymbol{w}^m) = \mathring{T}_k^{\pm}(\pi^m,\boldsymbol{w}^m) \to \mathring{T}_k^{\pm}\left(\mathring{\Pi}^d(\mu\boldsymbol{\tau}), \mathring{\boldsymbol{W}}(\mu\boldsymbol{\tau})\right) = \mathring{\mathcal{L}}_k^{\pm}(\mu\boldsymbol{\tau}).$$

This implies (4.58). □

Similar to [9, Corollary 4.7], the next assertion follows from Theorems 4.8 and 4.5.

Corollary 4.9. *Let S_1 be a non-empty submanifold of $\partial\Omega$ with smooth boundary and Conditions 2.1 and 4.4 hold. Then, the operators*

$$\widehat{\boldsymbol{\mathcal{L}}} : \widetilde{\boldsymbol{H}}^{1/2}(S_1) \to \boldsymbol{H}^{-1/2}(\partial\Omega), \quad (\boldsymbol{\mathcal{L}}^{\pm} - \widehat{\boldsymbol{\mathcal{L}}}) : \widetilde{\boldsymbol{H}}^{1/2}(S_1) \to \boldsymbol{H}^{1/2}(\partial\Omega), \tag{4.60}$$

are continuous and the operators

$$(\boldsymbol{\mathcal{L}}^{\pm} - \widehat{\boldsymbol{\mathcal{L}}}) : \widetilde{\boldsymbol{H}}^{1/2}(S_1) \to \boldsymbol{H}^{-1/2}(\partial\Omega), \tag{4.61}$$

are compact.

For bounded domains, we had compactness of the remainder operators $\boldsymbol{\mathcal{R}}$ and $\mathcal{R}^{\bullet}$ implied by the Rellich compact embedding theorem, which does not hold for exterior (unbounded) domains considered in this paper. To overcome this issue, we prove that the for exterior domains the operators $\boldsymbol{\mathcal{R}}$ and $\mathcal{R}^{\bullet}$ are limits of some sequences of compact operators and thus are also compact. We will require the following condition.

Condition 4.10. $\lim_{|x|\to\infty}\rho(\boldsymbol{x})\nabla\mu(\boldsymbol{x}) = 0.$

The proof of the following assertion is similar to [6, Lemma 7.4] for the corresponding scalar case.

Lemma 4.11. *Let Conditions 2.1 and 4.10 hold. For any sufficiently large $\eta>0$,*
(i) the operator $\boldsymbol{\mathcal{R}}$ can be represented as $\boldsymbol{\mathcal{R}}=\boldsymbol{\mathcal{R}}_{s,\eta}+\boldsymbol{\mathcal{R}}_{c,\eta}$, where $\|\boldsymbol{\mathcal{R}}_{s,\eta}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)} \to 0$ as $\eta\to\infty$, while $\boldsymbol{\mathcal{R}}_{c,\eta} : \boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)$ is compact;
(ii) the operator $\mathcal{R}^{\bullet}$ can be represented as $\mathcal{R}^{\bullet}=\mathcal{R}^{\bullet}_{s,\eta}+\mathcal{R}^{\bullet}_{c,\eta}$, where $\|\mathcal{R}^{\bullet}_{s,\eta}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)\to L_2(\Omega)} \to 0$ as $\eta\to\infty$, while $\mathcal{R}^{\bullet}_{c,\eta} : \boldsymbol{\mathcal{H}}^1(\Omega)\to L_2(\Omega)$ is compact.

Proof. (i) Let $B(\mathbf{0},\eta)$ be a ball with centre in $\mathbf{0}\in\mathbb{R}^3$ and radius $\eta>0$ large enough such that $\partial\Omega\subseteq B(\mathbf{0},\eta)$. Consider a cut-ff function $\chi\in\mathcal{D}(B(\mathbf{0},2\eta))$ such that $0\le\chi\le 1$ in $\mathbb{R}^3$ and $\chi=1$ in $\overline{B}(\mathbf{0},\eta)$. Let us now define two operators,

$$\boldsymbol{\mathcal{R}}_{c,\eta}\boldsymbol{g} := \boldsymbol{\mathcal{R}}(\chi\boldsymbol{g}), \quad \boldsymbol{\mathcal{R}}_{s,\eta}\boldsymbol{g} := \boldsymbol{\mathcal{R}}((1-\chi)\boldsymbol{g}) \quad \textit{for } \boldsymbol{g}\in\boldsymbol{\mathcal{H}}^1(\Omega). \tag{4.62}$$

Taking into account (4.6), the divergence theorem (cf. (4.29)) and that $1-\chi=0$ on $\partial\Omega$, we obtain,

$$\begin{aligned}\|\boldsymbol{\mathcal{R}}_{s,\eta}\boldsymbol{g}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)} = \|\boldsymbol{\mathcal{R}}((1-\chi)\boldsymbol{g})\|_{\boldsymbol{\mathcal{H}}^1(\Omega)} &= \left\|\frac{1}{\mu}\left(\partial_j\mathring{\mathcal{U}}_{\cdot i}h_{ij} + \partial_i\mathring{\mathcal{U}}_{\cdot j}h_{ij} - \mathring{\mathcal{Q}}_{\cdot}h_{jj}\right)\right\|_{\boldsymbol{\mathcal{H}}^1(\Omega)}\\ &= \left\|\frac{1}{\mu}\left(\mathring{\mathcal{U}}_{\cdot i}(\partial_j h_{ij}) + \mathring{\mathcal{U}}_{\cdot j}(\partial_i h_{ij}) - \mathring{\mathcal{Q}}_{\cdot}h_{jj}\right)\right\|_{\boldsymbol{\mathcal{H}}^1(\Omega)}\\ &\le C_4(\mu)\big(\|\mathring{\mathcal{U}}\|_{\widetilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)}\|\partial_i(h_{i\cdot}+h_{\cdot i})\|_{\widetilde{\mathcal{H}}^{-1}(\Omega)}\end{aligned}$$

$$+ \|\mathring{\mathcal{Q}}\|_{L_2(\rho;\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)}\|\partial_i\|h_{jj}\|_{L_2(\Omega)}\big),$$

where $h_{ij} := (1-\chi)g_j\partial_i\mu$, $C_4(\mu)$ is from Remark 2.2 We also have the following estimates,

$$\begin{aligned}\|\partial_i(h_{i\cdot}+h_{\cdot i})\|_{\widetilde{\mathcal{H}}^{-1}(\Omega)} &\le \|(h+h^\top)\|_{L_2(\Omega)^{3\times3}} \le 2\|[(1-\chi)\boldsymbol{g}\otimes\nabla\mu]\|_{L_2(\Omega)^{3\times3}}\\ &\le 2\,\|\boldsymbol{g}\|_{\boldsymbol{L}_2(\rho^{-1};\Omega)}\|\rho\nabla\mu\|_{\boldsymbol{L}_\infty(\mathbb{R}^3\setminus B(\boldsymbol{0},\eta))}\\ &\le 2\|\boldsymbol{g}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)}\|\rho\nabla\mu\|_{\boldsymbol{L}_\infty(\mathbb{R}^3\setminus B(\boldsymbol{0},\eta))},\end{aligned}$$

$$\begin{aligned}\|h_{jj}\|_{L_2(\Omega)} &= \|(1-\chi)\boldsymbol{g}\cdot\nabla\mu\|_{L_2(\Omega)} \le \|\boldsymbol{g}\|_{\boldsymbol{L}_2(\rho^{-1};\Omega)}\|\rho\nabla\mu\|_{\boldsymbol{L}_\infty(\mathbb{R}^3\setminus B(\boldsymbol{0},\eta))}\\ &\le \|\boldsymbol{g}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)}\|\rho\nabla\mu\|_{\boldsymbol{L}_\infty(\mathbb{R}^3\setminus B(\boldsymbol{0},\eta))}.\end{aligned}$$

Then we obtain the following estimate for the norm of $\boldsymbol{\mathcal{R}}_{s,\eta}\boldsymbol{g}$

$$\begin{aligned}&\|\boldsymbol{\mathcal{R}}_{s,\eta}\boldsymbol{g}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)} \\ &\le C_4(\mu)\left(2\mathring{\mathcal{U}}\|_{\widetilde{\mathcal{H}}^{-1}(\Omega)\to\mathcal{H}^1(\Omega)} + \|\mathring{\mathcal{Q}}\|_{L_2(\rho;\Omega)\to\mathcal{H}^1(\Omega)}\right) \times \|\boldsymbol{g}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)}\|\rho\nabla\mu\|_{\boldsymbol{L}_\infty(\mathbb{R}^3\setminus B(\boldsymbol{0},\eta))}.\end{aligned} \tag{4.63}$$

Taking the limit as $\eta\to\infty$ in (4.63), we have $\|\rho\nabla\mu\|_{\boldsymbol{L}_\infty(\mathbb{R}^3\setminus B(\boldsymbol{0},\eta))}\to 0$ by virtue of Condition 4.10. Therefore $\|\boldsymbol{\mathcal{R}}_{s,\eta}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)}\to 0$ as $\eta\to\infty$, which completes the proof for operator $\boldsymbol{\mathcal{R}}_{s,\eta}$.

Let us prove now that $\boldsymbol{\mathcal{R}}_{c,\eta}\boldsymbol{g}$ is compact. By definition, $\chi = 0$ if $\boldsymbol{y}$ in $\mathbb{R}^3\setminus\overline{B}(\boldsymbol{0},2\eta)$ and hence the multiplication by χ is a continuous mapping from $\boldsymbol{\mathcal{H}}^1(\Omega)$ to $\widetilde{\boldsymbol{H}}^1(\Omega)$. Consequently, the operator $\boldsymbol{\mathcal{R}}_{c,\eta}:\boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)$, satisfies the following relations

$$\boldsymbol{\mathcal{R}}_{c,\eta}\boldsymbol{g} = \boldsymbol{\mathcal{R}}(\chi\boldsymbol{g}) = \boldsymbol{\mathcal{R}}\mathring{E}_{\Omega_{2\eta}}r_{\Omega_{2\eta}}(\chi\boldsymbol{g}) = \boldsymbol{\mathcal{R}}\mathring{E}_{\Omega_{2\eta}}\mathfrak{E}\,r_{\Omega_{2\eta}}(\chi\boldsymbol{g}). \tag{4.64}$$

Here $r_{\Omega_{2\eta}}$ is the continuous restriction operator from $\widetilde{\boldsymbol{H}}^1(\Omega)$ to $\mathring{\boldsymbol{H}}^1(\Omega_{2\eta})$, where $\mathring{\boldsymbol{H}}^1(\Omega_{2\eta})\subset\boldsymbol{H}^1(\Omega_{2\eta})$ is the completion of space $\boldsymbol{\mathcal{D}}(\Omega_{2\eta})$ in the norm of $\boldsymbol{H}^1(\Omega_{2\eta})$, $\mathring{E}_{\Omega_{2\eta}}$ is the operator of extension by zero outside $\Omega_{2\eta}$ for functions defined in $\Omega_{2\eta}$ and it is a continuous operator from $\mathring{\boldsymbol{H}}^1(\Omega_{2\eta})$ to $\boldsymbol{\mathcal{H}}^1(\Omega)$ and from $\boldsymbol{L}_2(\Omega_{2\eta})$ to $\boldsymbol{L}_2(\Omega)$, $\mathfrak{E}:\boldsymbol{H}^1(\Omega_{2\eta})\to\boldsymbol{L}_2(\Omega_{2\eta})$ is the embedding operator that is compact on the bounded domain $\Omega_{2\eta}$ due to the Rellich compact embedding theorem. The operator $\boldsymbol{\mathcal{R}}:\boldsymbol{L}_2(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)$ is continuous by virtue of Theorem 4.2. Then the continuity of all operators in the right-hand side of (4.64) and the compactness of one of them imply (see e.g. [4, Proposition 6.3]) that the operator $\boldsymbol{\mathcal{R}}_{c,\eta}:\boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)$ is compact.

(ii) Reasoning in a similar way, we can obtain the corresponding result for the remainder pressure operator. To this end, let us define the operators

$$\mathcal{R}^\bullet_{c,\eta}\boldsymbol{g} := \mathcal{R}^\bullet(\chi\boldsymbol{g}),\quad \mathcal{R}^\bullet_{s,\eta}\boldsymbol{g} := \mathcal{R}^\bullet((1-\chi)\boldsymbol{g})\quad \text{for } \boldsymbol{g}\in\boldsymbol{\mathcal{H}}^1(\Omega). \tag{4.65}$$

Taking into account the relations (4.8), we can obtain the following inequality

$$\begin{aligned}\|\mathcal{R}^\bullet_{s,\eta}\boldsymbol{g}\|_{L_2(\Omega)} &= \|\mathcal{R}^\bullet((1-\chi)\boldsymbol{g})\|_{L_2(\Omega)} = 2\|\partial_i\mathring{\mathcal{Q}}_j h_{ij} + h_{jj}\|_{L_2(\Omega)}\\ &\le 2\|\mathring{\mathcal{Q}}_j h_{\cdot j}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)} + 2\|h_{jj}\|_{L_2(\Omega)} \le 2(\|\mathring{Q}\|_{L_2(\Omega)\to\mathcal{H}^1(\Omega)}+1)\|h\|_{L_2(\Omega)^{3\times3}},\end{aligned}$$

where

$$\begin{aligned}\|h\|_{L_2(\Omega)^{3\times3}} &= \|[(1-\chi)\boldsymbol{g}\otimes\nabla\mu]\|_{L_2(\Omega)^{3\times3}}\\ &\le \|\boldsymbol{g}\|_{\boldsymbol{L}_2(\rho^{-1};\Omega)}\|\rho\nabla\mu\|_{\boldsymbol{L}_\infty(\mathbb{R}^3\setminus B(\boldsymbol{0},\eta))} \le \|\boldsymbol{g}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)}\|\rho\nabla\mu\|_{\boldsymbol{L}_\infty(\mathbb{R}^3\setminus B(\boldsymbol{0},\eta))}.\end{aligned}$$

Hence

$$\|\mathcal{R}^\bullet_{s,\eta}\boldsymbol{g}\|_{L_2(\Omega)} \le 2(\|\mathring{Q}\|_{L_2(\Omega)\to\mathcal{H}^1(\Omega)}+1)\|\boldsymbol{g}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)}\|\rho\nabla\mu\|_{\boldsymbol{L}_\infty(\mathbb{R}^3\setminus B(\boldsymbol{0},\eta))}. \tag{4.66}$$

Since $\|\rho\nabla\mu\|_{\boldsymbol{L}_\infty(\mathbb{R}^3\setminus B(\boldsymbol{0},\eta))}\to 0$ as $\eta\to\infty$ by Condition 4.10, inequality (4.66) implies that $\|\mathcal{R}^\bullet_{s,\eta}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)}\to 0$ as $\eta\to\infty$, which completes the proof for operator $\mathcal{R}^\bullet_{s,\eta}$.

Let us prove now that $\mathcal{R}^\bullet_{c,\eta}\boldsymbol{g}$ is compact. First, the operator $\mathcal{R}^\bullet_{c,\eta}:\boldsymbol{\mathcal{H}}^1(\Omega)\to L_2(\Omega)$, satisfies the following relations

$$\mathcal{R}^\bullet_{c,\eta}\boldsymbol{g}=\mathcal{R}^\bullet(\chi\boldsymbol{g})=\mathcal{R}^\bullet\mathring{E}_{\Omega_{2\eta}}r_{\Omega_{2\eta}}(\chi\boldsymbol{g})=\mathcal{R}^\bullet\mathring{E}_{\Omega_{2\eta}}\mathfrak{E}\,r_{\Omega_{2\eta}}(\chi\boldsymbol{g}). \tag{4.67}$$

Here $r_{\Omega_{2\eta}}$ is the continuous restriction operator from $\widetilde{\boldsymbol{H}}^1(\Omega)$ to $\mathring{\boldsymbol{H}}^1(\Omega_{2\eta})$, $\mathring{E}_{\Omega_{2\eta}}$ is the operator of extension by zero outside $\Omega_{2\eta}$ for functions defined in $\Omega_{2\eta}$ and it is a continuous operator from $\mathring{\boldsymbol{H}}^1(\Omega_{2\eta})$ to $\boldsymbol{\mathcal{H}}^1(\Omega)$ and from $\boldsymbol{L}_2(\Omega_{2\eta})$ to $\boldsymbol{L}_2(\rho^{-1};\Omega)$, $\mathfrak{E}:\boldsymbol{H}^1(\Omega_{2\eta})\to\boldsymbol{L}_2(\Omega_{2\eta})$ is the embedding operator that is compact on the bounded domain $\Omega_{2\eta}$ due to the Rellich compact embedding theorem. The operator $\mathcal{R}^\bullet:\boldsymbol{L}_2(\rho^{-1};\Omega)\to L_2(\Omega)$ is continuous by virtue of Theorem 4.2. Then the continuity of all operators in the right-hand side of (4.67) and the compactness of one of them imply that the operator $\mathcal{R}^\bullet_{c,\eta}:\boldsymbol{\mathcal{H}}^1(\Omega)\to L_2(\Omega)$ is compact. □

Theorem 4.12. *Let Conditions 2.1 and 4.10 hold. Then the following operators are compact,*

$$\boldsymbol{\mathcal{R}}:\boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega),\quad \mathcal{R}^\bullet:\boldsymbol{\mathcal{H}}^1(\Omega)\to L_2(\Omega). \tag{4.68}$$

Proof. By Lemma 4.11,

$$\begin{aligned}&\|\boldsymbol{\mathcal{R}}-\boldsymbol{\mathcal{R}}_{c,\eta}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)}=\|\boldsymbol{\mathcal{R}}_{s,\eta}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)}\to 0,\\ &\|\mathcal{R}^\bullet-\mathcal{R}^\bullet_{c,\eta}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)}=\|\mathcal{R}^\bullet_{s,\eta}\|_{\boldsymbol{\mathcal{H}}^1(\Omega)\to\boldsymbol{\mathcal{H}}^1(\Omega)}\to 0\quad\text{as }\eta\to\infty.\end{aligned}$$

Hence the sequence of compact operators $\boldsymbol{\mathcal{R}}_{c,\eta}$ converges to the operator $\boldsymbol{\mathcal{R}}$ and the sequence of compact operators $\mathcal{R}^\bullet_{c,\eta}$ converges to the operator $\mathcal{R}^\bullet$. Then (see e.g. [4, Theorem 6.1]) both operators in (4.68) are compact as well. □

5. **The Third Green identities.** The following assertion presents the third Green identities based on the parametrix $(q^k,\boldsymbol{u}^k)$, in the exterior domain. Its proof is word-for-word with the proof of corresponding result in [9, Theorem 5.1] for the bounded domain, if we replace there $\boldsymbol{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ by $\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ and employ the second Green identity (2.11) and the density Theorem 2.3 in the exterior domain.

Theorem 5.1. *For any $(p,\boldsymbol{v})\in\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$, the following third Green identities hold under Condition 2.1,*

$$p+\mathcal{R}^\bullet\boldsymbol{v}-\Pi^s\boldsymbol{T}^+(p,\boldsymbol{v})+\Pi^d\boldsymbol{\gamma}^+\boldsymbol{v}=\mathring{\mathcal{Q}}\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})+\frac{4\mu}{3}\text{div}\boldsymbol{v}\quad \textit{in }\Omega, \tag{5.1}$$

$$\boldsymbol{v}+\boldsymbol{\mathcal{R}}\boldsymbol{v}-\boldsymbol{V}\boldsymbol{T}^+(p,\boldsymbol{v})+\boldsymbol{W}\boldsymbol{\gamma}^+\boldsymbol{v}=\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{A}}(p,\boldsymbol{v})-\boldsymbol{\mathcal{Q}}\,\text{div}\boldsymbol{v}\quad \textit{in }\Omega. \tag{5.2}$$

If the couple $(p,\boldsymbol{v})\in\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ is a solution of the Stokes PDE system (2.12a)-(2.12b) with variable coefficient, then (5.1) and (5.2) give

$$p+\mathcal{R}^\bullet\boldsymbol{v}-\Pi^s\boldsymbol{T}^+(p,\boldsymbol{v})+\Pi^d\boldsymbol{\gamma}^+\boldsymbol{v}=\mathring{\mathcal{Q}}\boldsymbol{f}+\frac{4\mu}{3}g\quad\text{in }\Omega, \tag{5.3}$$

$$\boldsymbol{v}+\boldsymbol{\mathcal{R}}\boldsymbol{v}-\boldsymbol{V}\boldsymbol{T}^+(p,\boldsymbol{v})+\boldsymbol{W}\boldsymbol{\gamma}^+\boldsymbol{v}=\boldsymbol{\mathcal{U}}\boldsymbol{f}-\boldsymbol{\mathcal{Q}}g\quad\text{in }\Omega. \tag{5.4}$$

Consequently under Condition 2.1 we can obtain the trace and under Conditions 2.1 and 4.4 the traction of the third Green identities for $(p,\boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$ on $\partial\Omega$.

$$\frac{1}{2}\gamma^+\boldsymbol{v} + \gamma^+\boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{\mathcal{V}}\boldsymbol{T}^+(p,\boldsymbol{v}) + \boldsymbol{\mathcal{W}}\gamma^+\boldsymbol{v} = \gamma^+\boldsymbol{\mathcal{U}}\boldsymbol{f} - \gamma^+\boldsymbol{\mathcal{Q}}g, \tag{5.5}$$

$$\frac{1}{2}\boldsymbol{T}^+(p,\boldsymbol{v}) + \boldsymbol{T}^+(\mathcal{R}^\bullet,\boldsymbol{\mathcal{R}})\boldsymbol{v} - \boldsymbol{\mathcal{W}}'\boldsymbol{T}^+(p,\boldsymbol{v}) + \boldsymbol{\mathcal{L}}^+\gamma^+\boldsymbol{v} = \boldsymbol{T}^+(\mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g, \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g). \tag{5.6}$$

Note that the traction operator $\boldsymbol{T}^+$ is well defined on all the potentials in (5.6) due to the mapping properties provided by Theorem 4.5.

The following two assertions are instrumental for the proof of equivalence of the BDIE systems and the BVPs.

Lemma 5.2. *Let conditions 2.1 and 4.4 hold. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{H}}^1(\Omega)$, $p \in L_2(\Omega)$, $g \in L_2(\Omega)$, $\boldsymbol{f} \in \boldsymbol{L}_2(\rho;\Omega)$, $\boldsymbol{\Psi} \in \boldsymbol{H}^{-1/2}(\partial\Omega)$ and $\boldsymbol{\Phi} \in \boldsymbol{H}^{1/2}(\partial\Omega)$ satisfy the equations*

$$p + \mathcal{R}^\bullet\boldsymbol{v} - \Pi^s\boldsymbol{\Psi} + \Pi^d\boldsymbol{\Phi} = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4\mu}{3}g \quad \text{in} \quad \Omega, \tag{5.7}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\Psi} + \boldsymbol{W}\boldsymbol{\Phi} = \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g \qquad \text{in} \quad \Omega. \tag{5.8}$$

Then $(p,\boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$ and the couple solves the equations

$$\boldsymbol{\mathcal{A}}(p,\boldsymbol{v}) = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{v} = g.$$

Moreover, the following relations hold true,

$$\Pi^s(\boldsymbol{\Psi} - \boldsymbol{T}^+(p,\boldsymbol{v})) - \Pi^d(\boldsymbol{\Phi} - \gamma^+\boldsymbol{v}) = 0 \quad \text{in} \quad \Omega, \tag{5.9}$$

$$\boldsymbol{V}(\boldsymbol{\Psi} - \boldsymbol{T}^+(p,\boldsymbol{v})) - \boldsymbol{W}(\boldsymbol{\Phi} - \gamma^+\boldsymbol{v}) = \boldsymbol{0} \quad \text{in} \quad \Omega. \tag{5.10}$$

Proof. By virtue of Theorem 4.5, it is easy to deduce that $(p,\boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega,\boldsymbol{\mathcal{A}})$. The remaining part of the proof follows word-for-word from [9, Lemma 5.3]. □

Lemma 5.3. *Let $\partial\Omega = \overline{S}_1 \cup \overline{S}_2$, where S_1 and S_2 are open non-empty non-intersecting simply connected submanifolds of $\partial\Omega$ with infinitely smooth boundaries. Let $\boldsymbol{\Psi}^* \in \widetilde{\boldsymbol{H}}^{-1/2}(S_1)$, $\boldsymbol{\Phi}^* \in \widetilde{\boldsymbol{H}}^{1/2}(S_2)$. If*

$$\Pi^s(\boldsymbol{\Psi}^*) - \Pi^d(\boldsymbol{\Phi}^*) = \boldsymbol{0}, \qquad \boldsymbol{V}\boldsymbol{\Psi}^*(\boldsymbol{x}) - \boldsymbol{W}\boldsymbol{\Phi}^*(\boldsymbol{x}) = \boldsymbol{0}, \quad \text{in } \Omega, \tag{5.11}$$

then $\boldsymbol{\Psi}^* = \boldsymbol{0}$ and $\boldsymbol{\Phi}^* = \boldsymbol{0}$ on $\partial\Omega$.

Proof. Let us employ relations (4.10) in the first equation in (5.11) to obtain

$$\mathring{\Pi}^s\boldsymbol{\Psi}^* - \mathring{\Pi}^d(\mu\boldsymbol{\Phi}^*) = 0 \quad \text{in } \Omega. \tag{5.12}$$

Multiplying the second equation in (5.11) by μ and applying relations (4.9), we obtain

$$\mathring{\boldsymbol{V}}\boldsymbol{\Psi}^* - \mathring{\boldsymbol{W}}(\mu\boldsymbol{\Phi}^*) = \boldsymbol{0} \quad \text{in } \Omega. \tag{5.13}$$

Then we apply the trace operator to both sides of the equation taking into account Theorem 4.7,

$$\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\Psi}^* + \frac{1}{2}\mu\boldsymbol{\Phi}^* - \mathring{\boldsymbol{\mathcal{W}}}(\mu\boldsymbol{\Phi}^*) = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{5.14}$$

Now we apply the traction operator to (5.12) as the pressure equation and to (5.13) as the velocity equation, keeping in mind relations (4.56) and (4.57), to obtain

$$\frac{1}{2}\boldsymbol{\Psi}^* + \mathring{\boldsymbol{\mathcal{W}}}'\boldsymbol{\Psi}^* - \mathring{\boldsymbol{\mathcal{L}}}(\mu\boldsymbol{\Phi}^*) = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{5.15}$$

To simplify the notation, let us denote $\widehat{\Phi} := (\mu \mathbf{\Phi}^*)$ and $\widehat{\mathbf{\Psi}} := \mathbf{\Psi}^*$. We consider now the system of equations given by restrictions of (5.14) to S_1 and (5.15) to S_2 taking into account that $\mathbf{\Phi}^* = 0$ on S_1 and $\mathbf{\Psi}^* = 0$ on S_2,

$$-\mathring{\boldsymbol{\mathcal{V}}}\widehat{\mathbf{\Psi}} + \mathring{\boldsymbol{\mathcal{W}}}\widehat{\mathbf{\Phi}} = \mathbf{0} \quad \text{on } S_1, \tag{5.16}$$

$$-\mathring{\boldsymbol{\mathcal{W}}}'\widehat{\mathbf{\Psi}} + \mathring{\boldsymbol{\mathcal{L}}}\widehat{\mathbf{\Phi}} = \mathbf{0} \quad \text{on } S_2. \tag{5.17}$$

A similar system has been considered in [17, Theorem 3.10], where, however, the operators $\mathring{\boldsymbol{\mathcal{W}}}$ and $\mathring{\boldsymbol{\mathcal{W}}}'$ were obtained using the normal vector directed outward of the bounded domain, which leads to the change of signs in front of these operators. Nevertheless, introducing the new variable $\widehat{\mathbf{\Psi}}_- = -\widehat{\mathbf{\Psi}}$, we again return to the system of the form (5.16)-(5.17) which by [17, Theorem 3.10] implies that it has only the trivial solution and hence $\mathbf{\Psi}^* = \mathbf{0}$, $\mathbf{\Phi}^* = \mathbf{0}$. $\square$

6. **BDIE systems.** We aim to obtain two different segregated BDIE systems for the mixed BVP (2.12) employing a procedure similar to [5, 25, 23] and references therein. To this end, let the functions $\mathbf{\Phi}_0 \in \boldsymbol{H}^{1/2}(\partial\Omega)$ and $\mathbf{\Psi}_0 \in \boldsymbol{H}^{-1/2}(\partial\Omega)$ be respective extensions of the boundary functions $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(\partial\Omega_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(\partial\Omega_N)$ in (2.12c) and (2.12d). Then we can represent

$$\gamma^+\boldsymbol{v} = \mathbf{\Phi}_0 + \boldsymbol{\varphi}, \quad \boldsymbol{T}^+(p, \boldsymbol{v}) = \mathbf{\Psi}_0 + \boldsymbol{\psi} \text{ on } \partial\Omega, \tag{6.1}$$

where $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D)$ are unknown boundary functions that will be considered as formally independent of (segregated from) from functions $(p, \boldsymbol{v})$ in the domain.

6.1. **BDIE system M11$_*$.** Let us now take equations (5.3) and (5.4) in the domain Ω and restrictions of the trace equation (5.5) and the conormal derivative equation (5.6) to the boundary parts $\partial\Omega_D$ and $\partial\Omega_N$, respectively. Substituting there representations (6.1) and considering further the unknown boundary functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ as formally independent of the unknown domain functions p and $\boldsymbol{v}$, we obtain the following system, M11$_*$, of four boundary-domain integral equations for four unknowns, $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D)$,

$$p + \mathcal{R}^{\bullet}\boldsymbol{v} - \Pi^s\boldsymbol{\psi} + \Pi^d\boldsymbol{\varphi} = F_0 \quad \text{in } \Omega, \tag{6.2a}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\psi} + \boldsymbol{W}\boldsymbol{\varphi} = \boldsymbol{F} \quad \text{in } \Omega, \tag{6.2b}$$

$$r_{\partial\Omega_D}\gamma^+\boldsymbol{\mathcal{R}}\boldsymbol{v} - r_{\partial\Omega_D}\boldsymbol{\mathcal{V}}\boldsymbol{\psi} + r_{\partial\Omega_D}\boldsymbol{\mathcal{W}}\boldsymbol{\varphi} = r_{\partial\Omega_D}\gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0 \quad \text{on } \partial\Omega_D, \tag{6.2c}$$

$$r_{\partial\Omega_N}\boldsymbol{T}^+(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}})\boldsymbol{v} - r_{\partial\Omega_N}\boldsymbol{\mathcal{W}}'\boldsymbol{\psi} + r_{\partial\Omega_N}\boldsymbol{\mathcal{L}}^+\boldsymbol{\varphi} = r_{\partial\Omega_N}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - \boldsymbol{\psi}_0 \text{ on } \partial\Omega_N, \tag{6.2d}$$

where

$$F_0 = \mathring{\mathcal{Q}}\boldsymbol{f} + \frac{4}{3}\mu g + \Pi^s\mathbf{\Psi}_0 - \Pi^d\mathbf{\Phi}_0, \quad \boldsymbol{F} = \boldsymbol{\mathcal{U}}\boldsymbol{f} - \boldsymbol{\mathcal{Q}}g + \boldsymbol{V}\mathbf{\Psi}_0 - \boldsymbol{W}\mathbf{\Phi}_0. \tag{6.3}$$

Applying Lemma 5.2 to (6.3), keeping in mind the equations (6.2a) and (6.2b), and taking into account the mapping properties delivered by Theorems 4.2, 4.3 and 4.5, we obtain that $(F_0, \boldsymbol{F}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$.

We denote the right hand side of BDIE system (6.2) as

$$\mathcal{F}_*^{11} := [F_0, \mathcal{F}^{11}] = [F_0, \boldsymbol{F}, r_{\partial\Omega_D}\gamma^+\boldsymbol{F} - \boldsymbol{\varphi}_0, r_{\partial\Omega_N}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - \boldsymbol{\psi}_0], \tag{6.4}$$

which implies $\mathcal{F}_*^{11} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(\partial\Omega_D) \times \boldsymbol{H}^{-1/2}(\partial\Omega_N)$.

Note that the domain equations, (6.2a) and (6.2b), look like equations of the second kind, while both boundary equations, (6.2c) and (6.2d), as equations of the first kind, which is hinted by the indices 11 in the notations of the system and the corresponding operators.

Note also that BDIE system (6.2) can be split into the BDIE system, M11, of 3 vector equations (6.2b), (6.2c), (6.2d) for 3 vector unknowns, $\boldsymbol{v}$, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, and the separate equation (6.2a) that can be used, after solving the system, to obtain the pressure, p. However since the couple $(p, \boldsymbol{v})$ shares the space $\boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, equations (6.2b), (6.2c), (6.2d) are not completely separate from equation (6.2a).

System M11$_*$ given by equations (6.2a)-(6.2d) can be written using matrix notations as

$$\mathcal{M}_*^{11}\mathcal{X} = \mathcal{F}_*^{11}, \tag{6.5}$$

where $\mathcal{X} = (p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ represents the 4-tuple containing the unknowns of the system. The matrix operator $\mathcal{M}_*^{11}$ is defined by

$$\mathcal{M}_*^{11} = \begin{bmatrix} I & \mathcal{R}^\bullet & -\Pi^s & \Pi^d \\ 0 & \boldsymbol{I} + \boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ 0 & r_{\partial\Omega_D}\boldsymbol{\gamma}^+\boldsymbol{\mathcal{R}} & -r_{\partial\Omega_D}\boldsymbol{\mathcal{V}} & r_{\partial\Omega_D}\boldsymbol{\mathcal{W}} \\ 0 & r_{\partial\Omega_N}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & -r_{\partial\Omega_N}\boldsymbol{\mathcal{W}}' & r_{\partial\Omega_N}\boldsymbol{\mathcal{L}} \end{bmatrix}. \tag{6.6}$$

We note that the mapping properties of the operators involved in the matrix imply the continuity of the operator

$$\begin{aligned}\mathcal{M}_*^{11} : L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)\\ \to L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N).\end{aligned}$$

The following result can be proved by applying an argument similar to [9, Remark 6.1].

Remark 6.1. *The term* $\mathcal{F}_*^{11} = 0$ *if and only if* $(\boldsymbol{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = \boldsymbol{0}$.

6.2. **BDIE system M22$_*$.** Let us now obtain the BDIE system of the second kind, which will be hinted by the indices 22 (although with the spaces for unknowns and right-hand sides coinciding only up to 'tilde'). To this end, let us take equations (5.4) and (5.3) in the domain Ω, as in M11$_*$, but, unlike M11$_*$, restriction of the conormal derivative equation (5.6) to the Dirichlet part of the boundary, $\partial\Omega_D$, and restriction of the trace equation (5.5) to the Neumann boundary part of the boundary, $\partial\Omega_N$. Substituting there representations (6.1) and considering the unknown boundary functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ again as formally independent of the unknown domain functions $\boldsymbol{v}$ and p, we obtain the following system, M22$_*$, of four boundary-domain integral equations for four unknowns, $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, $\boldsymbol{\varphi} \in \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ and $\boldsymbol{\psi} \in \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D)$,

$$p + \mathcal{R}^\bullet\boldsymbol{v} - \Pi^s\boldsymbol{\psi} + \Pi^d\boldsymbol{\varphi} = F_0 \qquad \text{in } \Omega \tag{6.7a}$$

$$\boldsymbol{v} + \boldsymbol{\mathcal{R}}\boldsymbol{v} - \boldsymbol{V}\boldsymbol{\psi} + \boldsymbol{W}\boldsymbol{\varphi} = \boldsymbol{F} \qquad \text{in } \Omega, \tag{6.7b}$$

$$\begin{aligned}\frac{1}{2}\boldsymbol{\psi} + r_{\partial\Omega_D}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}})\boldsymbol{v} - r_{\partial\Omega_D}\boldsymbol{\mathcal{W}}'\boldsymbol{\psi} + r_{\partial\Omega_D}\boldsymbol{\mathcal{L}}^+\boldsymbol{\varphi}&\\ = r_{\partial\Omega_D}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - r_{\partial\Omega_D}\boldsymbol{\Psi}_0 \qquad &\text{on } \partial\Omega_D,\end{aligned} \tag{6.7c}$$

$$\begin{aligned}\frac{1}{2}\boldsymbol{\varphi} + r_{\partial\Omega_N}\boldsymbol{\gamma}^+\boldsymbol{\mathcal{R}}\boldsymbol{v} - r_{\partial\Omega_N}\boldsymbol{\mathcal{V}}\boldsymbol{\psi} + r_{\partial\Omega_N}\boldsymbol{\mathcal{W}}\boldsymbol{\varphi}&\\ = r_{\partial\Omega_N}\boldsymbol{\gamma}^+\boldsymbol{F} - r_{\partial\Omega_N}\boldsymbol{\Phi}_0 \qquad &\text{on } \partial\Omega_N.\end{aligned} \tag{6.7d}$$

where the terms in the right hand sides F_0 and $\boldsymbol{F}$ are given by (6.3).

Note that BDIE system (6.7a)-(6.7d) can be split into the BDIE system, M22, of 3 vector equations, (6.7b)-(6.7d), for 3 vector unknowns, $\boldsymbol{v}$, $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, and the separate equation (6.7a) that can be used, after solving the system, to obtain the pressure, p. However, since the couple $(p, \boldsymbol{v})$ shares the space $\boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$, equations (6.7b), (6.7c) and (6.7d) are not completely separate from equation (6.7a).

System M22$_*$ can be written using matrix notations as

$$\mathcal{M}_*^{22}\mathcal{X} = \mathcal{F}_*^{22}, \tag{6.8}$$

where the matrix operator $\mathcal{M}_*^{22}$ is defined by

$$\mathcal{M}_*^{22} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\Pi^s & \Pi^d \\ \mathbf{0} & \boldsymbol{I}+\boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ \mathbf{0} & r_{\partial\Omega_D}\boldsymbol{T}^+(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}}) & r_{\partial\Omega_D}\left(\dfrac{1}{2}\boldsymbol{I}-\boldsymbol{\mathcal{W}}'\right) & r_{\partial\Omega_D}\boldsymbol{\mathcal{L}}^+ \\ \mathbf{0} & r_{\partial\Omega_N}\boldsymbol{\gamma}^+\boldsymbol{\mathcal{R}} & -r_{\partial\Omega_N}\boldsymbol{\mathcal{V}} & r_{\partial\Omega_N}\left(\dfrac{1}{2}\boldsymbol{I}+\boldsymbol{\mathcal{W}}\right) \end{bmatrix}, \tag{6.9}$$

the 4-tuple $\mathcal{X} = (p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ represents the unknowns of the system, and the 4-tuple

$$\mathcal{F}_*^{22} = [F_0, \boldsymbol{F}, r_{\partial\Omega_D}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - r_{\partial\Omega_D}\boldsymbol{\Psi}_0, r_{\partial\Omega_N}\boldsymbol{\gamma}^+\boldsymbol{F} - r_{\partial\Omega_N}\boldsymbol{\Phi}_0]$$

is the right hand side and $\mathcal{F}_*^{22} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N)$.

Due to the mapping properties of the operators involved in (6.9), we have the continuous mapping

$$\begin{aligned} \mathcal{M}_*^{22} : L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N) \\ \to L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N). \end{aligned}$$

The following result can be proved by applying an argument similar to [9, Remark 6.2].

Remark 6.2. *The term $\mathcal{F}_*^{22} = 0$ if and only if $(\boldsymbol{f}, g, \boldsymbol{\Phi}_0, \boldsymbol{\Psi}_0) = 0$.*

7. **Equivalence theorem.** The following result is analogous to the equivalence theorems proven for bounded domains in [9, Theorem 6.3] (cf. also the equivalence result for boundary integral equations associated with the mixed problem for strongly elliptic systems in bounded domains in [21, Theorem 7.9]).

Theorem 7.1 (Equivalence Theorem)**.** *Let $\boldsymbol{f} \in \boldsymbol{L}_2(\rho; \Omega)$, $g \in L_2(\Omega)$ and let $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{-1/2}(\partial\Omega)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-1/2}(\partial\Omega)$ be some fixed extensions of $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(\partial\Omega_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(\partial\Omega_N)$ respectively. Let conditions 2.1 and 4.4 hold.*

(i) If some $(p, \boldsymbol{v}) \in L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)$ solves the mixed BVP (2.12), then the set $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})$, with

$$\boldsymbol{\varphi} = \boldsymbol{\gamma}^+\boldsymbol{v} - \boldsymbol{\Phi}_0, \quad \boldsymbol{\psi} = \boldsymbol{T}^+(p, \boldsymbol{v}) - \boldsymbol{\Psi}_0 \quad \textit{on } \partial\Omega, \tag{7.1}$$

belongs to $\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ and solves BDIE systems (6.2) and (6.7).

(ii) If $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ solves one of the BDIE systems, (6.2) or (6.7), then it also solves the other BDIE systems. Furthermore, the pair $(p, \boldsymbol{v})$ belongs to $\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ and solves the mixed BVP (2.12), while $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ satisfy (7.1).

(iii) BDIE systems (6.2) and (6.7) have at most one solution $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})$ *in the space* $L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$.

Proof. (i) Let $(p, \boldsymbol{v}) \in L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)$ be a solution of BVP (2.12). Since $\boldsymbol{f} \in \boldsymbol{L}_2(\rho; \Omega)$, then $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$. Let us define the functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ by (7.1). By the BVP boundary conditions, $\gamma^+\boldsymbol{v} = \boldsymbol{\varphi}_0 = r_{\partial\Omega_D}\boldsymbol{\Phi}_0$ on $\partial\Omega_D$ and $\boldsymbol{T}^+(p, \boldsymbol{v}) = \boldsymbol{\psi}_0 = r_{\partial\Omega_N}\boldsymbol{\Psi}_0$ on $\partial\Omega_N$. Then (7.1) implies that $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$. Taking into account the third Green identities (5.3)-(5.6), we obtain that $(p, \boldsymbol{v}, \boldsymbol{\varphi}, \boldsymbol{\psi})$ solves BDIE systems (6.2) and (6.7).

(ii-11) Let $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ solve BDIE system (6.2). Then equations (6.2a), (6.2b) and Theorem 4.5 imply that $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A})$ and the canonical conormal derivative $\boldsymbol{T}^+(p, \boldsymbol{v})$ is well defined. If we take the trace of (6.2b) restricted to $\partial\Omega_D$, use the jump relations (4.55) for the trace of $\boldsymbol{V}$ and $\boldsymbol{W}$ in Theorem 4.7, and subtract it from (6.2c), we arrive at $r_{\partial\Omega_D}\gamma^+\boldsymbol{v} - \frac{1}{2}r_{\partial\Omega_D}\boldsymbol{\varphi} = \boldsymbol{\varphi}_0$ on $\partial\Omega_D$. Since $\boldsymbol{\varphi}$ vanishes on $\partial\Omega_D$, this implies that the Dirichlet condition (2.12c) is satisfied.

Repeating the same procedure but now taking the traction of (6.2a) and (6.2b), restricted to $\partial\Omega_N$, using jump relations (4.56) and (4.46) for the tractions of $(\Pi^s, \boldsymbol{v})$ and $(\Pi^d, \boldsymbol{W})$, and subtracting it from (6.2d), we arrive at $r_{\partial\Omega_N}\boldsymbol{T}(p, \boldsymbol{v}) - \frac{1}{2}r_{\partial\Omega_N}\boldsymbol{\psi} = \boldsymbol{\psi}_0$ on $\partial\Omega_N$. Since $\boldsymbol{\psi}$ vanishes on $\partial\Omega_N$,, this implies that the Neumann condition (2.12d) is satisfied.

Because $\boldsymbol{\Phi}_0 = \boldsymbol{\varphi}_0$, on $\partial\Omega_D$ and $\boldsymbol{\Psi}_0 = \boldsymbol{\psi}_0$, on $\partial\Omega_N$, we also obtain the inclusions

$$\boldsymbol{\Psi}^* := \boldsymbol{\psi} + \boldsymbol{\Psi}_0 - \boldsymbol{T}^+(p, \boldsymbol{v}) \in \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D), \tag{7.2}$$

$$\boldsymbol{\Phi}^* := \boldsymbol{\varphi} + \boldsymbol{\Phi}_0 - \gamma^+\boldsymbol{v} \in \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N). \tag{7.3}$$

By relations (6.2a) and (6.2b) the hypotheses of Lemma 5.2 are satisfied with $\boldsymbol{\Psi} = \boldsymbol{\psi} + \boldsymbol{\Psi}_0$ and $\boldsymbol{\Phi} = \boldsymbol{\varphi} + \boldsymbol{\Phi}_0$. As a result, we obtain that the couple $(p, \boldsymbol{v})$ satisfies (2.12a) and (2.12b), moreover,

$$\Pi^s(\boldsymbol{\Psi}^*) - \Pi^d(\boldsymbol{\Phi}^*) = 0, \quad \boldsymbol{V}(\boldsymbol{\Psi}^*) - \boldsymbol{W}(\boldsymbol{\Phi}^*) = \boldsymbol{0} \quad \text{in } \Omega \tag{7.4}$$

Due to inclusions (7.2)-(7.3) and relations (7.4), Lemma 5.3 for $S_1 = \partial\Omega_D$, and $S_2 = \partial\Omega_N$ implies $\boldsymbol{\Psi}^* = \boldsymbol{0}$ and $\boldsymbol{\Phi}^* = \boldsymbol{0}$ on $\partial\Omega$ and thus relations (7.1) hold.

Hence, by item (i) the set $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})$ solves also BDIE system (6.7).

(ii-22) Let now $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in L_2(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ solve BDIE system (6.7). Then equations (6.7a), (6.7b) and Theorem 4.5 imply that $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A})$ and the canonical conormal derivative $\boldsymbol{T}^+(p, \boldsymbol{v})$ is well defined. Applying Lemma 5.2 with $\boldsymbol{\Psi} = \boldsymbol{\psi} + \boldsymbol{\Psi}_0$ and $\boldsymbol{\Phi} = \boldsymbol{\varphi} + \boldsymbol{\Phi}_0$ to BDIEs (6.7a)-(6.7b), we deduce that the couple $(p, \boldsymbol{v})$ solves PDE system (2.12a)-(2.12b) and

$$\Pi^s(\boldsymbol{\Psi}^*) - \Pi^d(\boldsymbol{\Phi}^*) = 0, \qquad \boldsymbol{V}(\boldsymbol{\Psi}^*) - \boldsymbol{W}(\boldsymbol{\Phi}^*) = \boldsymbol{0}, \quad \text{in } \Omega, \tag{7.5}$$

where

$$\boldsymbol{\Psi}^* := \boldsymbol{\psi} + \boldsymbol{\Psi}_0 - \boldsymbol{T}^+(p, \boldsymbol{v}), \boldsymbol{\Phi}^* := \boldsymbol{\varphi} + \boldsymbol{\Phi}_0 - \gamma^+\boldsymbol{v}, \qquad \textit{on} \quad \partial\Omega. \tag{7.6}$$

Taking the traction of (6.7a) and (6.7b) restricted to $\partial\Omega_D$ and subtracting it from (6.7c) we get

$$r_{\partial\Omega_D}\boldsymbol{T}^+(p, \boldsymbol{v}) - r_{\partial\Omega_D}\boldsymbol{\Psi}_0 = \boldsymbol{\psi}, \quad \textit{on} \quad \partial\Omega_D. \tag{7.7}$$

Taking the trace of (6.7b) restricted to $\partial\Omega_N$ and subtracting it from (6.7d) we get

$$r_{\partial\Omega_N}\gamma^+\boldsymbol{v} - r_{\partial\Omega_N}\boldsymbol{\Phi}_0 = \boldsymbol{\varphi}, \quad on \quad \partial\Omega_N. \tag{7.8}$$

Due to (7.7) and (7.8), we have $\boldsymbol{\Psi}^* \in \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D)$ and $\boldsymbol{\Phi}^* \in \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$. Now, we can apply Lemma 5.3 with $S_1 = \partial\Omega_D$ and $S_2 = \partial\Omega_N$, to obtain $\boldsymbol{\Psi}^* = \mathbf{0}$ and $\boldsymbol{\Phi}^* = \mathbf{0}$ on $\partial\Omega$, which by (7.6) imply relations (7.1). Since $r_{\partial\Omega_D}\boldsymbol{\Phi}_0 = \boldsymbol{\varphi}_0$ and $r_{\partial\Omega_N}\boldsymbol{\Psi}_0 = \boldsymbol{\psi}_0$, relations (7.1) imply boundary conditions (2.12c) and (2.12d). Thus the couple $(p, \boldsymbol{v})$ is a solution of BVP (2.12) and hence, by item (i) the set $(p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})$ solves also BDIE system (6.2).

Finally, item (iii) follows from items (i) and (ii) and the fact that the BVP (2.12) has at most one solution, see Theorem 2.4. □

8. Isomorphism results.

In addition to the "narrow" spaces, $\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})\times\widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D)\times\widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$, mostly considered up to now, we now prove the boundary-domain integral operator isomorphism properties also in the "wider" spaces, with $L_2(\Omega)\times\boldsymbol{\mathcal{H}}^1(\Omega)$ instead of $\boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$. To this end, we introduce the notations

$$\begin{aligned}
&\mathbb{X} := \boldsymbol{\mathcal{H}}^1(\Omega)\times\widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D)\times\widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N), && \mathbb{X}_* := L_2(\Omega)\times\mathbb{X},\\
&\mathbb{Y}^{11} := \boldsymbol{\mathcal{H}}^1(\Omega)\times\boldsymbol{H}^{1/2}(\partial\Omega_D)\times\boldsymbol{H}^{-1/2}(\partial\Omega_N), && \mathbb{Y}^{11}_* := L_2(\Omega)\times\mathbb{Y}^{11},\\
&\mathbb{Y}^{22} := \boldsymbol{\mathcal{H}}^1(\Omega)\times\boldsymbol{H}^{-1/2}(\partial\Omega_D)\times\boldsymbol{H}^{1/2}(\partial\Omega_N), && \mathbb{Y}^{22}_* = L_2(\Omega)\times\mathbb{Y}^{22}.
\end{aligned}$$

8.1. Isomorphism properties of BDIE operator $\mathcal{M}^{11}_*$.

Recall that operator $\mathcal{M}^{11}_*$ is given by (6.6).

Theorem 8.1. *Let conditions 2.1, 4.4 and 4.10 hold. Then, the operator*

$$\mathcal{M}^{11}_* : \mathbb{X}_* \to \mathbb{Y}^{11}_*, \tag{8.1}$$

where $\mathcal{M}^{11}_*$ is presented by (6.6), is an isomorphism.

Proof. Let $\widetilde{\mathcal{M}}^{11}_* : \mathbb{X}_* \to \mathbb{Y}^{11}_*$ be the matrix operator

$$\widetilde{\mathcal{M}}^{11}_* := \begin{bmatrix} I & \mathcal{R}^\bullet & -\Pi^s & \Pi^d \\ \mathbf{0} & \boldsymbol{I} & -\boldsymbol{V} & \boldsymbol{W} \\ \mathbf{0} & \mathbf{0} & -r_{\partial\Omega_D}\boldsymbol{\mathcal{V}} & r_{\partial\Omega_D}\boldsymbol{\mathcal{W}} \\ \mathbf{0} & \mathbf{0} & -r_{\partial\Omega_N}\mathring{\boldsymbol{\mathcal{W}}}' & r_{\partial\Omega_N}\widehat{\boldsymbol{\mathcal{L}}} \end{bmatrix}.$$

Note that the operator $\widetilde{\mathcal{M}}^{11}_*$ is an upper block-triangular matrix operator. The first two diagonal blocks are given by the identity operators I and $\boldsymbol{I}$, whereas the third diagonal block can be represented as

$$\begin{bmatrix} -r_{\partial\Omega_D}\boldsymbol{\mathcal{V}} & r_{\partial\Omega_D}\boldsymbol{\mathcal{W}} \\ -r_{\partial\Omega_N}\mathring{\boldsymbol{\mathcal{W}}}' & r_{\partial\Omega_N}\widehat{\boldsymbol{\mathcal{L}}} \end{bmatrix} = \mathrm{diag}(\frac{1}{\mu}\boldsymbol{I},\boldsymbol{I})\begin{bmatrix} -r_{\partial\Omega_D}\mathring{\boldsymbol{\mathcal{V}}} & r_{\partial\Omega_D}\mathring{\boldsymbol{\mathcal{W}}} \\ -r_{\partial\Omega_N}\mathring{\boldsymbol{\mathcal{W}}}' & r_{\partial\Omega_N}\mathring{\boldsymbol{\mathcal{L}}} \end{bmatrix}\mathrm{diag}(\boldsymbol{I},\mu\boldsymbol{I})\circ \tag{8.2}$$

Taking into account [17, Theorem 3.10] (cf. also the argument at the end of Lemma 5.3 proof), block (8.2) is an isomorphism and thus the operator $\widetilde{\mathcal{M}}^{11}_* : \mathbb{X}_* \to \mathbb{Y}^{11}_*$ is an isomorphism and hence is a Fredholm operator with zero index.

The operator $\mathcal{M}^{11}_* - \widetilde{\mathcal{M}}^{11}_* : \mathbb{X}_* \to \mathbb{Y}^{11}_*$ has the form

$$\mathcal{M}^{11}_* - \widetilde{\mathcal{M}}^{11}_* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{0} & \boldsymbol{\mathcal{R}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{\partial\Omega_D}\gamma^+\boldsymbol{\mathcal{R}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{\partial\Omega_N}\boldsymbol{T}^+(\mathcal{R}^\bullet,\boldsymbol{\mathcal{R}}) & r_{\partial\Omega_N}(\mathring{\boldsymbol{\mathcal{W}}}' - \boldsymbol{\mathcal{W}}') & r_{\partial\Omega_N}(\boldsymbol{\mathcal{L}}^+ - \widehat{\boldsymbol{\mathcal{L}}}) \end{bmatrix}.$$

and is compact due to the compactness of operators $\boldsymbol{\mathcal{R}}$ and $\mathcal{R}^\bullet$ given by Theorem 4.12 and of operators $\boldsymbol{\mathcal{W}'}$, $\mathring{\boldsymbol{\mathcal{W}}}'$ and $\boldsymbol{\mathcal{L}}^+ - \widehat{\boldsymbol{\mathcal{L}}}$ given by Theorem 4.6 and Corollary 4.9.

Therefore, the operator $\mathcal{M}^{11}_* : \mathbb{X}_* \to \mathbb{Y}^{11}_*$ is Fredholm with zero index. Moreover, this operator is also injective by virtue of Theorem 7.1, which implies that it is an isomorphism. □

Theorem 8.2. *Let conditions 2.1, 4.4 and 4.10 hold. Then the operator*

$$
\begin{aligned}
\mathcal{M}^{11}_* :& \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N) \\
&\to \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(\partial\Omega_D) \times \boldsymbol{H}^{-1/2}(\partial\Omega_N)
\end{aligned}
\tag{8.3}
$$

is an isomorphism.

Proof. Let us consider the solution $\mathcal{X} = (\mathcal{M}^{11}_*)^{-1}\mathcal{F}^{11}_* \in \mathbb{X}_*$ of the system (6.5). Here $\mathcal{F}^{11}_* \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(\partial\Omega_D) \times \boldsymbol{H}^{-1/2}(\partial\Omega_N)$ is an arbitrary right hand side and $(\mathcal{M}^{11}_*)^{-1}$ is the inverse of operator (8.1), which exists by virtue of Theorem 8.1.

Applying Lemma 5.2 to the first two equations of the system M11$_*$, we get that $\mathcal{X} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ if $\mathcal{F}^{11}_* \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}}) \times \boldsymbol{H}^{1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N)$. Consequently, the operator $(\mathcal{M}^{11}_*)^{-1}$ is also the continuous inverse of the operator (8.3). □

8.2. **Solvability of the mixed BVP and equivalence of constant-coefficient BIEs.** The proved results for the BDIE system M11$_*$ immediately lead to the following assertion for the mixed BVP (note that in a more general setting the mixed BVP solvability for the variable-coefficient Stokes system in exterior domains was considered in [16]).

Theorem 8.3. *Let $\boldsymbol{f} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$, $g \in L_2(\rho;\Omega)$, $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(\partial\Omega_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(\partial\Omega_N)$. In addition, let conditions 2.1, 4.4 and 4.10 hold. Then, the BVP (2.12) is uniquely solvable in $\boldsymbol{H}^{1,0}(\Omega;\boldsymbol{\mathcal{A}})$. Furthermore, the mixed BVP operator (2.13) is an isomorphism.*

Proof. Let $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{1/2}(\partial\Omega)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-1/2}(\partial\Omega)$ be some extensions of $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(\partial\Omega_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(\partial\Omega_N)$, respectively.

The BDIE system M11$_*$ is uniquely solvable by Theorem 8.2 and is equivalent to the BVP (2.12) by Theorem 7.1. In addition, as operator (8.3) is an isomorphism. The BVP solution uniqueness (that is, independence of the chosen extensions $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Psi}_0$) is implied by Theorem 2.4, while the continuity of the extension operators, well known for smooth domain boundary $\partial\Omega$ and smooth interfaces between $\partial\Omega_D$ and $\partial\Omega_N$, completes the continuity of the inverse to BVP operator (2.13). □

When $\mu = 1$, the operator $\boldsymbol{\mathcal{A}}$ becomes $\mathring{\boldsymbol{\mathcal{A}}}$, $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}^\bullet \equiv 0$ and the boundary-domain integral equations system (6.2) becomes a BIE system with 2 vector equations and 2 vector unknowns on the boundary $\partial\Omega$,

$$
r_{\partial\Omega_D}\left(\frac{1}{2}\boldsymbol{\psi} - \mathring{\boldsymbol{\mathcal{W}}}'\boldsymbol{\psi} + \mathring{\boldsymbol{\mathcal{L}}}\boldsymbol{\varphi}\right) = r_{\partial\Omega_D}\boldsymbol{T}^+(F_0,\boldsymbol{F}) - r_{\partial\Omega_D}\boldsymbol{\Psi}_0, \ \ on\ \partial\Omega_D, \tag{8.4}
$$

$$
r_{\partial\Omega_N}\left(\frac{1}{2}\boldsymbol{\varphi} - \mathring{\mathcal{V}}\boldsymbol{\psi} + \mathring{\boldsymbol{\mathcal{W}}}\boldsymbol{\varphi}\right) = r_{\partial\Omega_N}\boldsymbol{\gamma}^+\boldsymbol{F} - r_{\partial\Omega_N}\boldsymbol{\Phi}_0, \ \ on\ \partial\Omega_N. \tag{8.5}
$$

and the representation formulas for $(p,\boldsymbol{v})$ in Ω,

$$
p = F_0 + \mathring{\Pi}^s\boldsymbol{\psi} - \mathring{\Pi}^d\boldsymbol{\varphi} \quad \text{in}\ \Omega, \tag{8.6}
$$

$$\boldsymbol{v} = \boldsymbol{F} + \mathring{\boldsymbol{V}}\boldsymbol{\psi} - \mathring{\boldsymbol{W}}\boldsymbol{\varphi} \quad \text{in } \Omega. \tag{8.7}$$

where the terms F_0 and $\boldsymbol{F}$ are given by (6.3).

By considering $\mu = 1$ in Theorem 8.1 and Corollary 8.3, we obtain the following assertion for the constant coefficient case.

Corollary 8.4. *Let $\mu = 1$ in Ω, $\boldsymbol{f} \in \boldsymbol{L}_2(\Omega)$ and $g \in L_2(\rho;\Omega)$. Moreover, let $\boldsymbol{\Phi}_0 \in \boldsymbol{H}^{1/2}(\partial\Omega)$ and $\boldsymbol{\Psi}_0 \in \boldsymbol{H}^{-1/2}(\partial\Omega)$ be some extensions of $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{1/2}(\partial\Omega_D)$ and $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-1/2}(\partial\Omega_N)$, respectively. Furthermore, let conditions 2.1, 4.4 and 4.10 hold.*

(i) If some $(p, \boldsymbol{v}) \in L_2(\rho;\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)$ solves the mixed BVP (2.12), then the solution is unique, the couple $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ given by

$$\boldsymbol{\varphi} = \gamma^+\boldsymbol{v} - \boldsymbol{\Phi}_0, \quad \boldsymbol{\psi} = \boldsymbol{T}^+(p, \boldsymbol{v}) - \boldsymbol{\Psi}_0 \quad \text{on } \partial\Omega, \tag{8.8}$$

solves the BIE system (8.4)-(8.5) and $(p, \boldsymbol{v})$ satisfies (8.6)-(8.7).

(ii) If a couple $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$ solves BIE system (8.4)-(8.5), then the couple $(p, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$ given by (8.6)-(8.7) solves the mixed BVP (2.12) and relations (8.8) hold. Moreover, the BDIE solution is unique in $\widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$.

8.3. **Isomorphism properties of BDIE operator $\mathcal{M}^{22}_*$.** Let us first provide two assertions to obtain some special operator representations for the right-hand sides of the considered BDIE system.

Lemma 8.5. *Let $S = \overline{S}_1 \cup \overline{S}_2$, where S_1 and S_2 are two open non-intersecting simply connected non-empty submanifolds of $\partial\Omega$ with infinitely smooth boundaries. For any 4-tuple*

$$\mathcal{F} = (F_0, \boldsymbol{F}, \boldsymbol{\Psi}, \boldsymbol{\Phi}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2)$$

there exists a unique 4-tuple

$$(g_*, \boldsymbol{f}_*, \boldsymbol{\Psi}_*, \boldsymbol{\Phi}_*) = \widetilde{\mathcal{C}}_{S_1,S_2}\mathcal{F} \in L_2(\rho;\Omega) \times \boldsymbol{L}_2(\rho;\Omega) \times \boldsymbol{H}^{-1/2}(\partial\Omega) \times \boldsymbol{H}^{1/2}(\partial\Omega) \tag{8.9}$$

such that

$$\mathring{\mathcal{Q}}\boldsymbol{f}_* + \frac{4}{3}\mu g_* + \Pi^s\boldsymbol{\Psi}_* - \Pi^d\boldsymbol{\Phi}_* = F_0 \quad \text{in} \quad \Omega, \tag{8.10a}$$

$$\boldsymbol{\mathcal{U}}\boldsymbol{f}_* - \boldsymbol{\mathcal{Q}}g_* + \boldsymbol{V}\boldsymbol{\Psi}_* - \boldsymbol{W}\boldsymbol{\Phi}_* = \boldsymbol{F} \quad \text{in} \quad \Omega, \tag{8.10b}$$

$$r_{S_1}\boldsymbol{\Psi}_* = \boldsymbol{\Psi} \quad \text{on} \quad S_1, \tag{8.10c}$$

$$r_{S_2}\boldsymbol{\Phi}_* = \boldsymbol{\Phi} \quad \text{on} \quad S_2, \tag{8.10d}$$

and

$$\widetilde{\mathcal{C}}_{S_1,S_2} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2) \to L_2(\Omega) \times \boldsymbol{L}_2(\Omega) \times \boldsymbol{H}^{-1/2}(\partial\Omega) \times \boldsymbol{H}^{1/2}(\partial\Omega) \tag{8.11}$$

is a continuous operator.

Proof. The proof is similar to the one for bounded domains in [9, Lemma 7.5]. Let $\boldsymbol{\Psi}^0 = E^{-1/2}_{S_1}\boldsymbol{\Psi} \in \boldsymbol{H}^{-1/2}(\partial\Omega)$ and $\boldsymbol{\Phi}^0 = E^{1/2}_{S_2}\boldsymbol{\Phi} \in \boldsymbol{H}^{1/2}(\partial\Omega)$ be extensions of $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ to the entire boundary $\partial\Omega$ from S_1 and S_2, respectively. Here $E^s_{S_i} : \boldsymbol{H}^s(S_i) \to \boldsymbol{H}^s(\partial\Omega)$, $i = \{1, 2\}$, $|s| \le 1$, are some linear continuous extension operators from S_i to $\partial\Omega$ (cf. [35, Subsection 4.2]). Then any other extensions, $\boldsymbol{\Psi}_*$ and $\boldsymbol{\Phi}_*$, of the $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ can be represented as

$$\boldsymbol{\Psi}_* = \boldsymbol{\Psi}^0 + \widetilde{\boldsymbol{\psi}}, \qquad \widetilde{\boldsymbol{\psi}} \in \widetilde{\boldsymbol{H}}^{-1/2}(S_2), \tag{8.12}$$

$$\mathbf{\Phi}_* = \mathbf{\Phi}^0 + \widetilde{\boldsymbol{\varphi}}, \qquad \widetilde{\boldsymbol{\varphi}} \in \widetilde{\boldsymbol{H}}^{1/2}(S_1). \tag{8.13}$$

The distributions $\mathbf{\Psi}_*$ and $\mathbf{\Phi}_*$ satisfy the conditions (8.10c) and (8.10d) for any $\widetilde{\boldsymbol{\psi}}$ and $\widetilde{\boldsymbol{\varphi}}$. Consequently, it is only necessary to choose $g_*, \boldsymbol{f}_*, \widetilde{\boldsymbol{\psi}}$ and $\widetilde{\boldsymbol{\varphi}}$ such that equations (8.10a)-(8.10b) are satisfied.

Applying relations (4.5)-(4.10), equations (8.10a)-(8.10b) are reduced to

$$\mathring{\mathcal{Q}} f_* + \frac{4}{3}\mu g_* + \mathring{\Pi}^s \left(\mathbf{\Psi}_0 + \widetilde{\boldsymbol{\psi}}\right) - \mathring{\Pi}^d \left(\mu \mathbf{\Phi}_0 + \mu\widetilde{\boldsymbol{\varphi}}\right) = F_0, \tag{8.14}$$

$$\mathring{\boldsymbol{\mathcal{U}}} f_* - \mathring{\boldsymbol{\mathcal{Q}}}(\mu g_*) + \mathring{\boldsymbol{V}} \left(\mathbf{\Psi}_0 + \widetilde{\boldsymbol{\psi}}\right) - \mathring{\boldsymbol{W}} \left(\mu \mathbf{\Phi}_0 + \mu\widetilde{\boldsymbol{\varphi}}\right) = \mu \boldsymbol{F}. \tag{8.15}$$

Applying the Stokes operator with constant viscosity $\mu = 1$, $\mathring{\boldsymbol{\mathcal{A}}}$, to equations (8.14) and (8.15), and the divergence operator to equation (8.15), we obtain

$$\boldsymbol{f}_* = \mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F}), \tag{8.16}$$

$$g_* = \frac{1}{\mu}\operatorname{div}(\mu\boldsymbol{F}), \tag{8.17}$$

which shows that the function $\boldsymbol{f}_*$ is uniquely determined by F_0 and $\boldsymbol{F}$ and belongs to $\boldsymbol{L}_2(\rho;\Omega)$ since $(F_0, \mu\boldsymbol{F}) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A})$ by virtue of the mapping properties given by Theorem 4.5. In addition, (8.17) shows that g_* is also uniquely determined by $\boldsymbol{F}$ and belongs to $L_2(\rho;\Omega)$ since $\mu\boldsymbol{F} \in \boldsymbol{\mathcal{H}}^1(\Omega)$.

Substituting (8.16) and (8.17) into equations (8.14)-(8.15) we obtain

$$\mathring{\Pi}^s \widetilde{\boldsymbol{\psi}} - \mathring{\Pi}^d(\mu\widetilde{\boldsymbol{\varphi}}) = J_0\mathcal{F}, \quad \mathring{\boldsymbol{V}}\widetilde{\boldsymbol{\psi}} - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}}) = \boldsymbol{J}\mathcal{F} \quad \text{in } \Omega, \tag{8.18}$$

where the operators J_0 and $\boldsymbol{J}$ are defined as

$$J_0\mathcal{F} := \left(F_0 - \frac{4}{3}\operatorname{div}(\mu\boldsymbol{F}) - \mathring{\mathcal{Q}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) - \mathring{\Pi}^s(E_{S_1}^{-1/2}\mathbf{\Psi}) + \mathring{\Pi}^d(\mu E_{S_2}^{1/2}\mathbf{\Phi})\right), \tag{8.19}$$

$$\boldsymbol{J}\mathcal{F} := \left(\mu\boldsymbol{F} - \mathring{\boldsymbol{\mathcal{U}}}\left(\mathring{\boldsymbol{\mathcal{A}}}(F_0, \mu\boldsymbol{F})\right) + \mathring{\boldsymbol{\mathcal{Q}}}\operatorname{div}(\mu\boldsymbol{F}) - \mathring{\boldsymbol{V}}(E_{S_1}^{-1/2}\mathbf{\Psi}) + \mathring{\boldsymbol{W}}(\mu E_{S_2}^{1/2}\mathbf{\Phi})\right), \tag{8.20}$$

Let $\widetilde{\boldsymbol{\psi}}$ and $\widetilde{\boldsymbol{\varphi}}$ satisfy (8.18). Then they also satisfy the system

$$r_{S_2}\boldsymbol{\gamma}^+\left(\mathring{\boldsymbol{V}}\widetilde{\boldsymbol{\psi}} - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}})\right) = r_{S_2}\left(\boldsymbol{\gamma}^+\boldsymbol{J}\mathcal{F}\right), \tag{8.21}$$

$$r_{S_1}\left[\mathring{\boldsymbol{T}}^+\left(\mathring{\Pi}^s(\widetilde{\psi}) - \mathring{\Pi}^d(\mu\widetilde{\boldsymbol{\varphi}}), \mathring{\boldsymbol{V}}\widetilde{\psi} - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}})\right)\right] = r_{S_1}\left(\mathring{\boldsymbol{T}}^+(J_0\mathcal{F}, \boldsymbol{J}\mathcal{F})\right). \tag{8.22}$$

Using matrix notations it can be written as follows

$$\begin{bmatrix} r_{S_2}\mathring{\boldsymbol{\mathcal{V}}} & r_{S_2}\boldsymbol{\gamma}^+\mathring{\boldsymbol{\mathcal{W}}} \\ r_{S_1}\boldsymbol{\mathcal{W}}' & r_{S_1}\mathring{\boldsymbol{\mathcal{L}}} \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{\psi}} \\ \mu\widetilde{\boldsymbol{\varphi}} \end{bmatrix} = \begin{bmatrix} r_{S_2}\left(\boldsymbol{\gamma}^+\boldsymbol{J}\mathcal{F}\right) \\ r_{S_1}\left(\mathring{\boldsymbol{T}}^+(J_0\mathcal{F}, \boldsymbol{J}\mathcal{F})\right) \end{bmatrix}. \tag{8.23}$$

The matrix operator given by the left-hand side of equation (8.23) is an isomorphism between the spaces $\widetilde{\boldsymbol{H}}^{-1/2}(S_2) \times \widetilde{\boldsymbol{H}}^{1/2}(S_1)$ and $\boldsymbol{H}^{1/2}(S_2) \times \boldsymbol{H}^{-1/2}(S_1)$ (see [17, Theorem 3.10] and the argument at the end of Lemma 5.3 proof).

Therefore the solution of system (8.23) can be written as $(\widetilde{\boldsymbol{\varphi}}, \widetilde{\boldsymbol{\psi}}) = \mathring{C}\mathcal{F}$, where $\mathring{C}$ is a continuous operator, which together with (8.16)-(8.17), (8.12)-(8.13) and continuity of the extension operators $E_{S_i}^{\pm 1/2}$ produces a linear continuous operator $\widetilde{\mathcal{C}}_{S_1,S_2}$ in (8.11). Hence we proved that if a 4-tuple $(g_*, \boldsymbol{f}_*, \mathbf{\Psi}_*, \mathbf{\Phi}_*)$ satisfying (8.10) does exist, it can be written as (8.9).

Let us prove that $\mathbf{\Psi}_*$ and $\mathbf{\Phi}_*$, obtained by substituting in (8.12) and (8.13) a solution $(\widetilde{\boldsymbol{\psi}}, \widetilde{\boldsymbol{\varphi}})$ of (8.23), and $\boldsymbol{f}_*$, g_*, given by (8.16)-(8.17), satisfy (8.10). Equations (8.10c) and (8.10d) are immediately implied by (8.12) and (8.13). The couple $(\mathring{\Pi}^s\widetilde{\boldsymbol{\psi}} - \mathring{\Pi}^d(\mu\widetilde{\boldsymbol{\varphi}}), \mathring{\boldsymbol{V}}\widetilde{\boldsymbol{\psi}} - \mathring{\boldsymbol{W}}(\mu\widetilde{\boldsymbol{\varphi}}))$ satisfies the incompressible homogeneous Stokes system with $\mu = 1$. It is easy to check that the same system is also satisfied by the couple $(J_0\mathcal{F}, \boldsymbol{J}\mathcal{F})$. By (8.21)-(8.22), the couples satisfy the same mixed boundary conditions and thus they coincide also in the domain Ω by the uniqueness Theorem 2.4 with $\mu = 1$, i.e., equations (8.18) hold and substitution of (8.19) and (8.20) into their right hand sides leads to (8.10a) and (8.10b).

Since the extension operators $E_{S_i}^{\pm 1/2}$ are not unique, we still need to prove that the operator $\widetilde{\mathcal{C}}_{S_1,S_2}$ is unique. To this end, let us consider system (8.10) with zero right-hand side $\mathcal{F}$. Then representations (8.16)-(8.17) imply $\boldsymbol{f}_* = \mathbf{0}$, $g_* = 0$, while (8.10c)-(8.10d) and (8.12)-(8.13) give $\mathbf{\Psi}_* = \widetilde{\boldsymbol{\psi}}$, $\mathbf{\Phi}_* = \widetilde{\boldsymbol{\varphi}}$ on $\partial\Omega$, and finally (8.23) implies $\widetilde{\boldsymbol{\psi}} = \mathbf{0}$, $\widetilde{\boldsymbol{\varphi}} = \mathbf{0}$. This means the solution $(g_*, \boldsymbol{f}_*, \mathbf{\Psi}_*, \mathbf{\Phi}_*)$ of inhomogeneous system (8.10) is unique, along with the uniqueness of operator $\widetilde{\mathcal{C}}_{S_1,S_2}$. □

Corollary 8.6. *For any*

$$\mathcal{F} = (\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1, \boldsymbol{\mathcal{F}}_2, \boldsymbol{\mathcal{F}}_3) \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2),$$

there exists a unique 4-tuple

$$(g_*, \boldsymbol{f}_*, \mathbf{\Psi}_*, \mathbf{\Phi}_*) = \mathcal{C}_{S_1,S_2}\mathcal{F} \in L_2(\rho; \Omega) \times \boldsymbol{L}_2(\rho; \Omega) \times \boldsymbol{H}^{-1/2}(\partial\Omega) \times \boldsymbol{H}^{1/2}(\partial\Omega),$$

such that

$$\mathcal{Q}\boldsymbol{f}_* + \frac{4}{3}\mu g_* + \Pi^s\mathbf{\Psi}_* - \Pi^d\mathbf{\Phi}_* = \mathcal{F}_0 \qquad \text{in } \Omega, \tag{8.24}$$

$$\boldsymbol{\mathcal{U}}\boldsymbol{f}_* + \mathring{\boldsymbol{\mathcal{Q}}}g_* + \boldsymbol{\mathcal{V}}\mathbf{\Psi}_* - \boldsymbol{\mathcal{W}}\mathbf{\Phi}_* = \boldsymbol{\mathcal{F}}_1 \qquad \text{in } \Omega, \tag{8.25}$$

$$r_{S_1}(\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) - \mathbf{\Psi}_*) = \boldsymbol{\mathcal{F}}_2 \qquad \text{on } S_1, \tag{8.26}$$

$$r_{S_2}(\gamma^+\boldsymbol{\mathcal{F}}_1 - \mathbf{\Phi}_*) = \boldsymbol{\mathcal{F}}_3 \qquad \text{on } S_2, \tag{8.27}$$

and

$$\begin{aligned}\mathcal{C}_{S_1,S_2} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(S_1) \times \boldsymbol{H}^{1/2}(S_2)\\ \to L_2(\rho; \Omega) \times \boldsymbol{L}_2(\rho; \Omega) \times \boldsymbol{H}^{-1/2}(\partial\Omega) \times \boldsymbol{H}^{1/2}(\partial\Omega)\end{aligned}$$

is a continuous operator.

Proof. Let us take $\mathbf{\Psi} := r_{S_1}\boldsymbol{T}^+(\mathcal{F}_0, \boldsymbol{\mathcal{F}}_1) - \boldsymbol{\mathcal{F}}_2$, which implies $\mathbf{\Psi} \in \boldsymbol{H}^{-1/2}(S_1)$. Firstly, $\boldsymbol{\mathcal{F}}_2 \in \boldsymbol{H}^{-1/2}(S_1)$. In a similar fashion, let $\mathbf{\Phi} := r_{S_2}\gamma^+\boldsymbol{\mathcal{F}}_1 - \boldsymbol{\mathcal{F}}_3$, which implies $\mathbf{\Phi} \in \boldsymbol{H}^{1/2}(S_2)$. Then the Corollary follows from Lemma 8.5. □

Recall that operator $\mathcal{M}_*^{22}$ is given by (6.9).

Theorem 8.7. *Let conditions 2.1, 4.4 and 4.10 hold. Then the operator*

$$\begin{aligned}\mathcal{M}_*^{22} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)\\ \to \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N)\end{aligned} \tag{8.28}$$

is an isomorphism.

Proof. Let us consider system (6.8) with an arbitrary right hand side

$$\mathcal{F}_*^{22} \in \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N).$$

By Corollary 8.6, the right hand side $\mathcal{F}_*^{22}$ can be written in form (8.24)-(8.27) with $S_1 = \partial\Omega_D$ and $S_2 = \partial\Omega_N$. In addition,

$$(g_*, \boldsymbol{f}_*, \boldsymbol{\Psi}_*, \boldsymbol{\Phi}_*) = \mathcal{C}_{\partial\Omega_D,\partial\Omega_N}\mathcal{F}_*^{22}$$

where the operator

$$\begin{aligned}\mathcal{C}_{\partial\Omega_D,\partial\Omega_N} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A}) \times \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N)&\\ \to L_2(\rho;\Omega) \times \boldsymbol{L}_2(\rho;\Omega) \times \boldsymbol{H}^{-1/2}(\partial\Omega) \times \boldsymbol{H}^{1/2}(\partial\Omega)&\end{aligned}$$

is continuous. Then by Corollary 8.3 and Theorem and 7.1(i,iii), there exists a unique solution of the equation $\mathcal{M}_*^{22}\mathcal{X} = \mathcal{F}_*^{22}$. This solution can be represented as $\mathcal{X} = (p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) = (\mathcal{M}_*^{22})^{-1}\mathcal{F}_*^{22}$ with the operator

$$\begin{aligned}(\mathcal{M}_*^{22})^{-1} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A}) \times \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N)&\\ \to \boldsymbol{\mathcal{H}}^{1,0}(\Omega;\mathcal{A}) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)&\end{aligned}$$

represented by

$$(p, \boldsymbol{v}) = \boldsymbol{\mathcal{A}}_M^{-1}(\boldsymbol{f}_*, g_*, r_{\partial\Omega_D}\boldsymbol{\Phi}_*, r_{\partial\Omega_N}\boldsymbol{\Psi}_*), \quad \boldsymbol{\psi} = \boldsymbol{T}^+(p, \boldsymbol{v}) - \boldsymbol{\Psi}_*, \quad \boldsymbol{\varphi} = \boldsymbol{\gamma}^+\boldsymbol{v} - \boldsymbol{\Phi}_*,$$

where the operator

$$\boldsymbol{\mathcal{A}}_M^{-1} : \boldsymbol{L}_2(\rho;\Omega) \times L_2(\Omega) \times \boldsymbol{H}^{1/2}(\partial\Omega_D) \times \boldsymbol{H}^{-1/2}(\partial\Omega_N) \to \boldsymbol{\mathcal{H}}^{1,0}(\Omega, \boldsymbol{\mathcal{A}})$$

is continuous, see Corollary 8.3. Consequently, the operator $(\mathcal{M}_*^{22})^{-1}$ is a right inverse of the operator (8.28). In addition, $(\mathcal{M}_*^{22})^{-1}$ is also the double sided inverse due to the injectivity of (8.28) given by Theorem 7.1(iii). $\square$

System (8.4)-(8.5) can be expressed using matrix notations as

$$\mathring{\mathcal{M}}^{22}\mathring{\mathcal{X}} = \mathring{\mathcal{F}}^{22}, \tag{8.29}$$

where $\mathring{\mathcal{X}} = (\boldsymbol{\psi}, \boldsymbol{\varphi}) \in \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N)$, the operator

$$\mathring{\mathcal{M}}^{22} : \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N) \to \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N), \tag{8.30}$$

is defined by

$$\mathring{\mathcal{M}}^{22} = \begin{bmatrix} r_{\partial\Omega_D}\left(\frac{1}{2}\boldsymbol{I} - \mathring{\boldsymbol{\mathcal{W}}}'\right) & r_{\partial\Omega_D}\mathring{\boldsymbol{\mathcal{L}}} \\ -r_{\partial\Omega_N}\mathring{\boldsymbol{\mathcal{V}}} & r_{\partial\Omega_N}\left(\frac{1}{2}\boldsymbol{I} + \mathring{\boldsymbol{\mathcal{W}}}\right) \end{bmatrix}, \tag{8.31}$$

and the right hand side $\mathring{\mathcal{F}}^{22} \in \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N)$ is given by

$$\mathring{\mathcal{F}}^{22} = \begin{bmatrix} r_{\partial\Omega_D}\left(\mathring{T}^+(F_0, \boldsymbol{F}) - \boldsymbol{\Psi}_0\right) \\ r_{\partial\Omega_N}\left(\gamma^+\boldsymbol{F} - \boldsymbol{\Phi}_0\right) \end{bmatrix}. \tag{8.32}$$

Operator (8.30) is evidently continuous and moreover, by Corollary 8.4(ii) it is also injective.

Theorem 8.8. *Let conditions 2.1, 4.4 and 4.10 hold. Then, the operator*

$$\mathring{\mathcal{M}}^{22} : \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N) \to \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N) \tag{8.33}$$

is an isomorphism.

Proof. A solution of the system (8.29) with an arbitrary right hand side

$$\mathring{\mathcal{F}}^{22} = [\mathring{\boldsymbol{\mathcal{F}}}_2^{22}, \mathring{\boldsymbol{\mathcal{F}}}_3^{22}] \in \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N)$$

is given by the pair $(\boldsymbol{\psi}, \boldsymbol{\varphi})$ which satisfies the following extended system, cf. (8.4)-(8.7),

$$\widehat{\mathcal{M}}^{22}\mathcal{X} = \widehat{\mathcal{F}}^{22} \tag{8.34}$$

where $\mathcal{X} = (p, \boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi})$, $\widehat{\mathcal{F}}^{22} = (0, \boldsymbol{0}, \mathring{\boldsymbol{\mathcal{F}}}_2^{22}, \mathring{\boldsymbol{\mathcal{F}}}_3^{22})$ and

$$\widehat{\mathcal{M}}^{22} = \begin{bmatrix} I & 0 & -\mathring{\Pi}^s & \mathring{\Pi}^d \\ 0 & \boldsymbol{I} & -\mathring{\boldsymbol{V}} & \mathring{\boldsymbol{W}} \\ 0 & 0 & r_{\partial\Omega_D}\left(\dfrac{1}{2}\boldsymbol{I} - \mathring{\boldsymbol{\mathcal{W}}}'\right) & r_{\partial\Omega_D}\mathring{\boldsymbol{\mathcal{L}}} \\ 0 & 0 & -r_{\partial\Omega_N}\mathring{\boldsymbol{\mathcal{V}}} & r_{\partial\Omega_N}\left(\dfrac{1}{2}\boldsymbol{I} + \mathring{\boldsymbol{\mathcal{W}}}\right) \end{bmatrix} \tag{8.35}$$

By Theorem 8.7 with $\mu = 1$, the operator

$$\begin{aligned} \widehat{\mathcal{M}}^{22} : \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N) \\ \to \boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A}) \times \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N) \end{aligned} \tag{8.36}$$

is an isomorphism. Hence system (8.34) always has a solution $\chi = (\widehat{\mathcal{M}}^{22})^{-1}\widehat{\mathcal{F}}^{22}$ and particularly $(\boldsymbol{\psi}, \boldsymbol{\varphi}) = \left(((\widehat{\mathcal{M}}^{22})^{-1}\widehat{\mathcal{F}}^{22})_3, (\widehat{\mathcal{M}}^{22})^{-1}\widehat{\mathcal{F}}^{22})_4\right)$. This implies that operator (8.33) is surjective. By Corollary 8.4 operator (8.33) is also injective and hence is an isomorphism. □

Let us prove that operator $\mathcal{M}_*^{22}$ is an isomorphism also in wider spaces.

Theorem 8.9. *Let conditions 2.1, 4.4 and 4.10 hold. Then, the operator*

$$\mathcal{M}_*^{22} : \mathbb{X}_* \to \mathbb{Y}_*^{22} \tag{8.37}$$

is an isomorphism

Proof. Let us consider the following operator

$$\widetilde{\mathcal{M}}_*^{22} = \begin{bmatrix} I & 0 & -\Pi^s & \Pi^d \\ 0 & \boldsymbol{I} & -\boldsymbol{V} & \boldsymbol{W} \\ 0 & 0 & r_{\partial\Omega_D}\left(\dfrac{1}{2}\boldsymbol{I} - \mathring{\boldsymbol{\mathcal{W}}}'\right) & r_{\partial\Omega_D}\widehat{\boldsymbol{\mathcal{L}}} \\ 0 & 0 & -r_{\partial\Omega_N}\boldsymbol{\mathcal{V}} & r_{\partial\Omega_N}\left(\dfrac{1}{2}\boldsymbol{I} + \boldsymbol{\mathcal{W}}\right) \end{bmatrix} \tag{8.38}$$

We can express the operator $\widetilde{\mathcal{M}}_*^{22}$ in the form

$$\widetilde{\mathcal{M}}_*^{22} = \mathrm{diag}(I, \frac{1}{\mu}\boldsymbol{I}, \boldsymbol{I}, \frac{1}{\mu}\boldsymbol{I})\widehat{\mathcal{M}}_*^{22}\mathrm{diag}(I, \mu\boldsymbol{I}, \boldsymbol{I}, \mu\boldsymbol{I}). \tag{8.39}$$

The operator $\widehat{\mathcal{M}}_*^{22}$ defined by (8.35) can be understood as a block triangular operator matrix with the following three diagonal block operators

$$\begin{aligned} I :&\ L_2(\Omega^+) \to L_2(\Omega^+), \\ \boldsymbol{I} :&\ \boldsymbol{\mathcal{H}}^1(\Omega^+) \to \boldsymbol{\mathcal{H}}^1(\Omega^+), \\ \mathring{\mathcal{M}}^{22} :&\ \widetilde{\boldsymbol{H}}^{-1/2}(\partial\Omega_D) \times \widetilde{\boldsymbol{H}}^{1/2}(\partial\Omega_N) \to \boldsymbol{H}^{-1/2}(\partial\Omega_D) \times \boldsymbol{H}^{1/2}(\partial\Omega_N). \end{aligned}$$

By Theorem 8.8, the operator $\mathring{\mathcal{M}}^{22}$ is an isomorphism. Consequently, $\widehat{\mathcal{M}}_*^{22}$ is an isomorphism as well. As μ is strictly positive, the diagonal matrices are invertible and the operator $\widetilde{\mathcal{M}}_*^{22}$ is an isomorphism.

The operator $\mathcal{M}_*^{22} - \widetilde{\mathcal{M}}_*^{22} : \mathbb{X}_* \to \mathbb{Y}_*^{22}$ has the form

$$\mathcal{M}_*^{22} - \widetilde{\mathcal{M}}_*^{22} = \begin{bmatrix} 0 & \mathcal{R}^\bullet & 0 & 0 \\ \mathbf{0} & \boldsymbol{\mathcal{R}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{\partial\Omega_D}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & r_{\partial\Omega_D}\left(\mathring{\boldsymbol{\mathcal{W}}}' - \boldsymbol{\mathcal{W}}'\right) & r_{\partial\Omega_D}\left(\boldsymbol{\mathcal{L}}^+ - \widehat{\boldsymbol{\mathcal{L}}}\right) \\ \mathbf{0} & r_{\partial\Omega_N}\gamma^+\boldsymbol{\mathcal{R}} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

and is compact due to the compactness of operators $\boldsymbol{\mathcal{R}}$ and $\mathcal{R}^\bullet$ given by Theorem 4.12 and of operators $\boldsymbol{\mathcal{W}}'$, $\mathring{\boldsymbol{\mathcal{W}}}'$ and $\boldsymbol{\mathcal{L}}^+ - \widehat{\boldsymbol{\mathcal{L}}}$ given by Theorem 4.6 and Corollary 4.9.

Therefore, the operator $\mathcal{M}_*^{22} : \mathbb{X}_* \to \mathbb{Y}_*^{22}$ is Fredholm with zero index. Moreover, this operator is also injective in virtue of Theorem 7.1 and the Remark 6.2, which implies that its an isomorphism. $\square$

8.4. **Isomorphism properties of split BDIE operators.** Since the unknown p appears only in the first equation and there is no interaction between p and $\boldsymbol{v}$ through the wider space $\mathbb{X}_*$ (in contrast to the narrow space containing $\boldsymbol{\mathcal{H}}^{1,0}(\Omega; \mathcal{A})$), we can split the BDIE system to the smaller system containing three vector equations for the unknown 3-tuple $\mathcal{X}^3 = (\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\varphi}) \in \mathbb{X}$ and the remaining first equation considered as the representation formula for p. Hence the operators that define the smaller split systems M11 and M22 are given by

$$\mathcal{M}^{11} = \begin{bmatrix} \boldsymbol{I} + \boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ r_{\partial\Omega_D}\gamma^+\boldsymbol{\mathcal{R}} & -r_{\partial\Omega_D}\boldsymbol{\mathcal{V}} & r_{\partial\Omega_D}\boldsymbol{\mathcal{W}} \\ r_{\partial\Omega_N}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & -r_{\partial\Omega_N}\boldsymbol{\mathcal{W}}' & r_{\partial\Omega_N}\boldsymbol{\mathcal{L}} \end{bmatrix},$$

$$\mathcal{M}^{22} = \begin{bmatrix} \boldsymbol{I} + \boldsymbol{\mathcal{R}} & -\boldsymbol{V} & \boldsymbol{W} \\ r_{\partial\Omega_D}\boldsymbol{T}^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & r_{\partial\Omega_D}\left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{\mathcal{W}}'\right) & r_{\partial\Omega_D}\boldsymbol{\mathcal{L}}^+ \\ r_{\partial\Omega_N}\gamma^+\boldsymbol{\mathcal{R}} & -r_{\partial\Omega_N}\boldsymbol{\mathcal{V}} & r_{\partial\Omega_N}\left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{\mathcal{W}}\right) \end{bmatrix}.$$

The corresponding right hand sides are given by

$$\mathcal{F}^{11} := [\boldsymbol{F}, r_{\partial\Omega_D}\boldsymbol{\gamma}^+\boldsymbol{F} - \boldsymbol{\varphi}_0, r_{\partial\Omega_N}\boldsymbol{T}^+(\boldsymbol{F}, F) - \boldsymbol{\psi}_0] \in \mathbb{Y}^{11},$$
$$\mathcal{F}^{22} := [\boldsymbol{F}, r_{\partial\Omega_D}\boldsymbol{T}^+(F_0, \boldsymbol{F}) - r_{\partial\Omega_D}\boldsymbol{\Psi}_0, r_{\partial\Omega_N}\gamma^+\boldsymbol{F} - r_{\partial\Omega_N}\boldsymbol{\Phi}_0] \in \mathbb{Y}^{22}.$$

Consequently, we can write the systems M11 and M22 as

$$\mathcal{M}^{11}\mathcal{X}^3 = \mathcal{F}^{11}, \qquad \mathcal{M}^{22}\mathcal{X}^3 = \mathcal{F}^{22}.$$

Since the pressure unknown only appears on the first equation of the BDIE systems M11$_*$ and M22$_*$, the invertibility of the operators $\mathcal{M}^{11}$ and $\mathcal{M}^{22}$ is implied by the invertibility of the operators $\mathcal{M}_*^{11}$ and $\mathcal{M}_*^{22}$, which leads us to the following assertion.

Corollary 8.10. *The operators*

$$\mathcal{M}^{11} : \mathbb{X} \to \mathbb{Y}^{11} \quad \textit{and} \quad \mathcal{M}^{22} : \mathbb{X} \to \mathbb{Y}^{22}$$

are isomorphisms.

9. **Conclusion.** In this paper, we considered the mixed problem for compressible Stokes system with a variable smooth viscosity coefficient in a three-dimensional exterior domain with a smooth boundary. The Stokes equations right-hand sides are from the weighted $L_2(\Omega)$ spaces, the Dirichlet data from the space $H^{\frac{1}{2}}(\partial\Omega_D)$ and the Neumann data from the space $H^{-\frac{1}{2}}(\partial\Omega_N)$. Introducing the third Green identity for exterior domains, the BVP was reduced to two systems of Boundary-Domain Integral Equations and their equivalence to the original BVP was shown. After showing compactness of the remainder operators in the exterior domain, where the Rellich compactness theorem is not directly applicable, we proved that the associated operators are isomorphisms in the corresponding weighted Sobolev spaces.

Employing methods similar to [24], this approach can be extended to Lipschitz domains, non-smooth coefficients, and more general PDE right-hand sides. Challenges of the mixed boundary conditions on Lipschitz boundaries can be addressed, e.g., as in [21, 28] and [18, Section 8].

REFERENCES

[1] F. Alliot and C. Amrouche, Weak solutions for the exterior Stokes problem in weighted Sobolev spaces, *Math. Meth. Appl. Sci.*, **23** (2000), 575–600.

[2] C. Amrouche, V. Girault and J. Giroire, Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator. An approach in weighted Sobolev spaces, *J. Math. Pures Appl.*, **76** (1997), 55–81.

[3] A. Bossavit, *Électromagnétisme, en Vue de la Modélisation*, Springer, Berlin, 2003.

[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[5] O. Chkadua, S. E. Mikhailov and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient, I: Equivalence and invertibility, *J. Integral Equ. Appl.*, **21** (2009), 499–543.

[6] O. Chkadua, S. E. Mikhailov and D. Natroshvili, Analysis of direct segregated boundary-domain integral equations for variable-coefficient mixed BVPs in exterior domains, *Anal. Appl.*, **11** (2013), 1350006, 33 pp.

[7] M. Costabel, Boundary integral operators on Lipschitz domains: Elementary results, *SIAM J. Math. Anal.*, **19** (1988), 613–626.

[8] T. T. Dufera and S. E. Mikhailov, Boundary-domain integral equations for variable-coefficient Dirichlet BVP in 2D unbounded domain, in *Analysis, Probability, Applications, and Computations* (eds. Lindahl et al), Springer Nature, Switzerland AG, 2019.

[9] C. Fresneda-Portillo and S. E. Mikhailov, Analysis of boundary-domain integral equations to the mixed BVP for a compressible Stokes system with variable viscosity, *Commun. Pure Appl. Anal.*, **18** (2019), 3059–3088.

[10] J. Giroire, *Étude de quelques problèmes aux limites extérieurs et résolution par équations intégrales*, Thése de Doctorat d'État, Université Pierre-et-Marie-Curie (Paris VI), 1987.

[11] J. Giroire and J. Nedelec, Numerical solution of an exterior Neumann problem using a double layer potential, *Math. Comp.*, **32** (1978), 973–990.

[12] R. Gutt, M. Kohr, S. E. Mikhailov and W. L. Wendland, On the mixed problem for the semilinear Darcy-Forchheimer-Brinkman PDE system in Besov spaces on creased Lipschitz domains, *Math. Methods in Appl. Sci.*, **40** (2017), 7780–7829.

[13] B. Hanouzet, Espaces de Sobolev avec Poids. Application au probleme de Dirichlet dans un demi espace, *Rendiconti del Seminario Matematico della Universita di Padova*, **46** (1971), 227–272.

[14] G. C. Hsiao and W. L. Wendland, *Boundary Integral Equations*, Springer, Berlin, 2008.

[15] M. Kohr, M. Lanza de Cristoforis, S. E. Mikhailov and W. L. Wendland, Integral potential method for transmission problem with Lipschitz interface in $\mathbb{R}^3$ for the Stokes and Darcy-Forchheimer-Brinkman PDE systems, *Z. Angew. Math. Phys.*, **67** (2016), 116, 30pp.

[16] M. Kohr, S. E. Mikhailov and W. L. Wendland, Variational approach for layer potentials of the Stokes system with L_∞ symmetrically elliptic coefficient tensor and applications to Stokes and Navier-Stokes boundary problems, `arXiv:2002.09990`.

[17] M. Kohr and W. L. Wendland, Variational boundary integral equations for the Stokes system, *Appl. Anal.*, **85** (2006), 1343–1372.

[18] M. Kohr and W. L. Wendland, Boundary value problems for the Brinkman system with L_∞ coefficients in Lipschitz domains on compact Riemannian manifolds. A variational approach, *J. Math. Pures Appl.*, **131** (2019), 17–63.

[19] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon & Breach, New York, 1969.

[20] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Springer, Berlin, 1973.

[21] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, UK, 2000.

[22] S. E. Mikhailov, Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains, *J. Math. Anal. Appl.*, **378** (2011), 324–342.

[23] S. E. Mikhailov, Localized boundary-domain integral formulations for problems with variable coefficients, *Eng. Anal. Bound. Elem.*, **26** (2002), 681–690.

[24] S. E. Mikhailov, Analysis of segregated boundary-domain integral equations for BVPs with non-smooth coefficient on Lipschitz domains, *Bound. Value Probl.*, **87** (2018), 1–52.

[25] S. E. Mikhailov and C. F. Portillo, BDIE system to the mixed BVP for the Stokes equations with variable viscosity, in *Integral Methods in Science and Engineering: Theoretical and Computational Advances* (eds. C. Constanda and A. Kirsh), Springer, Boston, 2015.

[26] S. E. Mikhailov and C. F. Portillo, A new family of boundary-domain integral equations for a mixed elliptic BVP with variable coefficient, in *Proceedings of the 10th UK Conference on Boundary Integral Methods* (ed. P. Harris), Brighton University Press, 2015.

[27] S. E. Mikhailov and C. F. Portillo, BDIEs for the compressible Stokes system with variable viscosity mixed BVP in bounded domains, in *Proceedings of the 11th UK Conference on Boundary Integral Methods* (ed. D.J. Chappell), Nottingham Trent Univ., 2017.

[28] I. Mitrea and M. Mitrea, The Poisson problem with mixed boundary conditions in Sobolev and Besov spaces in non-smooth domains, *T. Am. Math. Soc.*, **359** (2007), 4143–4182.

[29] J. Nedelec, *Acoustic and Electromagnetic Equations*, Springer-Verlag, New York, 2001.

[30] J. Nedelec and J. Planchard, Une méthode variationnelle d'éléments finis pour la résolution numérique d'un problème extérieur dans $\mathbb{R}^3$, *RAIRO*, **7** (1973), 105–129.

[31] J. B. Neto, Inhomogeneous boundary value problems in a half space, *Ann. Sc. Sup. Pisa*, **19** (1965), 331–365.

[32] B. Reidinger and O. Steinbach, A symmetric boundary element method for the Stokes problem in multiple connected domains, *Math. Meth. Appl. Sci.*, **26** (2003), 77–93.

[33] O. Steinbach, *Numerical Approximation Methods for Elliptic Boundary Value Problems*, Springer, Berlin, 2007.

[34] R. Temam, *Navier-Stokes Equations*, AMS Chelsea Edition, American Mathematical Society, 2001.

[35] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

[36] W. Varnhorn, *The Stokes Equations*, Akademie Verlag, Berlin, 1994.

[37] W. L. Wendland and J. Zhu, The boundary element method for three dimensional Stokes flow exterior to an open surface, *Math. Comput. Model.*, **6** (1991), 19–42.

Received August 2020; revised November 2020.

E-mail address: sergey.mikhailov@brunel.ac.uk
E-mail address: cfresneda@uloyola.es