MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2006; **29**:715–739
Published online 20 December 2005 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/mma.706
MOS subject classification: 47 G; 45 A 05; 45 K 05; 35 J 25

Analysis of united boundary-domain integro-differential and integral equations for a mixed BVP with variable coefficient

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Communicated by M. Mitrea

SUMMARY

The mixed (Dirichlet–Neumann) boundary-value problem for the 'Laplace' linear differential equation with variable coefficient is reduced to boundary-domain integro-differential or integral equations (BDIDEs or BDIEs) based on a specially constructed parametrix. The BDIDEs/BDIEs contain integral operators defined on the domain under consideration as well as potential-type operators defined on open sub-manifolds of the boundary and acting on the trace and/or co-normal derivative of the unknown solution or on an auxiliary function. Some of the considered BDIDEs are to be supplemented by the original boundary conditions, thus constituting boundary-domain integro-differential problems (BDIDPs). Solvability, solution uniqueness, and equivalence of the BDIEs/BDIDEs/BDIDPs to the original BVP, as well as invertibility of the associated operators are investigated in appropriate Sobolev spaces. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: integral equations; integro-differential equations; parametrix; partial differential equations; variable coefficients; mixed boundary-value problem; Sobolev spaces; equivalence; invertibility

1. INTRODUCTION

It is well known that reduction of boundary-value problems (BVP) with arbitrarily variable coefficients to boundary integral equations is usually not effective for numerical implementations. This is due to the fact that the fundamental solution necessary for such reduction is generally not available in an analytical form (except some special dependence of the coefficients on co-ordinates, see, e.g. Reference [1]). Using a parametrix (Levi function) introduced in References [2,3], as a substitute of a fundamental solution, it is possible, however, to reduce such a BVP to a boundary-domain integral equation (BDIE) (see, e.g. References [4, Section 18], [5,6], where the Dirichlet, Neumann and Robin problems were reduced to indirect BDIEs).

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Received 8 October 2004
Revised 26 October 2005
Accepted 30 October 2005

In this paper we consider a three-dimensional mixed (Dirichlet–Neumann) BVP for the variable-coefficient 'Laplace' equation. Such problems appear, e.g. in electrostatics, stationary heat transfer and other diffusion problems for inhomogeneous media.

In References [7,8], the BVP was reduced to four different systems of BDIEs, which do not involve derivatives of the unknown solution u. This was done, however, at the expense of considering the boundary traces of the solution and its co-normal derivatives as functions independent of (segregated from) the solution inside the domain.

On the other hand, it seems natural in numerical implementations to employ the relations between the boundary and internal values of the solution. To create an analytical ground of such united approach, in this paper we consider reduction of the BVP to three different *united* boundary-domain *integro-differential* equations/problems, and one *partly segregated* BDIE. Some of them are associated with the BDIDE/BDIE formulated in Reference [9]. Equivalence of the considered BDIDPs/BDIDEs/BDIEs to the original mixed BVP is proved along with their solvability and solution uniqueness as well as invertibility of the associated operators in corresponding Sobolev–Slobodetski spaces.

2. FORMULATION OF THE BVP

Let Ω^+ be a bounded open three-dimensional region of $\mathbb{R}^3$ and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. For simplicity, we assume that the boundary $S := \partial\Omega^+$ is a simply connected, closed, infinitely smooth surface. Moreover, $S = \overline{S}_\mathrm{D} \cup \overline{S}_\mathrm{N}$ where S_D and S_N are non-empty, non-intersecting ($S_\mathrm{D} \cap S_\mathrm{N} = \emptyset$), simply connected sub-manifolds of S with infinitely smooth boundary curve $\ell := \partial S_\mathrm{D} = \partial S_\mathrm{N} \in C^\infty$. Let $a \in C^\infty(\mathbb{R}^3)$, $a(x) > 0$. Let also $\partial_j = \partial_{x_j} := \partial/\partial x_j$ $(j = 1, 2, 3)$, $\partial_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

We consider the following scalar elliptic differential equation:

$$Lu(x) := L(x, \partial_x)u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x), \quad x \in \Omega^\pm \tag{1}$$

where u is an unknown function and f is a given function in Ω^+.

In what follows $H^s(\Omega^+) = H_2^s(\Omega^+)$ and, respectively, $H^s_{\mathrm{loc}}(\Omega^-)$, $H^s_{\mathrm{comp}}, (\Omega^-)$, $H^s(S)$ are the Bessel potential spaces, where s is a real number (see, e.g. References [10,11]). We recall that H^s coincide with the Sobolev–Slobodetski spaces W_2^s for any non-negative or integer s.

For a linear operator L_*, we introduce the following subspace of $H^s(\Omega)$, c.f. Reference [12]:

$$H^{s,0}(\Omega; L_*) := \{g : g \in H^s(\Omega),\ L_* g \in L_2(\Omega)\}$$

provided with the norm

$$\|g\|_{H^{s,0}(\Omega;L_*)} := \|g\|_{H^s(\Omega)} + \|L_* g\|_{L_2(\Omega)}$$

In this paper, we will particularly use the space $H^{s,0}(\Omega; L_*)$ for L_* being either the operator L from (1) or the Laplace operator Δ. Further we have,

$$Lu - \Delta u = \sum_{i=1}^{3} \frac{\partial a}{\partial x_i}\frac{\partial u}{\partial x_i}$$

Then $Lu - \Delta u \in L_2(\Omega)$ for $u \in H^s(\Omega)$, and $H^{s,0}(\Omega; L) = H^{s,0}(\Omega; \Delta)$ for $s \geqslant 1$.

For $S_1 \subset S$, we will use the subspace $\tilde{H}^s(S_1) = \{g : g \in H^s(S), \operatorname{supp} g \subset \overline{S_1}\}$ of $H^s(S)$, while $H^s(S_1) = \{r_{S_1} g : g \in H^s(S)\}$ denotes the space of restriction on S_1 of functions from $H^s(S)$, where r_{S_1} denotes the restriction operator on S_1.

From the trace theorem (see References [10,11,13]) for $u \in H^s(\Omega^+)$ $(u \in H^s_{\mathrm{loc}}(\Omega^-))$, $s > \frac{1}{2}$, it follows that $u|_S^{\pm} := \tau_S^{\pm} u \in H^{s-(1/2)}(S)$, where $\tau_S^{\pm}$ is the trace operator on S from $\Omega^{\pm}$. We will use also notations $u^{\pm}$ or $[u]^{\pm}$ for the traces $u|_S^{\pm}$, when this will cause no confusion.

For $u \in H^s(\Omega^+)$ $(u \in H^s_{\mathrm{loc}}(\Omega^-))$, $s > \frac{3}{2}$, we can denote by $T^{\pm}$ the corresponding co-normal differentiation operator on S in the sense of traces,

$$T^{\pm}(x, n^{\pm}(x), \partial_x) u(x) := \sum_{i=1}^{3} a(x) n_i^{\pm}(x) \left(\frac{\partial u(x)}{\partial x_i} \right)^{\pm} = a(x) \left(\frac{\partial u(x)}{\partial n^{\pm}(x)} \right)^{\pm}$$

where $n^{\pm}(x)$ is the exterior (to $\Omega^{\pm}$) unit normal vectors at the point $x \in S$.

Let $u \in H^{s,0}(\Omega^+; \Delta)$ $[u \in H^{s,0}_{\mathrm{loc}}(\Omega^-; \Delta)]$, $1 \leqslant s < \frac{3}{2}$. We can correctly define the generalized co-normal derivative $T^{\pm} u \in H^{s-(3/2)}(S)$ with the help of the first Green's formula (c.f., e.g. References [12], [13, Lemma 4.3]),

$$\langle T^{\pm} u, v^{\pm} \rangle_S := \int_{\Omega^{\pm}} v(x) L u(x) \, \mathrm{d}x + \mathscr{E}^{\pm}(u, v) \quad \forall v \in H^{2-s}(\Omega^+) \quad [v \in H^{2-s}_{\mathrm{comp}}(\overline{\Omega^-})] \tag{2}$$

where

$$\mathscr{E}^{\pm}(u, v) := \sum_{i=1}^{3} \langle a \partial_i u, \partial_i v \rangle_{\Omega^{\pm}}$$

and $\langle \cdot , \cdot \rangle_S$ denotes the duality brackets between the spaces $H^{s-(3/2)}(S)$ and $H^{(3/2)-s}(S)$, while $\langle \cdot , \cdot \rangle_{\Omega^{\pm}}$ denotes the duality brackets between the spaces $H^{s-1}(\Omega^{\pm})$ and $H^{1-s}(\Omega^{\pm})$, extending the usual L_2 scalar product.

If $u, v \in H^1(\Omega^+)$ $[u \in H^1_{\mathrm{loc}}(\Omega^-),\ v \in H^1_{\mathrm{comp}}(\overline{\Omega^-})]$, then

$$\mathscr{E}^{\pm}(u, v) := \int_{\Omega^{\pm}} \sum_{i=1}^{3} a(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} \, \mathrm{d}x \tag{3}$$

We will consider the following *mixed BVP* for $1 \leqslant s < \frac{3}{2}$. Find a function $u \in H^s(\Omega^+)$ satisfying the conditions

$$Lu = f \quad \text{in } \Omega^+ \tag{4}$$

$$r_{S_{\mathrm{D}}} u^+ = \varphi_0 \quad \text{on } S_{\mathrm{D}} \tag{5}$$

$$r_{S_{\mathrm{N}}} T^+ u = \psi_0 \quad \text{on } S_{\mathrm{N}} \tag{6}$$

where $\varphi_0 \in H^{s-(1/2)}(S_{\mathrm{D}})$, $\psi_0 \in H^{s-(3/2)}(S_{\mathrm{N}})$ and $f \in L_2(\Omega^+)$.

Equation (4) is understood in the distributional sense, that is,

$$\mathscr{E}^+(u, v) = -\int_{\Omega^+} f(x) v(x) \, \mathrm{d}x \quad \forall v \in \tilde{H}^{2-s}(\Omega^+)$$

condition (5) is understood in the trace sense, while equality (6) is understood in the functional sense (2).

We have the following uniqueness theorem.

Theorem 2.1
The homogeneous version of BVP (4)–(6), i.e. with $f=0, \varphi_0=0, \ \psi_0=0$, has only the trivial solution in $H^s(\Omega^+), \ 1\leqslant s<\frac{3}{2}$.

Proof
The proof immediately follows from Green's formula (2) with $v=u$ as a solution of the homogeneous mixed BVP, taking into account (3). □

Thus, non-homogeneous problems (4)–(6) may possess at most one solution due to the problem linearity.
Moreover, the following existence theorem for the original BVP holds true.

Theorem 2.2
Let $1\leqslant s<\frac{3}{2}$, $\varphi_0\in H^{s-(1/2)}(S_\mathrm{D})$, $\psi_0\in H^{s-(3/2)}(S_\mathrm{N})$ and $f\in L_2(\Omega^+)$. Then mixed BVP (4)–(6) is uniquely solvable in $H^{s,0}(\Omega^+;\Delta)$.

Proof
Follows, e.g. from Corollary C.4 of Appendix C. □

3. PARAMETRIX AND POTENTIAL-TYPE OPERATORS

3.1. Parametrix

We will say, a function $P(x,y)$ of two variables $x,y\in\Omega$ is a parametrix (the Levi function) for the operator $L(x,\partial_x)$ in $\mathbb{R}^3$ if (see, e.g. References [2–4])

$$L(x,\partial_x)P(x,y)=\delta(x-y)+R(x,y) \tag{7}$$

where $\delta(\cdot)$ is the Dirac distribution and $R(x,y)$ possesses a weak (integrable) singularity at $x=y$, i.e.

$$R(x,y)=\mathcal{O}(|x-y|^{-\varkappa}) \quad \text{with } \varkappa<3 \tag{8}$$

It is easy to see that for the operator $L(x,\partial_x)$ given by (1), the function

$$P(x,y)=\frac{-1}{4\pi a(y)|x-y|}, \quad x,y\in\mathbb{R}^3 \tag{9}$$

is a parametrix and the corresponding remainder function is

$$R(x,y)=\sum_{i=1}^{3}\frac{x_i-y_i}{4\pi a(y)|x-y|^3}\,\frac{\partial a(x)}{\partial x_i}, \quad x,y\in\mathbb{R}^3 \tag{10}$$

and satisfies estimate (8) with $\varkappa=2$, due to the smoothness of the function $a(x)$.
Evidently, the parametrix $P(x,y)$ given by (9) is a fundamental solution to the operator $L(y,\partial_x):=a(y)\Delta(\partial_x)$ with 'frozen' coefficient $a(x)=a(y)$, i.e.

$$L(y,\partial_x)P(x,y)=\delta(x-y)$$

Note that remainder (10) is not smooth enough for the parametrix (9) and the corresponding potential operators to be treated as in Reference [13].

Let us introduce the single and the double layer surface potential operators,

$$Vg(y) := -\int_S P(x,y)g(x)\,\mathrm{d}S_x, \quad y \notin S \tag{11}$$

$$Wg(y) := -\int_S [T(x,n(x),\partial_x)P(x,y)]g(x)\,\mathrm{d}S_x, \quad y \notin S \tag{12}$$

where g is some scalar function, and the integrals are understood in the distributional sense if g is not integrable.

The corresponding boundary integral (pseudodifferential) operators of direct surface values of the simple layer potential $\mathscr{V}$ and of the double layer potential $\mathscr{W}$, the co-normal derivatives of the simple layer potential $\mathscr{W}'$ and of the double layer potential $\mathscr{L}^{\pm}$ are

$$\mathscr{V}g(y) := -\int_S P(x,y)g(x)\,\mathrm{d}S_x \tag{13}$$

$$\mathscr{W}g(y) := -\int_S [T(x,n(x),\partial_x)P(x,y)]g(x)\,\mathrm{d}S_x \tag{14}$$

$$\mathscr{W}'g(y) := -\int_S [T(y,n(y),\partial_y)P(x,y)]g(x)\,\mathrm{d}S_x \tag{15}$$

$$\mathscr{L}^{\pm}g(y) := [T(y,n(y),\partial_y)Wg(y)]^{\pm} \tag{16}$$

where $y \in S$.

The parametrix-based volume potential operator and the remainder potential operator, corresponding to parametrix (9) and to remainder (10) are

$$\mathscr{P}g(y) := \int_{\Omega^+} P(x,y)g(x)\,\mathrm{d}x \tag{17}$$

$$\mathscr{R}g(y) := \int_{\Omega^+} R(x,y)g(x)\,\mathrm{d}x \tag{18}$$

Note that if $g \in H^s(\Omega^+)$ for $\frac{1}{2} < s < \frac{3}{2}$, then (18) can be rewritten in the form

$$\mathscr{R}g = -\mathscr{P}\tilde{L}g - V\left(g^+\frac{\partial a}{\partial n}\right) \tag{19}$$

where

$$\tilde{L}g := \sum_{j=1}^{3} \frac{\partial}{\partial x_i}\left(g\frac{\partial a}{\partial x_i}\right)$$

and, evidently, $\tilde{L}g \in H^{s-1}(\Omega^+) = \tilde{H}^{s-1}(\Omega^+)$.

The mapping properties of potentials and operators (11)–(18) and the jump properties of surface potentials of type (11)–(12), connecting them with (13)–(16) are well known for the case $a = \text{const}$ in H^s. They were extended to the case of variable coefficient $a(x)$ in

Reference [7], where the invertibility results for the operators $\mathscr{V}$ and $\mathscr{L}^+$ on a part S_1 of S are also obtained. We provide some of the latter results in Appendices A and B along with several of their counterparts in the space $H^{s,0}(\Omega, \Delta)$.

4. GREEN IDENTITIES AND INTEGRAL RELATIONS

Let $u \in H^{s,0}(\Omega^+; \Delta)$, $v \in H^{2-s,0}(\Omega^+; \Delta)$ be some real functions, $1 \leqslant s < \frac{3}{2}$. Then, subtracting (2) from its counterpart with exchanged roles of u and v, we obtain the so-called second Green identity for the operator $L(x, \partial_x)$,

$$\int_{\Omega^+} [vL(x,\partial_x)u - uL(x,\partial_x)v]\,\mathrm{d}x = \langle T^+u, v^+\rangle_S - \langle u^+, T^+v\rangle_S \tag{20}$$

For $u \in H^{s,0}(\Omega^+; \Delta)$, $1 \leqslant s < \frac{3}{2}$, and $v(x) = P(x, y)$, where the parametrix $P(x, y)$ is given by (9), we obtain from (20), (7) by the standard limiting procedures (c.f. Reference [4]), the third Green identity,

$$u(y) + \mathscr{R}u(y) - VT^+u(y) + Wu^+(y) = \mathscr{P}Lu(y), \quad y \in \Omega^+ \tag{21}$$

If $u \in H^{s,0}(\Omega^+; \Delta)$, $1 \leqslant s < \frac{3}{2}$, is a solution of Equation (1), then (21) gives

$$Gu := u + \mathscr{R}u - VT^+u + Wu^+ = \mathscr{P}f \quad \text{in } \Omega^+ \tag{22}$$

$$\mathscr{G}u := \tfrac{1}{2}u^+ + [\mathscr{R}u]^+ - \mathscr{V}T^+u + \mathscr{W}u^+ = [\mathscr{P}f]^+ \quad \text{on } S \tag{23}$$

$$\mathscr{T}u := \tfrac{1}{2}T^+u + T^+\mathscr{R}u - \mathscr{W}'T^+u + \mathscr{L}^+u^+ = T^+\mathscr{P}f \quad \text{on } S \tag{24}$$

For some functions f, Ψ, Φ, let us consider a more general 'indirect' integral relation, associated with (22),

$$u(y) + \mathscr{R}u(y) - V\Psi(y) + W\Phi(y) = \mathscr{P}f(y), \quad y \in \Omega^+ \tag{25}$$

Lemma 4.1
Let $1 \leqslant s < \frac{3}{2}$. Suppose some functions $u \in H^s(\Omega^+)$, $\Psi \in H^{s-(3/2)}(S)$, $\Phi \in H^{s-(1/2)}(S)$, $f \in L_2(\Omega^+)$ satisfy (25). Then $u \in H^{s,0}(\Omega^+, \Delta)$, it is a solution of PDE (4) in Ω^+ and

$$V(\Psi - T^+u)(y) - W(\Phi - u^+)(y) = 0, \quad y \in \Omega^+$$

Proof
First of all, Equation (25) and mapping properties of the operators $\mathscr{R}$, $\mathscr{P}$, V and W, see Appendix B and Theorem A.1, imply $u \in H^{s,0}(\Omega^+; \Delta)$. The rest of the lemma claims follow from its counterpart proved in Reference [7, Lemma 4.1] for $s = 1$. □

Lemma 4.2
Let $s \geqslant 1$.

(i) Let $\Psi^* \in H^{s-(3/2)}(S)$. If $V\Psi^*(y) = 0$, $y \in \Omega^+$, then $\Psi^* = 0$.
(ii) Let $\Phi^* \in H^{s-(1/2)}(S)$. If $W\Phi^*(y) = 0$, $y \in \Omega^+$, then $\Phi^* = 0$.

(iii) Let $S=\bar{S}_1\cup\bar{S}_2$, where S_1 and S_2 are non-intersecting simply connected non-empty sub-manifolds of S with infinitely smooth boundaries. Let $\Psi^*\in\tilde{H}^{s-(3/2)}(S_1)$, $\Phi^*\in\tilde{H}^{s-(1/2)}(S_2)$. If

$$V\Psi^*(y)-W\Phi^*(y)=0,\quad y\in\Omega^+$$

then $\Psi^*=0$, $\Phi^*=0$.

Proof

For $s=1$ the proof is provided in Reference [7, Lemma 4.2], which evidently implies the lemma claims also for $s>1$. □

5. BOUNDARY-DOMAIN INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS AND PROBLEMS

5.1. United boundary-domain integro-differential problem (GDN)

Theorem 5.1

Let $f\in L_2(\Omega^+)$. A function $u\in H^{s,0}(\Omega^+;\Delta)$, $1\leqslant s<\frac{3}{2}$, is a solution PDE (4) in Ω^+ if and only if it is a solution of BDIDE (22).

Proof

If $u\in H^{s,0}(\Omega^+;\Delta)$, $1\leqslant s$, solves PDE (4) in Ω^+, then, as follows from (20), it satisfies (22). On the other hand, if $u\in H^{s,0}(\Omega^+;\Delta)$, $1\leqslant s<\frac{3}{2}$, solves BDIDE (22), then using Lemma 4.1 for $\Psi=T^+u$, $\Phi=u^+$ completes the proof. □

The proved equivalence of the PDE and the BDIDE now allows to supplement BDIDE (22) with the original mixed boundary conditions (5)–(6) and arrive at the following BDIDP (*GDN*):

$$\mathscr{A}^{\mathrm{GDN}}u=\mathscr{F}^{\mathrm{GDN}}\tag{26}$$

where

$$\mathscr{A}^{\mathrm{GDN}}:=\begin{bmatrix} I+\mathscr{R}-VT^++W\tau^+\\ r_{S_{\mathrm{D}}}\tau^+\\ r_{S_{\mathrm{N}}}T^+\end{bmatrix},\quad \mathscr{F}^{\mathrm{GDN}}=\begin{bmatrix}\mathscr{P}f\\ \varphi_0\\ \psi_0\end{bmatrix}$$

The BDIDP is equivalent to the mixed BVP (4)–(6) in Ω^+, in the following sense.

Theorem 5.2

Let $f\in L_2(\Omega^+)$, $\varphi_0\in H^{s-(1/2)}(S_{\mathrm{D}})$, $\psi_0\in H^{s-(3/2)}(S_{\mathrm{N}})$, $1\leqslant s<\frac{3}{2}$.

(i) There exists a unique solution $u\in H^{s,0}(\Omega^+;\Delta)$ of mixed BVP (4)–(6) in Ω^+. The function u solves also BDIDP (26).

(ii) There exists a unique solution $u\in H^{s,0}(\Omega^+;\Delta)$ of BDIDP (26). The function u solves also mixed BVP (4)–(6) in Ω^+.

Proof

A solution of BVP (4)–(6) does exist and is unique due to Theorem 2.2 and provides a solution to BDIDP (26) due to Theorem 5.1. On the other hand, any solution of BDIDP (26) satisfies also (4) due to the same Theorem 5.1.

Since $\Delta(a\mathscr{P}f)=f$ in Ω^+ for any $f\in L_2(\Omega^+)$, then $\mathscr{P}f=0$ implies $f=0$. Thus, uniqueness of solution to BDIDP (26) follows from uniqueness of solution to BVP (4)–(6), Theorem 2.1. $\square$

Due to the mapping properties of operators V, W, $\mathscr{P}$ and $\mathscr{R}$, see Appendices, we have $\mathscr{F}^{\mathrm{GDN}}\in H^{s,0}(\Omega^+;\Delta)\times H^{s-(1/2)}(S_{\mathrm{D}})\times H^{s-(3/2)}(S_{\mathrm{N}})$, and the operator $\mathscr{A}^{\mathrm{GDN}}: H^{s,0}(\Omega^+;\Delta)\to H^{s,0}(\Omega^+;\Delta)\times H^{s-(1/2)}(S_{\mathrm{D}})\times H^{s-(3/2)}(S_{\mathrm{N}})$ is continuous. It is also injective due to Theorem 5.2.

Let us introduce a more convenient space to describe properties of the operator $\mathscr{A}^{\mathrm{GDN}}$.

Definition 5.3
Let $s\leqslant 2$. The space $Y_1^s(\Omega^+;L)$ consists of the functions of the form

$$\mathscr{F}_*=\mathscr{P}f_* \quad \text{in } \Omega^+ \tag{27}$$

with $f_*\in L_2(\Omega^+)$ and is provided with the norm of space $H^{s,0}(\Omega^+;\Delta)$, $\|\mathscr{F}_*\|_{Y_1^s(\Omega^+;L)}:=\|\mathscr{F}_*\|_{H^{s,0}(\Omega^+;\Delta)}$.

The mapping properties of the operator $\mathscr{P}$, see Remark B.2, imply that $Y_1^s(\Omega^+;L)$ is a subset of $H^{s,0}(\Omega^+;\Delta)$ for $s\leqslant 2$. Completeness of $Y_1^s(\Omega^+;L)$ is proved in Lemma 5.6 below.

Let us give another characterization of the space $Y_1^s(\Omega^+;L)$. Let T_Δ^+, V_Δ, W_Δ and $\mathscr{P}_\Delta$ denote the operators of co-normal derivative, simple layer potential, double layer potential and volume potential associated with the Laplace operator, that is, for the coefficient $a=1$.

Remark 5.4
Let $1\leqslant s<\frac{3}{2}$. A function $\mathscr{F}_*\in H^{s,0}(\Omega^+;\Delta)$ belongs to $Y_1^s(\Omega^+;L)$ if and only if

$$V_\Delta T_\Delta^+(a\mathscr{F}_*)-W_\Delta(a\mathscr{F}_*)^+=0 \quad \text{in } \Omega^+ \tag{28}$$

or, the same,

$$V\left[T^+\mathscr{F}_*+\mathscr{F}_*^+\frac{\partial a}{\partial n^+}\right]-W(\mathscr{F}_*)^+=0 \quad \text{in } \Omega^+ \tag{29}$$

Proof
Condition (27) can be rewritten as

$$a\mathscr{F}_*=\mathscr{P}_\Delta f_* \quad \text{in } \Omega^+ \tag{30}$$

Third Green's identity (21) for $u=a\mathscr{F}_*$ and for the potentials associated with the operator Δ gives

$$a\mathscr{F}_*-V_\Delta T_\Delta^+(a\mathscr{F}_*)+W(a\mathscr{F}_*)^+=\mathscr{P}_\Delta\Delta(a\mathscr{F}_*) \quad \text{in } \Omega^+ \tag{31}$$

Thus, (28) implies (30) with $f_*=\Delta(a\mathscr{F}_*)$.

On the other hand, if (30) is satisfied, then application of the Laplace operator to it gives $\Delta(a\mathscr{F}_*)=f_*$ in Ω^+, which substitution into (31) and comparison with (30) implies (28).

Condition (29) follows from (28) and the definitions of V and W. □

To realize, how narrow is the subspace $Y_1^s(\Omega^+;L)$, let us prove the following statement.

Lemma 5.5

For any function $\mathscr{F}_* \in H^{s,0}(\Omega;\Delta)$, $s \geqslant 1$, there exists a unique couple $(f_*,\Psi_*)=\mathscr{C}_\Psi\mathscr{F}_* \in L_2(\Omega)\times H^{s-(3/2)}(S)$ such that

$$\mathscr{F}_*(y)=\mathscr{P}f_*(y)+V\Psi_*(y), \quad y\in\Omega \tag{32}$$

and $\mathscr{C}_\Psi : H^{s,0}(\Omega;\Delta)\to L_2(\Omega)\times H^{s-(3/2)}(S)$ is a linear bounded operator.

Proof

We adapt here the proof scheme from Reference [8, Lemma 5.2]. Suppose first there exist some functions $f_*(y)$, $\Psi_*(y)$ satisfying (32) and find their expressions in terms of $\mathscr{F}_*(y)$. Taking into account definitions (17) and (11) for the volume and single layer potentials, ansatz (32) can be rewritten as

$$a(y)\mathscr{F}_*(y)=\mathscr{P}_\Delta f_*(y)+V_\Delta\Psi_*(y), \quad y\in\Omega \tag{33}$$

Applying the Laplace operator to (33) we obtain that

$$f_*=\Delta(a\mathscr{F}_*) \quad \text{in } \Omega \tag{34}$$

Then (33) can be rewritten as

$$V_\Delta\Psi_*(y)=Q(y), \quad y\in\Omega \tag{35}$$

where

$$Q(y):=a(y)\mathscr{F}_*(y)-\mathscr{P}_\Delta[\Delta(a\mathscr{F}_*)](y), \quad y\in\Omega \tag{36}$$

The trace of (35) on the boundary gives

$$\mathscr{V}_\Delta\Psi_*(y)=Q^+(y), \quad y\in S \tag{37}$$

where $\mathscr{V}_\Delta:=\mathscr{V}|_{a=1}$ is the direct value on S of the single layer operator associated with the Laplace operator.

Since $\mathscr{V}_\Delta : H^s(S)\to H^{s+1}(S)$, $s\in\mathbb{R}$, is isomorphism (c.f., e.g. Reference [14, Chapter XI, Part B, Section 2, Remark 1]), we obtain the following expression for Ψ_*:

$$\Psi_*(y)=\mathscr{V}_\Delta^{-1}Q^+(y), \quad y\in S \tag{38}$$

Relations (34) and (38) imply uniqueness of the couple f_*,Ψ_*. Now we have to prove that $f_*(y)$, $\Psi_*(y)$ given by (34) and (38) do satisfy (32). Indeed, the potential $V_\Delta\Psi_*(y)$ with $\Psi_*(y)$ given by (38) is a harmonic function, and one can check that Q given by (36) is also harmonic. Since (37) implies $V_\Delta\Psi_*(y)$ and $Q(y)$ coincide on the boundary, the two harmonic functions should coincide also in the domain, i.e. (35) holds true, which implies (32).

Thus, (34), (38), (36) give bounded operator

$$\mathscr{C}_\Psi = [\Delta,\ \mathscr{V}_\Delta^{-1}\tau_S^+(I-\mathscr{P}_\Delta\Delta)]a\circ : H^{s,0}(\Omega;\Delta)\to L_2(\Omega)\times H^{s-(3/2)}(S)$$

mapping $\mathscr{F}_*$ to (f_*,Ψ_*). □

Lemma 5.5 implies that ansatz (27) does not cover the whole space $H^{s,0}(\Omega^+;\Delta)$, i.e. $Y_1^s(\Omega^+;L)$ is more narrow than the space $H^{s,0}(\Omega^+;\Delta)$. Let us prove $Y_1^s(\Omega^+;L)$ a closed subspace of $H^{s,0}(\Omega^+;\Delta)$.

Lemma 5.6
Let $1\leqslant s\leqslant 2$. The space $Y_1^s(\Omega^+;L)$ is complete.

Proof
Let $\mathscr{F}_1^{(n)}$, $n=1,2,\dots$ be a Cauchy sequence in $Y_1^s(\Omega^+;L)$. Then $\mathscr{F}_1^{(n)}=\mathscr{P}f_*^{(n)}$ in Ω^+ for some $f_*^{(n)}\in L_2(\Omega^+)$. Due to Lemma 5.5, $f_*^{(n)}=\mathscr{C}_{\Psi 1}(S_\mathrm{D},S_\mathrm{N})\mathscr{F}_*^{(n)}$, where $\mathscr{C}_{\Psi 1}:H^{s,0}(\Omega^+;\Delta)\to L_2(\Omega^+)$ is a linear bounded operator, which implies $f_*^{(n)}$ is a Cauchy sequence in $L_2(\Omega^+)$. Since $L_2(\Omega^+)$ is complete, the sequence has a limit $f_*\in L_2(\Omega^+)$. Due to Remark B.2, the operator $\mathscr{P}:L_2(\Omega^+)\to H^{s,0}(\Omega^+;\Delta)$ is bounded, implying $\mathscr{F}_1^{(n)}$ converges to $\mathscr{F}_*=\mathscr{P}f_*$ in $H^{s,0}(\Omega^+;\Delta)$, which completes the proof. □

Now we are in a position to prove the invertibility theorem.

Theorem 5.7
Let $1\leqslant s<\frac{3}{2}$. The operator

$$\mathscr{A}^{\mathrm{GDN}}:H^{s,0}(\Omega^+;\Delta)\to Y_1^s(\Omega^+;L)\times H^{s-(1/2)}(S_\mathrm{D})\times H^{s-(3/2)}(S_\mathrm{N}) \qquad (39)$$

is continuous and continuously invertible.

Proof
If $u\in H^{s,0}(\Omega^+;\Delta)$, then the third Green identity (21) implies $\mathscr{A}^{\mathrm{GDN}}u=(\mathscr{P}Lu, r_{S_\mathrm{D}}\tau^+u, r_{S_\mathrm{N}}T^+u)^\top$, i.e. operator (39) is continuous.

On the other hand, if $\mathscr{F}\in Y_1^s(\Omega^+;L)\times H^{s-(1/2)}(S_\mathrm{D})\times H^{s-(3/2)}(S_\mathrm{N})$, then $\mathscr{F}_1=\mathscr{P}f_*$. Due to Lemma 5.5, $f_*=\mathscr{C}_{\Psi 1}\mathscr{F}_1$, where $\mathscr{C}_{\Psi 1}:H^{s,0}(\Omega^+;\Delta)\to L_2(\Omega^+)$ and consequently $\mathscr{C}_{\Psi 1}:Y_1^s(\Omega^+;L)\to L_2(\Omega^+)$ is a linear bounded operator. Then the equivalence Theorem 5.2 and invertibility of the BVP operator given by Corollary C.4 imply that equation $\mathscr{A}^{\mathrm{GDN}}u=\mathscr{F}$ has a unique solution $u=(A^{\mathrm{DN}})^{-1}(f_*,\mathscr{F}_2,\mathscr{F}_3)^\top=(A^{\mathrm{DN}})^{-1}\,\mathrm{diag}(\mathscr{C}_{\Psi 1},I,I)\mathscr{F}$. Here,

$$(A^{\mathrm{DN}})^{-1}:L_2(\Omega^+)\times H^{s-(1/2)}(S_\mathrm{D})\times H^{s-(3/2)}(S_\mathrm{N})\to H^{s,0}(\Omega^+;\Delta)$$

is a bounded inverse to the operator A^{DN} of the mixed BVP from (C8). Thus, $(A^{\mathrm{DN}})^{-1}\,\mathrm{diag}(\mathscr{C}_{\Psi 1},I,I)$ is a bounded inverse to operator (39). □

5.2. United boundary-domain integro-differential problem ($\tilde{G}N$)

Departing from the BDIDP (26), one can formulate another BDIDP, which does not include the explicit Dirichlet boundary condition. For $S_1\subseteq S$ and $\frac{1}{2}<s<\frac{3}{2}$, we will use the following subspaces of $H^s(\Omega^\pm)$:

$$\tilde{H}^s_{S_1}(\Omega^\pm) := \{g : g \in H^s(\Omega^\pm), r_{S_1} g^\pm = 0\}$$

$$\tilde{H}^{s,0}_{S_1}(\Omega^\pm; L_*) := \{g : g \in H^{s,0}(\Omega^\pm; L_*), r_{S_1} g^\pm = 0\}$$

provided with the norms of spaces $H^s(\Omega^\pm)$ and $H^{s,0}(\Omega^\pm; L_*)$, respectively. Evidently, if $g \in \tilde{H}^s_{S_1}(\Omega^\pm)$, then $g^\pm \in \tilde{H}^{s-(1/2)}(S \backslash S_1)$.

Let $u_0 \in H^{s,0}(\Omega^+; \Delta)$ be a fixed extension of the given boundary function $\varphi_0 \in H^{s-(1/2)}(S_D)$ into the domain Ω^+. Existence of such a functions is provided, e.g. by Theorem 2.2. Denoting $\tilde{u} = u - u_0$ and substituting it into BDIDE (22) and boundary condition (6), lead to the following BDIDP $(\tilde{G}N)$ for $\tilde{u} \in \tilde{H}^{s,0}_{S_D}(\Omega^+; \Delta)$:

$$\tilde{u} + \mathscr{R}\tilde{u} - VT^+\tilde{u} + W\tilde{u}^+(y) = \mathscr{P}f - F_0 \quad \text{on } \Omega^+ \tag{40}$$

$$r_{S_N} T^+ \tilde{u} = \psi_0 - r_{S_N} T^+ u_0 \quad \text{in } S_N \tag{41}$$

where

$$F^0 := u_0 + \mathscr{R}u_0 - VT^+u_0 + Wu_0^+ \tag{42}$$

Note that

$$F^0 = \mathscr{P}Lu_0 \tag{43}$$

due to the third Green identity (21) applied to u_0.

BDIDP (40)–(41) is only a reformulation of the GBDIDP, thus the following reformulation of Theorem 5.2 holds true.

Theorem 5.8
Let $f \in L_2(\Omega^+)$, $\varphi_0 \in H^{s-(1/2)}(S_D)$, $\psi_0 \in H^{s-(3/2)}(S_N)$, $1 \leqslant s < \frac{3}{2}$. Let $u_0 \in H^{s,0}(\Omega^+; \Delta)$ be an extension of φ_0.

(i) There exists a unique solution $u \in H^{s,0}(\Omega^+; \Delta)$ of mixed BVP (4)–(6) in Ω^+. The function $\tilde{u} = u - u_0 \in \tilde{H}^{s,0}_{S_D}(\Omega^+; \Delta)$ is a solution of BDIDP (40)–(41).

(ii) There exists a unique solution $\tilde{u} \in \tilde{H}^{s,0}_{S_D}(\Omega^+; \Delta)$ of BDIDP (40)–(41). The function $u = \tilde{u} + u_0 \in H^{s,0}(\Omega^+; \Delta)$ is a solution of mixed BVP (4)–(6) in Ω^+.

BDIDP (40), (41) can be written in the form

$$\mathscr{A}^{\tilde{G}N}\tilde{u} = \mathscr{F}^{\tilde{G}N} \tag{44}$$

where

$$\mathscr{A}^{\tilde{G}N} := \begin{bmatrix} I + \mathscr{R} - VT^+ + W\tau^+ \\ r_{S_N} T^+ \end{bmatrix}; \quad \mathscr{F}^{\tilde{G}N} := \begin{bmatrix} \mathscr{P}f - F^0 \\ \psi_0 - r_{S_N} T^+ u_0 \end{bmatrix}$$

Relation (43) implies the membership $\mathscr{F}^{\tilde{G}N} \in Y^s_1(\Omega^+; L) \times H^{s-(3/2)}(S_N)$.

Now we can state the invertibility theorem.

Theorem 5.9
Let $1 \leqslant s < \frac{3}{2}$. The operator

$$\mathscr{A}^{\tilde{G}N} : \tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta) \to Y_1^s(\Omega^+;L) \times H^{s-(3/2)}(S_N) \tag{45}$$

is continuous and continuously invertible.

Proof
If $\tilde{u} \in \tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta)$, then the third Green identity (21) implies $\mathscr{A}^{\tilde{G}N}\tilde{u} = (\mathscr{P}L\tilde{u},\ r_{S_N}T^+\tilde{u})^\top$, i.e. operator (45) is continuous.

Now, let $\mathscr{A}^{\tilde{G}N*} := [(\mathscr{A}^{\mathrm{GDN}})_1^{-1}, (\mathscr{A}^{\mathrm{GDN}})_3^{-1}]$, where $(\mathscr{A}^{\mathrm{GDN}})^{-1} = [(\mathscr{A}^{\mathrm{GDN}})_1^{-1}, (\mathscr{A}^{\mathrm{GDN}})_2^{-1}, (\mathscr{A}^{\mathrm{GDN}})_3^{-1}]$ is the operator inverse to operator $\mathscr{A}^{\mathrm{GDN}}$ from (26). Then for any $\mathscr{F} \in Y_1^s(\Omega^+;L) \times H^{s-(3/2)}(S_N)$, the function $\tilde{u} = \mathscr{A}^{\tilde{G}N*}\mathscr{F} \in H^{s,0}(\Omega^+;\Delta)$ satisfies system (26) with $\mathscr{F}_1^{\mathrm{GDN}} = \mathscr{F}_1$, $\mathscr{F}_2^{\mathrm{GDN}} = r_{S_D}\tilde{u}^+ = 0$, $\mathscr{F}_3^{\mathrm{GDN}} = \mathscr{F}_2$. Thus, $\tilde{u}$ belongs to $\tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta)$ and solves the system $\mathscr{A}^{\tilde{G}N}\tilde{u} = \mathscr{F}$, i.e. $\mathscr{A}^{\tilde{G}N*}$ is a right inverse to (45) and is continuous due to Theorem 5.7. Due to the unique solvability of the system $\mathscr{A}^{\tilde{G}N}\tilde{u} = \mathscr{F}$ implied by Theorem 5.8(ii), we have that $\mathscr{A}^{\tilde{G}N*}$ is in fact a two-side inverse to $\mathscr{A}^{\tilde{G}N}$. □

5.3. *United boundary-domain integro-differential equation* ($\tilde{G}$)

In this section, we will get rid of the remaining Neumann boundary condition to deal with only one integro-differential equation. Let $1 < s < \frac{3}{2}$, $\psi_0 \in H^{s-(3/2)}(S_N)$, $\varphi_0 \in H^{s-(1/2)}(S_D)$, and $u_0 \in H^{s,0}(\Omega^+;\Delta)$ be an extension of φ_0 into the domain Ω^+. If $u \in H^{s,0}(\Omega^+;\Delta)$, then $\tilde{u} = u - u_0 \in \tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta)$, $r_{S_N}T^+u_0 \in H^{s-(3/2)}(S_N)$, $r_{S_D}T^+\tilde{u} \in H^{s-(3/2)}(S_D)$. Since $H^{s-(3/2)}(S_N) = \tilde{H}^{s-(3/2)}(S_N)$, $H^{s-(3/2)}(S_D) = \tilde{H}^{s-(3/2)}(S_D)$ for $1 < s < \frac{3}{2}$, the surface potentials and the corresponding surface pseudodifferential operators for functions from these spaces have the nice mapping properties described in Appendix A.

Substituting the Neumann boundary condition (41) into (40) leads to the following BDIE ($\tilde{G}$) for $\tilde{u} \in \tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta)$, c.f. Reference [9, Equation (16)]:

$$\mathscr{A}^{\tilde{G}}\tilde{u} := \tilde{u} + \mathscr{R}\tilde{u} - Vr_{S_D}T^+\tilde{u} + W\tilde{u}^+ = \mathscr{F}^{\tilde{G}} \quad \text{in } \Omega^+ \tag{46}$$

where

$$\mathscr{F}^{\tilde{G}} = \mathscr{P}f - F^0 + V(\psi_0 - r_{S_N}T^+u_0) \tag{47}$$

and F^0 is given by (42).

Let us prove the equivalence of the BDIDE to the BVP (4)–(6).

Theorem 5.10
Let $f \in L_2(\Omega^+)$, $\varphi_0 \in H^{s-(1/2)}(S_D)$, $\psi_0 \in H^{s-(3/2)}(S_N)$, $1 < s < \frac{3}{2}$. Let $u_0 \in H^{s,0}(\Omega^+;\Delta)$ be an extension of φ_0.

(i) There exists a unique solution $u \in H^{s,0}(\Omega^+;\Delta)$ of mixed BVP (4)–(6) in Ω^+. The function $\tilde{u} = u - u_0 \in \tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta)$ is a solution of BDIDE (46).

(ii) There exists a unique solution $\tilde{u}\in\tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta)$ of BDIDE (46). The function $u=\tilde{u}+u_0\in H^{s,0}(\Omega^+;\Delta)$ is a solution of mixed BVP (4)–(6) in Ω^+.

Proof
Any solution of BVP (4)–(6) solves BDIDE (46) due to the third Green formula (22).

On the other hand, if $\tilde{u}$ is a solution of BDIDE (46), then Lemma 4.1 with account of (43) implies that $u=u_0+\tilde{u}$ satisfies Equation (4) and $V(\psi_0-r_{S_N}T^+u)=0$ in Ω^+. Lemma 4.2(i) then implies that Neumann's boundary condition (6) is satisfied for u. The Dirichlet boundary condition for u is satisfied due to the chosen space for $\tilde{u}$ and the extension property of the function u_0. Thus any solution $\tilde{u}$ of BDIDE (46) generates a solution $u_0+\tilde{u}$ to BVP (4)–(6).

To prove the unique solvability of BDIDE (46), let us consider its homogeneous counterpart. Since $\mathscr{F}^{\tilde{G}}=0$ can be associated with $f=0$, $u_0=0$, $\psi_0=0$, any solution $\tilde{u}$ of homogeneous BDIDE (46), according to the previous paragraph, is a solution to the homogeneous BVP (4)–(6), which is trivial due to Theorem 2.1. □

The mapping properties of operators V, W, $\mathscr{P}$ and $\mathscr{R}$ imply the membership $\mathscr{F}^{\tilde{G}}\in H^{s,0}(\Omega^+;\Delta)$ and continuity of the operator $\mathscr{A}^{\tilde{G}}:\tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta)\to H^{s,0}(\Omega^+;\Delta)$, while Theorem 5.10 implies its injectivity. To consider invertibility of operator $\mathscr{A}^{\tilde{G}}$, we introduce another one space.

Definition 5.11
Let S_* be an open sub-manifold of S, and $s\leqslant 2$. The space $Y^s_2(\Omega^+,S_*;L)$ consists of the functions of the form

$$\mathscr{F}_*=\mathscr{P}f_*+V\Psi_*\quad\text{in }\Omega^+\tag{48}$$

with $f_*\in L_2(\Omega^+)$, $\Psi_*\in\tilde{H}^{s-(3/2)}(S_*)$, and is provided with the norm of space $H^{s,0}(\Omega^+;\Delta)$, $\|\mathscr{F}_*\|_{Y^s_2(\Omega^+;L)}:=\|\mathscr{F}_*\|_{H^{s,0}(\Omega^+;\Delta)}$.

Let $1\leqslant s\leqslant 2$. The mapping properties of the operators $\mathscr{P}$ and V, see Remark B.2 and Theorem A.1, imply that $Y^s_2(\Omega^+,S_*;L)$ is a subset of $H^{s,0}(\Omega^+;\Delta)$, while Lemma 5.5 implies $Y^s_2(\Omega^+,S_*;L)$ does not cover all space $H^{s,0}(\Omega^+;\Delta)$ since Ψ_* from (48) is zero on $S\backslash S_*$. Completeness of $Y^s_2(\Omega^+,S_*;L)$ is proved similar to Lemma 5.6.

Let us give another characterization of the space $Y^s_2(\Omega^+,S_*;L)$.

Remark 5.12
Let $1\leqslant s<\frac{3}{2}$. A function $\mathscr{F}_*\in H^{s,0}(\Omega^+;\Delta)$ belongs to $Y^s_2(\Omega^+,S_*;L)$ if and only if there exists $\Psi_{**}\in\tilde{H}^{s-(3/2)}(S_*)$ such that

$$V_\Delta[T^+_\Delta(a\mathscr{F}_*)-\Psi_{**}]-W_\Delta(a\mathscr{F}_*)^+=0\quad\text{in }\Omega^+\tag{49}$$

or, the same,

$$V\left[T^+\mathscr{F}_*+\mathscr{F}^+_*\frac{\partial a}{\partial n^+}-\Psi_{**}\right]-W\mathscr{F}^+_*=0\quad\text{in }\Omega^+\tag{50}$$

Proof
Condition (48) can be rewritten as

$$a\mathscr{F}_* = \mathscr{P}_\Delta f_* + V_\Delta \Psi_* \quad \text{in } \Omega^+ \tag{51}$$

Third Green's identity (21) for $u = a\mathscr{F}_*$ and for the potentials associated with the Laplace operator Δ gives

$$a\mathscr{F}_* - V_\Delta T_\Delta^+(a\mathscr{F}_*) + W(a\mathscr{F}_*)^+ = \mathscr{P}_\Delta \Delta(a\mathscr{F}_*) \quad \text{in } \Omega^+ \tag{52}$$

Thus, (49) implies (51) with $f_* = \Delta(a\mathscr{F}_*)$ and $\Psi_* = \Psi_{**}$.

On the other hand, if (51) is satisfied, then application of the Laplace operator to it gives $\Delta(a\mathscr{F}_*) = f_*$ in Ω^+, which substitution into (52) and subtraction from (51) implies (29) with $\Psi_{**} = \Psi_*$.

Condition (50) follows from (49) and definitions of T, V and W. □

For $\psi_0 \in \tilde{H}^{s-(3/2)}(S_{\mathrm{N}})$, $u_0 \in H^{s,0}(\Omega^+;\Delta)$, $1 < s < \frac{3}{2}$, relation (47) and (43) imply the membership $\mathscr{F}^{\tilde{G}} \in Y_2^s(\Omega^+, S_{\mathrm{N}};\Delta)$.

Let us state the invertibility theorem.

Theorem 5.13
Let $1 < s < \frac{3}{2}$. The operator

$$\mathscr{A}^{\tilde{G}} : \tilde{H}^{s,0}_{S_{\mathrm{D}}}(\Omega^+;\Delta) \to Y_2^s(\Omega^+, S_{\mathrm{N}};\Delta) \tag{53}$$

is continuous and continuously invertible.

Proof
If $\tilde{u} \in \tilde{H}^{s,0}_{S_{\mathrm{D}}}(\Omega^+;\Delta)$, then the third Green identity (21) implies $\mathscr{A}^{\tilde{G}}\tilde{u} = \mathscr{P}L\tilde{u} + Vr_{S_{\mathrm{N}}}T^+\tilde{u}$, i.e. operator (53) is continuous.

On the other hand, if $\mathscr{F}_* \in Y_2^s(\Omega^+, S_{\mathrm{N}};L)$, then $\mathscr{F}_* = \mathscr{P}f_* + V\Psi_*$. Due to Lemma 5.5, $(f_*, \Psi_*)^\top = (\mathscr{C}_{\Psi 1}, \mathscr{C}_{\Psi 2})^\top \mathscr{F}_*$, where

$$(\mathscr{C}_{\Psi 1}, \mathscr{C}_{\Psi 2})^\top : H^{s,0}(\Omega^+;\Delta) \to L_2(\Omega^+) \times H^{s-(3/2)}(S)$$

and consequently

$$(\mathscr{C}_{\Psi 1}, \mathscr{C}_{\Psi 2})^\top : Y_2^s(\Omega^+, S_{\mathrm{N}};L) \to L_2(\Omega^+) \times \tilde{H}^{s-(3/2)}(S_{\mathrm{N}})$$

is a linear bounded operator. Then the equivalence Theorem 5.10 with $\varphi_0 = 0$, $u_0 = 0$, $\psi_0 = \Psi_*$ and invertibility of the BVP operator given by Corollary C.4 imply that equation $\mathscr{A}^{\tilde{G}}\tilde{u} = \mathscr{F}_*$ has a unique solution

$$\tilde{u} = (A^{\mathrm{DN}})^{-1}(f_*, 0, \Psi_*)^\top = [(A^{\mathrm{DN}})_1^{-1}\mathscr{C}_{\Psi 1} + (A^{\mathrm{DN}})_3^{-1}\mathscr{C}_{\Psi 2}]\mathscr{F}_*$$

Here,

$$(A^{\mathrm{DN}})^{-1} : L_2(\Omega^+) \times H^{s-(1/2)}(S_{\mathrm{D}}) \times H^{s-(3/2)}(S_{\mathrm{N}}) \to H^{s,0}(\Omega^+;\Delta)$$

is a bounded inverse to the operator A^{DN} of the mixed BVP from (C8), which implies

$$[(A^{\mathrm{DN}})_1^{-1}, (A^{\mathrm{DN}})_3^{-1}] : L_2(\Omega^+) \times \tilde{H}^{s-(3/2)}(S_{\mathrm{N}}) \to \tilde{H}^{s,0}_{S_{\mathrm{D}}}(\Omega^+;\Delta)$$

is also bounded. Thus, $(A^{\mathrm{DN}})_1^{-1}\mathscr{C}_{\Psi 1} + (A^{\mathrm{DN}})_3^{-1}\mathscr{C}_{\Psi 2}$ is a bounded inverse to operator (53). □

5.4. Partly segregated boundary-domain integral equation ($\tilde{G}_{\mathrm{D}}$)

In this section, we consider the *integral* only equation, without the differential term, where the trace of solution is used in the boundary integrals (unlike Reference [7]) but the unknown co-normal derivative is replaced by an auxiliary boundary function.

Let in this section $1 \leqslant s < \frac{3}{2}$, $\psi_0 \in H^{s-(3/2)}(S_{\mathrm{N}})$, $\varphi_0 \in H^{s-(1/2)}(S_{\mathrm{D}})$, and $u_0 \in H^{s,0}(\Omega^+;\Delta)$ be an extension of φ_0 into the domain Ω^+. If $u \in H^{s,0}(\Omega^+;\Delta)$ and satisfies the Dirichlet condition (5), then $\tilde{u} = u - u_0 \in \tilde{H}^{s,0}_{S_{\mathrm{D}}}(\Omega^+;\Delta)$, $\psi_0 - r_{S_{\mathrm{N}}} T^+ u_0 \in H^{s-(3/2)}(S_{\mathrm{N}})$, $r_{S_{\mathrm{D}}} T^+ \tilde{u} \in H^{s-(3/2)}(S_{\mathrm{D}})$.

Let $\Psi_0 \in H^{s-(3/2)}(S)$ be a fixed extension of the given function $\psi_0 - r_{S_{\mathrm{N}}} T^+ u_0 \in H^{s-(3/2)}(S_{\mathrm{N}})$ from S_{N} to the whole of S. Note that if $1 < s < \frac{3}{2}$, then $H^{s-(3/2)}(S_{\mathrm{N}}) = \tilde{H}^{s-(3/2)}(S_{\mathrm{N}})$ and one may simply choose Ψ_0 as the canonical extension of $\psi_0 - r_{S_{\mathrm{N}}} T^+ u_0$ from S_{N} to the whole of S by zero. An arbitrary extension $\Psi \in H^{s-(3/2)}(S)$ of $\psi_0 - r_{S_{\mathrm{N}}} T^+ u_0$ preserving the function space can be then represented as $\Psi = \Psi_0 + \psi$ with $\psi \in \tilde{H}^{s-(3/2)}(S_{\mathrm{D}})$.

To reduce BVP (4)–(6) to a pure integral equation, let us consider (22) in Ω^+ and replace there u with $u_0 + \tilde{u}$ in Ω^+ and $T^+\tilde{u}$ with $\Psi_0 + \psi$ on S, where $\psi \in \tilde{H}^{s-(3/2)}(S_{\mathrm{D}})$ is a new auxiliary function. This leads to the following BDIE ($\tilde{G}_{\mathrm{D}}$), c.f. Reference [9, Equation (14)]:

$$\tilde{u} + \mathscr{R}\tilde{u} + W\tilde{u}^+ - V\psi = \mathscr{F}^{\tilde{G}_{\mathrm{D}}} \quad \text{in } \Omega^+ \tag{54}$$

for the couple $(\tilde{u}, \psi) \in \tilde{H}^{s,0}_{S_{\mathrm{D}}}(\Omega^+;\Delta) \times \tilde{H}^{s-(3/2)}(S_{\mathrm{D}})$, where

$$\mathscr{F}^{\tilde{G}_{\mathrm{D}}} := \mathscr{P}f - F^0 + V\Psi_0 \tag{55}$$

and F^0 is given by (42).

Let us prove the following equivalence statement.

Theorem 5.14

Let $f \in L_2(\Omega^+)$, $\varphi_0 \in H^{s-(1/2)}(S_{\mathrm{D}})$, $\psi_0 \in H^{s-(3/2)}(S_{\mathrm{N}})$, $1 \leqslant s < \frac{3}{2}$. Let $u_0 \in H^{s,0}(\Omega^+;\Delta)$ be an extension of φ_0 and $\Psi_0 \in H^{s-(3/2)}(S)$ be an extension of $\psi_0 - r_{S_{\mathrm{N}}} T^+ u_0$.

(i) There exists a unique solution $u \in H^{s,0}(\Omega^+;\Delta)$ of mixed BVP (4)–(6) in Ω^+. The couple $(\tilde{u}, \psi) \in \tilde{H}^{s,0}_{S_{\mathrm{D}}}(\Omega^+;\Delta) \times \tilde{H}^{s-(3/2)}(S_{\mathrm{D}})$, where

$$\tilde{u} = u - u_0 \quad \text{in } \Omega^+ \tag{56}$$

$$\psi = T^+[u - u_0] - \Psi_0 \quad \text{on } S \tag{57}$$

is a solution of BDIE system (54)–(55).

(ii) There exists a unique solution $(\tilde{u}, \psi) \in \tilde{H}^{s,0}_{S_{\mathrm{D}}}(\Omega^+;\Delta) \times \tilde{H}^{s-(3/2)}(S_{\mathrm{D}})$ of BDIE (54) with the right-hand side (55). The function u defined by (56) is a solution of mixed BVP (4)–(6) in Ω^+, and Equation (57) holds.

Proof

(i) The unique solvability of BVP (4)–(6) is implied, e.g. by Theorem 2.2. For any solution u of the BVP, the couple $(\tilde{u}, \psi)$ defined by (56)–(57) solves BDIE system (54)–(55) due to the third Green formula (22). Thus, point (i) is proved.

(ii) Existence of a solution to BDIE system (54)–(55) is implied by point (i).

Let $(\tilde{u},\psi)\in\tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta)\times\tilde{H}^{s-(3/2)}(S_D)$ be a solution of (54)–(55). The function u defined by (56) evidently satisfies Dirichlet's boundary condition (5).

Equation (54) and Lemma 4.1 for $\tilde{u}$, $\Psi=\psi+\Psi_0$ and $\Phi=\tilde{u}^+$ with account of (43) imply that u is a solution of PDE (4) in Ω^+, and

$$V\Psi^* - W\Phi^* = 0 \quad \text{in } \Omega^+$$

where $\Psi^*=\psi+\Psi_0-T^+\tilde{u}$ and $\Phi^*=0$. Eventually, Lemma 4.2(i) implies $\psi=T^+\tilde{u}-\Psi_0$ on S, i.e. condition (57) is satisfied. Its restriction on S_N implies also the Neumann boundary condition (6) for u if one takes into consideration that $\psi=0$ and $\Psi_0=\psi_0-r_{S_N}T^+u_0$ on S_N.

To prove the unique solvability of BDIDE (54), let us consider its homogeneous counterpart. Since $\mathscr{F}^{\tilde{G}_D}=0$ can be associated with $f=0$, $u_0=0$, $\psi_0=0$, any solution $(\tilde{u},\psi)$ of homogeneous BDIDE (54), according to (56), (57), gives a solution $\tilde{u}$ to the homogeneous BVP (4)–(6), which is trivial due to Theorem 2.1. □

BDIE (54) can be rewritten in the form

$$\mathscr{A}^{\tilde{G}_D}\tilde{\mathscr{U}}=\mathscr{F}^{\tilde{G}_D} \tag{58}$$

where $\tilde{\mathscr{U}}:=[\tilde{u},\psi]^\top$ and

$$\mathscr{A}^{\tilde{G}_D}:=[I+\mathscr{R}+W\tau^+,-V] \tag{59}$$

The mapping properties of the operators involved in (55) and (59), outlined in Appendices A and B imply $\mathscr{F}^{\tilde{G}_D}\in H^{s,0}(\Omega^+;\Delta)$, and the operator $\mathscr{A}^{\tilde{G}_D}:\tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta)\times\tilde{H}^{s-(3/2)}(S_D)\to H^{s,0}(\Omega^+;\Delta)$ is continuous, while Theorem 5.14 implies its injectivity.

Let us prove the following invertibility theorem.

Theorem 5.15
Let $1\leqslant s<\frac{3}{2}$. The operator

$$\mathscr{A}^{\tilde{G}_D}:\tilde{H}^{s,0}_{S_D}(\Omega^+;\Delta)\times\tilde{H}^{s-(3/2)}(S_D)\to H^{s,0}(\Omega^+;\Delta) \tag{60}$$

is continuous and continuously invertible.

Proof
Since continuity of the operator (60) has been already discussed, let us prove its continuous invertibility.

Let $\mathscr{F}_*$ be an arbitrary function from $H^{s,0}(\Omega^+;\Delta)$. Due to Lemma 5.5, it can presented as

$$\mathscr{F}_*(y)=\mathscr{P}f_*(y)+V\Psi_*(y), \quad y\in\Omega \tag{61}$$

where $(f_*,\Psi_*)=\mathscr{C}_\Psi\mathscr{F}_*\in L_2(\Omega)\times H^{s-(3/2)}(S)$, and $\mathscr{C}_\Psi:H^{s,0}(\Omega;\Delta)\to L_2(\Omega)\times H^{s-(3/2)}(S)$ is a linear bounded operator.

In addition to Equation (58), let us consider its trace on S_D and its co-normal derivative on S_N, replacing $\tilde{u}^+$ by an auxiliary function $\varphi\in\tilde{H}^{s-(1/2)}(S_N)$ everywhere in the three equations, and $T^+\tilde{u}$ in the third equation by Ψ_* from (61). Then we obtain the following system:

$$\mathscr{A}^{G\mathscr{G}\mathscr{T}}\mathscr{U}=\mathscr{F}^{G\mathscr{G}\mathscr{T}} \tag{62}$$

where $\mathscr{U}^\top := (u, \psi, \varphi)^\top \in H^{s,0}(\Omega^+;\Delta) \times \tilde{H}^{s-(3/2)}(S_D) \times \tilde{H}^{s-(1/2)}(S_N)$ and

$$\mathscr{A}^{\mathscr{G}\mathscr{G}\mathscr{T}} := \begin{bmatrix} I + \mathscr{R} & -V & W \\ r_{S_D}\mathscr{R}^+ & -r_{S_D}\mathscr{V} & r_{S_D}\mathscr{W} \\ r_{S_N}T^+\mathscr{R} & -r_{S_N}\mathscr{W}' & r_{S_N}\mathscr{L}^+ \end{bmatrix}, \quad \mathscr{F}^{\mathscr{G}\mathscr{G}\mathscr{T}} := \begin{bmatrix} \mathscr{F}_* \\ r_{S_D}\tau^+\mathscr{F}_* \\ r_{S_N}[T^+\mathscr{F}_* - \Psi_*] \end{bmatrix}$$

It was proved in Reference [7] that the operator $\mathscr{A}^{\mathscr{G}\mathscr{G}\mathscr{T}} : H^s(\Omega^+) \times \tilde{H}^{s-(3/2)}(S_D) \times \tilde{H}^{s-(1/2)}(S_N) \to H^s(\Omega^+) \times H^{s-(1/2)}(S_D) \times H^{s-(3/2)}(S_N)$ is continuous and invertible for $s = 1$. Theorem C.2 from Appendix C extends that proof to $1 \leqslant s < \frac{3}{2}$, while Theorem C.3 affirms that the operator $\mathscr{A}^{\mathscr{G}\mathscr{G}\mathscr{T}} : H^{s,0}(\Omega^+;\Delta) \times \tilde{H}^{s-(3/2)}(S_D) \times \tilde{H}^{s-(1/2)}(S_N) \to H^{s,0}(\Omega^+;\Delta) \times H^{s-(1/2)}(S_D) \times H^{s-(3/2)}(S_N)$ is also continuous and invertible for $1 \leqslant s < \frac{3}{2}$. Thus, a unique solution of system (62) is

$$(u, \psi, \varphi)^\top = (\mathscr{A}^{\mathscr{G}\mathscr{G}\mathscr{T}})^{-1}\mathscr{F}^{\mathscr{G}\mathscr{G}\mathscr{T}} \tag{63}$$

where $(\mathscr{A}^{\mathscr{G}\mathscr{G}\mathscr{T}})^{-1}$ is a bounded inverse to the operator $\mathscr{A}^{\mathscr{G}\mathscr{G}\mathscr{T}} : H^{s,0}(\Omega^+;\Delta) \times \tilde{H}^{s-(3/2)}(S_D) \times \tilde{H}^{s-(1/2)}(S_N) \to H^{s,0}(\Omega^+;\Delta) \times H^{s-(1/2)}(S_D) \times H^{s-(3/2)}(S_N)$. If we prove that its solution is such that $r_{S_D}u^+ = 0$ and $r_{S_N}u^+ = \varphi$, then it delivers a solution of system (58).

Indeed, subtracting the trace of the first equation of (62) on S_D from the second equation, we obtain

$$r_{S_D}u^+ = 0 \tag{64}$$

Subtracting the co-normal derivative of the first equation of (62) on S_N from the third equation, we obtain

$$r_{S_N}T^+u = r_{S_N}\Psi_* \tag{65}$$

The first equation of (62) and Lemma 4.1 with $\Psi = \psi + \Psi_*$, $\Phi = \varphi$ imply that

$$V\Psi^*(y) - W\Phi^*(y) = 0, \quad y \in \Omega^+$$

where $\Psi^* = \Psi_* + \psi - T^+u$ and $\Phi^* = \varphi - u^+$. Further, $\Phi^* \in \tilde{H}^{s-(1/2)}(S_N)$ due to (64), and $\Psi^* \in \tilde{H}^{s-(3/2)}(S_D)$ due to (65). Lemma 4.2(iii) with $S_1 = S_D$, $S_2 = S_N$ implies $\Phi^* = 0$ on S, i.e. $r_{S_N}u^+ = \varphi$, which implies that $\tilde{\mathscr{U}} = (u, \psi)$ given by (63) provide a solution of the equation $\mathscr{A}^{\tilde{G}_D}\tilde{\mathscr{U}} = \mathscr{F}_*$. Uniqueness of the equation solution is implied by the injectivity of operator (60). This means the operator

$$[(\mathscr{A}^{\mathscr{G}\mathscr{G}\mathscr{T}})^{-1}_{i1} + (\mathscr{A}^{\mathscr{G}\mathscr{G}\mathscr{T}})^{-1}_{i2} r_{S_D}\tau^+ + (\mathscr{A}^{\mathscr{G}\mathscr{G}\mathscr{T}})^{-1}_{i3} r_{S_N}(T^+ - \mathscr{C}_{\Psi 2})], \quad i = 1, 2$$

is a bounded inverse to operator (60). □

CONCLUDING REMARKS

A mixed BVP for a variable-coefficient PDE with a right-hand side function from $L_2(\Omega^+)$, and with the Dirichlet and the Neumann data from the spaces $H^{s-(1/2)}(S_D)$ and $H^{s-(3/2)}(S_N)$, respectively, was considered in this paper for $1 \leqslant s < \frac{3}{2}$. It was shown that the BVP can be equivalently reduced to two direct united boundary-domain integro-differential problems, or to a united BDIDE or to a partly segregated BDIE. This implied *unique solvability* of the BDIDPs/BDIDE/BDIEs with the right-hand sides generated by the considered BVP. *The continuity and continuous invertibility* of the left-hand side operators of all the considered BDIDPs/BDIDE/BDIEs was proved in appropriate spaces.

The BDIDE $(\tilde{G})$ has the form of operator equation of the second kind but the domain of the left-hand side operator does not include the range of the operator. Although the resolvent theory and the Neumann series method (c.f. References [15,16] and references therein) are then not directly applicable to the equation solution, a further analysis is needed to find out whether it might be possible after an appropriate modification of the operator and/or the spaces (c.f. References [17,18]).

By the same approach, the corresponding BDIDEs/BDIDPs for unbounded domains can be analysed as well. The approach can be extended also to more general PDEs and to systems of PDEs, while smoothness of the variable coefficients and the boundary can be essentially relaxed, and the PDE right-hand side can be considered in more general spaces, c.f. Reference [19].

This study can serve as a starting point for approaching BDIDEs/BDIDPs based on the *localized* parametrices, leading after discretization to sparsely populated systems of linear algebraic equations, attractive for computations, c.f. Reference [9].

APPENDIX A: PROPERTIES OF THE SURFACE POTENTIALS

The mapping and jump properties of the potentials of type (11)–(12) and the corresponding boundary integral and pseudodifferential operators in the Hölder ($C^{k+\alpha}$), Bessel potential (H^s_p) and Besov ($B^s_{p,q}$) spaces are well studied nowadays for the constant coefficient, $a=\text{const}$, (see, e.g. list of references in Reference [7]). Some of them were extended in Reference [7] to the case of variable positive coefficient $a\in C^\infty(\mathbb{R})$ and are provided in the appendix for convenience (without proves). Several of those results are also reformulated and proved below in some different spaces employed in the main text.

Theorem A.1

The following operators are continuous for $s\geqslant 1$:

$$V : H^{s-(3/2)}(S)\to H^{s,0}(\Omega^+;\Delta)\quad [H^{s-(3/2)}(S)\to H^{s,0}_{\text{loc}}(\Omega^-;\Delta)]$$

$$W : H^{s-(1/2)}(S)\to H^{s,0}(\Omega^+;\Delta)\quad [H^{s-(1/2)}(S)\to H^{s,0}_{\text{loc}}(\Omega^-;\Delta)]$$

Proof

$$V\Psi(y)=\frac{1}{a(y)}V_\Delta\Psi(y),\quad V_\Delta\Psi(y):=\int_S P_\Delta(x,y)\Psi(x)\,\mathrm{d}x \tag{A1}$$

$$W\Phi(y)=\frac{1}{a(y)}W_\Delta[a\Phi](y),\quad W_\Delta[a\Phi](y):=\int_S \frac{\partial P_\Delta(x,y)}{\partial n(x)}a(x)\Phi(x)\,\mathrm{d}x \tag{A2}$$

where $P_\Delta(x,y):=-(4\pi)^{-1}|x-y|^{-1}$ is the fundamental solution to the Laplace equation.

This is well known that the operators

$$V_\Delta : H^{s-(3/2)}(S) \to H^s(\Omega^+) \quad [H^{s-(3/2)}(S) \to H^s_{\mathrm{loc}}(\Omega^-)] \tag{A3}$$

$$W_\Delta : H^{s-(1/2)}(S) \to H^s(\Omega^+) \quad [H^{s-(1/2)}(S) \to H^s_{\mathrm{loc}}(\Omega^-)] \tag{A4}$$

are continuous for any $s \in \mathbb{R}$ (see, e.g. the above references). Since $a(x) \neq 0$ and $a \in C^\infty(\mathbb{R})$, equalities (A1), (A2) imply the similar properties hold true for the corresponding operators V and W.

On the other hand,

$$[\Delta V\Psi](y) = \left[\Delta \frac{1}{a(y)}\right] V_\Delta \Psi(y) + \sum_{i=1}^{3} \frac{\partial}{\partial y_i} \left[\frac{1}{a(y)}\right] \frac{\partial V_\Delta \Psi(y)}{\partial y_i}$$

$$[\Delta W\Phi](y) = \left[\Delta \frac{1}{a(y)}\right] V_\Delta [a\Phi](y) + \sum_{i=1}^{3} \frac{\partial}{\partial y_i} \left[\frac{1}{a(y)}\right] \frac{\partial W_\Delta [a\Phi](y)}{\partial y_i}$$

since $\Delta V_\Delta \Psi(y) = \Delta W_\Delta [a\Phi](y) = 0$ for $y \in \Omega^\pm$.

Due to the continuity of operators (A3), (A4), this implies the operators

$$\Delta V : H^{s-(3/2)}(S) \to H^{s-1}(\Omega^+) \quad [H^{s-(3/2)}(S) \to H^{s-1}_{\mathrm{loc}}(\Omega^-)]$$

$$\Delta W : H^{s-(1/2)}(S) \to H^{s-1}(\Omega^+) \quad [H^{s-(1/2)}(S) \to H^{s-1}_{\mathrm{loc}}(\Omega^-)]$$

are continuous for $s \in \mathbb{R}$. Since $H^{s-1}(\Omega) \subseteq L_2(\Omega)$ for $s \geqslant 1$, this completes the theorem. □

Theorem A.2
Let $s \in \mathbb{R}$. The following pseudodifferential operators are continuous

$$\mathscr{V} : H^s(S) \to H^{s+1}(S)$$

$$\mathscr{W}, \mathscr{W}' : H^s(S) \to H^{s+1}(S)$$

$$\mathscr{L}^\pm : H^s(S) \to H^{s-1}(S)$$

Theorem A.3
Let $g_1 \in H^{s-(3/2)}(S)$, and $g_2 \in H^{s-(1/2)}(S)$, $1 \leqslant s$. Then there hold the jump relations on S

$$[Vg_1(y)]^\pm = \mathscr{V} g_1(y)$$

$$[Wg_2(y)]^\pm = \mp\tfrac{1}{2} g_2(y) + \mathscr{W} g_2(y)$$

$$[T(y, n(y), \partial_y) V g_1(y)]^\pm = \pm\tfrac{1}{2} g_1(y) + \mathscr{W}' g_1(y)$$

where $y \in S$.

Proof
For $s=1$ the proof follows from the jump properties of the corresponding surface potentials associated with the Laplace fundamental solution, which evidently imply the case $s>1$. □

Theorem A.4
Let $s \in \mathbb{R}$, and S_1 and S_2 be non-empty open sub-manifolds of S. The operators

$$r_{S_2}\mathscr{V} : \tilde{H}^s(S_1) \to H^s(S_2)$$

$$r_{S_2}\mathscr{W} : \tilde{H}^s(S_1) \to H^s(S_2)$$

$$r_{S_2}\mathscr{W}' : \tilde{H}^s(S_1) \to H^s(S_2)$$

are compact.

Theorem A.5
Let S_1 be a non-empty, simply connected sub-manifold of S with infinitely smooth boundary curve, and $0<s<1$. Then the pseudodifferential operator

$$r_{S_1}\mathscr{V} : \tilde{H}^{s-1}(S_1) \to H^s(S_1)$$

is invertible, while

$$r_{S_1}\mathscr{L}^{\pm} : \tilde{H}^s(S_1) \to H^{s-1}(S_1)$$

is a Fredholm pseudodifferential operator of index zero.

Corollary A.6
Let S_1 and $S\backslash\overline{S}_1$ be non-empty, open simply connected sub-manifolds of S with an infinitely smooth boundary curve, and $0 < s < 1$. Then the pseudodifferential operator

$$\hat{\mathscr{L}} := \left[\mathscr{L}^{\pm} + \frac{\partial a}{\partial n}\left(\mp\frac{1}{2}I + \mathscr{W}\right)\right] : \tilde{H}^s_2(S_1) \to H^{s-1}_2(S_1) \tag{A5}$$

is invertible and the operator

$$\mathscr{L}^{\pm} - \hat{\mathscr{L}} : \tilde{H}^s_2(S_1) \to H^{s-1}_2(S_1)$$

is compact.

APPENDIX B: PROPERTIES OF THE VOLUME POTENTIALS

The following mapping properties were proved in Reference [7].

Theorem B.1
Let Ω^+ be a bounded open three-dimensional region of $\mathbb{R}^3$ with a simply connected, closed, infinitely smooth boundary. The following operators are continuous:

$$\mathscr{P} : \tilde{H}^s(\Omega^+) \to H^{s+2}(\Omega^+), \quad s \in \mathbb{R} \tag{B1}$$

$$: H^s(\Omega^+) \to H^{s+2}(\Omega^+), \quad s > -\tfrac{1}{2} \tag{B2}$$

$$\mathscr{R} : \tilde{H}^s(\Omega^+) \to H^{s+1}(\Omega^+), \quad s \in \mathbb{R} \tag{B3}$$

$$: H^s(\Omega^+) \to H^{s+1}(\Omega^+), \quad s > -\tfrac{1}{2} \tag{B4}$$

$$\mathscr{P}^+ : \tilde{H}^s(\Omega^+) \to H^{s+(3/2)}(S), \quad s > -\tfrac{3}{2} \tag{B5}$$

$$: H^s(\Omega^+) \to H^{s+(3/2)}(S), \quad s > -\tfrac{1}{2} \tag{B6}$$

$$\mathscr{R}^+ : \tilde{H}^s(\Omega^+) \to H^{s+(1/2)}(S), \quad s > -\tfrac{1}{2} \tag{B7}$$

$$: H^s(\Omega^+) \to H^{s+(1/2)}(S), \quad s > -\tfrac{1}{2} \tag{B8}$$

$$T^+\mathscr{P} : \tilde{H}^s(\Omega^+) \to H^{s+(1/2)}(S), \quad s > -\tfrac{1}{2} \tag{B9}$$

$$: H^s(\Omega^+) \to H^{s+(1/2)}(S), \quad s > -\tfrac{1}{2} \tag{B10}$$

$$T^+\mathscr{R} : \tilde{H}^s(\Omega^+) \to H^{s-(1/2)}(S), \quad s > \tfrac{1}{2} \tag{B11}$$

$$: H^s(\Omega^+) \to H^{s-(1/2)}(S), \quad s > \tfrac{1}{2} \tag{B12}$$

Remark B.2
Due to (B2) and (B4), the operators

$$\mathscr{P} : H^{s-1}(\Omega^+) \to H^{s+1,0}(\Omega^+; \Delta)$$

$$\mathscr{R} : H^s(\Omega^+) \to H^{s+1,0}(\Omega^+; \Delta)$$

are continuous for $s \geqslant 1$.

Corollary B.3
The operators

$$\mathscr{R} : H^s(\Omega^+) \to H^s(\Omega^+), \quad s > -\tfrac{1}{2} \tag{B13}$$

$$: H^s(\Omega^+) \to H^{s,0}(\Omega^+; \Delta), \quad s > 1 \tag{B14}$$

$$r_{S_1}\mathscr{R}^+ : H^s(\Omega^+) \to H^{s-(1/2)}(S_1), \quad s > -\tfrac{1}{2} \tag{B15}$$

$$r_{S_1}T^+\mathscr{R} : H^s(\Omega^+) \to H^{s-(3/2)}(S_1), \quad s > \tfrac{1}{2} \tag{B16}$$

are compact for any non-empty, open sub-manifold S_1 of S with an infinitely smooth boundary curve.

Proof
Compactness of the operators (B13), (B15) and (B16) follow from (B4), (B8), and (B12), respectively, and the Rellich compact imbedding theorem. Then (B13) and (B4) imply (B14). □

APPENDIX C: SEGREGATED BOUNDARY-DOMAIN INTEGRAL EQUATION SYSTEM ($G\mathscr{G}\mathscr{T}$)

Let us consider a segregated purely *integral* boundary-domain formulation introduced and analysed in Reference [7] in the space $H^1(\Omega^+)$ for u. In this section, we will extend those results to $u\in H^s(\Omega^+)$ and $u\in H^{s,0}(\Omega^+;\Delta)$, $1\leqslant s<\frac{3}{2}$.

Let $\Phi_0\in H^{s-(1/2)}(S)$ be a fixed extension of a given function $\varphi_0\in H^{s-(1/2)}(S_\mathrm{D})$ from the sub-manifold S_D to the whole of S (see the Dirichlet condition (5)). An arbitrary extension $\Phi\in H^{s-(1/2)}(S)$ preserving the function space can be then represented as $\Phi=\Phi_0+\varphi$ with some $\varphi\in\tilde{H}^{s-(1/2)}(S_\mathrm{N})$.

Analogously, let $\Psi_0\in H^{s-(3/2)}(S)$ be a fixed extension of a given distribution $\psi_0\in H^{s-(3/2)}(S_\mathrm{N})$ from the sub-manifold S_N to the whole of S (see the Neumann condition (6)). An arbitrary extension $\Psi\in H^{s-(3/2)}(S)$ preserving the function space can be then represented as $\Psi=\Psi_0+\psi$ with $\psi\in\tilde{H}^{s-(3/2)}(S_\mathrm{D})$.

If $\psi_0\in\tilde{H}^{s-(3/2)}(S_\mathrm{N})$ or $\varphi_0\in\tilde{H}^{s-(1/2)}(S_\mathrm{D})$, one may choose $\Psi_0=\psi_0$ or $\Phi_0=\varphi_0$, respectively. Particularly, if $1<s<\frac{3}{2}$, then $H^{s-(3/2)}(S_\mathrm{N})=\tilde{H}^{s-(3/2)}(S_\mathrm{N})$, and one may choose the extension of $\psi_0\in H^{s-(3/2)}(S_\mathrm{N})$ by zero as Ψ_0.

Reducing BVP (4)–(6) to a BDIE system in this section, we will use Equation (22) in Ω^+, the restriction of Equation (23) on S_D, and the restriction of Equation (24) on S_N, where $\Phi_0+\varphi$ is substituted for u^+ and $\Psi_0+\psi$ for T^+u. Then we arrive at the system

$$u(y)+\mathscr{R}u(y)-V\psi(y)+W\varphi(y)=F_0(y),\quad y\in\Omega^+ \tag{C1}$$

$$r_{S_\mathrm{D}}\mathscr{R}^+u(y)-r_{S_\mathrm{D}}\mathscr{V}\psi(y)+r_{S_\mathrm{D}}\mathscr{W}\varphi(y)=r_{S_\mathrm{D}}F_0^+(y)-\varphi_0(y),\quad y\in S_\mathrm{D} \tag{C2}$$

$$r_{S_\mathrm{N}}T^+\mathscr{R}u(y)-r_{S_\mathrm{N}}\mathscr{W}'\psi(y)+r_{S_\mathrm{N}}\mathscr{L}^+\varphi(y)=r_{S_\mathrm{N}}T^+F_0(y)-\psi_0(y),\quad y\in S_\mathrm{N} \tag{C3}$$

where

$$F_0(y):=\mathscr{P}f(y)+V\Psi_0(y)-W\Phi_0(y),\quad y\in\Omega^+$$

Note that for $f\in L_2(\Omega^+)$, $\Psi_0\in H^{s-(3/2)}(S)$, and $\Phi_0\in H^{s-(1/2)}(S)$, we have the inclusion $F_0\in H^{s,0}(\Omega^+;\Delta)$ due to the mapping properties of the Newtonian (volume) and layer potentials.

The second and the third equations of the system are associated with the operator $\mathscr{G}$ on S_D and with the operator $\mathscr{T}$ on S_N, respectively.

Let us prove that BVP (4)–(6) in Ω^+ is equivalent to the system of BDIEs (C1)–(C3).

Theorem C.1

Let $f\in L_2(\Omega^+)$. Let $\Phi_0\in H^{s-(1/2)}(S)$ and $\Psi_0\in H^{s-(3/2)}(S)$ be some extensions of $\varphi_0\in H^{s-(1/2)}(S_\mathrm{D})$ and $\psi_0\in H^{s-(3/2)}(S_\mathrm{N})$, respectively, for $s\geqslant 1$.

(i) If some $u\in H^{s,0}(\Omega^+;\Delta)$ solves the mixed BVP (4)–(6) in Ω^+, then the solution is unique and the triple $(u,\psi,\varphi)\in H^{s,0}(\Omega^+;\Delta)\times\tilde{H}^{s-(3/2)}(S_\mathrm{D})\times\tilde{H}^{s-(1/2)}(S_\mathrm{N})$, where

$$\psi(y)=T^+u(y)-\Psi_0(y),\quad \varphi(y)=u^+(y)-\Phi_0(y),\ \ y\in S \tag{C4}$$

solves BDIE system (C1)–(C3);

(ii) If a triple $(u,\psi,\varphi)\in H^{s,0}(\Omega^+;\Delta)\times\tilde{H}^{s-(3/2)}(S_\mathrm{D})\times\tilde{H}^{s-(1/2)}(S_\mathrm{N})$ solves BDIE system (C1)–(C3), then the solution is unique, u solves BVP (4)–(6), and ψ, φ satisfy (C4).

Proof
The theorem claims were proved in Reference [7, Theorem 4.4] for $s=1$ and $u\in H^1(\Omega^+)$. From that result and particularly from relations (C4), the theorem claims follow also for $s\geqslant 1$ and $u\in H^s(\Omega^+)$. Eventually, the mapping properties of the surface and volume potentials imply that if $u\in H^s(\Omega^+)$ satisfies Equation (C1), then $u\in H^{s,0}(\Omega^+;\Delta)$, which completes the proof. □

System (C1)–(C3) can be rewritten in the form

$$\mathscr{A}^{G\mathscr{G}\mathscr{T}}\mathscr{U}=\mathscr{F}^{G\mathscr{G}\mathscr{T}} \tag{C5}$$

where $\mathscr{U}^\top:=(u,\psi,\varphi)\in H^{s,0}(\Omega^+;\Delta)\times\tilde{H}^{s-(3/2)}(S_D)\times\tilde{H}^{s-(1/2)}(S_N)$

$$\mathscr{A}^{G\mathscr{G}\mathscr{T}}:=\begin{bmatrix} I-\mathscr{R} & -V & W \\ r_{S_D}\mathscr{R}^+ & -r_{S_D}\mathscr{V} & r_{S_D}\mathscr{W} \\ r_{S_N}T^+\mathscr{R} & -r_{S_N}\mathscr{W}' & r_{S_N}\mathscr{L}^+ \end{bmatrix},\quad \mathscr{F}^{G\mathscr{G}\mathscr{T}}:=\begin{bmatrix} F_0 \\ r_{S_D}F_0^+-\varphi_0 \\ r_{S_N}T^+F_0-\psi_0 \end{bmatrix}$$

Due to the mapping properties of operators V, $\mathscr{V}$, W, $\mathscr{W}$, $\mathscr{P}$, $\mathscr{R}$ and $\mathscr{R}^+$ described in Appendices A and B, we have $\mathscr{F}^{G\mathscr{G}\mathscr{T}}\in H^{s,0}(\Omega^+;\Delta)\times H^{s-(1/2)}(S_D)\times H^{s-(3/2)}(S_N)$, and the operator $\mathscr{A}^{G\mathscr{G}\mathscr{T}}$: $H^{s,0}(\Omega^+;\Delta)\times\tilde{H}^{s-(3/2)}(S_D)\times\tilde{H}^{s-(1/2)}(S_N)\to H^{s,0}(\Omega^+;\Delta)\times H^{s-(1/2)}(S_D)\times H^{s-(3/2)}(S_N)$ is continuous and, due to Theorem C.1, injective for $s\geqslant 1$.

The proof of the following invertibility theorem practically coincides with that of Reference [7, Theorem 4.4] extending, however, the result of Reference [7] from $s=1$ to $1\leqslant s<\frac{3}{2}$.

Theorem C.2
The operator $\mathscr{A}^{G\mathscr{G}\mathscr{T}}:H^s(\Omega^+)\times\tilde{H}^{s-(3/2)}(S_D)\times\tilde{H}^{s-(1/2)}(S_N)\to H^s(\Omega^+)\times H^{s-(1/2)}(S_D)\times H^{s-(3/2)}(S_N)$ is continuous and continuously invertible for $1\leqslant s<\frac{3}{2}$.

Proof
The continuity and injectivity is proved above. To prove the invertibility, let us consider the following operator:

$$\mathscr{A}_0^{G\mathscr{G}\mathscr{T}}:=\begin{bmatrix} I & -V & W \\ 0 & -r_{S_D}\mathscr{V} & r_{S_D}\mathscr{W} \\ 0 & 0 & r_{S_N}\hat{\mathscr{L}} \end{bmatrix}$$

where $\hat{\mathscr{L}}$ is given by (A5). As a result of compactness properties (B13)–(B16) and Theorem A.4, the operator $\mathscr{A}_0^{G\mathscr{G}\mathscr{T}}$ is a compact perturbation of the operator $\mathscr{A}^{G\mathscr{G}\mathscr{T}}$.

Due to Theorem A.5 for $\mathscr{V}$ and Corollary A.6 for $\hat{\mathscr{L}}$, the operator $\mathscr{A}_0^{G\mathscr{G}\mathscr{T}}$ is an upper triangular matrix operator with the following scalar diagonal invertible operators:

$$I:H^s(\Omega^+)\to H^s(\Omega^+)$$

$$r_{S_D}\mathscr{V}:\tilde{H}^{s-(3/2)}(S_D)\to H^{s-(1/2)}(S_D)$$

$$r_{S_N}\hat{\mathscr{L}}:\tilde{H}^{s-(1/2)}(S_N)\to H^{s-(3/2)}(S_N)$$

for $\frac{1}{2} < s < \frac{3}{2}$. This implies that

$$\mathscr{A}_0^{\mathscr{GGT}} : H^s(\Omega^+) \times \tilde{H}^{s-(3/2)}(S_D) \times \tilde{H}^{s-(1/2)}(S_N) \to H^s(\Omega^+;\Delta) \times H^{s-(1/2)}(S_D) \times H^{s-(3/2)}(S_N)$$

is an invertible operator for $\frac{1}{2} < s < \frac{3}{2}$. This implies operator $\mathscr{A}^{\mathscr{GGT}}$ possesses the Fredholm property and its index is zero for such s.

The injectivity of the operator $\mathscr{A}^{\mathscr{GGT}}$ already proved for $1 \leqslant s < \frac{3}{2}$ completes the theorem proof. □

Theorem C.3
The operator $\mathscr{A}^{\mathscr{GGT}} : H^{s,0}(\Omega^+;\Delta) \times \tilde{H}^{s-(3/2)}(S_D) \times \tilde{H}^{s-(1/2)}(S_N) \to H^{s,0}(\Omega^+;\Delta) \times H^{s-(1/2)}(S_D) \times H^{s-(3/2)}(S_N)$ is continuous and invertible for $1 \leqslant s < \frac{3}{2}$.

Proof
According to Theorem C.2, the triple $(u,\psi,\varphi)^\top = (\mathscr{A}^{\mathscr{GGT}})^{-1}\mathscr{F}^{\mathscr{GGT}} \in H^s(\Omega^+) \times \tilde{H}^{s-(3/2)}(S_D) \times \tilde{H}^{s-(1/2)}(S_N)$ satisfies equation

$$u(y) + \mathscr{R}u(y) - V\psi(y) + W\varphi(y) = \mathscr{F}_1^{\mathscr{GGT}}(y) \tag{C6}$$

for any $\mathscr{F}^{\mathscr{GGT}} \in H^{s,0}(\Omega^+;\Delta) \times H^{s-(1/2)}(S_D) \times H^{s-(3/2)}(S_N)$ and any $1 \leqslant s < \frac{3}{2}$. Due to Theorem A.1 and Remark B.2, this equation implies $u \in H^{s,0}(\Omega^+;\Delta)$ and the operator $(\mathscr{A}^{\mathscr{GGT}})^{-1} : H^{s,0}(\Omega^+;\Delta) \times H^{s-(1/2)}(S_D) \times H^{s-(3/2)}(S_N) \to H^{s,0}(\Omega^+;\Delta) \times \tilde{H}^{s-(3/2)}(S_D) \times \tilde{H}^{s-(1/2)}(S_N)$ is bounded. □

Original BVP (4)–(6) can be written in the form

$$A^{\mathrm{DN}}u = F^{\mathrm{DN}} \tag{C7}$$

where

$$A^{\mathrm{DN}} := \begin{bmatrix} L \\ r_{S_D}\tau^+ \\ r_{S_N}T^+ \end{bmatrix}, \quad F^{\mathrm{DN}} = \begin{bmatrix} f \\ \varphi_0 \\ \psi_0 \end{bmatrix} \tag{C8}$$

The operator $A^{\mathrm{DN}} : H^{s,0}(\Omega^+;\Delta) \to L_2(\Omega^+) \times H^{s-(1/2)}(S_D) \times H^{s-(3/2)}(S_N)$ for $1 \leqslant s < \frac{3}{2}$ is evidently continuous and due to the uniqueness theorem for the BVP is also injective.

The invertibility of the operator $\mathscr{A}^{\mathscr{GGT}}$ from Theorem C.3 and equivalence Theorem C.1 lead to the following invertibility result, c.f. Reference [8, Corollary 5.5].

Corollary C.4
The operator $A^{\mathrm{DN}} : H^{s,0}(\Omega^+;\Delta) \to L_2(\Omega^+) \times H^{s-(1/2)}(S_D) \times H^{s-(3/2)}(S_N)$ for $1 \leqslant s < \frac{3}{2}$ is continuous and continuously invertible.

Note that the corollary evidently implies Theorem 2.2.

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