O. Chkadua, S. E. Mikhailov, and D. Natroshvili

LOCALIZED DIRECT SEGREGATED BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR VARIABLE COEFFICIENT TRANSMISSION PROBLEMS WITH INTERFACE CRACK

Dedicated to the 120-th birthday anniversary of academician N. Muskhelishvili
Abstract. Some transmission problems for scalar second order elliptic partial differential equations are considered in a bounded composite domain consisting of adjacent anisotropic subdomains having a common interface surface. The matrix of coefficients of the differential operator has a jump across the interface but in each of the adjacent subdomains is represented as the product of a constant matrix by a smooth variable scalar function. The Dirichlet or mixed type boundary conditions are prescribed on the exterior boundary of the composite domain, the Neumann conditions on the the interface crack surfaces and the transmission conditions on the rest of the interface. Employing the parametrix-based localized potential method, the transmission problems are reduced to the localized boundary-domain integral equations. The corresponding localized boundary-domain integral operators are investigated and their invertibility in appropriate function spaces is proved.

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1. Introduction

We consider the basic, mixed and crack type transmission problems for scalar second order elliptic partial differential equations with variable coefficients and develop the generalized potential method based on the localized parametrix method.

For simplicity and detailed illustration of our approach we consider the simplest case when two adjacent domains under consideration, $\Omega_1$ and $\Omega_2$, have a common simply connected boundary $S_i$ called interface surface. The matrix of coefficients of the elliptic scalar operator in each domain is represented as the product of a constant matrix by a smooth variable scalar function. These coefficients are discontinuous across the interface surface.

We deal with the case when the Dirichlet or mixed type boundary conditions on the exterior boundary $S_e$ of the composite domain $\Omega_1 \cup \Omega_2$, the Neumann conditions on the the interface crack surfaces and the transmission conditions on the rest of the interface are prescribed.

The transmission problems treated in the paper can be investigated in by the variational methods, and the corresponding uniqueness and existence results can be obtained similar to e.g., [13], [14], [15], [16].

For special cases when the fundamental solution is available the Dirichlet and Neumann type boundary value problems were also investigated by the classical potential method (see [3], [13], [16], [23]) and the references therein).

Our goal here is to show that the transmission problems in question can be equivalently reduced to some localized boundary-domain integral equations (LBDIE) and that the corresponding localized boundary-domain integral operators (LBDIO) are invertible, which beside a pure mathematical interest may have also some applications in numerical analysis for construction of efficient numerical algorithms (see, e.g., [17], [21], [27], [30], [31] and the references therein). In our case, the localized parametrix $P_q(x-y,y)$, $q = 1, 2$, is represented as the product of a Levi function $P^{(1)}_q(x-y,y)$ of the differential operator under consideration by an appropriately chosen cut-off function $\chi_q(x-y)$ supported on some neighbourhood of the origin. Clearly, the kernels of the corresponding localized potentials are supported in some neighbourhood of the reference point $y$ (assuming that $x$ is an integration variable) and they do not solve the original differential equation.

In spite of the fact that the localized potentials preserve almost all mapping properties of the classical non-localized ones (cf. [7]), some unusual properties of the localized potentials appear due to the localization of the kernel functions which have no counterparts in classical potential theory and which need special consideration and analysis.

By means of the direct approach based on Green’s representation formula we reduce the transmission problems to the localized boundary-domain integral equation (LBDIE) system. First we establish the equivalence between the original transmission problems and the corresponding LBDIEs systems.
which proved to be a quite nontrivial problem and plays a crucial role in our analysis. Afterwards we investigate Fredholm properties of the LBDIOs and prove their invertibility in appropriate function spaces. This paper is heavily based and essentially develops methods and results of [5], [6], [7], [8], [19].

2. Transmission Problems

Let $\Omega$ and $\Omega_1$ be bounded open domains in $\mathbb{R}^3$ and $\overline{\Omega}_1 \subset \Omega$. Denote $\Omega_2 := \Omega \setminus \overline{\Omega}_1$, and $S_1 := \partial \Omega_1$, $S_c := \partial \Omega$. Clearly, $\partial \Omega_2 = S_1 \cup S_c$. We assume that the interface surface $S_1$ and the exterior boundary $S_c$ of the composite body $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ are sufficiently smooth, say $C^\infty$-regular if not otherwise stated.

Throughout the paper $n^{(q)} = n^{(q)}(x)$ denotes the unit normal vector to $\partial \Omega_q$ directed outward the domains $\Omega_q$. Clearly, $n^{(1)}(x) = -n^{(2)}(x)$ for $x \in S_1$.

By $H^r(\Omega') = H_2^r(\Omega')$ and $H^r(S) = H_2^r(S), \ r \in \mathbb{R}$, we denote the Bessel potential spaces on a domain $\Omega'$ and on a closed manifold $S$ without boundary. The subspace of $H^r(\mathbb{R}^3)$ of functions with compact support is denoted by $H^r_{comp}(\mathbb{R}^3)$. Recall that $H^0(\Omega') = L_2(\Omega')$ is a space of square integrable functions in $\Omega'$.

For a smooth proper submanifold $\mathcal{M} \subset S$ we denote by $\tilde{H}^r(\mathcal{M})$ the subspace of $H^r(S)$,

$$\tilde{H}^r(\mathcal{M}) := \{ g : \ g \in H^r(S), \ \text{supp} \ g \subset \overline{\mathcal{M}} \},$$

while $H^r(\mathcal{M})$ denotes the spaces of restrictions on $\mathcal{M}$ of functions from $H^r(S)$,

$$H^r(\mathcal{M}) := \{ r_\mathcal{M} f : \ f \in H^r(S) \},$$

where $r_\mathcal{M}$ is the restriction operator onto $\mathcal{M}$.

Let us consider the differential operators in the domains $\Omega_q$

$$A_q(x, \partial_x)u(x) := \sum_{j,k=1}^3 \partial_{x_j} \left[ a_{kj}^{(q)}(x) \partial_{x_k} u(x) \right], \quad q = 1, 2, \quad (2.1)$$

where $\partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial / \partial x_j, \ j = 1, 2, 3$, and

$$a_{kj}^{(q)}(x) = a_{jk}^{(q)}(x) = a_q(x) a_{kj}^{(q)*}, \quad (2.2)$$

$$a_q(x) := [a_{kj}^{(q)}(x)]_{3 \times 3} = a_q(x) [a_{kj}^{(q)*}]_{3 \times 3}, \quad a_q^* := [a_{kj}^{(q)*}]_{3 \times 3}. \quad (2.3)$$

Here $a_{kj}^{(q)}$ are constants and the matrix $a_q^* := [a_{kj}^{(q)*}]_{3 \times 3}$ is positive definite. Moreover, we assume that

$$a_q \in C^\infty(\mathbb{R}^3), \quad 0 < c_0 \leq a_q(x) \leq c_1 < \infty, \quad q = 1, 2. \quad (2.4)$$

Further, for sufficiently smooth functions (from the space $H^2(\Omega_q)$ say) we introduce the co-normal derivative operator on $\partial \Omega_q$, $q = 1, 2$, in the usual
trace sense:
\[ T_q^+(x, \partial_x)u(x) := \sum_{k,j=1}^{3} a_{kj}^{(q)}(x) u_k^{(q)}(x) \gamma_q[\partial_x u(x)], \quad x \in \partial \Omega_q, \quad (2.5) \]

where the symbol \( \gamma_q \equiv \gamma_q^+ \) denotes the trace operator on \( \partial \Omega_q \) from the interior of \( \Omega_q \). Analogously is defined the external co-normal derivative operator \( T_q^- (x, \partial_x)w \) with the help of the exterior trace operator \( \gamma_q^- \) on \( \partial \Omega_q \) denoting the limiting value on \( \partial \Omega_q \) from the exterior domain \( \Omega_q^e := \mathbb{R}^3 \setminus \overline{\Omega}_q \):
\[ T_q^- (x, \partial_x)w(x) := \sum_{k,j=1}^{3} a_{kj}^{(q)}(x) u_k^{(q)}(x) \gamma_q^-[\partial_x w(x)], \quad x \in \partial \Omega_q. \]

We set
\[ H^{1,0}(\Omega_q; A_q) := \{ v \in H^1(\Omega_q) : A_q v \in H^0(\Omega_q) \}, \quad q = 1, 2. \quad (2.6) \]

One can correctly define the generalized (canonical) co-normal derivatives \( T_q u \equiv T_q^+ u \in H^{-\frac{1}{2}}(\partial \Omega_q) \) (cf., for example, [9, Lemma 3.2], [16, Lemma 4.3], [20, Definition 3.3]),
\[ \langle T_q u, w \rangle_{\partial \Omega_q} \equiv \langle T_q^+ u, w \rangle_{\partial \Omega_q} := \int_{\partial \Omega_q} \left[ (\ell_q w) A_q u + E_q(u, \ell_q w) \right] \ dx \quad \forall \ w \in H^{\frac{1}{2}}(\partial \Omega_q), \quad (2.7) \]

where \( \ell_q \) is a continuous linear extension operator, \( \ell_q : H^{\frac{1}{2}}(\partial \Omega_q) \to H^1(\Omega_q) \) which is a right inverse to the trace operator \( \gamma_q \),
\[ E_q(u, v) := \sum_{i,j=1}^{3} a_{ij}^{(q)}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} \equiv \nabla_x u \cdot a_q(x) \nabla_x v, \quad \nabla_x := (\partial_1, \partial_2, \partial_3)^T. \]

Here and in what follows the central dot denotes the scalar product in \( \mathbb{R}^3 \) or in \( \mathbb{C}^3 \). In (2.7), the symbol \( \langle g_1, g_2 \rangle_{\partial \Omega_q} \) denotes the duality brackets between the spaces \( H^{-\frac{1}{2}}(\partial \Omega_q) \) and \( H^{\frac{1}{2}}(\partial \Omega_q) \), coinciding with \( \int_{\partial \Omega_q} g_1(x) g_2(x) \, dS \) if \( g_1, g_2 \in L^2(\partial \Omega_q) \). Below for such dualities we will use sometimes the usual integral symbols when they do not cause confusion. The canonical co-normal derivative operators \( T_q : H^{1,0}(\Omega_q; A_q) \to H^{-\frac{1}{2}}(\partial \Omega_q) \) defined by (2.7) are continuous extensions of the classical co-normal derivative operators from (2.5), and the second Green identity
\[ \int_{\partial \Omega_q} [v A_q u - u A_q v] \, dx = \int_{\partial \Omega_q} [(\gamma_q v)T_q u - (\gamma_q u)T_q v] \, dS, \quad q = 1, 2, \quad (2.8) \]
holds for \( u, v \in H^{1,0}(\Omega_q; A_q) \).

Now we formulate the following Dirichlet, Neumann and mixed type transmission problems:
Find functions \( u_1 \in H^{1,0}(\Omega_1; A_1) \) and \( u_2 \in H^{1,0}(\Omega_2; A_2) \) satisfying the differential equations
\[
A_q(x, \partial)u_q = f_q \quad \text{in} \quad \Omega_q, \; q = 1, 2, \tag{2.9}
\]
the transmission conditions on the interface surface
\[
\gamma_1 u_1 - \gamma_2 u_2 = \varphi_0 \quad \text{on} \quad S_i, \tag{2.10}
\]
\[
T_1 u_1 + T_2 u_2 = \psi_0 \quad \text{on} \quad S_i, \tag{2.11}
\]
and one of the following conditions on the exterior boundary:
the Dirichlet boundary condition
\[
\gamma_2 u_2 = \varphi_{0e} \quad \text{on} \quad S_e; \tag{2.12}
\]
or the Neumann boundary condition
\[
T_2 u_2 = \psi_{0e} \quad \text{on} \quad S_e, \tag{2.13}
\]
or mixed type boundary conditions
\[
\gamma_2 u_2 = \varphi_{0e}^{(M)} \quad \text{on} \quad S_{eD}, \tag{2.14}
\]
\[
T_2 u_2 = \psi_{0e}^{(M)} \quad \text{on} \quad S_{eN}, \tag{2.15}
\]
where \( S_{eD} \) and \( S_{eN} \) are smooth disjoint submanifolds of \( S_e; \) \( S_e = S_{eD} \cup S_{eN} \) and \( S_{eD} \cap S_{eN} = \emptyset. \)

We will call these boundary transmission problems as (TD), (TN) and (TM) problems.

For the data in the above formulated problems we assume
\[
\varphi_0 \in H^\frac{1}{2}(S_i), \; \psi_0 \in H^{-\frac{1}{2}}(S_i), \; \varphi_{0e} \in H^\frac{1}{2}(S_e), \; \psi_{0e} \in H^{-\frac{1}{2}}(S_e), \tag{2.16}
\]
\[
\varphi_{0e}^{(M)} \in H^1(S_{eD}), \; \psi_{0e}^{(M)} \in H^{-1}(S_{eN}), \; \lambda_q \in H^0(\Omega_q), \; q = 1, 2.
\]
Equations (2.1) are understood in the distributional sense, the Dirichlet type boundary value and transmission conditions are understood in the usual trace sense, while the Neumann type boundary value and transmission conditions for the co-normal derivatives are understood in the sense of the canonical co-normal derivatives defined by (2.7).

We recall that the normal vectors \( n^{(1)} \) and \( n^{(2)} \) in the definitions of the co-normal derivatives \( T_1 u \) and \( T_2 u \) on \( S_i \) have opposite directions.

Further, for the case when the interface crack is present, let the interface \( S_i \) be a union of smooth disjoint proper submanifolds, the interface crack part \( S_i^{(c)} \) and the transmission part \( S_i^{(t)} \), i.e., \( S_i = S_i^{(c)} \cup S_i^{(t)} \) and \( S_i^{(c)} \cap S_i^{(t)} = \emptyset. \)

Let us set the following interface crack type transmission problems for the composite domain \( \Omega = \Omega_1 \cup \Omega_2; \)
Find functions \( u_1 \in H^{1,0}(\Omega_1; A_1) \) and \( u_2 \in H^{1,0}(\Omega_2; A_2) \) satisfying the differential equations (2.9) in \( \Omega_1 \) and \( \Omega_2 \) respectively, one of the boundary conditions
\[ (2.12), \text{ or } (2.13), \text{ or } (2.14)-(2.15) \text{ on the exterior boundary } S_e, \text{ the transmission conditions on } S^{(t)}_i \]
\[
\begin{align*}
\gamma_1 u_1 - \gamma_2 u_2 &= \varphi^{(t)}_0 \quad \text{on } S^{(t)}_i, \\
T_1 u_1 + T_2 u_2 &= \psi^{(t)}_0 \quad \text{on } S^{(b)}_i,
\end{align*}
\]
and the crack type conditions on \( S^{(c)}_i \)
\[
\begin{align*}
T_1 u_1 &= \psi'_0 \quad \text{on } S^{(c)}_i, \\
T_2 u_2 &= \psi''_0 \quad \text{on } S^{(c)}_i.
\end{align*}
\]
We will call these crack type boundary transmission problems as (CTD), (CTN) and (CTM) problems, respectively.

Along with the conditions (2.16), for the data in the above formulated crack type problems we require that
\[
\begin{align*}
\varphi^{(t)}_0 &\in H^{1/2}(S^{(t)}_i),
\psi^{(t)}_0 &\in H^{-1/2}(S^{(t)}_i), \\
\psi'_0 &\in H^{-1/2}(S^{(c)}_i),
\psi''_0 &\in H^{-1/2}(S^{(c)}_i).
\end{align*}
\]
It is easy to see that for the function
\[
\psi_0 := \begin{cases} 
\varphi^{(t)}_0 & \text{on } S^{(t)}_i, \\
\psi'_0 & \text{on } S^{(c)}_i,
\end{cases}
\]
the following embedding
\[
\psi_0 \in H^{-1/2}(S_i)
\]
is a necessary compatibility condition for the above formulated interface crack problems to be solvable in the space \( H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \) since
\[
\psi_0 = T_1 u_1 + T_2 u_2 \text{ on } S_i.
\]
In what follows we assume that for \( \psi_0 \) given by (2.22) the condition (2.23) is satisfied.

As we have mentioned in the introduction, all the above formulated transmission problems can be investigated by the functional-variational methods and the corresponding uniqueness and existence results can be obtained similar to e.g., [13], [15], [16]. In particular, there holds the following proposition which can be proved on the basis of the Lax-Milgram theorem.

**Theorem 2.1.** If the conditions (2.16), (2.21), and (2.23) are satisfied, then

(i) The transmission problems (TD), (TM), (CTD), and (CTM) are uniquely solvable in the space \( H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \).

(ii) The following condition
\[
\int_{\Omega_1} f_1 \, dx + \int_{\Omega_2} f_2 \, dx = \int_{S_i} \psi_0 \, dS + \int_{S_e} \psi_0 e \, dS
\]
(2.25)
is necessary and sufficient for the transmission problem (TN) to be solvable in the space $H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$. The same condition (2.25) with the function $\psi_{i0}$ defined by (2.22) is necessary and sufficient for the crack type transmission problem (CTN) to be solvable in the space $H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$. In both cases a solution pair $(u_1, u_2)$ is defined modulo a constant summand $(c, c)$.

We recall that our goal here is to show that the above transmission problems can be equivalently reduced to some segregated LBDIEs and to perform full analysis of the corresponding LBDIOs.

3. Properties of Localized Potentials

It is well known that the fundamental solution-function of the elliptic operator with constant coefficients

$$A_{q\ast}(\partial) := \sum_{i,j=1}^{3} a_{kj\ast}^{(q)} \partial_k \partial_j$$

is written as (see, e.g., [22], [23])

$$P_{q\ast}(x) = \frac{\alpha_q}{(x \cdot a_{q\ast}^{-1} x)^{\frac{d}{2}}} \text{ with } \alpha_q = -\frac{1}{4\pi [\det a_{q\ast}]^{\frac{1}{2}}}, \quad a_{q\ast} = [a_{kj\ast}^{(q)}]_{3 \times 3}. \quad (3.2)$$

Here $a_{q\ast}^{-1}$ stands for the inverse matrix to $a_{q\ast}$. Clearly, $a_{q\ast}^{-1}$ is symmetric and positive definite. Therefore there is a symmetric positive definite matrix $d_{q\ast}$ such that $a_{q\ast}^{-1} = d_{q\ast}^2$ and

$$(x \cdot a_{q\ast}^{-1} x) = |d_{q\ast} x|^2, \quad \det d_{q\ast} = [\det a_{q\ast}]^{-\frac{1}{2}}. \quad (3.3)$$

Throughout the paper the subscript $\ast$ means that the corresponding operator, matrix or function is related to the operator with constant coefficients (3.1).

Note that

$$A_{q\ast}(\partial_x) P_{q\ast}(x - y) = \delta(x - y), \quad (4.4)$$

where $\delta(\cdot)$ is the Dirac distribution.

Now we introduce the localized parametrix (localized Levi function) for the operator $A_q$,

$$P_{q}(x - y) \equiv P_{q\ast}(x - y) := \frac{1}{a_q(y)} \chi_q(x - y) P_{q\ast}(x - y), \quad q = 1, 2, \quad (3.5)$$

where $\chi$ is a localizing cut-off function (see Appendix A)

$$\chi_q(x) := \chi(d_{q\ast} x) = \chi((d_{q\ast} x)^1/2) = \chi(x \cdot a_{q\ast}^{-1} x)^{1/2}, \quad \chi \in \mathcal{X}^k, \quad k \geq 1. \quad (3.6)$$

Throughout the paper we assume that the condition (3.6) is satisfied if not otherwise stated.
One can easily check the following relations
\[ A_q(x, \partial_x)u(x) = a_q(x)A_q(x)u(x) + \nabla_x a_q(x) \cdot a_q \nabla_x u(x), \quad (3.7) \]
\[ A_q(x, \partial_x)P_q(x - y, y) = \delta(x - y) + R_q(x, y), \quad q = 1, 2, \] \( (3.8) \)
where
\[ R_q(x, y) = \]
\[ = \frac{a_q(x)}{a_q(y)} \left[ P_q(x - y, y)A_q(\partial_x)\chi_q(x - y) + 2\nabla_x \chi_q(x - y) \cdot a_q \nabla_x P_q(x - y) \right] + \]
\[ + \frac{1}{a_q(y)} \left( \nabla_x a_q(x) \cdot a_q \nabla_x [\chi_q(x - y)P_q(x - y)] \right). \quad (3.9) \]
The function \( R_q(x, y) \) possesses a weak singularity of type \( O(|x - y|^{-2}) \) as \( x \to y \) if \( \chi_q \) is smooth enough, e.g., if \( \chi_q \in \mathcal{C}^2 \).

Let us introduce the localized surface and volume potentials, based on the localized parametrix \( P_q \),
\[ V^{(q)}_S g(x) := -\int_S P_q(x - y, y)g(x) dS_y, \quad (3.10) \]
\[ W^{(q)}_S g(x) := -\int_S \left[ T_q(x, \partial_x)P_q(x - y, y) \right] g(x) dS_y, \quad (3.11) \]
\[ \mathcal{P}_q f(y) := \int_{\partial\Omega_q} P_q(x - y, y) f(x) dx, \quad (3.12) \]
\[ \mathcal{R}_q f(y) := \int_{\partial\Omega_q} R_q(x, y) f(x) dx. \quad (3.13) \]
Here and further on \( S \in \{ S_1, S_{2e}, \partial\Omega_2 \} \).

Note that for layer potentials we drop the subindex \( S \) when \( S = \partial\Omega_q \), i.e., \( V^{(q)} := V^{(q)}_{\partial\Omega_q} \), \( W^{(q)} := W^{(q)}_{\partial\Omega_q} \). If the domain of integration in (3.12) is the whole space \( \Omega_q = \mathbb{R}^3 \), we employ the notation \( P_q f = \mathcal{P}_q f \).

Let us also define the corresponding boundary operators generated by the direct values of the localized single and double layer potentials and their co-normal derivatives on \( S \),
\[ V^{(q)}_S g(x) := -\int_S P_q(x - y, y)g(x) dS_y, \quad (3.14) \]
\[ W^{(q)}_S g(x) := -\int_S \left[ T_q(x, \partial_x)P_q(x - y, y) \right] g(x) dS_y, \quad (3.15) \]
\[ W^{(q)}_S g(x) := -\int_S \left[ T_q(y, \partial_y)P_q(x - y, y) \right] g(x) dS_y. \quad (3.16) \]
\text{\begin{equation}
\mathcal{L}^{(q)\pm}_s g(y) := T^{\pm}_q(y, \partial_y) W^{(q)}_s g(y).
\end{equation}}
\text{(3.17)}

For the pseudodifferential operator in (3.17), we employ also the notation \( \mathcal{L}^{(q)} := \mathcal{L}^{(q)\pm} \).

Note that the kernel functions of the operators (3.15) and (3.16) are at most weakly singular if the cut-of function \( \chi \in X^2 \) and the surface \( S \) is \( C^{1,\alpha} \) smooth with \( \alpha > 0 \):
\text{\begin{align*}
T_q(x, \partial_x) P_q(x - y, y) &= O(|x - y|^{-2+\alpha}), \\
T_q(y, \partial_y) P_q(x - y, y) &= O(|x - y|^{-2+\alpha})
\end{align*}}
\text{(3.18)}

for sufficiently small \(|x - y|\) (cf. [23], [22], [7]). We will also need a localized parametrix of the constant-coefficient differential operator \( A_q^\ast (\partial) \),
\text{\begin{equation}
P_{q^\ast}(x - y) := \chi_q(x - y) P_{q^\ast 1}(x - y) = a_q(y) P_q(x - y, y).
\end{equation}}
\text{(3.19)}

We have
\text{\begin{equation}
A_q^\ast (\partial_x) P_{q^\ast}(x - y) = \delta(x - y) + R_{q^\ast}(x, y),
\end{equation}}
\text{(3.20)}

where
\text{\begin{equation}
R_{q^\ast}(x, y) = P_{q^\ast 1}(x - y) A_q^\ast (\partial_x) \chi_q(x - y) + 2 \nabla_x \chi_q(x - y) \cdot a_q \nabla_x P_{q^\ast 1}(x - y).
\end{equation}}
\text{(3.21)}

Denote the surface and volume potentials constructed with the help of the localized parametrix \( P_{q^\ast} \) by the symbols \( \mathcal{V}^{(q)}_{S^\ast} \), \( \mathcal{W}^{(q)}_{S^\ast} \), \( \mathcal{P}_{q^\ast} \) and \( \mathcal{R}_{q^\ast} \),
\text{\begin{align*}
\mathcal{V}^{(q)}_{S^\ast} g(y) &:= - \int_S P_{q^\ast}(x - y) g(x) dS_x, \\
\mathcal{W}^{(q)}_{S^\ast} g(y) &:= - \int_S [T_{q^\ast}(x, \partial_x) P_{q^\ast}(x - y)] g(x) dS_x, \\
\mathcal{P}_{q^\ast} f(y) &:= \int_{\Omega_q} P_{q^\ast}(x - y) f(x) dx, \\
\mathcal{R}_{q^\ast} f(y) &:= \int_{\Omega_q} R_{q^\ast}(x - y) f(x) dx.
\end{align*}}
\text{(3.22)-(3.25)}

Here \( T_{q^\ast} \) stands for the co-normal derivative operator corresponding to the constant coefficient differential operator \( A_q^\ast (\partial) \), which for sufficiently smooth \( u \) takes form
\text{\begin{equation}
T_{q^\ast}(x, \partial_x) u(x) \equiv T_{q^\ast}^\ast(x, \partial_x) u(x) := 3 \sum_{k,j} a_{kj}^{(q)}(x) n_k^{(q)}(x) \gamma_k \partial_x u(x), \quad x \in \partial \Omega_q,
\end{equation}}
\text{(3.26)}
that can be continuously extended to \( u \in H^{1,0}(\Omega_q; A_{q*}) \) similar to (2.7). Note that

\[
H^{1,0}(\Omega_q; A_q) = H^{1,0}(\Omega_q; A_{q*}) \quad \text{and} \quad T_q(x, \partial_x)u(x) = a_q(x)T_q(x, \partial_x)u(x)
\]
due to (2.5) and (3.26). Again, if the domain of integration in (3.24) is the whole space \( \Omega_q = \mathbb{R}^3 \), we employ the notation \( P_{q*} f = P_{q*} f \).

Further, we introduce the boundary operators generated by the direct values of the localized layer potentials (3.22) and (3.23), and their co-normal derivatives on \( S \),

\[
V^{(q)}_{\sigma*} g(y) := -\int_S P_{q*}(x-y)g(x)dS_x, \quad (3.27)
\]

\[
W^{(q)}_{\sigma*} g(y) := -\int_S [T_{q*}(x, \partial_x)P_{q*}(x-y)]g(x)dS_x, \quad (3.28)
\]

\[
W'^{(q)}_{\sigma*} g(y) := -\int_S [T_{q*}(y, \partial_y)P_{q*}(x-y)]g(x)dS_x, \quad (3.29)
\]

\[
L^{(q)\pm}_{\sigma*} g(y) := T^{\pm}_{q*}(y, \partial_y)W^{(q)}_{\sigma*} g(y), \quad (3.30)
\]

For the pseudodifferential operator in (3.30), we employ also the notation \( L^{(q)}_{\sigma*} := L^{(q)\pm}_{\sigma*} \).

In view of the relations (3.5) and (3.19) it follows that

\[
V^{(q)}_{\sigma*} g(y) = a^{-1}_q(y)V^{(q)}_{\sigma*} g(y), \quad (3.31)
\]

\[
W^{(q)}_{\sigma*} g(y) = a^{-1}_q(y)W^{(q)}_{\sigma*}(a_q g)(y), \quad (3.32)
\]

\[
P_{q*} f(y) = a^{-1}_q(y)P_{q*} f(y). \quad (3.33)
\]

Therefore, the potentials with and without subscript “\( \sigma* \)” have exactly the same mapping and smoothness properties for sufficiently smooth variable coefficients \( a_q \).

Before we go over to the localized boundary-domain integral formulation of the above stated transmission problems we derive some basic properties of the layer and volume potentials corresponding to the localized parametrix \( P_{q*} \) needed in our further analysis (cf. [7], [13]).

To this end let us note that the volume potential \( P_{q*} f \), as a convolution of \( P_{q*} \) and \( f \), can be represented as a pseudodifferential operator

\[
P_{q*} f(y) = \mathcal{F}^{-1}_{\xi \rightarrow y} \left[ \hat{P}_{q*} \hat{f}(\xi) \right], \quad (3.34)
\]

where \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) stand for the generalized direct and inverse Fourier transform operators, respectively, and overset “tilde” denotes the direct Fourier
transform,
\[\mathfrak{F}_{x \rightarrow \xi}[f] \equiv \tilde{f}(\xi) := \int_{\mathbb{R}^3} f(x)e^{ix \cdot \xi} \, dx,\]
\[\mathfrak{F}_{\xi^{-1} \rightarrow y}[f] := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f(\xi)e^{-iy \cdot \xi} \, d\xi.\] (3.35)

The properties of the symbol function \(\widetilde{P}_{q^*}(\xi)\) of the pseudodifferential operator \(P_{q^*}\) is described by the following assertion.

**Lemma 3.1.**

(i) Let \(\chi \in X^k, \, k \geq 0\). Then \(\widetilde{P}_{q^*}(\xi) \in C(\mathbb{R}^3)\) and for \(\xi \neq 0\) the following expansion holds
\[\widetilde{P}_{q^*}(\xi) = \sum_{m=0}^{k^*} \frac{(-1)^{m+1}}{|\xi \cdot a_{q^*}^\dagger\xi|^{m+1}} \hat{\chi}^{(2m)}(0) - \frac{1}{|\xi \cdot a_{q^*}^\dagger\xi|^{(k+1)/2}} \int_0^\infty \sin\left(\frac{|\xi|}{\rho} + \frac{k\pi}{2}\right) \hat{\chi}^{(k)}(\rho) \, d\rho,\] (3.36)

where \(k^*\) is the integer part of \((k-1)/2\) and the sum disappears in (3.36) if \(k^* < 0\), i.e., if \(k = 0\).

(ii) If \(\chi \in X^1_1\), then
\[\widetilde{P}_{q^*}(\xi) < 0 \text{ for almost all } \xi \in \mathbb{R}^3.\] (3.37)

(iii) If \(\chi \in X^1_1\) and \(\sigma_\chi(\omega) > 0\) for all \(\omega \in \mathbb{R}\) (see Definition A.1), then \(\widetilde{P}_{q^*}(\xi) < 0\) for all \(\xi \in \mathbb{R}^3\) and there are positive constants \(c_1\) and \(c_2\) such that
\[c_1 \frac{1}{1 + |\xi|^2} \leq |\widetilde{P}_{q^*}(\xi)| \leq c_2 \frac{1}{1 + |\xi|^2} \text{ for all } \xi \in \mathbb{R}^3.\] (3.38)

**Proof.** By formulas (3.2) and (3.3) we have
\[\widetilde{P}_{q^*}(\xi) = \int_{\mathbb{R}^3} \frac{\alpha_q \chi(d_{q^*}x)}{(x \cdot a_{q^*}d_{q^*}x)^2} e^{ix \cdot \xi} \, dx = \int_{\mathbb{R}^3} \frac{\alpha_q \chi(d_{q^*}x)}{|d_{q^*}x|^2} e^{ix \cdot \xi} \, dx = \frac{\alpha_q}{\det d_{q^*}} \int_{\mathbb{R}^3} \frac{\chi(\eta)}{|\eta|^2} e^{i\eta \cdot a_{q^*}^\dagger\xi} \, d\eta = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\chi(\eta)}{|\eta|^2} e^{i\eta \cdot d_{q^*}^{-1}a_{q^*}^\dagger\xi} \, d\eta = -\frac{1}{4\pi} \delta_{|\eta|^2 - a_{q^*}^\dagger\xi} \left[\frac{\chi(\eta)}{|\eta|}\right] = -\frac{1}{4\pi} \int_0^\infty \chi(\rho) \sin(\rho|\xi|) \, d\rho = \frac{\hat{\chi}(\xi - a_{q^*}^\dagger\xi)}{|\xi|}, \] with \(\xi = d_{q^*}^{-1}a_{q^*}^\dagger\xi\).

(3.39)

Now (3.36) can be easily obtained from (3.39) by the integration by parts formula taking into account that \(\hat{\chi}^{(k-1)}(\rho) \to 0\) as \(\rho \to \infty\) if \(\hat{\chi} \in W_1^k(0, \infty).\)
Further, since $|\zeta|^2 = |d_{q*}^{-1}\xi|^2 = \xi \cdot a_{q*} \xi$, the proof of items (ii) and (iii) follow from (3.40), (3.36) and Definition A.1.

By positive definiteness of the matrices $a_{q*}$ and in view of the equality (3.33), $P_q = a_q^{-1} P_{q*}$, Lemma 3.1(i) implies the following important assertion.

**Theorem 3.2.** There exists a positive constant $c_1$ such that
\[
|\tilde{P}_{q*}(\xi)| \leq c_1 (1 + |\xi|^2)^{-\frac{k+1}{2}} \quad \text{for all } \xi \in \mathbb{R}^3 \text{ if } \chi \in X^k, \quad k = 0, 1, \quad (3.41)
\]

and the operators
\[
P_q, P_{q*} : H^t(\mathbb{R}^3) \longrightarrow H^{t+k+1}(\mathbb{R}^3) \quad \forall t \in \mathbb{R} \text{ if } \chi \in X^k, \quad k = 0, 1, \quad (3.42)
\]
are continuous.

In particular, we see that the operators
\[
P_{q*}, P_q : H^0(\Omega_q) \longrightarrow H^2(\mathbb{R}^3) \quad (3.43)
\]
are continuous for arbitrary bounded domain $\Omega_q \subset \mathbb{R}^3$ if $\chi \in X^1$.

More restrictions on $\chi$ lead to the following counterpart of [7, Corollary 5.2(ii)].

**Lemma 3.3.** Let $\chi \in X_1^1$ and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$ (see Definition A.1). Then the operator
\[
P_{q*} : H^r(\mathbb{R}^3) \longrightarrow H^{r+2}(\mathbb{R}^3), \quad r \in \mathbb{R}, \quad q = 1, 2, \quad (3.44)
\]
is invertible and the inverse operator $P_{q*}^{-1}$ is a pseudodifferential operator with the symbol $\tilde{P}_{q*}^{-1}(\xi)$.

Moreover, if $\chi \in X_1^1$, then
\[
\tilde{P}_{q*}^{-1}(\xi) = -\xi \cdot a_{q*} \xi - \nu_{q*}(\xi), \quad (3.45)
\]
where
\[
\nu_{q*}(\xi) = \mathcal{O}(1), \quad \nu_{q*}(\xi) \geq 0 \quad \text{for all } \xi \in \mathbb{R}^3. \quad (3.46)
\]
The pseudodifferential operator $P_{q*}^{-1}$ can be decomposed as
\[
P_{q*}^{-1} = A_{q*}(\partial) - N_{q*} \quad (3.47)
\]
where $A_{q*}(\partial)$ is a partial differential operator with constant coefficients defined by (3.1) and $N_{q*}$ is a pseudodifferential operator with the symbol $\nu_{q*}(\xi)$.

**Proof.** It is an immediate consequence of Lemma 3.1(iii) except the inequality in (3.46) which follows from the imbedding $\chi \in X_1^1$. In fact, we have
\[
\nu_{q*}(\xi) = -\tilde{P}_{q*}^{-1}(\xi) - \xi \cdot a_{q*} \xi = -\frac{1 + (\xi \cdot a_{q*} \xi)\tilde{P}_{q*}(\xi)}{\tilde{P}_{q*}(\xi)} \quad \text{for all } \xi \in \mathbb{R}^3. \quad (3.48)
\]
Let for (3.20) we have

\[ \nu_{q^*}(\xi) = \left[ 1 - |\xi| \chi_{\omega}(\xi) \right] \frac{|\xi|}{\chi_{\omega}(\xi)} = \frac{1 - |\xi| \chi_{\omega}(\xi)}{\sigma_{\omega}(\xi)} \text{ for all } \xi \in \mathbb{R}^3. \]  

(3.49)

Now the desired inequality follows due to the relations (A.5) and equality (3.51) gives the estimates, \( |\xi| \chi_{\omega}(\xi) \).

Let us also denote,

\[ R_{q^*} f := \int_{\mathbb{R}^3} R_{q^*}(x - y) f(x) \, dx = \mathcal{F}^{-1}(\tilde{R}_{q^*} f), \]

where the kernel \( R_{q^*}(x - y) \) is given by (3.20)–(3.21) and \( \tilde{R}_{q^*} = \mathcal{F} R_{q^*} \).

**Theorem 3.4.** Let \( \chi \in X^k, \ k \geq 1 \). Then

\[ \tilde{R}_{q^*}(\xi) = - (\xi \cdot a_{q^*} \xi) \tilde{P}_{q^*} - 1 = |\xi| \tilde{X}_{\omega}(\xi) - 1 = \]

\[ = \sum_{m=1}^{k^*} \frac{(-1)^{m+1}}{|\xi| \cdot a_{q^*} \xi|^{m}} \tilde{X}^{(2m)}(0) - \frac{1}{|\xi| \cdot a_{q^*} \xi^{(k-1)/2}} \int_{0}^{\infty} \sin \left( |\xi| \varphi + \frac{k\pi}{2} \right) \tilde{\chi}^{(k)}(\varphi) \, d\varphi, \]

(3.51)

where \( \zeta = d_{q^*}^{-1} \xi, \ k^* \) is the integer part of \( (k - 1)/2 \), and the sum disappears in (3.51) if \( k^* < 1 \), i.e., \( k < 3 \).

Moreover,

(i) for \( s \in \mathbb{R} \) and \( k = 1, 2, 3 \), the following operator is continuous

\[ R_{q^*} : H^s(\mathbb{R}^3) \rightarrow H^{s+k-1}(\mathbb{R}^3); \]

(ii) if \( \chi \in X^k, \ k \geq 1 \), then \( \tilde{R}_{q^*}(\xi) \leq 0 \) for all \( \xi \in \mathbb{R}^3 \).

**Proof.** By (3.20) we have \( \tilde{R}_{q^*}(\xi) = - (\xi \cdot a_{q^*} \xi) \tilde{P}_{q^*} - 1 \) and Lemma 3.1 implies (3.50) and (3.51). Equality (3.51) gives the estimates,

\[ |\tilde{R}_{q^*}(\xi)| \leq c(1 + |\xi|^2)^{-\frac{k-1}{2}} \text{ for all } \xi \in \mathbb{R}^3 \text{ if } \chi \in X^k, \ k = 1, 2, 3, \]

which imply (3.52). Finally, (A.5) implies item (ii). \( \square \)

Taking into account that

\[ P_{q^*} f = P_{q^*} f, \quad R_{q^*} f = R_{q^*} f \quad \text{for } f \in \tilde{H}^s(\Omega_q), \ s \in \mathbb{R}, \]

(3.53)

we can write down the mapping properties for \( P_{q^*} \) and \( R_{q^*} \).

**Theorem 3.5.** The following operators are continuous

\[ P_{q^*}, P_{q^*} : H^s(\Omega_q) \rightarrow H^{s+2}(\Omega_q), \ s \in \mathbb{R}, \ \chi \in X^1, \]

(3.54)

\[ : H^s(\Omega_q) \rightarrow H^{s+2}(\Omega_q), \ -\frac{1}{2} < s < \frac{2k-1}{2}, \ \chi \in X^k, \ k = 1, 3, \]

(3.55)
LBDIE for Transmission Problems with Interface Crack

\[ R_{q^*} : \tilde{H}^s(\Omega_q^*) \longrightarrow H^{s+k-1}(\Omega_q), \quad s \in \mathbb{R}, \quad \chi \in X^k, \quad k = 1, 2, 3, \quad (3.56) \]

\[ : H^s(\Omega_q) \longrightarrow H^{s+\frac{1}{2} - \varepsilon}(\Omega_q), \quad \frac{1}{2} \leq s, \quad \chi \in X^k, \quad k = 2, 3, \quad (3.57) \]

where \( \varepsilon \) is an arbitrarily small positive number.

\textbf{Proof.} Due to the equality (3.33) it suffices to prove the mapping properties in (3.54)–(3.55) only for the operator \( P_{q^*} \). The mapping property (3.54) is implied by the first relation in (3.53) and Theorem 3.2. Then (3.55) for \( k = 1 \) follows since in this case \( H^s(\Omega_q) = \tilde{H}^s(\Omega_q) \). Similarly, (3.56) is implied by the second relation in (3.53) and Theorem 3.4(i).

To show the property (3.55) for \( k = 2, 3 \) we proceed as follows. From (3.36) and (3.50), (3.51) we get

\[ \tilde{P}_{q^*}(\xi) = -\frac{1}{\xi \cdot a_{q^*} \xi} + \tilde{Q}_q(\xi), \quad \xi \in \mathbb{R}^3 \setminus \{0\}, \quad (3.58) \]

with

\[ \tilde{Q}_q(\xi) = -\frac{\tilde{R}_{q^*}(\xi)}{(\xi \cdot a_{q^*} \xi)^2} = \mathcal{O}(|\xi|^{-k-1}) \quad \text{as} \quad |\xi| \to \infty, \quad k = 1, 2, 3, \quad (3.59) \]

The first summand in (3.58), \( \tilde{P}_{q^*} := -1/(\xi \cdot a_{q^*} \xi) \), is the symbol of the pseudodifferential operator \( P_{q^*} \) of the volume Newton type potential without localization, based on the fundamental solution (3.2). Since the symbol is of rational type of order \(-2\) possessing the transmission property, \( P_{q^1} \) maps \( H^s(\Omega_q) \) into \( H^{s+2}(\Omega_q) \) for \( s > -\frac{1}{2} \) due to [2, Section 2] and Theorem 8.6.1 in [13]. More precisely,

\[ r_{\ell_0} P_{q^1} \ell_0 : H^s(\Omega_q) \longrightarrow H^{s+2}(\Omega_q) \quad \text{for} \quad s > -\frac{1}{2}, \quad (3.60) \]

where \( \ell_0 \) is an extension by zero operator from \( \Omega_q \) onto the compliment domain \( \Omega_q^* = \mathbb{R}^3 \setminus \Omega_q \).

Further, by (3.59) we see that the corresponding pseudodifferential operator \( r_{\ell_0} Q_q \) with symbol \( \tilde{Q}_q(\xi) \) has the following mapping properties

\[ r_{\ell_0} Q_q \ell_0 : H^s(\Omega_q) \longrightarrow H^{s+k+1}(\Omega_q) \quad \text{if} \quad -\frac{1}{2} < s < \frac{1}{2}, \quad (3.61) \]

\[ r_{\ell_0} Q_q \ell_0 : H^s(\Omega_q) \longrightarrow H^{s+\delta_0}(\Omega_q) \quad \text{if} \quad s \geq \frac{1}{2} \quad \text{for all} \quad s_0 < \frac{1}{2} + k + 1. \quad (3.62) \]

Therefore

\[ r_{\ell_0} (P_{q^1} + Q_q) \ell_0 : H^s(\Omega_q) \longrightarrow H^{s+k}(\Omega_q) \quad \text{for} \quad s > -\frac{1}{2}, \quad k = 2, 3, \quad (3.63) \]

where

\[ s_2 = s + 2 \quad \text{if} \quad -\frac{1}{2} < s < \frac{3}{2}, \quad s_2 = 3 + \frac{1}{2} - \varepsilon \quad \text{if} \quad s > \frac{3}{2}, \quad (3.64) \]

\[ s_3 = s + 2 \quad \text{if} \quad -\frac{1}{2} < s < \frac{5}{2}, \quad s_3 = 4 + \frac{1}{2} - \varepsilon \quad \text{if} \quad s > \frac{5}{2}; \]

here \( \varepsilon \) is an arbitrarily small positive number.
The following operators are continuous

\[ \mathcal{P}_{q^*} = r_{\Omega} (\mathcal{P}_{q^*1} + \mathcal{Q}_q) \ell_0 \] due to (3.58) and the property (3.55) follows.

Finally, the property (3.57) follows from (3.51) and (3.56) since for \( s \geq 1/2 \) we have \( H^s(\Omega_q) \subset H^t(\Omega_q) \) with arbitrary \( t \in (-1/2, 1/2) \).

With the help of (3.9), (3.19) and (3.21) we have

\[ R_q(x, y) = \frac{a_q(x)}{a_q(y)} R_q(x, y) + \frac{1}{a_q(y)} \nabla_x a_q(x) \cdot a_{q^n} \nabla_x p_{q^n}(x - y) = \]

\[ = \frac{a_q(x)}{a_q(y)} R_q(x, y) - \frac{1}{a_q(y)} \nabla_x a_q(x) \cdot a_{q^n} \nabla_y p_{q^n}(x - y), \quad (3.65) \]

and consequently we get the following representation for the operator \( R_q \),

\[ \mathcal{R}_q f(y) := \frac{1}{a_q(y)} \left[ \mathcal{R}_{q^*}(a_q f) - \sum_{k,j=1}^3 \frac{\partial}{\partial y_k} \mathcal{P}_{q^n}(f a_{k,j} \partial_j a_q) \right], \quad (3.66) \]

Therefore from Theorem 3.5 immediately follows

**Theorem 3.6.** The following operators are continuous

\[ \mathcal{R}_q : \tilde{H}^s(\Omega_q) \longrightarrow H^s(\Omega_q), \quad s \in \mathbb{R}, \quad \chi \in X^1, \quad (3.67) \]

\[ : H^s(\Omega_q) \longrightarrow H^{k-\frac{1}{2} - \epsilon}(\Omega_q), \quad \frac{1}{2} \leq s, \quad \chi \in X^k, \quad k = 2, 3, \quad (3.68) \]

where \( \epsilon \) is an arbitrarily small positive number.

In view of compactness of the imbedding \( H^s(\Omega_q) \subset H^t(\Omega_q) \) for \( s > t \) and bounded \( \Omega_q \) from Theorem 3.6 we obtain the following statement.

**Lemma 3.7.** The operators

\[ \mathcal{R}_q : H^1(\Omega_q) \longrightarrow H^t(\Omega_q), \quad t < \frac{3}{2}, \quad \chi \in X^2, \quad (3.69) \]

\[ \mathcal{R}_q \mathcal{R}_q : H^1(\Omega_q) \longrightarrow H^{t-\frac{1}{2}}(\partial \Omega_q), \quad t < \frac{3}{2}, \quad \chi \in X^2, \quad (3.70) \]

\[ T_q \mathcal{R}_q : H^1(\Omega_q) \longrightarrow H^{t-\frac{1}{2}}(\partial \Omega_q), \quad t < \frac{3}{2}, \quad \chi \in X^3, \quad (3.71) \]

are compact.

Now we study the mapping properties and jump relations of the localized layer potentials.

First of all let us note that for the single layer potential we have the following representation (cf. [7])

\[ V^{(q)}_\delta \psi(y) = -\langle \gamma_\delta P_{q^n}(\cdot - y), \psi \rangle_S = -\langle \psi, P_{q^n}(\cdot - y), \gamma_\delta^* \psi \rangle_{\mathbb{R}^3} = \]

\[ = -\left[ P_{q^n} \ast (\gamma_\delta^* \psi) \right](y) = -P_{q^n}(\gamma_\delta^* \psi)(y), \quad (3.72) \]

where \ast denotes the convolution operator. The operator \( \gamma_\delta^* \) is adjoint to the trace operator \( \gamma_\delta : H^t(\mathbb{R}^3) \longrightarrow H^{t-\frac{1}{2}}(S), \ t > 1/2, \ i.e., \) is defined by the
relation
\[
\langle \gamma^*_S \psi, h \rangle := \langle \psi, \gamma_S h \rangle \quad \text{for all } h \in H^t(\mathbb{R}^3), \ \psi \in H^{\frac{1}{2}-t}(S), \ \ t > \frac{1}{2}, \quad (3.73)
\]
and thus the operator
\[
\gamma^*_S : H^{\frac{1}{2}-t}(S) \rightarrow H^{-t}(\mathbb{R}^3), \ \ t > \frac{1}{2}
\]
is continuous. Since \( \gamma_S h = 0 \) for any \( h \in C^\infty(\mathbb{R}^3, S) \), then \( \gamma^*_S \psi \in S \), i.e. in fact the operator
\[
\gamma^*_S : H^{\frac{1}{2}-t}(S) \rightarrow H^{-t}_{S^\dagger} := \{ f \in H^{-t}(\mathbb{R}^3) : \ \text{supp} f \subseteq S \}
\]
is also continuous for \( t > \frac{1}{2} \).

Quite analogously, for the double layer potential we have the following representation
\[
W^{(q)}_{S^\dagger}(\varphi, h)_{\mathbb{R}^3} := \langle \varphi, T q^* S h \rangle_{\mathbb{R}^3} \quad \text{for any } h \in H^t(\mathbb{R}^3), \ \varphi \in H^{\frac{1}{2}-t}(S)
\]
and thus the operator
\[
T q^* S : H^{\frac{1}{2}-t}(S) \rightarrow H^{-t}(\mathbb{R}^3), \ \ t > \frac{3}{2}
\]
is continuous. Since \( T q^* S h = 0 \) for any \( h \in C^\infty(\mathbb{R}^3 \setminus S) \), then \( T q^* S \varphi \in S \), i.e. in fact the operator
\[
T q^* S : H^{\frac{1}{2}-t}(S) \rightarrow H^{-t}_{S^\dagger}
\]
is also continuous for \( t > \frac{3}{2} \).

**Theorem 3.8.** If \( \chi \in X^k, \ k = 2, 3 \), then the following operators are continuous
\[
V^{(q)}_{S^\dagger} : H^s(S) \rightarrow H^{s+k-\frac{1}{2}}(\Omega^S) \quad \text{for } s < k - 1,
\]
\[
A q^* V^{(q)}_{S^\dagger} : H^s(S) \rightarrow H^{s+k-\frac{1}{2}}(\Omega^S) \quad \text{for } s < 0,
\]
\[
A q^* V^{(q)}_{S^\dagger} : H^s(S) \rightarrow H^{-n+k-\frac{1}{2}}(\Omega^S) \quad \text{for } s \geq 0, \ \forall \epsilon > 0,
\]
\[
W^{(q)}_{S^\dagger} : H^s(S) \rightarrow H^{s+k-\frac{1}{2}}(\Omega^S) \quad \text{for } s < k - 1,
\]
\[
A q^* W^{(q)}_{S^\dagger} : H^s(S) \rightarrow H^{s+k-\frac{1}{2}}(\Omega^S) \quad \text{for } s < 0,
\]
\[
A q^* W^{(q)}_{S^\dagger} : H^s(S) \rightarrow H^{-n+k-\frac{1}{2}}(\Omega^S) \quad \text{for } s \geq 0, \ \forall \epsilon > 0,
\]
where \( \Omega^S \) is an interior or exterior domain bounded by \( S \).
For $\lambda \in X^k$, $k = 2, 3$, by Lemma 3.1 we have $\tilde{P}_q \in C(\mathbb{R}^3)$ and in view of (3.58) we have
\[ \tilde{P}_q(\xi) = -|\xi|^{-2} + \tilde{Q}_q(\xi), \tag{3.86} \]
where $\xi = d_\gamma^{-1}\xi \in \mathbb{R}^3 \setminus \{0\}$, and $\tilde{Q}_q(\xi)$ is defined in (3.59).

Note that the symbol of the localized operator $P_q$ is of neither classical nor rational type, in general. Therefore we can not apply directly the well known theorems for pseudodifferential operators with rational type symbols (see, e.g. [4], [13], [25]).

However, due to (3.72), ansatz (3.86) gives us possibility to represent the localized single layer potential $V^{(*)}_S(\psi)$ as
\[ V^{(*)}_S(\psi) = V^{(*)}_{S\lambda}(\psi) + Q_1^* \gamma_3^* \psi, \tag{3.87} \]
where $V^{(*)}_S(\psi)$ is the non-localized single layer potential constructed by the fundamental solution $P_{q1}(x - y)$,
\[ V^{(*)}_{S\lambda}(\psi) = - \int_S P_{q\lambda}(x - y)\psi(x) dS = - P_{q1}^* \gamma_3^* \psi, \tag{3.88} \]
where the symbol of the operator $P_{q1}$ is $-|\xi|^{-2}$, while $Q_1$ is pseudodifferential operator with the symbol $\tilde{Q}_q$.

The principal homogeneous symbol of the pseudodifferential operator $P_{q1}$ is rational function in $\xi$, and due to equality (3.88) and [4, Ch. 5, Theorem 2.4] (see also [13, Theorem 8.5.8]) we have
\[ \mu V^{(*)}_{S\lambda} : H^s(S) \rightarrow H^{s+\frac{1}{2}}(\Omega^S) \quad \text{for} \quad s \in \mathbb{R}, \quad \forall \mu \in C_{\text{comp}}(\Omega^S). \tag{3.89} \]
On the other hand, the asymptotic relation (3.59) and mapping property (3.74) imply continuity of the mapping
\[ \mu Q_1^* \gamma_3^* : H^s(S) \rightarrow H^{s+k+\frac{1}{2}}(\mathbb{R}^3) \quad \text{for} \quad s < 0, \quad \forall \mu \in C_{\text{comp}}(\mathbb{R}^3), \tag{3.90} \]
and thus also of the mapping
\[ \mu Q_1^* \gamma_3^* : H^s(S) \rightarrow H^{s+k+\frac{1}{2} - \epsilon}(\mathbb{R}^3) \quad \text{for} \quad s \geq 0, \quad \forall \mu \in C_{\text{comp}}(\mathbb{R}^3) \tag{3.91} \]
for $k = 2, 3$ and $\epsilon > 0$.

Let first $\Omega^S$ be a bounded domain. Then (3.80) follows from (3.87) by (3.89), (3.90) and (3.91). Since $A_q V^{(*)}_{S\lambda}(\psi) = 0$ in $\Omega^S$, we have, $A_q V^{(*)}_{S\lambda}(\psi) = A_q Q_1^* \gamma_3^* \psi$ in $\Omega^S$, which by (3.90) and (3.91) also implies (3.81) and (3.82).

Let now $\Omega^S$ be an unbounded domain. Let $\lambda \in C_{\text{comp}}(\mathbb{R}^3)$ be such that $\lambda(0) = 1$ and represent $\chi = \chi_0 + \chi_\infty$, where $\chi_0 = \lambda \chi$, $\chi_\infty = (1 - \lambda) \chi$. Then evidently $V^{(*)}_{S\lambda}(\psi)$ is represented in terms of the potentials with the localizing functions $\chi_0$ and $\chi_\infty$, respectively,
\[ V^{(*)}_{S\lambda}(\psi) = V^{(*)}_{S\chi_0}(\psi) + V^{(*)}_{S\chi_\infty}(\psi). \]

Let us analyze the potential $V^{(*)}_{S\chi_0}(\psi)$ first. Follow the same arguments as above, we split it in two parts as in (3.87) and arrive at the continuity of
the localized single and double layer potentials possess domains $\Omega$ implies the counterparts of mappings (3.80)–(3.82) for $\mu$, that is, $\mu$ can be dropped in the mappings similar to (3.89)–(3.91) for them. This implies the counterparts of mappings (3.80)–(3.82) for $V_{S^\Omega}^{(q)}$ in unbounded domains $\Omega^S$.

Let us now analyze the potential $V_{S^\Omega}^{(q)}(\psi)$. Since $\chi_\infty(0) = 0$ the term with $m = 0$ in the sum in the representation (3.36) for the symbol $\tilde{P}_{q\chi_\infty}$ of the corresponding volume potential $P_{q\chi_\infty}$ vanishes, and we have the estimate

$$|\tilde{P}_{q\chi_\infty}(\xi)| \leq c(1 + |\xi|^2)^{-k+1} \text{ for all } \xi \in \mathbb{R}^3 \text{ if } \chi \in X^k, \ k = 2, 3.$$  

This implies continuity of the mapping

$$V_{S^\Omega}^{(q)} = P_{q\chi_\infty} \gamma_q^\ast : H^s(S) \rightarrow H^{s+k+\frac{3}{2}}(\mathbb{R}^3) \text{ for } s < 0, \ k = 2, 3,$$

and thus also of the mapping

$$V_{S^\Omega}^{(q)} = P_{q\chi_\infty} \gamma_q^\ast : H^s(S) \rightarrow H^{k+\frac{3}{2}-\epsilon}(\mathbb{R}^3) \text{ for } s \geq 0, \ k = 2, 3, \ \forall \epsilon > 0,$$

which give the counterparts of mappings (3.80)–(3.82) for $V_{S^\Omega}^{(q)}$ and thus mappings (3.80)–(3.82) for $V^{(q)}$ in unbounded domains $\Omega^S$.

To show the mapping properties (3.83)–(3.85), we rewrite (3.23) in the form

$$W_{S^\Omega}^{(q)}(y) = \int_S \left[T_p(x, \partial_y)P_{q\ast}(x - y)\right]g(x)\ dS_x =$$

$$= \sum_{k,j}a_{kj}^{(q)} \frac{\partial}{\partial y_j} \int_S P_{q\ast}(x - y)[r_k^{(q)}(x)g(x)]\ dS_x =$$

$$= \sum_{k,j}a_{kj}^{(q)} \frac{\partial}{\partial y_j} W_{S^\Omega}^{(q)}(y), \ \forall \epsilon > 0, \ \forall \epsilon > 0.$$

(3.92)

Whence (3.83)–(3.85) follow from (3.80)–(3.82).

From Theorem 3.8 we have the following assertion.

**Theorem 3.9.** The localized single and double layer potentials possess the following mapping properties

$$V^{(q)} : H^{-\frac{1}{2}}(\partial \Omega_q) \rightarrow H^{1,0}(\Omega_q; A_q), \ \chi \in X^2.,$$

(3.93)

$$W^{(q)} : H^{\frac{1}{2}}(\partial \Omega_q) \rightarrow H^{1,0}(\Omega_q; A_q), \ \chi \in X^3.$$

(3.94)

Moreover, the operators

$$r_{S^\Omega}^\gamma 2V_{S^\Omega}^{(2)} : H^{-\frac{1}{2}}(S_i) \rightarrow H^{\frac{1}{2}}(S_i), \ \chi \in X^2.$$

(3.95)
In particular, for \( \alpha = 1 \) and \( S = \partial \Omega_q \) the following equalities hold
\[
T_q^+ W^{(q)} h - T_q^- W^{(q)} h = : L^{q} h, \quad h \in X^3, \quad q = 1, 2.
\] (3.104)

The following statement is implied by Theorems 3.10 and 3.9, and the relations (3.18).

**Theorem 3.11.** The following boundary operators are continuous,
\[
V_S^{(q)} : H^{-\frac{1}{2}}(S) \longrightarrow H^{1/2}(S), \quad \chi \in X^2,
\] (3.105)
\[
W_S^{(q)} : H^{-\frac{1}{2}}(S) \longrightarrow H^{1/2}(S), \quad \chi \in X^2,
\] (3.106)
\[
W_S^{(q)} : H^{1/2}(S) \longrightarrow H^{-1/2}(S), \quad \chi \in X^3,
\] (3.107)
\[
L_S^{(q)} : H^{1/2}(S) \longrightarrow H^{-1/2}(S), \quad \chi \in X^3.
\] (3.108)

Moreover, the operators (3.106) and (3.107) are compact.

**Proof.** The continuity of the operators (3.105)–(3.108) follows from the mapping properties (3.93)–(3.94). On the other hand, from the relations (3.18) it follows that the kernels of the integral operators \( W_S^{(q)} \) and \( W_S^{(q)} \) are weakly singular of type \( O(|x - y|^{-2+\alpha}) \). Therefore, \( W_S^{(q)} \) and \( W_S^{(q)} \) are compact.
pseudodifferential operators on $S$ of order $-\alpha < 0$ and possess the following mapping properties

$$W^q_{S}: H^{-\frac{1}{2}}(S) \rightarrow H^{\frac{1}{2}+\alpha}(S), \quad \chi \in X^2,$$

$$W^q_{S}: H^{\frac{3}{2}}(S) \rightarrow H^{\frac{3}{2}+\alpha}(S), \quad \chi \in X^3,$$

implying the compactness of the operators (3.106) and (3.107) due to the Rellich compact imbedding theorem. □

Taking $v(x) := P_q(x - y, y)$ and $u = u_q \in H^{1,0}(\Omega_q; A_q)$ in the second Green identity (2.8), by the standard limiting procedure (see, e.g., [23]), we obtain the following third Green identity based on the localized parametrix,

$$u_q + R_q u_q - W^q T_q u_q + W^q \gamma_q u_q = P_q A_q u_q \in \Omega_q. \quad (3.109)$$

Recall that for layer potentials we drop the subindex $S$ when $S = \partial \Omega_q$.

Taking in mind the properties of the localized potentials, the trace and co-normal derivative of (3.109) have the following form,

$$\frac{1}{2} \gamma_q u_q + \gamma_q R_q u_q - W^q T_q u_q + W^q \gamma_q u_q = \gamma_q P_q A_q u_q \text{ on } \partial \Omega_q, \quad (3.110)$$

$$\frac{1}{2} T_q u_q + T_q R_q u_q - W^q T_q u_q + L^q \gamma_q u_q = T_q P_q A_q u_q \text{ on } \partial \Omega_q. \quad (3.111)$$

Recall that $L^q_S := L^q_S^+ \neq L^q_S^-$ if $a_q$ is not a constant function (see Theorem 3.10).

With the help of these relations we will construct various types of localized boundary domain integral equation systems for the above formulated Dirichlet and mixed type transmission BVPs with and without crack.

4. Some Injectivity Results

Before formulating the boundary-domain integral equations, we present in this section some auxiliary lemmata which play a crucial role in our analysis.

**Lemma 4.1.** If $\chi \in X^k$, $k \geq 1$, and $s \geq -1$, then the operator

$$-P^q_{q_1} : H^s(\mathbb{R}^3) \rightarrow H^{-s}(\mathbb{R}^3), \quad q = 1, 2, \quad (4.1)$$

is positive, i.e.,

$$-\langle P^q_{q_1} g, g \rangle_{\mathbb{R}^3} > 0 \quad \forall \ g \in H^s(\mathbb{R}^3), \quad g \neq 0,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ denotes the duality brackets between the spaces $H^{-s}(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3)$.

**Proof.** The continuity of operator (4.1) is implied by Theorem 3.2. For any $g \in H^s(\mathbb{R}^3)$, $s \geq -1$, we have,

$$\langle P^q_{q_1} g, g \rangle_{\mathbb{R}^3} = \langle F^{-1} [P^q_{q_1} g], g \rangle_{\mathbb{R}^3} =$$
We set
\[ a = (2\pi)^{-3}(\tilde{P}_q, \tilde{g}, \tilde{g})_{\mathbb{R}^3} = (2\pi)^{-3} \int_{\mathbb{R}^3} \tilde{P}_q(x)\tilde{g}(x)^2 \, dx. \] (4.2)

By Lemma 3.1(ii) \( \tilde{P}_q(\xi) < 0 \) for a.e. \( \xi \in \mathbb{R}^3 \). Hence the conclusion. \( \Box \)

Throughout the rest of this section and in the main statements further on we assume that the following relation holds on \( S_i \)
\[ a_2(x) = xa_1(x) \quad \text{for} \quad x \in S_i, \quad x = \text{const} > 0. \] (4.3)

**Lemma 4.2.** Let \( \chi \in C_0^\infty, G_q \in H^0(\Omega_q), g_1 \in H^{\frac{1}{2}}(S_i), g_2 \in H^{\frac{1}{2}}(S_i), g_e \in H^{-\frac{1}{2}}(S_e) \) and condition (4.3) hold. Further let
\[ V^{(1)}_{S_i}(g_1) + W^{(1)}_{S_i}(a_1g_2) + P_a(G_1) = 0 \quad \text{in} \quad \Omega_1, \] (4.4)
\[ V^{(2)}_{S_i}(g_1) - W^{(2)}_{S_i}(a_2g_2) + V^{(2)}_{S_e}(g_e) + P_a(G_2) = 0 \quad \text{in} \quad \Omega_2. \] (4.5)
Then
\[ G_q = 0 \quad \text{in} \quad \Omega_q, \quad q = 1, 2, \quad g_1 = 0, \quad g_2 = 0 \quad \text{on} \quad S_i, \quad \text{and} \quad g_e = 0 \quad \text{on} \quad S_e. \] (4.6)

**Proof.** We set
\[ U_1 := V^{(1)}_{S_i}(g_1) + W^{(1)}_{S_i}(a_1g_2) + P_a(G_1) \quad \text{in} \quad \mathbb{R}^3 \setminus \partial \Omega_1, \] (4.7)
\[ U_2 := V^{(2)}_{S_i}(g_1) - W^{(2)}_{S_i}(a_2g_2) + V^{(2)}_{S_e}(g_e) + P_a(G_2) \quad \text{in} \quad \mathbb{R}^3 \setminus \partial \Omega_2. \] (4.8)

Due to (4.4) and (4.5),
\[ U_q = 0 \quad \text{in} \quad \Omega_q, \quad q = 1, 2. \] (4.9)

In view of the restrictions on the density functions \( G_q, g_q, q = 1, 2, \) and \( g_e \), and on the localizing function \( \chi \) and due to mapping properties (3.43), (3.93) and (3.94) we have
\[ U_q \in H^{1,0}(\mathbb{R}^3 \setminus \partial \Omega_q; A_q^\star). \] (4.10)

Then we can write the following Green’s formulas
\[
\begin{align*}
\int_{\mathbb{R}^3 \setminus \Omega_1} (A_1 U_1) U_1 \, dx + \int_{\Omega_1} E_{q^*} U_1 U_1 \, dx &= - (T^{1*}_{q^*} U_1, \gamma_1 U_1)_{S_i}, \quad \text{(4.11)} \\
\int_{\mathbb{R}^3 \setminus (\Omega_1 \cup \Omega_2)} (A_2 U_2) U_2 \, dx + \int_{\Omega_1} E_{2*} U_2 U_2 \, dx &= - (T^{2*}_{q^*} U_2, \gamma_2 U_2)_{S_i}, \quad \text{(4.12)} \\
\int_{\mathbb{R}^3 \setminus (\Omega_1 \cup \Omega_2)} (A_2 U_2) U_2 \, dx + \int_{\Omega_1} E_{2*} U_2 U_2 \, dx &= - (T^{2*}_{q^*} U_2, \gamma_2 U_2)_{S_e}, \quad \text{(4.13)}
\end{align*}
\]
where
\[ E_{q^*}(U_q, U_q) := \sum_{k,j=1}^3 a_{kj}^{(q)} \partial_k U_q \partial_j U_q \geq c |\nabla U_q|^2, \quad q = 1, 2, \] (4.14)

with some positive constant \( c > 0 \) due to the positive definiteness of the matrix \( a_{kj}^{(q)} = [a^{(q)}_{kj}]_{3 \times 3} \).
With the help of the jump relations and the mapping properties of the localized layer potentials (3.100)–(3.103) we get

\[
\gamma_1^+ U_1 - \gamma_1^- U_1 = -a_1 g_1, \quad \gamma_2^+ U_2 - \gamma_2^- U_2 = a_2 g_2 \text{ on } S_i,
\]
\[
T_3^+ U_1 - T_3^- U_1 = T_3^+ U_2 - T_3^- U_2 = g_1 \text{ on } S_i,
\]
\[
\gamma_2^+ U_2 = \gamma_2^- U_2 = 0, \quad T_3^+ U_2 - T_3^- U_2 = g_e \text{ on } S_e.
\]

Therefore, from (4.11)–(4.13) with the help of (4.3), (4.9) and (4.15) we derive

\[
x \int_{\mathbb{R}^3 \setminus \Pi_1} [(A_{1x} U_1) U_1 + E_{1x} (U_1, U_1)] \, dx + \int_{\mathbb{R}^3 \setminus \Pi_2} [(A_{2x} U_2) U_2 + E_{2x} (U_2, U_2)] \, dx = 0. \tag{4.16}
\]

Further we proceed as follows. Denote by \( \tilde{G}_q := \ell_{0q} G_q \in \tilde{H}^0(\Omega_q) \) the extensions of the functions \( G_q \) onto the whole of \( \mathbb{R}^3 \) by zero. Then clearly \( S_q \cdot G_q = \tilde{G}_q \) and in view of formulas (3.72), (3.76) we can rewrite (4.7) and (4.8) as

\[
U_q = \mathbf{P} q \cdot F_q \text{ in } \mathbb{R}^3, \quad q = 1, 2, \tag{4.17}
\]

in the distributional sense, where the distributions \( F_1 \) and \( F_2 \) on \( \mathbb{R}^3 \) read as

\[
F_1 = \tilde{G}_1 - \gamma_{S_i} g_1 - T_{3S_i} (a_1 g_2),
\]
\[
F_2 = \tilde{G}_2 - \gamma_{S_i} g_1 + T_{3S_i} (a_2 g_2) - \gamma_{S_e} g_e,
\]

and thus \( F_q \in \tilde{H}^{-2}(\Omega_q) \) by (3.75) and (3.79). Whence in view of (3.20) we have

\[
A_{qx} U_q = F_q + R_{qx} F_q = R_{qx} F_q \text{ in } \mathbb{R}^3 \setminus \Pi_q, \tag{4.19}
\]

and \( R_{qx} F_q \in H^0(\mathbb{R}^3) \) by Theorem 3.4. Consequently, from (4.16) we derive

\[
\sum_{q=1}^{2} \omega_q \int_{\mathbb{R}^3 \setminus \Pi_q} [(R_{qx} F_q)(\mathbf{P} q \cdot F_q) + E_{qx} (U_q, U_q)] \, dx = 0, \tag{4.20}
\]

where \( \omega_1 = x \) and \( \omega_2 = 1 \).

Keeping in mind that \( \mathbf{P} q \cdot F_q \in H^0(\mathbb{R}^3) \) and \( \mathbf{P} q \cdot F_q = U_q = 0 \) in \( \Omega_q \), we can extend the integration to the whole space \( \mathbb{R}^3 \) and apply Parseval’s formula to obtain

\[
\int_{\mathbb{R}^3 \setminus \Pi_q} (R_{qx} F_q)(\mathbf{P} q \cdot F_q) \, dx = \int_{\mathbb{R}^3} (R_{qx} F_q)(\mathbf{P} q \cdot F_q) \, dx =
\]
\[
= \int_{\mathbb{R}^3} R_{qx} \mathbf{P} q \cdot F_q \mathbf{F} q \, d\xi \geq 0 \tag{4.21}
\]
As in the proof of Lemma 4.2 here we set

\[ U_1 = C_1 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}, \quad U_2 = C_2 \text{ in } \mathbb{R}^3 \setminus (\overline{\Omega} \cup \overline{\Omega_2}), \]

and taking into account that \( C_3 = r_s \gamma \overline{U}_2 = 0 \). Thus \( U_\eta = 0 \) in \( \mathbb{R}^3 \setminus \overline{\Omega} \) and in view of (4.17) we have \( U_\eta = 0, q = 1, 2, \) in \( \mathbb{R}^3 \).

Now taking jumps of traces and co-normal derivatives of (4.7) and (4.8) on \( \partial \Omega_1 \) and \( \partial \Omega_2 \), respectively, gives \( g_1 = 0 \) and \( g_2 = 0 \) on \( S_i \), and \( g_e = 0 \) on \( S_e \) (see (4.15)). Finally Lemma 4.1 implies \( g_\eta = 0 \) in \( \mathbb{R}^3 \).

**Lemma 4.3.** Let \( \chi \in X^{3, 2}_i \), \( G_\eta \in H^0(\Omega_\eta) \), \( g_i \in H^{-\frac{1}{2}}(S_i), g_{i2} \in H^{\frac{1}{2}}(S_i) \), \( g_e \in H^{\frac{1}{2}}(S_e) \) and condition (4.3) hold. Further let

\[
V^{(1)}_{s_i}(g_1) + W^{(1)}_{s_i}(a_1 g_{i2}) + P_{\nu}(G_1) = 0 \text{ in } \Omega_1, \tag{4.22}
\]

\[
V^{(2)}_{s_i}(g_1) - W^{(2)}_{s_i}(a_2 g_{i2}) + W^{(2)}_{s_i}(g_e) + P_{\nu}(G_2) = 0 \text{ in } \Omega_2. \tag{4.23}
\]

Then \( G_\eta = 0 \) in \( \Omega_\eta \), \( q = 1, 2, \ g_1 = 0, \ g_{i2} = 0 \) on \( S_i \), and \( g_e = 0 \) on \( S_e \). \( \tag{4.24} \)

**Proof.** As in the proof of Lemma 4.2 here we set

\[
U_1 := V^{(1)}_{s_i}(g_1) + W^{(1)}_{s_i}(a_1 g_{i2}) + P_{\nu}(G_1) \text{ in } \mathbb{R}^3 \setminus S_i, \tag{4.25}
\]

\[
U_2 := V^{(2)}_{s_i}(g_1) - W^{(2)}_{s_i}(a_2 g_{i2}) + W^{(2)}_{s_i}(g_e) + P_{\nu}(G_2) \text{ in } \mathbb{R}^3 \setminus (S_i \cup S_e). \tag{4.26}
\]

Again, by the assumptions stated in the lemma and the mapping properties of the localized volume and surface potentials we have

\[
U_\eta \in H^{1, 0}(\mathbb{R}^3 \setminus \partial \Omega_\eta; A_{\eta}^*), \tag{4.27}
\]

and we can write Green’s formulas (4.11)–(4.13). By relations

\[
\gamma_1^+ U_1 - \gamma_1^- U_1 = a_1 g_{i2}, \quad \gamma_2^+ U_2 - \gamma_2^- U_2 = a_2 g_{i2} \text{ on } S_i, \tag{4.28}
\]

\[
T_{s_i}^+ U_1 - T_{s_i}^- U_1 = T_{s_i}^+ U_2 - T_{s_i}^- U_2 = g_1 \text{ on } S_i, \tag{4.29}
\]

\[
\gamma_2^+ U_2 - \gamma_2^- U_2 = -g_e, \quad T_{s_e}^+ U_2 = T_{s_e}^- U_2 = 0 \text{ on } S_e, \tag{4.30}
\]

and taking into account that \( U_\eta = 0 \) in \( \Omega_\eta \) along with the relation (4.3), we arrive at the formula (4.16). By the word for word arguments from the proof of Lemma 4.2 we complete the proof. \( \square \)

**Lemma 4.4.** Let \( \chi \in X^{3, 2}_i \), \( G_\eta \in H^0(\Omega_\eta) \), \( g_i \in H^{-\frac{1}{2}}(S_i), g_{i2} \in H^{\frac{1}{2}}(S_i), g_e \in H^{-\frac{1}{2}}(S_e), g_{eD} \in H^{\frac{1}{2}}(S_e), g_{eN} \in H^\frac{1}{2}(S_e) \) and condition (4.3) hold. Further let

\[
V^{(1)}_{s_i}(g_1) + W^{(1)}_{s_i}(a_1 g_{i2}) + P_{\nu}(G_1) = 0 \text{ in } \Omega_1, \tag{4.29}
\]

\[
V^{(2)}_{s_i}(g_1) - W^{(2)}_{s_i}(a_2 g_{i2}) + V^{(2)}_{s_i}(g_{eD}) + W^{(2)}_{s_i}(g_{eN}) + P_{\nu}(G_2) = 0 \text{ in } \Omega_2. \tag{4.30}
\]

Then \( G_\eta = 0 \) in \( \Omega_\eta \), \( q = 1, 2, \ g_1 = 0 \) and \( g_{i2} = 0 \) on \( S_i \), \( g_{eD} = 0 \) and \( g_{eN} = 0 \) on \( S_e \).
Proof. As in the proof of Lemma 4.2 here we set
\[ U_1 := V_{s_1^*}(g_{11}) + W_{s_1^*}(a_1 g_{22}) + \mathcal{P}_{s_1^*}(G_1) \quad \text{in} \quad \mathbb{R}^3 \setminus S_1, \]
\[ U_2 := V_{s_2^*}(g_{11}) - W_{s_1^*}(a_2 g_{22}) + V_{s_2^*}(g_D) + W_{s_2^*}(g_{2N}) + \mathcal{P}_{s_2^*}(G_2) \quad \text{in} \quad \mathbb{R}^3 \setminus (S_1 \cup S_2). \]
(4.31)
(4.32)
Again, in view of the assumptions stated in the lemma and with the help of the mapping properties of the localized volume and surface potentials we have
\[ U_q \in H^{1,0}(\mathbb{R}^3 \setminus \partial \Omega_q; A_{q*}), \]
(4.33)
and we can write Green’s formulas (4.11)–(4.13). By relations
\[ \gamma_1^1 U_1 - \gamma_2^1 U_1 = a_1 g_{22}, \quad \gamma_2^2 U_2 - \gamma_2^2 U_2 = a_2 g_{22} \quad \text{on} \quad S_1, \]
\[ T_{s_1}^* U_1 - T_{s_1}^* U_1 = T_{s_2}^* U_2 - T_{s_2}^* U_2 = g_{11} \quad \text{on} \quad S_1, \]
\[ \gamma_2^2 U_2 - \gamma_2^2 U_2 = -g_{2N}, \quad T_{s_1}^* U_2 - T_{s_2}^* U_2 = g_D \quad \text{on} \quad S_e, \]
\[ r_{s_{eD}}^2 \gamma_2^2 U_2 = r_{s_{eD}}^2 \gamma_2^2 U_2 = 0 \quad \text{on} \quad S_{eD}, \]
\[ r_{s_{eN}}^2 \gamma_2^2 U_2 = r_{s_{eN}}^2 T_{s_e}^* U_2 = 0 \quad \text{on} \quad S_{eN}, \]
(4.34)
and taking into account that \( U_q = 0 \) in \( \Omega_q \) along with the relation (4.3), we easily arrive at the formula (4.16). By the word for word arguments applied in the proof of Lemma 4.2 we complete the proof. \( \square \)

Lemma 4.5. Let \( \chi \in X^3_{s_1}, \) condition (4.3) hold and
\[ G_q \in H^0(\Omega_q), \quad g_{11} \in H^{-\frac{1}{2}}(S_1^{(1)}), \quad g_{22}, g_{33} \in H^{\frac{1}{2}}(S_1), \]
\[ g_{22} - g_{33} \in H^{-\frac{1}{2}}(S_e^{(c)}), \quad g_c \in H^{-\frac{1}{2}}(S_c). \]
Further let
\[ V_{s_1^*}(g_{11}) + W_{s_1^*}(a_1 g_{22}) + \mathcal{P}_{s_1^*}(G_1) = 0 \quad \text{in} \quad \Omega_1, \]
\[ -V_{s_2^*}(g_{11}) + W_{s_2^*}(a_2 g_{22}) + V_{s_2^*}(g_D) + W_{s_2^*}(g_{2N}) + \mathcal{P}_{s_2^*}(G_2) = 0 \quad \text{in} \quad \Omega_2. \]
(4.35)
(4.36)
Then \( g_{11} = g_{22} = g_{33} = 0 \) on \( S_1, \)
\( g_c = 0 \) on \( S_c \) and \( G_q = 0 \) in \( \Omega_q, \) \( q = 1, 2. \)

Proof. Introduce the functions
\[ U_1 := V_{s_1^*}(g_{11}) + W_{s_1^*}(a_1 g_{22}) + \mathcal{P}_{s_1^*}(G_1) \quad \text{in} \quad \mathbb{R}^3 \setminus S_1, \]
\[ U_2 := -V_{s_2^*}(g_{11}) + W_{s_2^*}(a_2 g_{22}) + V_{s_2^*}(g_D) + \mathcal{P}_{s_2^*}(G_2) \quad \text{in} \quad \mathbb{R}^3 \setminus (S_1 \cup S_2). \]
(4.37)
(4.38)
Clearly \( U_q = 0 \) in \( \Omega_q, \) \( q = 1, 2. \) Denote again by \( \hat{G}_q := \ell_{u_q} G_q \in \tilde{H}^0(\Omega_q) \) the extensions of the functions \( G_q \) by zero on the whole of \( \mathbb{R}^3. \) Then \( \mathcal{P}_{q*} \hat{G}_q = \mathcal{P}_{q*} G_q, \) \( q = 1, 2. \)
In view of the assumptions stated in the lemma and with the help of the mapping properties of the localized volume and surface potentials we have
\[ U_q \in H^{1,0}(\mathbb{R}^3 \setminus \partial \Omega_q; A_{q*}), \quad q = 1, 2. \]
(4.39)
Therefore we can write Green’s formulas (4.11)–(4.13). Note that with the help of the jump relations of the localized layer potentials we get
\[\gamma_1^+ U_1 - \gamma_1^- U_1 = -a_1 g_{i1}, \quad \gamma_2^+ U_2 - \gamma_2^- U_2 = -a_2 g_{i3}\] on \(S_i\),
\[T_{1*}^+ U_1 - T_{1*}^- U_1 = g_{i1}, \quad T_{2*}^+ U_2 - T_{2*}^- U_2 = g_{i1}\] on \(S_i\),
\[\gamma_2^+ U_2 = \gamma_2^- U_2 = 0, \quad T_{2*}^+ U_2 - T_{2*}^- U_2 = g_c\] on \(S_c\).

Thus from (4.11)–(4.13) due to the lemma hypothesis and (4.3), we derive
\[\frac{1}{2} \int \left[ (A_1, U_1) + E_{1*} (U_1, U_1) \right] dx + \int \left[ (A_2, U_2) + E_{2*} (U_2, U_2) \right] dx = 0 \text{ on } \Omega\]
\[= - \langle g_{i1}, \gamma_1 U_1 \rangle_{\Omega} + \langle g_{i1}, a_2 g_{i3} \rangle_{\Omega} = 0. \quad (4.41)\]

Now, applying the same arguments as in the proof of Lemma 4.2 we conclude \(U_1 = C_1\) in \(\mathbb{R}^3 \setminus \Omega_1\), \(U_2 = C_2\) in \(\mathbb{R}^3 \setminus (\Omega_1 \cup \Omega_2)\), and \(U_3 = C_3\) in \(\Omega_3\), where \(C_j, j = 1, 2, 3\), are arbitrary constants. Since \(U_q \in H^1(\mathbb{R}^3 \setminus (\Omega_1 \cup \Omega_2))\), we get \(C_1 = C_2 = 0\), implying \(U_1 = 0\) in \(\mathbb{R}^3\) and \(U_2 = 0\) in \(\mathbb{R}^3 \setminus \Omega_1\). Consequently, \(g_{i2} = 0\) on \(S_i\).

Further, since \(g_{i2} - g_{i3} \in \tilde{H}^\frac{1}{2}(S_i^{(c)})\), from the second equation in (4.40) we derive
\[r_{s(i)} (\gamma_2 U_2) = r_{s(i)} (a_2 g_{i3}) = 0 \text{ on } S_i^{(t)}.\]
Then it follows that \(C_3 = r_{s(i)} \gamma_2 U_2 = 0\). Thus \(U_q = 0\) in \(\mathbb{R}^3, q = 1, 2\), and the relations (4.40) and Lemma 4.1 complete the proof. \(\square\)

In view of formulas (3.31)–(3.33) the above lemmata lead to the following corollaries.

**Corollary 4.6.** Let \(\chi \in X_{i*}^3, G_q \in H^0(\Omega_q), g_{i1} \in H^{-\frac{1}{2}}(S_i), g_{i2} \in H^\frac{1}{2}(S_i), g_c \in H^\frac{1}{2}(S_c)\), and condition (4.3) hold. Further let
\[V_{s_i}^{(1)} (g_{i1}) + W_{s_i}^{(1)} (g_{i2}) + P_1 (G_1) = 0 \text{ in } \Omega_1, \quad (4.42)\]
\[V_{s_i}^{(2)} (g_{i1}) - W_{s_i}^{(2)} (g_{i2}) + V_{s_c}^{(2)} (g_c) + P_2 (G_2) = 0 \text{ in } \Omega_2. \quad (4.43)\]
Then \(g_{i1} = 0, g_{i2} = 0\) on \(S_i\), \(g_c = 0\) on \(S_c\) and \(G_2 = 0\) in \(\Omega_q, q = 1, 2\).

**Corollary 4.7.** Let \(\chi \in X_{i*}^3, G_q \in H^0(\Omega_q), g_{i1} \in H^{-\frac{1}{2}}(S_i), g_{i2} \in H^\frac{1}{2}(S_i), g_c \in H^\frac{1}{2}(S_c)\) and condition (4.3) hold. Further let
\[V_{s_i}^{(1)} (g_{i1}) + W_{s_i}^{(1)} (g_{i2}) + P_1 (G_1) = 0 \text{ in } \Omega_1, \quad (4.44)\]
\[V_{s_i}^{(2)} (g_{i1}) - W_{s_i}^{(2)} (g_{i2}) + W_{s_c}^{(2)} (g_c) + P_2 (G_2) = 0 \text{ in } \Omega_2. \quad (4.45)\]
Then \(g_{i1} = 0, g_{i2} = 0\) on \(S_i\), \(g_c = 0\) on \(S_c\) and \(G_2 = 0\) in \(\Omega_q, q = 1, 2\).


Let \( \chi \in X^0_{\mathrm{ec}}, \) \( G_q \in H^0(\Omega_q), \) \( g_{i1} \in H^{-\frac{1}{2}}(S_i), \) \( g_{i2} \in H^{-\frac{1}{2}}(S_i), \) \( g_{eD} \in \tilde{H}^{-\frac{1}{2}}(S_{eD}), \) \( g_{eN} \in \tilde{H}^{\frac{1}{2}}(S_{eN}), \) and condition (4.3) hold. Further let
\[
V^{(1)}_{s_i}(g_{i1}) + W^{(1)}_{s_i}(g_{i2}) + P_1(G_1) = 0 \quad \text{in} \quad \Omega_1, \tag{4.46}
\]
\[
V^{(2)}_{s_i}(g_{i1}) - W^{(2)}_{s_i}(g_{i2}) + V^{(2)}_{s_e}(g_{eD}) + W^{(2)}_{s_e}(g_{eN}) + P_2(G_2) = 0 \quad \text{in} \quad \Omega_2. \tag{4.47}
\]
Then \( g_{i1} = 0 \) and \( g_{i2} = 0 \) on \( S_i, \) \( g_{eD} = 0, \) \( g_{eN} = 0 \) on \( S_e \) and \( G_q = 0 \) in \( \Omega_q, \) \( q = 1, 2. \)

**Corollary 4.8.** Let \( \chi \in X^0_{\mathrm{ec}}, \) \( G_q \in H^0(\Omega_q), \) \( g_{i1} \in H^{-\frac{1}{2}}(S_i), \) \( g_{i2} \in H^{-\frac{1}{2}}(S_i), \) \( g_{eD} \in \tilde{H}^{-\frac{1}{2}}(S_{eD}), \) \( g_{eN} \in \tilde{H}^{\frac{1}{2}}(S_{eN}), \) and condition (4.3) hold. Further let
\[
V^{(1)}_{s_i}(g_{i1}) + W^{(1)}_{s_i}(g_{i2}) + P_1(G_1) = 0 \quad \text{in} \quad \Omega_1, \tag{4.48}
\]
\[
-V^{(2)}_{s_i}(g_{i1}) + W^{(2)}_{s_i}(g_{i2}) + V^{(2)}_{s_e}(g_e) + P_2(G_2) = 0 \quad \text{in} \quad \Omega_2. \tag{4.49}
\]
Then \( g_{i1} = g_{i2} = g_{i3} = 0 \) on \( S_i, \) \( g_e = 0 \) on \( S_e \) and \( G_q = 0 \) in \( \Omega_q, \) \( q = 1, 2. \)

5. **LBDIE Systems for the Transmission-Dirichlet Problem**

Let a pair \( (u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \) be a solution to the transmission Dirichlet problem (2.9)–(2.12), i.e., Problem (TD). Assume that the problem right hand sides satisfy the imbeddings
\[
\varphi_{i0} \in H^\frac{1}{2}(S_i), \quad \psi_{i0} \in H^{-\frac{1}{2}}(S_i), \quad \varphi_{e0} \in H^{\frac{1}{2}}(S_e), \quad f_q \in H^0(\Omega_q), \quad q = 1, 2. \tag{5.1}
\]
Let us introduce the following combinations of the unknown boundary functions
\[
\psi_i = \frac{1}{2}(T_1 u_1 - T_2 u_2), \quad \varphi_i = \frac{1}{2}(\gamma_1 u_1 + \gamma_2 u_2), \quad \psi_e = T_2 u_2. \tag{5.2}
\]
Then evidently \( \psi_i \in H^{-\frac{1}{2}}(S_i), \) \( \varphi_i \in H^{\frac{1}{2}}(S_i), \) \( \psi_e \in H^{-\frac{1}{2}}(S_e). \)

5.1. **LBDIE system (TD1).** Let us introduce the vector function
\[
U^{(TD)} := (u_1, u_2, \psi_i, \varphi_i, \psi_e) \in \mathbb{H}^{(TD)}, \tag{5.3}
\]
where
\[
\mathbb{H}^{(TD)} := H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{-\frac{1}{2}}(S_i) \times H^{\frac{1}{2}}(S_i) \times H^{-\frac{1}{2}}(S_e). \tag{5.4}
\]
and assume formally that the components of \( U^{(TD)} \) are unrelated to each other (i.e., segregated).

Further, let us employ the third Green identities (3.109) in \( \Omega_1 \) and \( \Omega_2, \) difference of their traces (3.110) and sum of their co-normal derivatives (3.111) on \( S_i, \) and also the trace (3.110) on \( S_e. \)
Then after substituting transmission and boundary conditions (2.10)–(2.12) and notations (5.2) we arrive at the following system of direct segregated LBDIEs (TD1) for the components of the vector function $U^{(TD)} = (u_1, u_2, \psi_1, \varphi_1, \varphi_e)$,

$$
\begin{align*}
&u_1 + R_1 u_1 - V_{s_1}^{(1)} \psi_i + W_{s_1}^{(1)} \varphi_i = F_1^{(TD)} \text{ in } \Omega_1, \\
&u_2 + R_2 u_2 + V_{s_1}^{(2)} \psi_i + W_{s_1}^{(2)} \varphi_i - V_{s_e}^{(2)} \varphi_e = F_2^{(TD)} \text{ in } \Omega_2,
\end{align*}
$$

(5.5)

$$
\begin{align*}
&\gamma_1 R_1 u_1 - \gamma_2 R_2 u_2 - (V_{s_1}^{(1)} + V_{s_2}^{(2)}) \psi_i + (W_{s_1}^{(1)} - W_{s_2}^{(2)}) \varphi_i + \gamma_2 V_{s_e}^{(2)} \varphi_e = \\
&= \gamma_1 F_1^{(TD)} - \gamma_2 F_2^{(TD)} - \varphi_{0_1} \text{ on } S_1,
\end{align*}
$$

(5.6)

$$
\begin{align*}
&T_1 R_1 u_1 + T_2 R_2 u_2 - (V_{s_1}^{(1)} - W_{s_1}^{(2)}) \psi_i + (C_{s_1}^{(1)} + C_{s_2}^{(2)}) \varphi_i - T_2 V_{s_e}^{(2)} \varphi_e = \\
&= T_1 F_1^{(TD)} + T_2 F_2^{(TD)} - \psi_{0_1} \text{ on } S_1,
\end{align*}
$$

(5.7)

$$
\begin{align*}
&\gamma_2 R_2 u_2 + \gamma_2 V_{s_2}^{(1)} \psi_i + \gamma_2 W_{s_2}^{(2)} \varphi_i - V_{s_e}^{(2)} \psi_e = \gamma_2 F_2^{(TD)} - \psi_{0_e} \text{ on } S_e,
\end{align*}
$$

(5.8)

where

$$
F_1^{(TD)} = p f_1 + \frac{1}{2} V_{s_1}^{(1)} \varphi_{0_1} - \frac{1}{2} W_{s_1}^{(1)} \varphi_{0_1},
$$

(5.9)

$$
F_2^{(TD)} = p f_2 + \frac{1}{2} V_{s_2}^{(1)} \varphi_{0_1} + \frac{1}{2} W_{s_2}^{(1)} \varphi_{0_1} - W_{s_e}^{(2)} \varphi_{0_e}.
$$

(5.10)

If we introduce the notation

$$
K^{(TD)} = [K_{kj}^{(TD)}]_{5 \times 5} := \text{diag}(r_{0_1}, r_{0_2}, r_{0_1}, r_{0_2}, r_{0_e}) \times \\
\begin{bmatrix}
I + R_1 & 0 & -V_{s_1}^{(1)} & W_{s_1}^{(1)} & 0 \\
0 & I + R_2 & V_{s_1}^{(2)} & W_{s_1}^{(2)} & -V_{s_2}^{(2)} \\
\gamma_1 R_1 & -\gamma_2 R_2 & -V_{s_1}^{(1)} - V_{s_2}^{(2)} & W_{s_1}^{(1)} - W_{s_2}^{(2)} & \gamma_2 V_{s_2}^{(2)} \\
T_1 R_1 & T_2 R_2 & -W_{s_2}^{(1)} + W_{s_2}^{(2)} & C_{s_1}^{(1)} + C_{s_2}^{(2)} & -T_2 V_{s_2}^{(2)} \\
0 & \gamma_2 R_2 & \gamma_2 V_{s_2}^{(1)} & \gamma_2 W_{s_2}^{(2)} & -V_{s_e}^{(2)}
\end{bmatrix}
$$

(5.11)

the LBDIEs system (5.5)–(5.9) can be rewritten as

$$
K^{(TD)} U^{(TD)} = F^{(TD)},
$$

(5.12)

where $U^{(TD)} \in H^{(TD)}$ is the unknown vector, while $F^{(TD)} \in F^{(TD)}$ is the known vector generated by the right hand side functions in (5.5)–(5.9) and

$$
F^{(TD)} := H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{\frac{1}{2}}(S_I) \times H^{-\frac{1}{2}}(S_E) \times H^{\frac{1}{2}}(S_e).
$$

5.2. LBDIE system (TD2). Alternatively, let us employ the third Green identities (3.109) in $\Omega_1$ and $\Omega_2$, difference of their co-normal derivatives (3.111) on $S$, and sum of their traces (3.110), and also the co-normal derivative (3.111) on $S_e$. Then after substituting transmission and boundary conditions (2.10)–(2.12) and notations (5.2) we arrive at the following system

$$
\begin{align*}
&u_1 + R_1 u_1 - V_{s_1}^{(1)} \psi_i + W_{s_1}^{(1)} \varphi_i = F_1^{(TD)} \text{ in } \Omega_1, \\
&u_2 + R_2 u_2 + V_{s_1}^{(2)} \psi_i + W_{s_1}^{(2)} \varphi_i - V_{s_e}^{(2)} \varphi_e = F_2^{(TD)} \text{ in } \Omega_2,
\end{align*}
$$

(5.13)
of direct segregated LBDIEs (TD2) of the second kind for the components of the vector function $U^{(TD)} = (u_1, u_2, \varphi_1, \psi_e) \in \mathbb{H}^{(TD)}$,

$$u_1 + \mathcal{R}_1 u_1 - V^{(1)}_{s_i} \varphi_1 + W^{(1)}_{s_i} \varphi_1 = e^{(TD)}_1 \text{ in } \Omega_1,$$  \hspace{1cm} (5.14)

$$u_2 + \mathcal{R}_2 u_2 + V^{(2)}_{s_i} \varphi_1 + W^{(2)}_{s_i} \varphi_1 - V^{(2)}_{s_e} \psi_e = e^{(TD)}_2 \text{ in } \Omega_2,$$  \hspace{1cm} (5.15)

$$\psi_1 + T_1 \mathcal{R}_1 u_1 - T_2 \mathcal{R}_2 u_2 - (W^{(1)}_{s_i} + W^{(2)}_{s_i}) \psi_1 + (L^{(1)}_{s_i} - L^{(2)}_{s_i}) \varphi_1 + T_2 V^{(2)}_{s_e} \psi_e =$$

$$\hspace{5cm} = T_1 F_1^{(TD)} - T_2 F_2^{(TD)} \text{ on } S_i,$$  \hspace{1cm} (5.16)

$$\varphi_1 + \gamma_1 \mathcal{R}_1 u_1 + \gamma_2 \mathcal{R}_2 u_2 - (V^{(1)}_{s_i} - V^{(2)}_{s_i}) \psi_1 + (W^{(1)}_{s_i} + W^{(2)}_{s_i}) \varphi_1 - \gamma_2 V^{(2)}_{s_e} \psi_e =$$

$$\hspace{5cm} = \gamma_1 F_1^{(TD)} + \gamma_2 F_2^{(TD)} \text{ on } S_i,$$  \hspace{1cm} (5.17)

$$\frac{1}{2} \psi_e + T_2 \mathcal{R}_2 u + T_2 V^{(2)}_{s_i} \psi_1 + T_2 V^{(2)}_{s_e} \varphi_1 - W^{(2)}_{s_e} \psi_e = T_2 F_2^{(TD)} \text{ on } S_e,$$  \hspace{1cm} (5.18)

where $F_1^{(TD)}$, $F_2^{(TD)}$ are given by (5.10), (5.11).

If we introduce the notations

$$K^{(TD2)} = [K^{(TD2)}_{ij}]_{5 \times 5} := \text{diag}(r_{i1}, r_{i2}, s_{i1}, s_{i2}, r_{se}) \times$$

$$\begin{bmatrix}
I + \mathcal{R}_1 & 0 & -V^{(1)}_{s_i} & W^{(1)}_{s_i} & 0 \\
0 & I + \mathcal{R}_2 & V^{(2)}_{s_i} & W^{(2)}_{s_i} & -V^{(2)}_{s_e} \\
T_1 \mathcal{R}_1 & -T_2 \mathcal{R}_2 & I - W^{(1)}_{s_i} & W^{(2)}_{s_i} & c^{(1)}_{s_i} - c^{(2)}_{s_i} + T_2 V^{(2)}_{s_e} \\
\gamma_1 \mathcal{R}_1 & \gamma_2 \mathcal{R}_2 & -V^{(1)}_{s_i} + V^{(2)}_{s_i} & I + W^{(1)}_{s_i} + W^{(2)}_{s_i} & -\gamma_2 V^{(2)}_{s_e} \\
T_2 \mathcal{R}_2 & T_2 V^{(2)}_{s_i} & T_2 V^{(2)}_{s_e} & T_2 V^{(2)}_{s_i} & \frac{1}{2} I - W^{(2)}_{s_e}
\end{bmatrix},$$  \hspace{1cm} (5.19)

the LBDIEs system (5.14)–(5.18) can be rewritten as

$$K^{(TD2)} U^{(TD)} = \mathcal{F}^{(TD2)},$$  \hspace{1cm} (5.20)

where $U^{(TD)} \in \mathbb{H}^{(TD)}$ is the unknown vector, while $\mathcal{F}^{(TD2)} \in \mathbb{F}^{(TD2)}$ is the known vector generated by the right hand side functions in (5.14)–(5.18) and

$$\mathbb{F}^{(TD2)} := H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{-\frac{1}{2}}(S_i) \times H^{\frac{1}{2}}(S_e) \times H^{-\frac{1}{2}}(S_e).$$

5.3. Main theorems for LBDIE systems (TD1) and (TD2). There holds the following equivalence theorem.

**Theorem 5.1.** Let conditions (5.1) hold and $\chi \in X^3_{\text{a}}$.

(i) If a pair $(u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$ solves the Problem (TD), then the vector $U^{(TD)} \in \mathbb{H}^{(TD)}$ given by (5.3), where $\psi_1$, $\varphi_1$, and $\psi_e$ are defined by (5.2), solves both LBDIE systems (TD1) and (TD2).

(ii) Vice versa, if a vector $U^{(TD)} \in \mathbb{H}^{(TD)}$ solves LBDIE system (TD1) or LBDIE system (TD2) and condition (4.3) holds, then $(u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$ solves Problem (TD) and relations (5.2) hold.
Proof. Claim (i) immediately follows from the deduction of (TD1) and (TD2).

Now, let a vector \( U^{(TD)} \in H^{(TD)} \) solves LBDIE system (TD1). Subtracting from equation (5.7) the trace \( \gamma_1 \) of equation (5.5) and adding the trace \( \gamma_2 \) of equation (5.6), we prove (2.10). Similarly, subtracting from equation (5.8) the co-normal derivative \( T_1 \) of equation (5.5) and the co-normal derivative \( T_2 \) of equation (5.6), we prove (2.11). At last, subtracting from equation (5.9) the trace \( \gamma_2 \) of equation (5.6), we prove (2.12). That is, the transmission conditions on \( S_i \) and the Dirichlet boundary condition on \( S_e \) are fulfilled.

It remains to show that \( u_q \) solve differential equations (2.9) and that the conditions (5.2) hold true. Due to the embedding \( U^{(TD)} \in H^{(TD)} \), the third Green identities (3.109) hold. Comparing these identities with the first two equations of the LBDIE system, (5.5) and (5.6), and taking into account transmission conditions (2.10)–(2.11) and the Dirichlet boundary condition (2.12) already proved, we arrive at the relations

\[
V^{(1)}_{s_i} \left( \frac{T_1 u_1 - T_2 u_2}{2} - \psi \right) + W^{(1)}_{s_i} \left( \varphi_i - \frac{\gamma_1 u_1 + \gamma_2 u_2}{2} \right) = P_1 (f_1 - A_1 u_1) \text{ in } \Omega_1,
\]

\[
V^{(2)}_{s_i} \left( \frac{T_1 u_1 - T_2 u_2}{2} - \psi \right) - W^{(2)}_{s_i} \left( \varphi_i - \frac{\gamma_1 u_1 + \gamma_2 u_2}{2} \right) + V^{(2)}_{s_e} (\psi_e - T_2 u_2) = P_2 (A_2 u_2 - f_2) \text{ in } \Omega_2.
\]

Whence by Corollary 4.6 we conclude that conditions (5.2) are satisfied and

\[
A_1 u_1 - f_1 = 0 \text{ in } \Omega_1, \quad A_2 u_2 - f_2 = 0 \text{ in } \Omega_2. \tag{5.21}
\]

This completes the proof of item (ii) for LBDIE system (TD1).

Let now a vector \( U^{(TD)} \in H^{(TD)} \) solve LBDIE system (TD2). Subtracting from equation (5.2) the co-normal derivative \( T_1 \) of equation (5.14) and adding the co-normal derivative \( T_2 \) of equation (5.15), we prove the first relation in (5.2). Similarly, subtracting from equation (5.2) the trace \( \gamma_1 \) of equation (5.14) and the trace \( \gamma_2 \) of equation (5.15), we prove the second relation in (5.2). At last, subtracting from equation (5.11) the co-normal derivative \( T_2 \) of equation (5.15), we prove the third relation in (5.2).

It remains to show that \( u_q \) solve differential equations (2.9) and that the transmission conditions on \( S_i \) and the Dirichlet boundary condition on \( S_e \) are fulfilled. Due to the embedding \( U^{(TD)} \in H^{(TD)} \), the third Green identities (3.109) hold. Comparing these identities with the first two equations of the LBDIEs system, (5.5) and (5.6), and taking into account relations (5.2) already proved, we arrive at the relations

\[
\frac{1}{2} V^{(1)}_{s_i} (T_1 u_1 + T_2 u_2 - \psi_0) + \frac{1}{2} W^{(1)}_{s_i} (\varphi_0 - \gamma_1 u_1 + \gamma_2 u_2) = P_1 (f_1 - A_1 u_1) \text{ in } \Omega_1,
\]

\[
\frac{1}{2} V^{(2)}_{s_i} (T_1 u_1 + T_2 u_2 - \psi_0) - \frac{1}{2} W^{(2)}_{s_i} (\varphi_0 - \gamma_1 u_1 + \gamma_2 u_2) +
\]

This completes the proof of item (ii) for LBDIE system (TD2).
\[+W^{(2)}_{se}(\varphi_{0e} - \gamma_{2}u_{2}) = P_{2}(f_{2} - A_{2}u_{2}) \quad \text{in} \quad \Omega_{2}.\]

Whence by Corollary 4.7 we conclude that the transmission conditions on \(S_{i}\) and the Dirichlet boundary condition on \(S_{e}\) are satisfied and

\[A_{1}u_{1} - f_{1} = 0 \quad \text{in} \quad \Omega_{1}, \quad A_{2}u_{2} - f_{2} = 0 \quad \text{in} \quad \Omega_{2}. \quad (5.22)\]

This completes the proof of item (ii) for LBDIE system \((TD2)\). \(\square\)

Due to this equivalence theorem we conclude that the LBDIE system \((5.5)-(5.9)\) with the special right hand side functions which belong to the space \(F^{(TD1)}\) is uniquely solvable in the space \(H^{(TD)}\) defined by \((5.4)\). In particular, the corresponding homogeneous LBDIE system possesses only the trivial solution. By the way, one can easily observe that the right hand side in LBDIE system \((5.5)-(5.9)\) vanishes if \(f_{q} = 0\) in \(\Omega_{q}, q = 1, 2, \varphi_{0i} = 0\) and \(\psi_{0i} = 0\) on \(S_{i}\), and \(\varphi_{0e} = 0\) on \(S_{e}\).

Our next aim is to establish the invertibility of the matrix operator generated by the left hand side expressions in the LBDIE system \((5.5)-(5.9)\) in two sets of function spaces

\[\mathbb{K}^{(TD1)} : \mathbb{H}^{(TD)} \rightarrow \mathbb{Y}^{(TD1)}, \quad (5.23)\]

\[\mathbb{X}^{(TD)} \rightarrow \mathbb{Y}^{(TD1)}, \quad (5.24)\]

where we introduced the following notations for the wider function spaces,

\[\mathbb{X}^{(TD)} := H^{1}(\Omega_{1}) \times H^{1}(\Omega_{2}) \times H^{-\frac{1}{2}}(S_{i}) \times H^{\frac{1}{2}}(S_{i}) \times H^{-\frac{1}{2}}(S_{e}), \quad (5.25)\]

\[\mathbb{Y}^{(TD1)} := H^{1}(\Omega_{1}) \times H^{1}(\Omega_{2}) \times H^{\frac{1}{2}}(S_{i}) \times H^{-\frac{1}{2}}(S_{i}) \times H^{\frac{1}{2}}(S_{e}). \quad (5.26)\]

Evidently \(\mathbb{H}^{(TD)} \subset \mathbb{X}^{(TD)}\) and \(\mathbb{F}^{(TD1)} \subset \mathbb{Y}^{(TD1)}\). Due to Theorems 3.6, 3.9 and 3.11 the operators \((5.25)\) and \((5.26)\) are bounded.

**Theorem 5.2.** Let \(\chi \in X^{3}_{3}\) and condition \((4.3)\) hold. Then the operators \((5.23)\) and \((5.24)\) are invertible.

**Proof.** We can easily see that the upper triangular matrix operator

\[
\mathbb{K}_{0}^{(TD1)} := \begin{bmatrix}
I & 0 & -r_{n1}V^{(1)}_{s_{i}} & r_{n1}W^{(1)}_{s_{i}} & 0 \\
0 & I & r_{o2}V^{(2)}_{s_{i}} & r_{o2}W^{(2)}_{s_{i}} & -r_{o2}V^{(2)}_{s_{e}} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathcal{L}^{(3)}_{s_{i}} + \mathcal{L}^{(2)}_{s_{i}} & 0 \\
0 & 0 & 0 & 0 & -\mathcal{L}^{(2)}_{s_{e}}
\end{bmatrix} \quad (5.27)
\]

possesses the same mapping properties as the operator \(\mathbb{K}^{(TD1)}\),

\[\mathbb{K}_{0}^{(TD1)} : \mathbb{X}^{(TD)} \rightarrow \mathbb{Y}^{(TD1)}, \quad (5.28)\]

and by Lemma 3.7 and Theorems 3.9 and 3.11 the operator \((5.28)\) is a compact perturbation of the operator \((5.24)\).
On the other hand, for \( q = 1, 2 \) the operators (3.105) are strongly elliptic pseudodifferential operators of order \(-1\) with strictly positive principal homogeneous symbol \( \sigma_{y(\cdot)}(y, \zeta') \), while (3.108) are strongly elliptic pseudodifferential operators of order \(+1\) with strictly positive principal homogeneous symbol \( \sigma_{y(\cdot)}(y, \zeta') \) for \( \zeta' \in \mathbb{R}^2 \setminus \{0\} \) and \( y \in \partial \Omega_0 \) (see formulas (B.8) and (B.9) in Appendix B). Therefore by standard arguments it can be shown that the operators on the main diagonal in (5.27) are Fredholm of zero index in the appropriate function spaces (see, e.g., [1]). Thus operator (5.24) is also Fredholm with zero index.

It remains to show that the null space of operator (5.24) is trivial. We proceed as follows. Let \( U^{(TD)} \in \mathcal{X}^{(TD)} \) be a solution to the homogeneous system of equations \( K^{(TD)} U^{(TD)} = 0 \). Then due Theorems 3.6 and 3.9 we see from the first two equations of the system that \( U^{(TD)} \in \mathbb{H}^{(TD)} \) and by the equivalence Theorem 5.1 we conclude \( U^{(TD)} = 0 \). Thus the kernel of the operator (5.24) is trivial and consequently (5.24) is invertible.

To prove invertibility of operator (5.23), we remark that for any \( \mathcal{F}^{(TD)} \in \mathcal{F}^{(TD)} \), a unique solution \( U^{(TD)} \in \mathcal{X}^{(TD)} \) of equation (5.13) is delivered by the inverse to the operator (5.24). On the other hand, since \( \mathcal{F}^{(TD)} \in \mathcal{F}^{(TD)} \), the first two lines of the matrix operator \( K^{(TD)} \) imply that in fact \( U^{(TD)} \in \mathbb{H}^{(TD)} \) and the mapping \( \mathcal{F}^{(TD)} \rightarrow \mathbb{H}^{(TD)} \) delivered by the inverse to the operator (5.24) is continuous, i.e., this operator is inverse to operator (5.23).

\[ \Box \]

6. THE TRANSMISSION MIXED PROBLEM (TM)

Let us consider the mixed type transmission problems (2.9), (2.10), (2.11), (2.14), (2.15), with the right hand sides

\[
\begin{align*}
\varphi_{0i} & \in H^{\frac{1}{2}}(S_i), \quad \psi_{0i} \in H^{-\frac{1}{2}}(S_i), \\
\varphi_{0c}^{(M)} & \in H^{\frac{1}{2}}(S_{cD}), \quad \psi_{0c}^{(M)} \in H^{-\frac{1}{2}}(S_{cN}), \quad f_q \in H^0(\Omega_q), \quad q = 1, 2.
\end{align*}
\]

(6.1)

Let us denote by \( \Phi_{0c} \in H^{\frac{1}{2}}(S_c) \) and \( \Psi_{0c} \in H^{-\frac{1}{2}}(S_c) \) some fixed extensions of the boundary functions \( \varphi_{0c}^{(M)} \) and \( \psi_{0c}^{(M)} \) from \( S_{cD} \) and \( S_{cN} \), respectively, onto the whole surface \( S_c \), preserving the space. Then \( r_{s_{cD}} \Phi_{0c} = \varphi_{0c}^{(M)} \), \( r_{s_{cN}} \Psi_{0c} = \psi_{0c}^{(M)} \).

Any other extensions \( \Phi \in H^{\frac{1}{2}}(S_c) \) and \( \Psi \in H^{-\frac{1}{2}}(S_c) \) can be evidently represented then in the form

\[
\Phi = \Phi_{0c} + \varphi_c, \quad \varphi_c \in \tilde{H}^{\frac{1}{2}}(S_{cN}); \quad \Psi = \Psi_{0c} + \psi_c, \quad \psi_c \in \tilde{H}^{-\frac{1}{2}}(S_{cD}).
\]

Similar to (5.2) for the Problem (TD), let us introduce the following combinations of the unknown boundary functions

\[
\begin{align*}
\psi_1 = \frac{1}{2}(T_1u_1 - T_2u_2) & \in H^{-\frac{1}{2}}(S_i), \quad \varphi_1 = \frac{1}{2}(\gamma_1u_1 + \gamma_2u_2) \in H^{\frac{1}{2}}(S_i), \\
\psi_c = T_2u_2 - \Phi_{0c} & \in \tilde{H}^{-\frac{1}{2}}(S_{cD}), \quad \varphi_c = \gamma_2u_2 - \Phi_{0c} \in \tilde{H}^{\frac{1}{2}}(S_{cN}).
\end{align*}
\]

(6.2)
Further, let us set

$$U^{(TM)} := (u_1, u_2, \psi_1, \psi_e, \psi_e) \in \mathbb{H}^{(TM)},$$

(6.3)

where

$$\mathbb{H}^{(TM)} := H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{-\frac{1}{2}}(S_i) \times \times H^{-\frac{1}{2}}(S_e) \times \tilde{H}^{-\frac{1}{2}}(S_{eN})$$

(6.4)

and we assume again that the components of the vector $U^{(TM)}$ are formally unrelated.

Let us employ the third Green identities (3.109) in $\Omega_1$ and $\Omega_2$, difference of their traces (3.110) and sum of their co-normal derivatives (3.111) on $S_i$, and also the trace (3.110) on $S_{eD}$ and the co-normal derivative (3.111) on $S_{eN}$. Then after substituting transmission conditions (2.10)–(2.11) and mixed boundary conditions (2.14)–(2.15) along with notations (6.2), we arrive at the following system of direct segregated LBDIEs for the components of the vector $U^{(TM)}$,

$$u_1 + R_1 u_1 - V^{(1)}_{S_i} \psi_i + W^{(1)}_{S_i} \psi_i = F_1^{(TM)} \text{ in } \Omega_1,$$

(6.5)

$$u_2 + R_2 u_2 + V^{(2)}_{S_i} \psi_i + W^{(2)}_{S_i} \psi_i - V^{(2)}_{S_e} \psi_e + W^{(2)}_{S_e} \psi_e = F_2^{(TM)} \text{ in } \Omega_2,$$

(6.6)

$$\gamma_1 R_1 u_1 - \gamma_2 R_2 u_2 - (V^{(1)}_{S_i} + V^{(2)}_{S_i}) \psi_i + (W^{(1)}_{S_i} - W^{(2)}_{S_i}) \psi_i +$$

$$+ \gamma_2 V^{(2)}_{S_e} \psi_e - \gamma_2 W^{(2)}_{S_e} \psi_e = \gamma_1 F_1^{(TM)} - \gamma_2 F_2^{(TM)} - \phi_{0i} \text{ on } S_i,$$

(6.7)

$$T_1 R_1 u_1 + T_2 R_2 u_2 - (W^{(1)}_{S_i} - W^{(2)}_{S_i}) \psi_i + (L^{(1)}_{S_i} + L^{(2)}_{S_i}) \psi_i -$$

$$- T_2 V^{(2)}_{S_e} \psi_e + T_2 W^{(2)}_{S_e} \psi_e = T_1 F_1^{(TM)} + T_2 F_2^{(TD)} - \psi_{0i} \text{ on } S_i,$$

(6.8)

$$\gamma_2 R_2 u_1 + \gamma_2 V^{(2)}_{S_i} \psi_i + \gamma_2 W^{(2)}_{S_i} \psi_i - V^{(2)}_{S_e} \psi_e + W^{(2)}_{S_e} \psi_e =$$

$$= \gamma_2 F_2^{(TM)} - \psi_{0e} \text{ on } S_{eD},$$

(6.9)

$$T_2 R_2 u_1 + T_2 V^{(2)}_{S_i} \psi_i + T_2 W^{(2)}_{S_i} \psi_i - W^{(2)}_{S_e} \psi_e + L^{(2)}_{S_e} \psi_e =$$

$$= T_2 F_2^{(TM)} - \psi_{0e} \text{ on } S_{eN},$$

(6.10)

where

$$F_1^{(TM)} = \mathbb{P}_1 f_1 + \frac{1}{2} V^{(1)}_{S_i} \psi_{0i} - \frac{1}{2} W^{(1)}_{S_i} \psi_{0i},$$

(6.11)

$$F_2^{(TM)} = \mathbb{P}_2 f_2 + \frac{1}{2} V^{(2)}_{S_i} \psi_{0i} + \frac{1}{2} W^{(2)}_{S_i} \psi_{0i} + V^{(2)}_{S_e} \psi_{0e} - W^{(2)}_{S_e} \psi_{0e}.$$ 

(6.12)

As in the case of the problem (TD), we have here the following equivalence theorem.

**Theorem 6.1.** Let $\chi \in X_{1s}^3$ and conditions (6.1) hold. Further, let $\Phi_{0e} \in H^{-\frac{1}{2}}(S_e)$ and $\Psi_{0e} \in H^{-\frac{1}{2}}(S_e)$ be some fixed extensions of the boundary functions $\varphi_{0e}^{(N)}$ and $\psi_{0e}$ from $S_{eD}$ and $S_{eN}$, respectively, onto the whole surface $S_e$. 
(i) If a pair \((u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)\) solves the transmission mixed problem (TM), then the vector \(U^{(TM)} \in \mathbb{H}^{(TM)}\) given by (6.3), where \(\psi_i, \varphi_i, \psi_e\) and \(\varphi_e\) are defined by (6.2), solves LBDIE system (6.5)–(6.12).

(ii) Vice versa, if a vector \(U^{(TM)} \in \mathbb{H}^{(TM)}\) solves the LBDIE system (6.5)–(6.12) and condition (4.3) holds, then the pair \((u_1, u_2)\) solves the Problem (TM) and relations (6.2) hold.

**Proof.** The claim (i) immediately follows from the deduction of (6.5)–(6.12).

Now, let a vector \(U^{(TM)}\) solve the LBDIE system (6.5)–(6.12). Subtracting from equation (6.7) the trace \(\gamma_1\) of equation (6.5) and adding the trace \(\gamma_2\) of equation (6.6), we prove (2.10). Similarly, subtracting from equation (6.8) the co-normal derivative \(T_1\) of equation (6.5) and the co-normal derivative \(T_2\) of equation (6.6), we prove (2.11). Subtracting from equation (6.9) the trace \(\gamma_2\) of equation (6.6), we prove (2.14). Similarly, subtracting from equation (6.10) the co-normal derivative \(T_2\) of equation (6.6), we prove (2.15). That is, the transmission conditions on \(S_i\) and the mixed boundary conditions on \(S_e\) are fulfilled.

It remains to show that equations (2.9) and the relations (6.2) hold true. Due to the embedding \(U^{(TM)} \in \mathbb{H}^{(TM)}\), the third Green identities (3.109) hold. Comparing these identities with the first two equations of the LBDIE system, (6.5) and (6.6), and taking into account transmission conditions (2.10)–(2.11) and mixed boundary conditions (2.14)–(2.15), already proved, we arrive at the relations

\[
\begin{align*}
&V^{(1)}(T_1u_1 - T_2u_2) + W^{(1)}(\varphi_1 - \frac{\gamma_1u_1 + \gamma_2u_2}{2}) = \mathcal{P}_1(f_1 - A_1u_1) \quad \text{in} \quad \Omega_1, \\
&V^{(2)}(T_1u_1 - T_2u_2) - W^{(2)}(\varphi_2 - \frac{\gamma_1u_1 + \gamma_2u_2}{2}) + V^{(2)}(-T_2u_2 + \psi_e + \Psi_{0e}) + W^{(2)}(\gamma_2u_2 - \psi_e - \Phi_{0e}) = \mathcal{P}_2(A_2u_2 - f_2) \quad \text{in} \quad \Omega_2.
\end{align*}
\]

Whence by Corollary 4.8 we conclude that (2.9) and (6.2) are satisfied. \(\Box\)

Denote by \(\mathcal{K}^{(TM)}\) the localized boundary-domain \(6 \times 6\) matrix integral operator generated by the left hand side expressions in (6.5)–(6.10),

\[
\mathcal{K}^{(TM)} = [\mathcal{K}_{kj}^{(TM)}]_{6 \times 6} := \text{diag}(r_{\Omega_1}, r_{\Omega_2}, r_{S_1}, r_{S_2}, r_{S_{1,1}}, r_{S_{1,2}}, r_{S_{1,3}}, r_{S_{2,1}}, r_{S_{2,2}}, r_{S_{2,3}}, r_{S_{2,4}}, r_{S_{2,5}}, r_{S_{2,6}}) \times
\]

\[
\begin{bmatrix}
I + \mathcal{R}_1 & 0 & -V^{(1)}_{s_1} & W^{(1)}_{s_1} & 0 & 0 \\
0 & I + \mathcal{R}_2 & V^{(2)}_{s_2} & W^{(2)}_{s_2} & -V^{(2)}_{s_e} & W^{(2)}_{s_e} \\
\gamma_{1} \mathcal{R}_1 - \gamma_{2} \mathcal{R}_2 & -V^{(1)}_{s_1} - V^{(2)}_{s_1} & W^{(1)}_{s_1} - W^{(2)}_{s_1} & -V^{(2)}_{s_1} & -\gamma_{2} W^{(2)}_{s_2} \\
T_1 \mathcal{R}_1 & T_2 \mathcal{R}_2 & W^{(1)}_{s_1} - W^{(2)}_{s_1} & \mathcal{L}^{(1)}_{s_1} + \mathcal{L}^{(2)}_{s_1} & -T_2 V^{(2)}_{s_1} & T_2 W^{(2)}_{s_2} \\
0 & \gamma_2 \mathcal{R}_2 & \gamma_2 V^{(2)}_{s_1} & \gamma_2 W^{(2)}_{s_1} - V^{(2)}_{s_1} - W^{(2)}_{s_1} & 0 \\
0 & T_2 \mathcal{R}_2 & T_2 V^{(2)}_{s_1} & T_2 W^{(2)}_{s_2} - W^{(2)}_{s_2} & \mathcal{L}^{(2)}_{s_2}
\end{bmatrix}
\]
and set
\[
\mathcal{F}(TM) := H^{1/2}(\Omega_1; A_1) \times H^{1/2}(\Omega_2; A_2) \times H^{1/2}(S_i) \times \\
\times H^{-1/2}(S_i) \times H^{1/2}(S_{eD}) \times H^{-1/2}(S_{eN}). \quad (6.14)
\]

Then the LBDIE system (6.5)–(6.10) can be written in matrix form as
\[
\mathbf{K}(TM) \mathbf{U}(TM) = \mathcal{F}(TM), \quad (6.15)
\]
where \(\mathbf{U}(TM)\) is the unknown vector function (6.3), while \(\mathcal{F}(TM) \in \mathcal{F}(TM)\) is the known vector function compiled by the right hand side functions in (6.5)–(6.12).

From Theorem 6.1 it follows that LBDIE system (6.5)–(6.10), i.e., equation (6.15) is uniquely solvable in the space \(H(TM)\) for the special right hand side vector-function (see the right hand side functions in (6.5)–(6.12)) which belong to the space \(\mathcal{F}(TM)\) defined by (6.14). One can easily observe that the right hand side expressions in LBDIE system (6.5)–(6.10) vanish if \(f_q = 0\) in \(\Omega_q, \ q = 1, 2, f_1 = 0\) and \(\psi_{0i} = 0\) on \(S_i, \Phi_{0e} = 0\) and \(\Psi_{0e} = 0\) on \(S_e\).

Now we establish that actually equation (6.15) is uniquely solvable in two sets of spaces. To this end let us consider the operators
\[
\mathbf{K}(TM) : \mathbb{H}(TM) \longrightarrow \mathcal{F}(TM), \quad (6.16)
\]
\[
\mathbf{Y}(TM) : \mathbb{X}(TM) \longrightarrow \mathbb{Y}(TM), \quad (6.17)
\]
where
\[
\mathbb{X}(TM) := H^{1}(\Omega_1) \times H^{1}(\Omega_2) \times H^{-1/2}(S_i) \times \tilde{H}^{1/2}(S_{eD}) \times \tilde{H}^{-1/2}(S_{eN}), \quad (6.18)
\]
\[
\mathbb{Y}(TM) := H^{1}(\Omega_1) \times H^{1}(\Omega_2) \times H^{-1/2}(S_i) \times \tilde{H}^{1/2}(S_{eD}) \times \tilde{H}^{-1/2}(S_{eN}). \quad (6.19)
\]
As follows from the mapping properties of the potentials (see Theorem 3.6, 3.9 and 3.11), the operators (6.16) and (6.17) are bounded. Further we show that the operator (6.17) is Fredholm with zero index and thus (6.17) and consequently (6.16) are invertible.

Consider the upper triangular operator
\[
\mathbf{K}_0(TM) := \\
\begin{bmatrix}
I & 0 & -r_{11} V^{(1)}_{S_i} & r_{12} W^{(1)}_{S_i} & 0 & 0 \\
0 & I & r_{21} V^{(2)}_{S_i} & r_{22} W^{(2)}_{S_i} & -r_{23} V^{(2)}_{S_e} & r_{24} W^{(2)}_{S_e} \\
0 & 0 & -\gamma^{(1)}_{S_i} & 0 & 0 & 0 \\
0 & 0 & 0 & L^{(1)}_{S_i} + L^{(2)}_{S_i} & 0 & 0 \\
0 & 0 & 0 & 0 & -r_{sD} V^{(2)}_{S_e} & 0 \\
0 & 0 & 0 & 0 & 0 & r_{sN} L^{(2)}_{S_e}
\end{bmatrix}. \quad (6.20)
\]

It is easy to see that, on the one hand, the operator
\[
\mathbf{K}_0(TM) : \mathbb{X}(TM) \longrightarrow \mathbb{Y}(TM), \quad (6.21)
\]
is bounded, while due to Lemma 3.7 and Theorems 3.9 and 3.11,
\[ K^{(TM)} - K_0^{(TM)} : X^{(TM)} \to Y^{(TM)} \] (6.22)
is a compact operator.

On the other hand, as it has been mentioned in the proof of Theorem 5.2, the third and forth operators in the main diagonal
\[-[\mathcal{V}^{(1)}_{S_i} + \mathcal{V}^{(2)}_{S_i}] : H^{-\frac{1}{2}}(S_i) \to H^{\frac{1}{2}}(S_i), \] (6.23)
\[ \mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} : H^{\frac{1}{2}}(S_i) \to H^{-\frac{1}{2}}(S_i), \] (6.24)
are Fredholm with zero index.

Moreover, applying the results of the theory of strongly elliptic pseudodifferential equations on manifolds with boundary (see, e.g., [3, Theorem 3.5], [6, Lemma 3.4]) we conclude that the operators on the main diagonal
\[ r^{(1)}_{S_{eD}} \mathcal{V}^{(2)}_{S_e} : \tilde{H}^{-\frac{1}{2}}(S_{eD}) \to \tilde{H}^{\frac{1}{2}}(S_{eD}), \] (6.25)
\[ r^{(2)}_{S_{eN}} \mathcal{L}^{(2)}_{S_e} : \tilde{H}^{\frac{1}{2}}(S_{eN}) \to \tilde{H}^{-\frac{1}{2}}(S_{eN}), \] (6.26)
are Fredholm with zero index.

Therefore, (6.21) and consequently (6.17) is a Fredholm operator with zero index. It remains to show that the null space of operator (6.17) is trivial. Let \( U_{TM} \) be a solution to the homogeneous equation \( K_{TM} U_{TM} = 0 \). Then due to the first two lines of the matrix equation and mapping properties (3.68), (3.93) and (3.94) we see that \( U_{TM} \) is in \( H_{TM} \) and by the equivalence Theorem 6.1 we conclude that the operator (6.17) is invertible.

To prove invertibility of operator (6.16), we remark that for any \( F^{TM} \) a unique solution \( U^{TM} \) of equation (6.15) is delivered by the inverse to the operator (6.17). On the other hand, since \( F^{TM} \) imply that in fact \( U^{TM} \) is in \( H^{TM} \) and the mapping \( F^{TM} \to H^{TM} \) delivered by the inverse to the operator (6.17) is continuous, i.e., this operator gives inverse to operator (6.16) as well.

Now we can summarize the results obtained above as the following

**Theorem 6.2.** Let \( \chi \in X_1^3 \) and condition (4.3) hold. Then the operators (6.16) and (6.17) are invertible.

7. Crack Type Transmission Dirichlet Problem (CTD)

Let a pair \((u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)\) be a solution to the problem (CTD) with the interface crack-transmission conditions (2.17)–(2.20) on \( S_i \) and the Dirichlet type boundary condition (2.12) on the exterior boundary \( S_e \), i.e.,
\[ A_q(x, \partial)u_q = f_q \text{ in } \Omega_q, \quad q = 1, 2, \] (7.1)
\[
\begin{align*}
\gamma_1 u_1 - \gamma_2 u_2 &= \varphi_{0i}^{(t)} \text{ on } S_i^{(t)}, \\
T_1 u_1 + T_2 u_2 &= \varphi_{0i}^{(t)} + \tilde{\psi}_i \text{ on } S_i^{(t)}, \\
T_1 u_1 &= \varphi_{0i}^{(c)} \text{ on } S_i^{(c)}, \\
T_2 u_2 &= \varphi_{0e}^{(c)} \text{ on } S_c.
\end{align*}
\]

Let \( \psi_{0i} \) be defined by (2.22). We assume that the conditions (2.21)-(2.23) are satisfied along with the conditions (2.16) for the function \( \varphi_{0c} \) and \( f_q \), \( q = 1, 2 \).

Denote by \( \Psi_{0i} \in H^{-\frac{1}{2}}(S_i) \) some fixed extension of the function \( \psi_{0i}^{(t)} - \psi_{0i}^{(c)} \) from \( S_i^{(c)} \) onto the whole of \( S_i \) preserving the function space. Analogously, let \( \Phi_{0i} \in H^{\frac{1}{2}}(S_i) \) be some fixed extension of the function \( \varphi_{0i}^{(t)} \) from \( S_i^{(c)} \) onto the whole of \( S_i \) preserving the function space. Then we can write the following relations on \( S_i \)

\[
\begin{align*}
T_1 u_1 &= \frac{1}{2} [T_1 u_1 + T_2 u_2] + \frac{1}{2} [T_1 u_1 - T_2 u_2] = \frac{1}{2} \psi_{0i} + \frac{1}{2} \Psi_{0i} + \tilde{\psi}_i, \\
T_2 u_2 &= \frac{1}{2} [T_1 u_1 + T_2 u_2] - \frac{1}{2} [T_1 u_1 - T_2 u_2] = \frac{1}{2} \psi_{0i} - \frac{1}{2} \Psi_{0i} - \tilde{\psi}_i, \\
\gamma_1 u_1 &= \frac{1}{2} [\gamma_1 u_1 + \gamma_2 u_2] + \frac{1}{2} [\gamma_1 u_1 - \gamma_2 u_2] = \frac{1}{2} \Phi_{0i} + \varphi_i + \tilde{\varphi}_i, \\
\gamma_2 u_2 &= \frac{1}{2} [\gamma_1 u_1 + \gamma_2 u_2] - \frac{1}{2} [\gamma_1 u_1 - \gamma_2 u_2] = \frac{1}{2} \Phi_{0i} + \varphi_i - \tilde{\varphi}_i,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{\psi}_i &= \frac{1}{2} [T_1 u_1 - T_2 u_2] - \frac{1}{2} \Psi_{0i} \in \tilde{H}^{-1/2}(S_i^{(t)}), \\
\varphi_i &= \frac{1}{2} [\gamma_1 u_1 + \gamma_2 u_2] \in H^{1/2}(S_c), \\
\tilde{\varphi}_i &= \frac{1}{2} [\gamma_1 u_1 - \gamma_2 u_2] - \frac{1}{2} \Phi_{0i} \in \tilde{H}^{1/2}(S_i^{(c)}),
\end{align*}
\]

are unknown functions. Let us introduce one more unknown function defined on \( S_c \)

\[
\psi_c := T_2 u_2 \in H^{-1/2}(S_c),
\]

and denote

\[
\begin{align*}
U^{(CTD)} &= (u_1, u_2, \tilde{\psi}_1, \varphi_i, \tilde{\varphi}_i, \psi_c) \in H^{(TD)}, \\
\mathbb{H}^{(CTD)} &= H^{1,0}(\Omega_1; L_1) \times H^{1,0}(\Omega_2; L_2) \times \tilde{H}^{-\frac{1}{2}}(S_i^{(t)}) \times \tilde{H}^{\frac{1}{2}}(S_i^{(c)}) \times \tilde{H}^{\frac{1}{2}}(S_i^{(c)}) \times H^{-\frac{1}{2}}(S_c),
\end{align*}
\]

We choose equations (3.109) in \( \Omega_1 \) and \( \Omega_2 \), difference of equations (3.110) for \( q = 1 \) and \( q = 2 \) on \( S_i^{(t)} \), sum of equations (3.111) for \( q = 1 \) and \( q = 2 \) on the whole of \( S_i \), difference of equations (3.111) for \( q = 1 \) and \( q = 2 \) on \( S_i^{(c)} \) and equation (3.111) for \( q = 2 \) on \( S_c \). Then after substituting there the
notation (7.7)–(7.10) and (7.14) and taking into consideration the relations (7.1)–(7.6), we arrive at the following system of direct segregated LBDIEs for the components of the vector \( \Psi^{(CTD)} = (u_1, u_2, \tilde{\psi}_1, \varphi_1, \tilde{\varphi}_1, \psi_1) \),

\[
\begin{align*}
    u_1 + R_1 u_1 - V^{(1)}_{s_1} \tilde{\psi}_1 + W^{(1)}_{s_1} \varphi_1 + W^{(1)}_{s_1} \tilde{\varphi}_1 &= F_1^{(CTD)} \quad \text{in } \Omega_1, \\
    u_2 + R_2 u_2 + V^{(2)}_{s_2} \psi_1 - W^{(2)}_{s_2} \psi_1 - V^{(2)}_{s_2} \tilde{\psi}_1 &= F_2^{(CTD)} \quad \text{in } \Omega_2, \\
    \gamma_1 R_1 u_1 - \gamma_2 R_2 u_2 - [V^{(1)}_{s_1} + V^{(2)}_{s_2}] \tilde{\psi}_1 + [W^{(1)}_{s_1} - W^{(2)}_{s_2}] \varphi_1 + [W^{(1)}_{s_1} + W^{(2)}_{s_2}] \tilde{\varphi}_1 + \\
    + \gamma_2 V^{(2)}_{s_2} \psi_1 &= \gamma_1 F_1^{(CTD)} - \gamma_2 F_2^{(CTD)} - \Phi_0 i \quad \text{on } S^{(i)}, \\
    T_1 R_1 u_1 + T_2 R_2 u_2 - [W^{(1)}_{s_1} - W^{(2)}_{s_2}] \tilde{\psi}_1 + [L^{(1)}_{s_1} + L^{(2)}_{s_2}] \varphi_1 + [L^{(1)}_{s_1} - L^{(2)}_{s_2}] \tilde{\varphi}_1 - \\
    - T_2 V^{(2)}_{s_2} \psi_1 &= T_1 F_1^{(CTD)} + T_2 F_2^{(CTD)} - \Phi_0 i \quad \text{on } S^{(c)}, \\
    \gamma_2 R_2 u + \gamma_2 V^{(2)}_{s_2} \psi_1 + \gamma_2 W^{(2)}_{s_2} \varphi_1 - \gamma_2 W^{(2)}_{s_2} \tilde{\varphi}_1 - V^{(2)}_{s_2} \psi_1 &= \\
    = \gamma_2 F_2^{(TM)} - \varphi_0 e \quad \text{on } S^e,
\end{align*}
\]

where

\[
\begin{align*}
    F_1^{(CTD)} &= P_1 f_1 + \frac{1}{2} V^{(1)}_{s_1} \psi_{0i} + \frac{1}{2} V^{(1)}_{s_1} \psi_{0i} - \frac{1}{2} W^{(1)}_{s_1} \Phi_{0i} \quad \text{in } \Omega_1, \\
    F_2^{(CTD)} &= P_2 f_2 + \frac{1}{2} V^{(2)}_{s_2} \psi_{0i} - \frac{1}{2} V^{(2)}_{s_2} \psi_{0i} + \frac{1}{2} W^{(2)}_{s_2} \Phi_{0i} - W^{(2)}_{s_2} \varphi_{0i} \quad \text{in } \Omega_2.
\end{align*}
\]

There holds the following equivalence theorem.

**Theorem 7.1.** Let \( \chi \in X^3_{1,s} \), conditions (2.21)–(2.23) be satisfied along with the conditions (2.16) for the functions \( \varphi_{0e} \) and \( f_q \), \( q = 1, 2 \), \( \psi_{0i} \) be defined by (2.22), and \( \Phi_{0i} \), \( \Phi_{0i} \), and \( \Phi_{0i} \) be the above introduced extended functions.

(i) If a pair \((u_1, u_2, \tilde{\psi}_1, \varphi_1, \tilde{\varphi}_1, \psi_1)\) solves the interface crack problem \((CTD)\), then the vector \((u_1, u_2, \tilde{\psi}_1, \varphi_1, \tilde{\varphi}_1, \psi_1)\), where \( \tilde{\psi}_1, \varphi_1, \tilde{\varphi}_1 \) and \( \psi_1 \) are defined by relations (7.11)–(7.14), solves LBDIE system (7.17)–(7.22).

(ii) Vice versa, if a vector \((u_1, u_2, \tilde{\psi}_1, \varphi_1, \tilde{\varphi}_1, \psi_1)\) \( \in \mathbb{E}^{(TD)} \) solves LBDIE system (7.17)–(7.22) and condition (4.3) holds, then the pair \((u_1, u_2)\) solves the problem \((CTD)\) and relations (7.7)–(7.14) hold true.

**Proof.** The proof of the claim (i) immediately follows from the deduction of system (7.17)–(7.22).

Now, let the vector (7.15) solve LBDIE system (7.17)–(7.22). One can easily verify that the boundary-transmission and crack conditions (7.2)–(7.6) are satisfied. To this end one needs, similar to the proof of Theorem 6.1, to take the traces and co-normal derivatives of the first two
equations (7.17) and (7.18) and compare them with the last four equations (7.19)–(7.22).

It remains to show that \( u_1 \) and \( u_2 \) solve the differential equations (7.1) and that the relations (7.7)–(7.14) hold true. Due to the embedding (7.16), we can write the third Green identities (3.109). Comparing these equalities with the first two equations of the LBDIE system, (7.17) and (7.18), and keeping in mind that for the functions \( u_1 \) and \( u_2 \) the boundary-transmission conditions (7.2)–(7.6) are already proved, we arrive at the relations

\[
V_{s_i}^{(1)}(g_{i1}) + W_{s_i}^{(1)}(g_{i2}) + P_1(G_1) = 0 \quad \text{in} \; \Omega_1, \\
V_{s_i}^{(2)}(g_{i1}) + W_{s_i}^{(2)}(g_{i4}) + P_2(G_2) = 0 \quad \text{in} \; \Omega_2,
\]

where

\[
G_1 := A_1 u_1 - f_1 \quad \text{in} \; \Omega_1, \quad G_2 := A_2 u_2 - f_2 \quad \text{in} \; \Omega_2,
\]

\[
g_{i1} := T_1 u_1 - \tilde{\psi}_i - \frac{1}{2} \psi_0 - \frac{1}{2} \Psi_0 \quad \text{on} \; S_i,
\]

\[
g_{i2} := \varphi_i + \tilde{\varphi}_i + \frac{1}{2} \Phi_0 - \gamma_1 u_1 \quad \text{on} \; S_i,
\]

\[
g_{i1}' := T_2 u_2 + \tilde{\psi}_i - \frac{1}{2} \psi_0 + \frac{1}{2} \Psi_0 \quad \text{on} \; S_i,
\]

\[
g_{i3} := \varphi_i - \tilde{\varphi}_i - \frac{1}{2} \Phi_0 - \gamma_2 u_2 \quad \text{on} \; S_i,
\]

\[
ge_e := T_3 u_2 - \psi_e \quad \text{on} \; S_e.
\]

Due to the boundary-transmission conditions (7.2)–(7.6) and equalities (2.22) we obtain,

\[
g_{i1} = -g_{i1}' \in \tilde{H}^{\frac{1}{2}}(S_i^{(t)}), \quad g_{i2} = g_{i3} \in \tilde{H}^{\frac{1}{2}}(S_i^{(e)}), \quad g_e \in H^{-\frac{1}{2}}(S_e). \tag{7.28}
\]

Therefore by Corollary 4.9 we have \( g_{i1} = g_{i1}' = g_{i2} = g_{i3} = 0 \) on \( S_i \), \( g_e = 0 \) on \( S_e \), and \( G_0 = 0 \) in \( \Omega_q \), \( q = 1, 2 \), which completes the proof.

Due to this equivalence theorem we conclude that the LBDIEs system (7.17)–(7.22) with the special right hand side functions which belong to the space

\[
\mathbb{H}^{(CTD)} := H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{\frac{1}{2}}(S_i^{(t)}) \times
\]

\[
\times H^{-\frac{1}{2}}(S_i) \times H^{-\frac{1}{2}}(S_i^{(e)}) \times H^{\frac{1}{2}}(S_e) \tag{7.29}
\]

is uniquely solvable in the space \( \mathbb{H}^{(CTD)} \) defined in (7.16). In particular, the corresponding homogeneous LBDIEs system possesses only the trivial solution. By the way, one can easily observe that the right hand side expressions in LBDIEs system (7.17)–(7.22) vanish if and only if \( f_q = 0 \) in \( \Omega_q \), \( q = 1, 2 \), \( \varphi_0 = \psi_0 = 0 \) on \( S_i^{(t)} \), \( \psi'_0 = \psi''_0 = 0 \) on \( S_i^{(e)} \) and \( \varphi_{0e} = 0 \) on \( S_2 \).

Our next aim is to establish that the matrix operator \( \mathcal{K}^{(CTD)} \) generated by the left hand side expressions in the LBDIEs system (7.17)–(7.22) is invertible in two sets spaces. We have
Let $K_{\gamma(CTD)} = [K_{\gamma(CTD)}]_{6 \times 6} \doteq \text{diag}(r_{n1}, r_{n2}, r_{s_1^{(1)}}, r_{s_1^{(c)}}, r_{s_2}) \times$

\[
\begin{bmatrix}
I + \mathcal{R}_1 & 0 & -V_{s_1}^{(1)} & W_{s_1}^{(1)} & 0 \\
0 & I + \mathcal{R}_2 & V_{s_1}^{(2)} & W_{s_1}^{(2)} & -W_{s_2}^{(2)} & -V_{s_2}^{(2)} \\
\gamma_1 \mathcal{R}_1 - \gamma_2 \mathcal{R}_2 & -V_{s_1}^{(1)} - V_{s_1}^{(2)} & W_{s_1}^{(1)} - W_{s_1}^{(2)} & W_{s_1}^{(1)} + W_{s_1}^{(2)} & \gamma_2 V_{s_1}^{(2)} & -T_1 \mathcal{R}_1 \gamma_2 V_{s_1}^{(2)} \\
T_1 \mathcal{R}_1 & T_2 \mathcal{R}_2 & -W_{s_1}^{(1)} + W_{s_1}^{(2)} & L_{s_1}^{(1)} + L_{s_1}^{(2)} & L_{s_1}^{(1)} - L_{s_1}^{(2)} & -T_2 V_{s_1}^{(2)} \\
0 & \gamma_2 \mathcal{R}_2 & \gamma_2 V_{s_1}^{(2)} & \gamma_2 W_{s_1}^{(2)} & -\gamma_2 W_{s_1}^{(2)} & -V_{s_2}^{(2)} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\] (7.30)

Introduce the function spaces

\[
\mathcal{X}(CTD) := H^1(\Omega_1) \times H^1(\Omega_2) \times \tilde{H}^{-\frac{1}{2}}(S_1^{(1)}) \times H^{-\frac{1}{2}}(S_1) \times \tilde{H}^{-\frac{1}{2}}(S_1^{(c)}) \times H^{-\frac{1}{2}}(S_1) \times H^{-\frac{1}{2}}(S_2),
\] (7.31)

\[
\mathcal{Y}(CTD) := H^1(\Omega_1) \times H^1(\Omega_2) \times \tilde{H}^{-\frac{1}{2}}(S_1^{(1)}) \times H^{-\frac{1}{2}}(S_1) \times \tilde{H}^{-\frac{1}{2}}(S_1^{(c)}) \times H^{-\frac{1}{2}}(S_1) \times H^{-\frac{1}{2}}(S_2).
\] (7.32)

By virtue of Theorems 3.9 and 3.11 we see that the operator $K(CTD)$ has the following mapping property

\[
K(CTD) : \mathcal{H}(CTD) \rightarrow \mathcal{Y}(CTD),
\] (7.33)

\[
: \mathcal{X}(CTD) \rightarrow \mathcal{Y}(CTD).
\] (7.34)

**Theorem 7.2.** Let $\chi \in X_{\chi}^2$ and condition (4.3) hold. Then operators (7.33) and (7.34) are invertible.

**Proof.** Due to compactness of the operators from Lemma 3.7 and Theorems 3.9 and 3.11, the upper block-triangular matrix operator

\[
K_0(CTD) := \text{diag}(r_{n1}, r_{n2}, r_{s_1^{(1)}}, r_{s_1^{(c)}}, r_{s_2}) \times
\begin{bmatrix}
I & 0 & -V_{s_1}^{(1)} & W_{s_1}^{(1)} & 0 \\
0 & I & V_{s_1}^{(2)} & W_{s_1}^{(2)} & -W_{s_2}^{(2)} & -V_{s_2}^{(2)} \\
0 & 0 & -V_{s_1}^{(1)} + V_{s_1}^{(2)} & 0 & 0 & 0 \\
0 & 0 & 0 & L_{s_1}^{(1)} + L_{s_1}^{(2)} & L_{s_1}^{(1)} - L_{s_1}^{(2)} & 0 \\
0 & 0 & 0 & L_{s_1}^{(1)} - L_{s_1}^{(2)} & L_{s_1}^{(1)} + L_{s_1}^{(2)} & 0 \\
0 & 0 & 0 & 0 & -V_{s_2}^{(2)} & 0
\end{bmatrix}
\] is a compact perturbation of the operator (7.34) and possesses the same mapping property,

\[
K_0(CTD) : \mathcal{X}(CTD) \rightarrow \mathcal{Y}(CTD).
\] (7.35)
Our goal is to show that the operator (7.35) is Fredholm with zero index. To this end, let us note that the operator (3.105) is a strongly elliptic pseudodifferential operator of order $-1$ with strictly positive principal homogenous symbol, while (3.108) is a strongly elliptic pseudodifferential operator of order $+1$ with strictly negative principal homogenous symbol. This can be shown by a standard approach since the principal homogeneous symbols of the localized operators and the corresponding non-localized ones coincide (cf. [7], [13]).

Therefore, applying the theory of pseudodifferential equations on manifolds with and/or without boundary ([11], [26]) one can show that the third and sixth operators in the main diagonal of $K_{0}^{(CTD)}$

$$r_{S_{i}}^{(1)}[Y_{S_{i}}^{(1)} + Y_{S_{i}}^{(2)}]: \tilde{H}^{-\frac{1}{2}}(S_{i}^{(1)}) \rightarrow H^{-\frac{1}{2}}(S_{i}^{(1)}),$$
$$Y_{S_{i}}^{(2)}: H^{-\frac{1}{2}}(S_{i}) \rightarrow H^{\frac{1}{2}}(S_{i})$$

are Fredholm with zero index.

Now let us consider the following $2 \times 2$ matrix operator block which stands in the main diagonal of the upper block-triangular matrix operator $K_{0}^{(CTD)}$

$$L := \begin{bmatrix} L_{S_{i}}^{(1)} + L_{S_{i}}^{(2)} & L_{S_{i}}^{(1)} - L_{S_{i}}^{(2)} \\ r_{S_{i}}^{(1)}[L_{S_{i}}^{(1)} - L_{S_{i}}^{(2)}] & r_{S_{i}}^{(2)}[L_{S_{i}}^{(1)} + L_{S_{i}}^{(2)}] \end{bmatrix}.$$  \hspace{1cm} (7.36)

Clearly,

$$L: H^{\frac{1}{2}}(S_{i}) \times \tilde{H}^{\frac{1}{2}}(S_{i}^{(c)}) \rightarrow H^{-\frac{1}{2}}(S_{i}) \times H^{\frac{1}{2}}(S_{i}^{(c)})$$  \hspace{1cm} (7.37)

is continuous. Denote by $\sigma^{(q)}(y, \xi'), y \in S_{i}, \xi' \in \mathbb{R}^2$, the principal homogeneous symbol of the operator $L_{S_{i}}^{(q)}, q = 1, 2$ (see formula (B.9)). As it is shown in Appendix B, $\sigma^{(q)}(y, \xi')$ is a homogeneous function in $\xi'$ of order 1 and $\sigma^{(q)}(y, \xi') < 0$ for all $\xi' \in \mathbb{R}^2 \setminus \{0\}$ and for all $y \in S_{i}$.

Therefore there is a compact operator $C: H^{\frac{1}{2}}(S_{i}) \rightarrow H^{-\frac{1}{2}}(S_{i})$ such that

$$L_{S_{i}}^{(1)} + L_{S_{i}}^{(2)} + C: H^{\frac{1}{2}}(S_{i}) \rightarrow H^{-\frac{1}{2}}(S_{i})$$  \hspace{1cm} (7.38)

is invertible. Denote the inverse operator by $[L_{S_{i}}^{(1)} + L_{S_{i}}^{(2)} + C]^{-1}$.

Further, let us introduce a compact perturbation of the operator $L$ in (7.36)–(7.37) defined by the relation

$$\tilde{L} := \begin{bmatrix} L_{S_{i}}^{(1)} + L_{S_{i}}^{(2)} + C & L_{S_{i}}^{(1)} - L_{S_{i}}^{(2)} \\ r_{S_{i}}^{(1)}[L_{S_{i}}^{(1)} - L_{S_{i}}^{(2)}] & r_{S_{i}}^{(2)}[L_{S_{i}}^{(1)} + L_{S_{i}}^{(2)}] \end{bmatrix}.$$  \hspace{1cm} (7.39)

It is easy to check that $\tilde{L}$ can be represented as the composition of two operators

$$\tilde{L} = \tilde{L}_{1}\tilde{L}_{2},$$

where

$$\tilde{L}_{1} := \begin{bmatrix} 0 & L_{S_{i}}^{(1)} + L_{S_{i}}^{(2)} + C \\ r_{S_{i}}^{(2)}[L_{S_{i}}^{(1)} - L_{S_{i}}^{(2)}] \\ r_{S_{i}}^{(3)}[L_{S_{i}}^{(1)} - L_{S_{i}}^{(2)}] \end{bmatrix}.$$  \hspace{1cm} (7.40)
with
\[ \mathcal{N}_{S_i} := \mathcal{L}_{S_i}^{(1)} + \mathcal{L}_{S_i}^{(2)} - [\mathcal{L}_{S_i}^{(1)} - \mathcal{L}_{S_i}^{(2)}] [\mathcal{L}_{S_i}^{(1)} + \mathcal{L}_{S_i}^{(2)} + \mathcal{C}]^{-1} [\mathcal{L}_{S_i}^{(1)} - \mathcal{L}_{S_i}^{(2)}] \] (7.41)
and
\[ \tilde{\mathcal{L}}_2 := \begin{bmatrix} 0 & I \\ I & [\mathcal{L}_{S_i}^{(1)} + \mathcal{L}_{S_i}^{(2)} + \mathcal{C}]^{-1} [\mathcal{L}_{S_i}^{(1)} - \mathcal{L}_{S_i}^{(2)}] \end{bmatrix}. \] (7.42)
Note that the operator
\[ \tilde{\mathcal{L}}_2 : H^{1/2} (S_i) \times \tilde{H}^{1/2} (S_i) \longrightarrow \tilde{H}^{1/2} (S_i) \times \tilde{H}^{1/2} (S_i), \] (7.43)
is invertible, while the operator
\[ \tilde{\mathcal{L}}_1 : \tilde{H}^{1/2} (S_i) \times \tilde{H}^{1/2} (S_i) \longrightarrow H^{-1/2} (S_i) \times H^{-1/2} (S_i, (7.44)) \]
is bounded. Due to the triangular structure of the operator \( \tilde{\mathcal{L}}_1 \) in (7.40) and in view of invertibility of the operator (7.38) we see that (7.44) is Fredholm with zero index if the pseudodifferential operator
\[ r_{\partial y, \partial \xi} : H^{1/2} (S_i) \longrightarrow H^{-1/2} (S_i) \] (7.45)
is Fredholm with zero index. Taking into consideration that \( \sigma^{(q)} (y, \xi') < 0 \) for all \( \xi' \in \mathbb{R}^2 \setminus \{0\} \) and for all \( y \in S_i \), we deduce that the principal homogeneous symbol \( \sigma_\lambda (y, \xi') \) of the operator \( \mathcal{N}_{S_i} \) is strictly negative,
\[ \sigma_\lambda (y, \xi') = \sigma^{(1)} (y, \xi') + \sigma^{(2)} (y, \xi') - \frac{[\sigma^{(1)} (y, \xi') - \sigma^{(2)} (y, \xi')]^2}{\sigma^{(1)} (y, \xi') + \sigma^{(2)} (y, \xi')} = -\frac{4\sigma^{(1)} (y, \xi') \sigma^{(2)} (y, \xi')}{\sigma^{(1)} (y, \xi') + \sigma^{(2)} (y, \xi')} < 0 \]
for all \( \xi' \in \mathbb{R}^2 \setminus \{0\} \) and for all \( y \in S_i \).

Therefore the pseudodifferential operator (7.45) and, consequently, (7.44) and (7.39) are Fredholm with zero index ([11], [26]). The operator (7.37) possesses the same property, since \( \mathcal{L} - \tilde{\mathcal{L}} \) is compact. This implies that the operator (7.35) is Fredholm with zero index and since
\[ \mathcal{K}^{(CTD)} = K_0 ^{(CTD)} : \mathcal{X}^{(CTD)} \longrightarrow \mathcal{Y}^{(CTD)} \]
is compact, the operator (7.34) is Fredholm with zero index as well.

It remains to show that the null space of the operator (7.34) is trivial. Let \( U_0 \in \mathcal{X}^{(CTD)} \) be a solution to the homogeneous equation \( K^{(CTD)} U_0 = 0 \). From equations (7.17) and (7.18) with zero right hand sides due to the mapping properties (3.68), (3.93) and (3.94) we then see that \( U_0 \in \mathcal{H}^{(CTD)} \).

By the equivalence Theorem 7.1 and the uniqueness Theorem 2.1 then it follows that \( U_0 = 0 \). Thus the kernel of the operator (7.34) is trivial and consequently it is invertible.

To prove invertibility of operator (7.33), we remark that for any \( \mathcal{F}^{(CTD)} \in \mathcal{F}^{(CTD)} \) a unique solution \( U^{(CTD)} \in \mathcal{X}^{(CTD)} \) of equation
\[ K^{(CTD)} U^{(CTD)} = \mathcal{F}^{(CTD)}, \] (7.46)
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is delivered by the inverse to the operator (7.34). On the other hand, since \( \mathcal{F}^{(CTD)} \in \mathbb{F}^{(CTD)} \), the first two lines of the matrix operator \( \mathcal{K}^{(CTD)} \) imply that in fact \( U^{(CTD)} \in \mathbb{H}^{(CTD)} \) and the mapping \( \mathbb{F}^{(CTD)} \to \mathbb{H}^{(CTD)} \) delivered by the inverse to the operator (7.34) is continuous, i.e., this operator gives inverse to operator (7.33) as well. \( \square \)

8. Appendix A: Classes of Localizing Functions

Let us introduce the classes for localizing functions.

**Definition A.1.**

(i) We say \( \chi \in X^k \) for integer \( k \geq 0 \) if \[ \chi(x) = \tilde{\chi}(|x|), \quad \tilde{\chi} \in W^k_1(0, \infty), \quad g\tilde{\chi}(\varrho) \in L_1(0, \infty). \] (A.1)

(ii) We say \( \chi \in X^k_1 \) for \( k \geq 1 \) if \( \chi \in X^k \), \( \chi(0) = 1 \) and \( \sigma_\chi(\omega) > 0 \) for a.e. \( \omega \in \mathbb{R} \),

where \[ \sigma_\chi(\omega) := \begin{cases} \frac{1}{\omega} \tilde{\chi}_s(\omega) & \text{for } \omega \in \mathbb{R} \setminus \{0\}, \\ \int_{0}^{\infty} g\tilde{\chi}(\varrho) \, d\varrho & \text{for } \omega = 0, \end{cases} \] (A.3)

and \( \tilde{\chi}_s(\omega) \) denotes the sine-transform of the function \( \tilde{\chi} \),

\[ \tilde{\chi}_s(\omega) := \int_{0}^{\infty} \tilde{\chi}(\varrho) \sin(\varrho\omega) \, d\varrho. \] (A.4)

(iii) We say \( \chi \in X^k_{1+} \) for \( k \geq 1 \) if \( \chi \in X^k \) and \( \omega \tilde{\chi}_s(\omega) \leq 1 \quad \forall \omega \in \mathbb{R}. \) (A.5)

Note that if \( \tilde{\chi} \) has a compact support, then the third condition in (A.1) is evidently satisfied. If \( \tilde{\chi} \in W^k(0, \infty), \) \( k \geq 1 \), then \( \tilde{\chi} \) is continuous due to the Sobolev embedding theorem, and \( \chi(0) = \tilde{\chi}(0) \) is well defined as the trace of \( \tilde{\chi} \). Evidently, we have the following embeddings, \( X^k_1 \subset X^{k_2}_1 \subset X^{k_2+}_1 \subset X^{k_2+}_{1+} \subset X^{k_2+}_{1+} \) for \( k_1 > k_2 \).

The class \( X^k_{1+} \) is defined in terms of the sine-transform. Since the classes \( X^k_{1+} \) and \( X^k_{1+} \) introduced in [7] are subsets of the corresponding classes \( X^k_{1+} \) and \( X^k_{1+} \), the following lemma implied by [7, Lemma 3.2] gives an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class.

**Lemma A.2.** If \( \chi \in X^k, \) \( k \geq 1, \) \( \tilde{\chi}(0) = 1, \) \( \tilde{\chi}(\varrho) \geq 0 \) for all \( \varrho \in (0, \infty), \) and \( \tilde{\chi} \) is a non-increasing function on \([0, +\infty), \) then \( \chi \in X^k_{1+}. \)
The following examples for $\chi$ are presented in [7],

\[
\chi_1(x) = \begin{cases} 
1 - \frac{|x|}{\varepsilon} & \text{for } |x| < \varepsilon, \\
0 & \text{for } |x| \geq \varepsilon,
\end{cases}
\]

\[
\chi_2(x) = \begin{cases} 
\exp \left[ \frac{|x|^2}{|x|^2 - \varepsilon^2} \right] & \text{for } |x| < \varepsilon, \\
0 & \text{for } |x| \geq \varepsilon,
\end{cases}
\]

One can observe that $\chi_1 \in X^k_1$, while $\chi_2 \in X^\infty_1$ due to Lemma A.2 and for them the inequality (A.2) holds for all $\omega \in \mathbb{R}$. Moreover, $\chi_1 \in X^k_2$ for $k = 2$ and $k = 3$. For details and further examples see [7].

9. Appendix B: Calculation of Symbols of Boundary Operators

Here we calculate the principal homogeneous symbols $\sigma_{\gamma(y, \xi)}(y, \xi')$ and $\sigma_{\gamma(y, \xi)}(y, \xi')$ of the boundary pseudodifferential operators $V^{(q)}_\gamma$ and $L^{(q)}_\gamma$, $q = 1, 2$, defined by formulas (3.14) and (3.17). Without loss of generality, we assume that the point $y \in \partial \Omega_q$ is the origin of some local co-ordinate system with the third co-ordinate axis coinciding with the outward unit normal vector $n(y)$. Due to the local principal technique (see, e.g. [11]), instead of $\Omega_q$, actually, we can consider the half-space $\mathbb{R}^3_+: = \{x \in \mathbb{R}^3 : x_3 < 0\}$ with the outward unit normal vector $n(y)(y) = (0, 0, 1)$ to the boundary $\partial \mathbb{R}^3_+$.

First we rewrite the fundamental solution (Levi function) of the operator $A_q(y, \partial x) = a_q(y)A_q(x, \partial x)$ (see (2.1) and (3.1)) in the following form

\[
P_{q1}(x, y) = a_q^{-1}(y)P_{q1*}(x, y) = a_q^{-1}(y)\delta_{\xi' = -x'}[A_q^{-1}(-i\xi)] = \\
= a_q^{-1}(y)\delta_{\xi' = -x'} \left[ \pm \frac{1}{2\pi} \int_{l^\pm} A_q^{-1}(-i\xi', -i\tau)e^{-i\tau x_3}d\tau \right],
\]

where $P_{q1*}(x, y)$ is defined by (3.2), the sign “+” corresponds to the case $x_3 < 0$, while the sign “−” corresponds to the case $x_3 > 0$. Here we use the notation: $x' = (x_1, x_2)$, $x = (x', x_3)$, $\xi' = (\xi_1, \xi_2)$, $\xi = (\xi', \xi_3)$, $l^\pm(l^1)$ is a closed contour orientated counterclockwise and enclosing all the roots of the polynomial $A_q(-i\xi', -i\tau)$ with respect to the variable $\tau$ in the half-plane $\text{Im} \tau > 0$ ($\text{Im} \tau < 0$).

Note that due to formulas (3.1) and (3.26)

\[
A_{q*}(\xi', \tau) = a_{q*}^{(q)}(\xi') + 2\tau \sum_{k=1}^{2} a_{k3}^{(q)} \xi_k + \sum_{k,j=1}^{2} a_{kj3}^{(q)} \xi_k \xi_j,
\]

\[
T_{q*}(\xi', \tau) = a_{q*}^{(q)}(\xi') + \sum_{k=1}^{2} a_{k3}^{(q)} \xi_k,
\]

since $n^{(q)} = (0, 0, 1)$.
Denote by $\tau_q^+$ and $\tau_q^-$ the zeros of the polynomial $A_q(\xi', \tau)$ with positive and negative imaginary parts respectively,

$$\tau_q^+(\xi') = \tau_{q1}(\xi') \pm i\tau_{q2}(\xi'), \quad \tau_{q2}(\xi') > 0,$$

where

$$\tau_{q1}(\xi') = -[a_{333}^{(q)}]^{-1} \sum_{k=1}^2 a_{k33}^{(q)} \xi_k,$$  \hspace{1cm} (B.5)

$$\tau_{q2}(\xi') = [a_{333}^{(q)}]^{-1} \sum_{k,j=1}^2 a_{k,j3}^{(q)} \xi_k \xi_j - \left( \sum_{k=1}^2 a_{k33}^{(q)} \xi_k \right)^2 > 0 \quad (B.6)$$

for all $\xi' \in \mathbb{R}^2 \setminus \{0\}.$

The latter inequality follows from the positive definiteness of the matrix $[a_{k,j3}^{(q)}]_{3 \times 3}.$

Now, in view of the representation (B.1) and formula (3.14), we get the following expression for the principal homogeneous symbol of the operator $\varphi(y):$

$$\sigma_{\varphi(y)}(y, \xi') = -\frac{1}{2\pi a_q(y)} \int_{i^+} A_q^{-1}(-i\xi', -i\tau) d\tau = \frac{1}{2\pi a_q(y)} \int_{i^+} \frac{d\tau}{A_q(\xi', \tau)} \quad (B.7)$$

and with the help of the residue theorem finally we deduce

$$\sigma_{\varphi(y)}(y, \xi') = \frac{i}{2a_q(y)} \frac{1}{a_{333}^{(q)} - \tau^+} \sum_{k=1}^2 a_{k33}^{(q)} \xi_k$$

$$= \frac{1}{2a_{333}^{(q)} a_q(y) \tau_{q2}(\xi')} > 0 \quad \text{for all} \quad \xi' \in \mathbb{R}^2 \setminus \{0\}. \quad (B.8)$$

Quite similarly, for the principal homogeneous symbol of the boundary pseudo-differential operator $\mathcal{L}^{(q)}$ with the help of (3.17) and (B.1) we get:

$$\sigma^{(q)}(y, \xi') \equiv \sigma_{\varphi^{(q)}}(y, \xi') = -\frac{1}{2\pi} \int_{i^+} T_q(y, -i\xi', -i\tau) T_q(y, i\xi', i\tau) A_q(-i\xi', -i\tau) d\tau =$$

$$= \frac{1}{2\pi} \int_{i^+} \frac{|T_q(y, \xi', \tau)|^2}{A_q(\xi', \tau)} d\tau = \frac{1}{2\pi} \int_{i^+} \frac{a_q^2(y)|T_q(y, \xi', \tau)|^2}{a_q(y) A_q(\xi', \tau)} d\tau =$$

$$= \frac{ia_q(y)}{2} \left[ a^{(q)}_{333} - \tau^+ + \sum_{k=1}^2 a_{k33}^{(q)} \xi_k \right] = -\frac{1}{2} a_{333}^{(q)} a_q(y) \tau_{q2}(\xi') < 0 \quad (B.9)$$

for all $\xi' \in \mathbb{R}^2 \setminus \{0\}.$

**Concluding Remarks**

Four segregated direct localized boundary-domain integral equation systems for several transmission problems for a scalar linear divergence PDE
with matrix variable coefficients of a special form were formulated and analyzed in the paper. They give some representative samples of different LBDIE systems that can be formulated and analyzed for such problems. The first two LBDIE systems, (TD1) and (TD2) are associated with the transmission-Dirichlet problem, where the boundary equations of the system (TD1) are of the first kind, while all the equations of the system (TD2) are of the second kind. The last two LBDIE systems are associated with the transmission-mixed problem and with the transmission-Dirichlet problem with the interface crack on a part of the interface. The boundary equations of the both these LBDIE systems are of the first kind.

Equivalence of the LBDIEs to the original variable-coefficient transmission-boundary-crack problems was proved in the case when right-hand side of the PDE is from $L^2(\Omega_q)$, and the Dirichlet and the Neumann data from the spaces $H^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}$, respectively, on the corresponding parts of the boundary. The invertibility of the operators for the LBDIE systems (TD1), (TM) and (CTD) was proved in the corresponding Sobolev spaces, employing the technique of pseudodifferential operators on manifolds. The main theorems for LBDIEs were proved under condition $\chi \in X^{-1+}_{\chi,1}$ on the localizing function, which is more relaxed than the condition $\chi \in X^{3+}_{\chi,1}$ from [7]. Condition (4.3) that the ratio of the coefficients on the interface should be constant appeared to be essential in the proof. A special consideration is needed to relax the latter condition.

Quite similarly the problems (TN), (CTN) and (CTM) can be reduced to the corresponding LBDIE systems which can be analyzed by the analogous arguments. By the same approach, the corresponding LBDIDE systems for unbounded domains can be analyzed as well. The approach can be extended also to more general PDEs and to systems of PDEs, while smoothness of the variable coefficients and the boundary can be essentially relaxed, and the PDE right hand side can be considered in more general spaces, c.f. [18, 19].

This study can serve as a basis for rigorous analysis of numerical, especially mesh-less methods for the LBDIEs that after discretization lead to sparsely populated systems of linear algebraic equations attractive for numerical computations (see e.g. [17, 21] for algorithm and implementation).

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Authors’ address:

O. Chkadua
Andrea Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: chkadua@rmi.ge

S. E. Mikhailov
Department of Mathematics
Brunel University West London
Uxbridge, UB8 3PH
UK
E-mail: Sergey.Mikhailov@brunel.ac.uk

D. Natroshvili
Department of Mathematics, Georgian Technical University
77, M. Kostava St., Tbilisi 0175
Georgia
E-mail: natrosh@hotmail.com