Memoirs on Differential Equations and Mathematical Physics
Volume 52, 2011, 17–64

O. Chkadua, S. E. Mikhailov, and D. Natroshvili

LOCALIZED DIRECT SEGREGATED BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR VARIABLE COEFFICIENT TRANSMISSION PROBLEMS WITH INTERFACE CRACK

Dedicated to the 120-th birthday anniversary
of academician N. Muskhelishvili

Abstract. Some transmission problems for scalar second order elliptic partial differential equations are considered in a bounded composite domain consisting of adjacent anisotropic subdomains having a common interface surface. The matrix of coefficients of the differential operator has a jump across the interface but in each of the adjacent subdomains is represented as the product of a constant matrix by a smooth variable scalar function. The Dirichlet or mixed type boundary conditions are prescribed on the exterior boundary of the composite domain, the Neumann conditions on the the interface crack surfaces and the transmission conditions on the rest of the interface. Employing the parametrix-based localized potential method, the transmission problems are reduced to the localized boundary-domain integral equations. The corresponding localized boundary-domain integral operators are investigated and their invertibility in appropriate function spaces is proved.

2010 Mathematics Subject Classification. 35J25, 31B10, 45P05, 45A05, 47G10, 47G30, 47G40.

Key words and phrases. Partial differential equation, transmission problem, interface crack problem, mixed problem, localized parametrix, localized boundary-domain integral equations, pseudo-differential equation.

რეზიუმე. ნაშრომში განხილულია ბზარის ტიპის ტრანსმისიის ამოცანები სკალარული, ცვლადკოეფიციენტებიანი, მეორე რიგის ელიფსური დიფერენციალური განტოლებისათვის საერთო საზღვრის მქონე კომპოზიტურ არეებში. თითოეულ არეში კოეფიციენტების მატრიცა წარმოდგება როგორც სკალარული ფუნქციისა და მუდმივი მატრიცის ნამრავლი, რომელთაც საკონტაქტო ზედაპირის გასწვრივ წყვეტა გააჩნია. კომპოზიტური არის გარე საზღვარზე მოცემულია ან დირიხლეს, ან ნეიმანის ან შერეული სასაზღვრო პირობები, ხოლო საკონტაქტო ზედაპირზე კი ან ტრანსმისიის პირობებია დასახელებული ან ტრანსმისიისა და საკონტაქტო ბზარის შერეული პირობები. ლოკალიზებული პარამეტრიქსის მეთოდის გამოყენებით ეს ამოცანები დაყვანილია ლოკალიზებულ ინტეგრალურ განტოლებათა სისტემაზე, რომელიც შეიცავს როგორც ზედაპირზე განსაზღვრულ ინტეგრალურ ოპერატორებს, ასევე არეზე განსაზღვრულ ინტეგრალურ ოპერატორებს. გამოკვლეულია მიღებული ლოკალიზებული ინტეგრალური ოპერატორების ასახვის თვისებები, დადგენილია მათი ფრედჰოლმურობა და დამტკიცებულია მათი შებრუნებადობა სობოლევის ფუნქციონალურ სივრცეებში.

1. Introduction

We consider the basic, mixed and crack type transmission problems for scalar second order elliptic partial differential equations with variable coefficients and develop the generalized potential method based on the *localized parametrix method.*

For simplicity and detailed illustration of our approach we consider the simplest case when two adjacent domains under consideration, Ω_1 and Ω_2, have a common simply connected boundary S_i called *interface surface.* The matrix of coefficients of the elliptic scalar operator in each domain is represented as the product of a constant matrix by a smooth variable scalar function. These coefficients are discontinuous across the interface surface.

We deal with the case when the Dirichlet or mixed type boundary conditions on the exterior boundary S_e of the composite domain $\overline{\Omega}_1 \cup \overline{\Omega}_2$, the Neumann conditions on the the interface crack surfaces and the transmission conditions on the rest of the interface are prescribed.

The transmission problems treated in the paper can be investigated in by the variational methods, and the corresponding uniqueness and existence results can be obtained similar to e.g., [13], [14], [15], [16].

For special cases when the fundamental solution is available the Dirichlet and Neumann type boundary value problems were also investigated by the classical potential method (see [3], [13], [16], [23]) and the references therein).

Our goal here is to show that the transmission problems in question can be equivalently reduced to some *localized boundary-domain integral equations* (LBDIE) and that the corresponding *localized boundary-domain integral operators* (LBDIO) are invertible, which beside a pure mathematical interest may have also some applications in numerical analysis for construction of efficient numerical algorithms (see, e.g., [17], [21], [27], [30], [31] and the references therein). In our case, the localized parametrix $P_{q\chi}(x-y,y)$, $q=1,2$, is represented as the product of a Levi function $P_{q1}(x-y,y)$ of the differential operator under consideration by an appropriately chosen cut-off function $\chi_q(x-y)$ supported on some neighbourhood of the origin. Clearly, the kernels of the corresponding localized potentials are supported in some neighbourhood of the reference point y (assuming that x is an integration variable) and they do not solve the original differential equation.

In spite of the fact that the localized potentials preserve almost all mapping properties of the classical non-localized ones (cf. [7]), some unusual properties of the localized potentials appear due to the localization of the kernel functions which have no counterparts in classical potential theory and which need special consideration and analysis.

By means of the direct approach based on Green's representation formula we reduce the transmission problems to the *localized boundary-domain integral equation* (LBDIE) system. First we establish the equivalence between the original transmission problems and the corresponding LBDIEs systems which proved to be a quite nontrivial problem and plays a crucial role in our analysis. Afterwards we investigate Fredholm properties of the LBDIOs and prove their invertibility in appropriate function spaces. This paper is heavily based and essentially develops methods and results of [5], [6], [7], [8], [19].

2. Transmission Problems

Let Ω and Ω_1 be bounded open domains in $\mathbb{R}^3$ and $\overline{\Omega}_1 \subset \Omega$. Denote $\Omega_2 := \Omega \setminus \overline{\Omega}_1$ and $S_i := \partial\Omega_1$, $S_e := \partial\Omega$. Clearly, $\partial\Omega_2 = S_i \cup S_e$. We assume that the *interface surface* S_i and the *exterior boundary* S_e of the composite body $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ are sufficiently smooth, say C^∞-regular if not otherwise stated.

Throughout the paper $n^{(q)} = n^{(q)}(x)$ denotes the unit normal vector to $\partial\Omega_q$ directed outward the domains Ω_q. Clearly, $n^{(1)}(x) = -n^{(2)}(x)$ for $x \in S_i$.

By $H^r(\Omega') = H^r_2(\Omega')$ and $H^r(S) = H^r_2(S)$, $r \in \mathbb{R}$, we denote the Bessel potential spaces on a domain Ω' and on a closed manifold S without boundary. The subspace of $H^r(\mathbb{R}^3)$ of functions with compact support is denoted by $H^r_{comp}(\mathbb{R}^3)$. Recall that $H^0(\Omega') = L_2(\Omega')$ is a space of square integrable functions in Ω'.

For a smooth proper submanifold $\mathcal{M} \subset S$ we denote by $\widetilde{H}^r(\mathcal{M})$ the subspace of $H^r(S)$,

$$\widetilde{H}^r(\mathcal{M}) := \big\{g :\ g \in H^r(S),\ \operatorname{supp} g \subset \overline{\mathcal{M}}\big\},$$

while $H^r(\mathcal{M})$ denotes the spaces of restrictions on $\mathcal{M}$ of functions from $H^r(S)$,

$$H^r(\mathcal{M}) := \big\{r_{\mathcal{M}} f :\ f \in H^r(S)\big\},$$

where $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$.

Let us consider the differential operators in the domains Ω_q

$$A_q(x, \partial_x)u(x) := \sum_{j,k=1}^{3} \partial_{x_k}\big[a^{(q)}_{kj}(x)\partial_{x_j} u(x)\big],\ \ q = 1, 2, \tag{2.1}$$

where $\partial_x = (\partial_1, \partial_2, \partial_3)$ $\partial_j = \partial_{x_j} = \partial/\partial x_j$, $j = 1, 2, 3$, and

$$a^{(q)}_{kj}(x) = a^{(q)}_{jk}(x) = a_q(x) a^{(q)}_{kj\star}, \tag{2.2}$$

$$\mathbf{a}_q(x) := [a^{(q)}_{kj}(x)]_{3\times 3} = a_q(x)[a^{(q)}_{kj\star}]_{3\times 3}, \quad \mathbf{a}_{q\star} := [a^{(q)}_{kj\star}]_{3\times 3}. \tag{2.3}$$

Here $a^{(q)}_{kj\star}$ are constants and the matrix $\mathbf{a}_{q\star} := [a^{(q)}_{kj\star}]_{3\times 3}$ is positive definite. Moreover, we assume that

$$a_q \in C^\infty(\mathbb{R}^3),\ \ 0 < c_0 \le a_q(x) \le c_1 < \infty,\ \ q = 1, 2. \tag{2.4}$$

Further, for sufficiently smooth functions (from the space $H^2(\Omega_q)$ say) we introduce the co-normal derivative operator on $\partial\Omega_q$, $q = 1, 2$, in the usual trace sense:

$$T_q(x,\partial_x)u(x) \equiv T_q^+(x,\partial_x)u(x) :=$$
$$:= \sum_{k,j=1}^{3} a_{kj}^{(q)}(x) n_k^{(q)}(x)\gamma_q[\partial_{x_j}u(x)], \quad x\in\partial\Omega_q, \tag{2.5}$$

where the symbol $\gamma_q \equiv \gamma_q^+$ denotes the trace operator on $\partial\Omega_q$ from the interior of Ω_q. Analogously is defined the external co-normal derivative operator $T_q^-(x,\partial_x)w$ with the help of the exterior trace operator γ_q^- on $\partial\Omega_q$ denoting the limiting value on $\partial\Omega_q$ from the exterior domain $\Omega_q^c := \mathbb{R}^3\setminus\overline{\Omega}_q$:

$$T_q^-(x,\partial_x)u(x) := \sum_{k,j=1}^{3} a_{kj}^{(q)}(x) n_k^{(q)}(x)\gamma_q^-[\partial_{x_j}u(x)], \quad x\in\partial\Omega_q.$$

We set

$$H^{1,0}(\Omega_q;A_q) := \{v\in H^1(\Omega_q):\ A_q v\in H^0(\Omega_q)\},\quad q=1,2. \tag{2.6}$$

One can correctly define the generalized (canonical) co-normal derivatives $T_q u \equiv T_q^+ u \in H^{-\frac12}(\partial\Omega_q)$ (cf., for example, [9, Lemma 3.2], [16, Lemma 4.3], [20, Definition 3.3]),

$$\langle T_q u, w\rangle_{\partial\Omega_q} \equiv \langle T_q^+ u, w\rangle_{\partial\Omega_q} :=$$
$$:= \int_{\Omega_q} \big[(\ell_q w)A_q u + E_q(u,\ell_q w)\big]\,dx \quad \forall\, w\in H^{\frac12}(\partial\Omega_q), \tag{2.7}$$

where ℓ_q is a continuous linear extension operator, $\ell_q : H^{\frac12}(\partial\Omega_q)\to H^1(\Omega_q)$ which is a right inverse to the trace operator γ_q,

$$E_q(u,v) := \sum_{i,j=1}^{3} a_{ij}^{(q)}(x)\,\frac{\partial u(x)}{\partial x_i}\,\frac{\partial v(x)}{\partial x_j} \equiv \nabla_x u\cdot \mathbf{a}_q(x)\nabla_x v,\quad \nabla_x := (\partial_1,\partial_2,\partial_3)^\top.$$

Here and in what follows the central dot denotes the scalar product in $\mathbb{R}^3$ or in $\mathbb{C}^3$. In (2.7), the symbol $\langle g_1,g_2\rangle_{\partial\Omega_q}$ denotes the duality brackets between the spaces $H^{-\frac12}(\partial\Omega_q)$ and $H^{\frac12}(\partial\Omega_q)$, coinciding with $\int_{\partial\Omega_q} g_1(x)g_2(x)\,dS$ if $g_1,\ g_2\in L_2(\partial\Omega_q)$. Below for such dualities we will use sometimes the usual integral symbols when they do not cause confusion. The canonical co-normal derivative operators $T_q : H^{1,0}(\Omega_q;A_q)\to H^{-\frac12}(\partial\Omega_q)$ defined by (2.7) are continuous extensions of the classical co-normal derivative operators from (2.5), and the second Green identity

$$\int_{\Omega_q} [vA_q u - uA_q v]\,dx = \int_{\partial\Omega_q} \big[(\gamma_q v)T_q u - (\gamma_q u)T_q v\big]\,dS,\quad q=1,2, \tag{2.8}$$

holds for $u,v\in H^{1,0}(\Omega_q;A_q)$.

Now we formulate the following Dirichlet, Neumann and mixed type transmission problems:

Find functions $u_1 \in H^{1,0}(\Omega_1; A_1)$ and $u_2 \in H^{1,0}(\Omega_2; A_2)$ satisfying the differential equations

$$A_q(x, \partial)u_q = f_q \ \text{ in } \ \Omega_q, \ \ q = 1, 2, \tag{2.9}$$

the transmission conditions on the interface surface

$$\gamma_1 u_1 - \gamma_2 u_2 = \varphi_{0i} \ \text{ on } \ S_i, \tag{2.10}$$

$$T_1 u_1 + T_2 u_2 = \psi_{0i} \ \text{ on } \ S_i, \tag{2.11}$$

and one of the following conditions on the exterior boundary:
the Dirichlet boundary condition

$$\gamma_2 u_2 = \varphi_{0e} \ \text{ on } \ S_e; \tag{2.12}$$

or the Neumann boundary condition

$$T_2 u_2 = \psi_{0e} \ \text{ on } \ S_e, \tag{2.13}$$

or mixed type boundary conditions

$$\gamma_2 u_2 = \varphi_{0e}^{(M)} \ \text{ on } \ S_{eD}, \tag{2.14}$$

$$T_2 u_2 = \psi_{0e}^{(M)} \ \text{ on } \ S_{eN}, \tag{2.15}$$

where S_{eD} and S_{eN} are smooth disjoint submanifolds of S_e: $S_e = \overline{S_{eD}} \cup \overline{S_{eN}}$ and $S_{eD} \cap S_{eN} = \varnothing$.

We will call these boundary transmission problems as (TD), (TN) and (TM) problems.

For the data in the above formulated problems we assume

$$\begin{gathered} \varphi_{0i} \in H^{\frac{1}{2}}(S_i), \ \ \psi_{0i} \in H^{-\frac{1}{2}}(S_i), \ \ \varphi_{0e} \in H^{\frac{1}{2}}(S_e), \ \ \psi_{0e} \in H^{-\frac{1}{2}}(S_e), \\ \varphi_{0e}^{(M)} \in H^{\frac{1}{2}}(S_{eD}), \ \ \psi_{0e}^{(M)} \in H^{-\frac{1}{2}}(S_{eN}), \ \ f_q \in H^0(\Omega_q), \ \ q = 1, 2. \end{gathered} \tag{2.16}$$

Equations (2.1) are understood in the distributional sense, the Dirichlet type boundary value and transmission conditions are understood in the usual trace sense, while the Neumann type boundary value and transmission conditions for the co-normal derivatives are understood in the sense of the *canonical* co-normal derivatives defined by (2.7).

We recall that the normal vectors $n^{(1)}$ and $n^{(2)}$ in the definitions of the co-normal derivatives $T_1 u$ and $T_2 u$ on S_i have opposite directions.

Further, for the case when the interface crack is present, let the interface S_i be a union of smooth disjoint proper submanifolds, the interface crack part $S_i^{(c)}$ and the transmission part $S_i^{(t)}$, i.e., $S_i = \overline{S_i^{(c)}} \cup \overline{S_i^{(t)}}$ and $S_i^{(c)} \cap S_i^{(t)} = \varnothing$.

Let us set the following interface crack type transmission problems for the composite domain $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$:
Find functions $u_1 \in H^{1,0}(\Omega_1; A_1)$ and $u_2 \in H^{1,0}(\Omega_2; A_2)$ satisfying the differential equations (2.9) in Ω_1 and Ω_2 respectively, one of the boundary conditions

(2.12), or (2.13), or (2.14)–(2.15) on the exterior boundary S_e, the transmission conditions on $S_i^{(t)}$

$$\gamma_1 u_1 - \gamma_2 u_2 = \varphi_{0i}^{(t)} \text{ on } S_i^{(t)}, \tag{2.17}$$

$$T_1 u_1 + T_2 u_2 = \psi_{0i}^{(t)} \text{ on } S_i^{(t)}, \tag{2.18}$$

and the crack type conditions on $S_i^{(c)}$

$$T_1 u_1 = \psi'_{0i} \text{ on } S_i^{(c)}, \tag{2.19}$$

$$T_2 u_2 = \psi''_{0i} \text{ on } S_i^{(c)}. \tag{2.20}$$

We will call these crack type boundary transmission problems as (CTD), (CTN) and (CTM) problems, respectively.

Along with the conditions (2.16), for the data in the above formulated crack type problems we require that

$$\begin{aligned} &\varphi_{0i}^{(t)} \in H^{\frac{1}{2}}(S_i^{(t)}), \quad \psi_{0i}^{(t)} \in H^{-\frac{1}{2}}(S_i^{(t)}), \\ &\psi'_{0i} \in H^{-\frac{1}{2}}(S_i^{(c)}), \quad \psi''_{0i} \in H^{-\frac{1}{2}}(S_i^{(c)}). \end{aligned} \tag{2.21}$$

It is easy to see that for the function

$$\psi_{0i} := \begin{cases} \psi_{0i}^{(t)} & \text{on } S_i^{(t)}, \\ \psi'_{0i} + \psi''_{0i} & \text{on } S_i^{(c)}, \end{cases} \tag{2.22}$$

the following embedding

$$\psi_{0i} \in H^{-1/2}(S_i) \tag{2.23}$$

is a necessary compatibility condition for the above formulated interface crack problems to be solvable in the space $H^{(1,0)}(\Omega_1; A_1) \times H^{(1,0)}(\Omega_2; A_2)$ since

$$\psi_{0i} = T_1 u_1 + T_2 u_2 \text{ on } S_i. \tag{2.24}$$

In what follows we assume that for ψ_{0i} given by (2.22) the condition (2.23) is satisfied.

As we have mentioned in the introduction, all the above formulated transmission problems can be investigated by the functional-variational methods and the corresponding uniqueness and existence results can be obtained similar to e.g., [13], [15], [16]. In particular, there holds the following proposition which can be proved on the basis of the Lax-Milgram theorem.

Theorem 2.1. *If the conditions* (2.16), (2.21), *and* (2.23) *are satisfied, then*

(i) *The transmission problems* (TD), (TM), (CTD), *and* (CTM) *are uniquely solvable in the space* $H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$.

(ii) *The following condition*

$$\int_{\Omega_1} f_1\, dx + \int_{\Omega_2} f_2\, dx = \int_{S_i} \psi_{0i}\, dS + \int_{S_e} \psi_{0e}\, dS \tag{2.25}$$

is necessary and sufficient for the transmission problem (TN) to be solvable in the space $H^{1,0}(\Omega_1;A_1)\times H^{1,0}(\Omega_2;A_2)$. *The same condition* (2.25) *with the function* ψ_{0i} *defined by* (2.22) *is necessary and sufficient for the crack type transmission problem (CTN) to be solvable in the space* $H^{1,0}(\Omega_1;A_1)\times H^{1,0}(\Omega_2;A_2)$. *In both cases a solution pair* (u_1,u_2) *is defined modulo a constant summand* (c,c).

We recall that our goal here is to show that the above transmission problems can be equivalently reduced to some segregated LBDIEs and to perform full analysis of the corresponding LBDIOs.

3. Properties of Localized Potentials

It is well known that the fundamental solution-function of the elliptic operator with constant coefficients

$$A_{q\star}(\partial):=\sum_{i,j=1}^{3}a^{(q)}_{kj\star}\partial_k\partial_j \tag{3.1}$$

is written as (see. e.g., [22], [23])

$$P_{q1\star}(x)=\frac{\alpha_q}{(x\cdot\mathbf{a}^{-1}_{q\star}x)^{\frac12}}\ \text{ with }\ \alpha_q=-\frac{1}{4\pi[\det\mathbf{a}_{q\star}]^{\frac12}}\,,\ \ \mathbf{a}_{q\star}=[a^{(q)}_{kj\star}]_{3\times3}. \tag{3.2}$$

Here $\mathbf{a}^{-1}_{q\star}$ stands for the inverse matrix to $\mathbf{a}_{q\star}$. Clearly, $\mathbf{a}^{-1}_{q\star}$ is symmetric and positive definite. Therefore there is a symmetric positive definite matrix $\mathbf{d}_{q\star}$ such that $\mathbf{a}^{-1}_{q\star}=\mathbf{d}^2_{q\star}$ and

$$(x\cdot\mathbf{a}^{-1}_{q\star}x)=|\mathbf{d}_{q\star}x|^2,\quad \det\mathbf{d}_{q\star}=[\det\mathbf{a}_{q\star}]^{-\frac12}\,. \tag{3.3}$$

Throughout the paper the subscript $\star$ means that the corresponding operator, matrix or function is related to the operator with constant coefficients (3.1).

Note that

$$A_{q\star}(\partial_x)P_{q1\star}(x-y)=\delta(x-y), \tag{3.4}$$

where $\delta(\cdot)$ is the Dirac distribution.

Now we introduce the *localized parametrix* (*localized Levi function*) for the operator A_q,

$$P_q(x-y,y)\equiv P_{q\chi}(x-y,y):=\frac{1}{a_q(y)}\,\chi_q(x-y)P_{q1\star}(x-y),\ \ q=1,2, \tag{3.5}$$

where χ is a localizing cut-off function (see Appendix A)

$$\chi_q(x):=\chi(\mathbf{d}_{q\star}x)=\breve\chi(|\mathbf{d}_{q\star}x|)=\breve\chi\big((x\cdot\mathbf{a}^{-1}_{q\star}x)^{1/2}\big),\ \ \chi\in X^k,\ \ k\ge1. \tag{3.6}$$

Throughout the paper we assume that the condition (3.6) is satisfied if not otherwise stated.

One can easily check the following relations

$$A_q(x,\partial_x)u(x) = a_q(x)A_{q\star}(\partial_x)u(x) + \nabla_x a_q(x)\cdot \mathbf{a}_{q\star}\nabla_x u(x), \quad (3.7)$$

$$A_q(x,\partial_x)P_q(x-y,y) = \delta(x-y) + R_q(x,y), \quad q=1,2, \quad (3.8)$$

where

$$R_q(x,y) =$$
$$= \frac{a_q(x)}{a_q(y)}\Big[P_{q1\star}(x-y)A_{q\star}(\partial_x)\chi_q(x-y) + 2\nabla_x\chi_q(x-y)\cdot \mathbf{a}_{q\star}\nabla_x P_{q1\star}(x-y)\Big] +$$
$$+ \frac{1}{a_q(y)}\Big(\nabla_x a_q(x)\cdot \mathbf{a}_{q\star}\nabla_x\big[\chi_q(x-y)P_{q1\star}(x-y)\big]\Big). \quad (3.9)$$

The function $R_q(x,y)$ possesses a weak singularity of type $\mathcal{O}(|x-y|^{-2})$ as $x\to y$ if χ_q is smooth enough, e.g., if $\chi_q \in X^2$.

Let us introduce the localized surface and volume potentials, based on the localized parametrix P_q,

$$V_S^{(q)}g(y) := -\int_S P_q(x-y,y)g(x)\,dS_x, \quad (3.10)$$

$$W_S^{(q)}g(y) := -\int_S \big[T_q(x,\partial_x)P_q(x-y,y)\big]g(x)\,dS_x, \quad (3.11)$$

$$\mathcal{P}_q f(y) := \int_{\Omega_q} P_q(x-y,y)f(x)\,dx, \quad (3.12)$$

$$\mathcal{R}_q f(y) := \int_{\Omega_q} R_q(x,y)f(x)\,dx. \quad (3.13)$$

Here and further on

$$S\in\{S_i, S_e, \partial\Omega_2\}.$$

Note that for layer potentials we drop the subindex S when $S=\partial\Omega_q$, i.e., $V^{(q)} := V^{(q)}_{\partial\Omega_q}$, $W^{(q)} := W^{(q)}_{\partial\Omega_q}$. If the domain of integration in (3.12) is the whole space $\Omega_q=\mathbb{R}^3$, we employ the notation $\mathcal{P}_q f = \mathbf{P}_q f$.

Let us also define the corresponding boundary operators generated by the direct values of the localized single and double layer potentials and their co-normal derivatives on S,

$$\mathcal{V}_S^{(q)}g(y) := -\int_S P_q(x-y,y)g(x)\,dS_x, \quad (3.14)$$

$$\mathcal{W}_S^{(q)}g(y) := -\int_S \big[T_q(x,\partial_x)P_q(x-y,y)\big]g(x)\,dS_x, \quad (3.15)$$

$$\mathcal{W}_S'^{(q)}g(y) := -\int_S \big[T_q(y,\partial_y)P_q(x-y,y)\big]g(x)\,dS_x, \quad (3.16)$$

$$\mathcal{L}_{S}^{(q)\pm} g(y) := T_q^{\pm}(y, \partial_y) W_{S}^{(q)} g(y). \tag{3.17}$$

For the pseudodifferential operator in (3.17), we employ also the notation $\mathcal{L}_{S}^{(q)} := \mathcal{L}_{S}^{(q)+}$.

Note that the kernel functions of the operators (3.15) and (3.16) are at most weakly singular if the cut-of function $\chi \in X^2$ and the surface S is $C^{1,\alpha}$ smooth with $\alpha > 0$:

$$\begin{aligned} T_q(x, \partial_x) P_q(x - y, y) &= \mathcal{O}(|x - y|^{-2+\alpha}), \\ T_q(y, \partial_y) P_q(x - y, y) &= \mathcal{O}(|x - y|^{-2+\alpha}) \end{aligned} \tag{3.18}$$

for sufficiently small $|x - y|$ (cf. [23], [22], [7]).

We will also need a localized parametrix of the constant-coefficient differential operator $A_{q\star}(\partial)$,

$$P_{q\star}(x - y) := \chi_{q}(x - y) P_{q1\star}(x - y) = a_q(y) P_q(x - y, y). \tag{3.19}$$

We have

$$A_{q\star}(\partial_x) P_{q\star}(x - y) = \delta(x - y) + R_{q\star}(x, y), \tag{3.20}$$

where

$$R_{q\star}(x, y) =$$
$$= P_{q1\star}(x-y) A_{q\star}(\partial_x) \chi_{q}(x-y) + 2\nabla_x \chi_{q}(x-y) \cdot \mathbf{a}_{q\star} \nabla_x P_{q1\star}(x-y). \tag{3.21}$$

Denote the surface and volume potentials constructed with the help of the localized parametrix $P_{q\star}$ by the symbols $V_{S\star}^{(q)}$, $W_{S\star}^{(q)}$, $\mathcal{P}_{q\star}$ and $\mathcal{R}_{q\star}$,

$$V_{S\star}^{(q)} g(y) := -\int_S P_{q\star}(x - y) g(x)\, dS_x, \tag{3.22}$$

$$W_{S\star}^{(q)} g(y) := -\int_S \big[T_{q\star}(x, \partial_x) P_{q\star}(x - y)\big] g(x)\, dS_x, \tag{3.23}$$

$$\mathcal{P}_{q\star} f(y) := \int_{\Omega_q} P_{q\star}(x - y) f(x)\, dx, \tag{3.24}$$

$$\mathcal{R}_{q\star} f(y) := \int_{\Omega_q} R_{q\star}(x - y) f(x)\, dx. \tag{3.25}$$

Here $T_{q\star}$ stands for the co-normal derivative operator corresponding to the constant coefficient differential operator $A_{q\star}(\partial)$, which for sufficiently smooth u takes form

$$T_{q\star}(x, \partial_x) u(x) \equiv$$
$$\equiv T_{q\star}^{+}(x, \partial_x) u(x) := \sum_{k,j}^{3} a_{kj\star}^{(q)} n_k^{(q)}(x) \gamma_q[\partial_{x_j} u(x)], \quad x \in \partial\Omega_q, \tag{3.26}$$

that can be continuously extended to $u \in H^{1,0}(\Omega_q; A_{q\star})$ similar to (2.7). Note that

$$H^{1,0}(\Omega_q; A_q) = H^{1,0}(\Omega_q; A_{q\star}) \text{ and } T_q(x, \partial_x)u(x) = a_q(x)T_{q\star}(x, \partial_x)u(x)$$

due to (2.5) and (3.26). Again, if the domain of integration in (3.24) is the whole space $\Omega_q = \mathbb{R}^3$, we employ the notation $\mathcal{P}_{q\star} f = \mathbf{P}_{q\star} f$.

Further, we introduce the boundary operators generated by the direct values of the localized layer potentials (3.22) and (3.23), and their co-normal derivatives on S,

$$\mathcal{V}_{S\star}^{(q)} g(y) := -\int\limits_S P_{q\star}(x-y)g(x)dS_x, \tag{3.27}$$

$$\mathcal{W}_{S\star}^{(q)} g(y) := -\int\limits_S \big[T_{q\star}(x, \partial_x)P_{q\star}(x-y)\big]g(x)dS_x, \tag{3.28}$$

$$\mathcal{W}_{S\star}^{\prime(q)} g(y) := -\int\limits_S \big[T_{q\star}(y, \partial_y)P_{q\star}(x-y)\big]g(x)\, dS_x, \tag{3.29}$$

$$\mathcal{L}_{S\star}^{(q)\pm} g(y) := T_{q\star}^{\pm}(y, \partial_y)W_{S\star}^{(q)} g(y). \tag{3.30}$$

For the pseudodifferential operator in (3.30), we employ also the notation $\mathcal{L}_{S\star}^{(q)} := \mathcal{L}_{S\star}^{(q)+}$.

In view of the relations (3.5) and (3.19) it follows that

$$V_S^{(q)} g(y) = a_q^{-1}(y)V_{S\star}^{(q)} g(y), \tag{3.31}$$

$$W_S^{(q)} g(y) = a_q^{-1}(y)W_{S\star}^{(q)}(a_q g)(y), \tag{3.32}$$

$$\mathcal{P}_q f(y) = a_q^{-1}(y)\mathcal{P}_{q\star} f(y). \tag{3.33}$$

Therefore, the potentials with and without subscript "$\star$" have exactly the same mapping and smoothness properties for sufficiently smooth variable coefficients a_q.

Before we go over to the localized boundary-domain integral formulation of the above stated transmission problems we derive some basic properties of the layer and volume potentials corresponding to the localized parametrix $P_{q\star}$ needed in our further analysis (cf. [7], [13]).

To this end let us note that the volume potential $\mathbf{P}_{q\star} f$, as a convolution of $P_{q\star}$ and f, can be represented as a pseudodifferential operator

$$\mathbf{P}_{q\star} f(y) = \mathfrak{F}^{-1}_{\xi\to y}\big[\widetilde{P}_{q\star}(\xi)\widetilde{f}(\xi)\big], \tag{3.34}$$

where $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ stand for the generalized direct and inverse Fourier transform operators, respectively, and overset "tilde" denotes the direct Fourier transform,

$$\begin{aligned} \mathfrak{F}_{x\to\xi}[f] \equiv \widetilde{f}(\xi) &:= \int\limits_{\mathbb{R}^3} f(x)e^{ix\cdot\xi}\,dx, \\ \mathfrak{F}^{-1}_{\xi\to y}[f] &:= \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} f(\xi)e^{-iy\cdot\xi}\,d\xi. \end{aligned} \tag{3.35}$$

The properties of the symbol function $\widetilde{P}_{q\star}(\xi)$ of the pseudodifferential operator $\mathbf{P}_{q\star}$ is described by the following assertion.

Lemma 3.1.

(i) *Let* $\chi \in X^k$, $k \geq 0$. *Then* $\widetilde{P}_{q\star}(\xi) \in C(\mathbb{R}^3)$ *and for* $\xi \neq 0$ *the following expansion holds*

$$\widetilde{P}_{q\star}(\xi) = \sum_{m=0}^{k^*} \frac{(-1)^{m+1}}{|\xi\cdot\mathbf{a}_{q\star}\xi|^{m+1}}\ \breve{\chi}^{(2m)}(0) - \\ - \frac{1}{|\xi\cdot\mathbf{a}_{q\star}\xi|^{(k+1)/2}}\int\limits_0^\infty \sin\Big(|\xi|\varrho+\frac{k\pi}{2}\Big)\breve{\chi}^{(k)}(\varrho)\,d\varrho, \tag{3.36}$$

where k^* *is the integer part of* $(k-1)/2$ *and the sum disappears in* (3.36) *if* $k^* < 0$, *i.e., if* $k = 0$.

(ii) *If* $\chi \in X^1_*$, *then*

$$\widetilde{P}_{q\star}(\xi) < 0 \ \ \textit{for almost all}\ \ \xi \in \mathbb{R}^3. \tag{3.37}$$

(iii) *If* $\chi \in X^1_*$ *and* $\sigma_\chi(\omega) > 0$ *for all* $\omega \in \mathbb{R}$ (*see Definition* A.1), *then* $\widetilde{P}_{q\star}(\xi) < 0$ *for all* $\xi \in \mathbb{R}^3$ *and there are positive constants* c_1 *and* c_2 *such that*

$$\frac{c_1}{1+|\xi|^2} \leq |\widetilde{P}_{q\star}(\xi)| \leq \frac{c_2}{1+|\xi|^2} \ \ \textit{for all}\ \ \xi \in \mathbb{R}^3. \tag{3.38}$$

Proof. By formulas (3.2) and (3.3) we have

$$\begin{aligned} \widetilde{P}_{q\star}(\xi) &= \int\limits_{\mathbb{R}^3} \frac{\alpha_q\chi(\mathbf{d}_{q\star}x)}{(x\cdot\mathbf{a}^{-1}_{q\star}x)^{\frac12}}e^{ix\cdot\xi}\,dx = \int\limits_{\mathbb{R}^3} \frac{\alpha_q\chi(\mathbf{d}_{q\star}x)}{|\mathbf{d}_{q\star}x|}e^{ix\cdot\xi}\,dx = \\ &= \frac{\alpha_q}{\det\mathbf{d}_{q\star}}\int\limits_{\mathbb{R}^3} \frac{\chi(\eta)}{|\eta|}e^{i\eta\cdot\mathbf{d}^{-1}_{q\star}\xi}\,d\eta = -\frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \frac{\chi(\eta)}{|\eta|}e^{i\eta\cdot\mathbf{d}^{-1}_{q\star}\xi}\,d\eta = \\ &= -\frac{1}{4\pi}\,\mathfrak{F}_{\eta\to\mathbf{d}^{-1}_{q\star}\xi}\Big[\frac{\chi(\eta)}{|\eta|}\Big] = -\frac{1}{|\zeta|}\int\limits_0^\infty \breve{\chi}(\varrho)\sin(\varrho|\zeta|)\,d\varrho = \qquad (3.39) \\ &= -\frac{\widehat{\chi}_s(|\zeta|)}{|\zeta|} \ \ \text{with}\ \ \zeta = \mathbf{d}^{-1}_{q\star}\xi. \qquad (3.40) \end{aligned}$$

Now (3.36) can be easily obtained from (3.39) by the integration by parts formula taking into account that $\breve{\chi}^{(k-1)}(\varrho) \to 0$ as $\rho \to \infty$ if $\breve{\chi} \in W^k_1(0,\infty)$.

Further, since $|\zeta|^2 = |\mathbf{d}_{q\star}^{-1}\xi|^2 = \xi \cdot \mathbf{a}_{q\star}\xi$, the proof of items (ii) and (iii) follow from (3.40), (3.36) and Definition A.1. □

By positive definiteness of the matrices $\mathbf{a}_{q\star}$ and in view of the equality (3.33), $\mathbf{P}_q = a_q^{-1}\mathbf{P}_{q\star}$, Lemma 3.1(i) implies the following important assertion.

Theorem 3.2. *There exists a positive constant c_1 such that*

$$|\widetilde{P}_{q\star}(\xi)| \le c_1\,(1+|\xi|^2)^{-\frac{k+1}{2}} \ \ \textit{for all}\ \ \xi \in \mathbb{R}^3 \ \ \textit{if}\ \ \chi \in X^k, \ \ k=0,1, \tag{3.41}$$

and the operators

$$\mathbf{P}_q,\, \mathbf{P}_{q\star} : H^t(\mathbb{R}^3) \longrightarrow H^{t+k+1}(\mathbb{R}^3) \ \ \forall\, t \in \mathbb{R} \ \ \textit{if}\ \ \chi \in X^k, \ \ k=0,1, \tag{3.42}$$

are continuous.

In particular, we see that the operators

$$\mathcal{P}_{q\star},\, \mathcal{P}_q : H^0(\Omega_q) \longrightarrow H^2(\mathbb{R}^3) \tag{3.43}$$

are continuous for arbitrary bounded domain $\Omega_q \subset \mathbb{R}^3$ if $\chi \in X^1$.

More restrictions on χ lead to the following counterpart of [7, Corollary 5.2(ii)].

Lemma 3.3. *Let $\chi \in X^1_*$ and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$ (see Definition* A.1*). Then the operator*

$$\mathbf{P}_{q\star} : H^r(\mathbb{R}^3) \longrightarrow H^{r+2}(\mathbb{R}^3), \ \ r \in \mathbb{R}, \ \ q=1,2, \tag{3.44}$$

is invertible and the inverse operator $\mathbf{P}_{q\star}^{-1}$ is a pseudodifferential operator with the symbol $\widetilde{P}_{q\star}^{-1}(\xi)$.

Moreover, if $\chi \in X^1_{1*}$, then

$$\widetilde{P}_{q\star}^{-1}(\xi) = -\xi \cdot \mathbf{a}_{q\star}\xi - \nu_{q\star}(\xi), \tag{3.45}$$

where

$$\nu_{q\star}(\xi) = \mathcal{O}(1), \ \ \nu_{q\star}(\xi) \ge 0 \ \ \textit{for all}\ \ \xi \in \mathbb{R}^3. \tag{3.46}$$

The pseudodifferential operator $\mathbf{P}_{q\star}^{-1}$ can be decomposed as

$$\mathbf{P}_{q\star}^{-1} = A_{q\star}(\partial) - \mathbf{N}_{q\star}, \tag{3.47}$$

where $A_{q\star}(\partial)$ is a partial differential operator with constant coefficients defined by (3.1) *and $\mathbf{N}_{q\star}$ is a pseudodifferential operator with the symbol $\nu_{q\star}(\xi)$.*

Proof. It is an immediate consequence of Lemma 3.1(iii) except the inequality in (3.46) which follows from the imbedding $\chi \in X^1_{1*}$. In fact, we have

$$\nu_{q\star}(\xi) = -\widetilde{P}_{q\star}^{-1}(\xi) - \xi \cdot \mathbf{a}_{q\star}\xi = -\frac{1 + (\xi \cdot \mathbf{a}_{q\star}\xi)\widetilde{P}_{q\star}(\xi)}{\widetilde{P}_{q\star}(\xi)} \ \ \text{for all}\ \ \xi \in \mathbb{R}^3. \tag{3.48}$$

Use the notation $\zeta = \mathbf{d}_{q\star}^{-1}\xi$, take into account the relations (A.4), (3.38), (3.39) and $|\mathbf{d}_{q\star}^{-1}\xi|^2 = \mathbf{a}_{q\star}\xi\cdot\xi$ to obtain

$$\nu_{q\star}(\xi) = \big[1 - |\zeta|\widehat{\chi}_s(|\zeta|)\big]\,\frac{|\zeta|}{\widehat{\chi}_s(|\zeta|)} = \frac{1 - |\zeta|\widehat{\chi}_s(|\zeta|)}{\sigma_\chi(|\zeta|)} \;\text{ for all }\; \xi\in\mathbb{R}^3. \tag{3.49}$$

Now the desired inequality follows due to the relations (A.5) and $\sigma_\chi(\omega) > 0$ for all $\omega\in\mathbb{R}$. □

Let us also denote,

$$\mathbf{R}_{q\star}f := \int_{\mathbb{R}^3} R_{q\star}(x-y)f(x)\,dx = \mathfrak{F}^{-1}(\widetilde{R}_{q\star}\widetilde{f}),$$

where the kernel $R_{q\star}(x-y)$ is given by (3.20)–(3.21) and $\widetilde{R}_{q\star} = \mathfrak{F}R_{q\star}$.

Theorem 3.4. *Let $\chi\in X^k$, $k\ge 1$. Then*

$$\widetilde{R}_{q\star}(\xi) = -(\xi\cdot\mathbf{a}_{q\star}\xi)\widetilde{P}_{q\star} - 1 = |\zeta|\hat{\chi}_s(|\zeta|) - 1 = \tag{3.50}$$

$$= \sum_{m=1}^{k^*}\frac{(-1)^{m+1}}{|\xi\cdot\mathbf{a}_{q\star}\xi|^m}\,\breve{\chi}^{(2m)}(0) - $$

$$- \frac{1}{|\xi\cdot\mathbf{a}_{q\star}\xi|^{(k-1)/2}}\int_0^\infty \sin\Big(|\zeta|\varrho + \frac{k\pi}{2}\Big)\breve{\chi}^{(k)}(\varrho)\,d\varrho, \tag{3.51}$$

where $\zeta = \mathbf{d}_{q\star}^{-1}\xi$, k^* is the integer part of $(k-1)/2$, and the sum disappears in (3.51) if $k^* < 1$, i.e., $k < 3$.

Moreover,

(i) *for $s\in\mathbb{R}$ and $k = 1,2,3$, the following operator is continuous*

$$\mathbf{R}_{q\star} : H^s(\mathbb{R}^3) \longrightarrow H^{s+k-1}(\mathbb{R}^3); \tag{3.52}$$

(ii) *if $\chi\in X^k_{1*}$, $k\ge 1$, then $\widetilde{R}_{q\star}(\xi)\le 0$ for all $\xi\in\mathbb{R}$.*

Proof. By (3.20) we have $\widetilde{R}_{q\star}(\xi) = -(\xi\cdot\mathbf{a}_{q\star}\xi)\widetilde{P}_{q\star} - 1$ and Lemma 3.1 implies (3.50) and (3.51). Equality (3.51) gives the estimates,

$$|\widetilde{R}_{q\star}(\xi)| \le c(1+|\xi|^2)^{-\frac{k-1}{2}} \;\text{ for all }\; \xi\in\mathbb{R}^3 \;\text{ if }\; \chi\in X^k, \;\; k = 1,2,3,$$

which imply (3.52). Finally, (A.5) implies item (ii). □

Taking into account that

$$\mathcal{P}_{q\star}f = \mathbf{P}_{q\star}f, \quad \mathcal{R}_{q\star}f = \mathbf{R}_{q\star}f \;\text{ for }\; f\in\widetilde{H}^s(\Omega_q), \;\; s\in\mathbb{R}, \tag{3.53}$$

we can write down the mapping properties for $\mathcal{P}_{q\star}$ and $\mathcal{R}_{q\star}$.

Theorem 3.5. *The following operators are continuous*

$$\mathcal{P}_q, \mathcal{P}_{q\star} : \widetilde{H}^s(\Omega_q) \longrightarrow H^{s+2}(\Omega_q), \;\; s\in\mathbb{R}, \;\; \chi\in X^1, \tag{3.54}$$

$$: H^s(\Omega_q) \longrightarrow H^{s+2}(\Omega_q), \; -\frac{1}{2} < s < \frac{2k-1}{2}\,, \; \chi\in X^k, \; k = \overline{1,3}, \tag{3.55}$$

$$\mathcal{R}_{q\star} : \widetilde{H}^s(\Omega_q) \longrightarrow H^{s+k-1}(\Omega_q), \quad s \in \mathbb{R}, \quad \chi \in X^k, \quad k = 1, 2, 3, \tag{3.56}$$

$$: H^s(\Omega_q) \longrightarrow H^{k-\frac{1}{2}-\varepsilon}(\Omega_q), \quad \frac{1}{2} \leq s, \quad \chi \in X^k, \quad k = 2, 3, \tag{3.57}$$

where ε is an arbitrarily small positive number.

Proof. Due to the equality (3.33) it suffices to prove the mapping properties in (3.54)–(3.55) only for the operator $\mathcal{P}_{q\star}$. The mapping property (3.54) is implied by the first relation in (3.53) and Theorem 3.2. Then (3.55) for $k = 1$ follows since in this case $H^s(\Omega_q) = \widetilde{H}^s(\Omega_q)$. Similarly, (3.56) is implied by the second relation in (3.53) and Theorem 3.4(i).

To show the property (3.55) for $k = 2, 3$ we proceed as follows. From (3.36) and (3.50), (3.51) we get

$$\widetilde{P}_{q\star}(\xi) = -\frac{1}{\xi \cdot \mathbf{a}_{q\star}\xi} + \widetilde{Q}_q(\xi), \quad \xi \in \mathbb{R}^3 \setminus \{0\}, \tag{3.58}$$

with

$$\widetilde{Q}_q(\xi) = -\frac{\widetilde{R}_{q\star}(\xi)}{(\xi \cdot \mathbf{a}_{q\star}\xi)^2} = \mathcal{O}(|\xi|^{-k-1}) \text{ as } |\xi| \to \infty, \quad k = 1, 2, 3, \tag{3.59}$$

The first summand in (3.58), $\widetilde{P}_{q1\star} := -1/(\xi \cdot \mathbf{a}_{q\star}\xi)$, is the symbol of the pseudodifferential operator $\mathbf{P}_{q1\star}$ of the volume Newton type potential without localization, based on the fundamental solution (3.2). Since the symbol is of rational type of order -2 possessing the transmission property, $\mathbf{P}_{q1\star}$ maps $H^s(\Omega_q)$ into $H^{s+2}(\Omega_q)$ for $s > -\frac{1}{2}$ due to [2, Section 2] and Theorem 8.6.1 in [13]. More precisely,

$$r_{\Omega_q} \mathbf{P}_{q1\star} \ell_0 : H^s(\Omega_q) \longrightarrow H^{s+2}(\Omega_q) \text{ for } s > -\frac{1}{2}\,, \tag{3.60}$$

where ℓ_0 is an extension by zero operator from Ω_q onto the compliment domain $\Omega_q^c = \mathbb{R}^3 \setminus \overline{\Omega}_q$.

Further, by (3.59) we see that the corresponding pseudodifferential operator $r_{\Omega_q} \mathbf{Q}_q$ with symbol $\widetilde{Q}_q(\xi)$ has the following mapping properties

$$r_{\Omega_q} \mathbf{Q}_q \ell_0 : H^s(\Omega_q) \longrightarrow H^{s+k+1}(\Omega_q) \text{ if } -\frac{1}{2} < s < \frac{1}{2}\,, \tag{3.61}$$

$$r_{\Omega_q} \mathbf{Q}_q \ell_0 : H^s(\Omega_q) \longrightarrow H^{s_0}(\Omega_q) \text{ if } s \geq \frac{1}{2} \text{ for all } s_0 < \frac{1}{2} + k + 1. \tag{3.62}$$

Therefore

$$r_{\Omega_q} (\mathbf{P}_{q1\star} + \mathbf{Q}_q) \ell_0 : H^s(\Omega_q) \longrightarrow H^{s_k}(\Omega_q) \text{ for } s > -\frac{1}{2}\,, \quad k = 2, 3, \tag{3.63}$$

where

$$\begin{aligned} &s_2 = s + 2 \text{ if } -\frac{1}{2} < s < \frac{3}{2}\,, \quad s_2 = 3 + \frac{1}{2} - \varepsilon \text{ if } s > \frac{3}{2}\,, \\ &s_3 = s + 2 \text{ if } -\frac{1}{2} < s < \frac{5}{2}\,, \quad s_3 = 4 + \frac{1}{2} - \varepsilon \text{ if } s > \frac{5}{2}\,; \end{aligned} \tag{3.64}$$

here ε is an arbitrarily small positive number.

Clearly, $\mathcal{P}_{q\star} = r_{\Omega_q}(\mathbf{P}_{q1\star} + \mathbf{Q}_q)\ell_0$ due to (3.58) and the property (3.55) follows.

Finally, the property (3.57) follows from (3.51) and (3.56) since for $s \geq 1/2$ we have $H^s(\Omega_q) \subset H^t(\Omega_q)$ with arbitrary $t \in (-1/2, 1/2)$. $\square$

With the help of (3.9), (3.19) and (3.21) we have

$$R_q(x,y) = \frac{a_q(x)}{a_q(y)}\, R_{q\star}(x,y) + \frac{1}{a_q(y)}\, \nabla_x a_q(x)\cdot \mathbf{a}_{q\star}\nabla_x P_{q\star}(x-y) = \\ = \frac{a_q(x)}{a_q(y)}\, R_{q\star}(x,y) - \frac{1}{a_q(y)}\, \nabla_x a_q(x)\cdot \mathbf{a}_{q\star}\nabla_y P_{q\star}(x-y), \qquad (3.65)$$

and consequently we get the following representation for the operator $\mathcal{R}_q$,

$$\mathcal{R}_q f(y) := \frac{1}{a_q(y)}\Big[\mathcal{R}_{q\star}(a_q f) - \sum_{k,j=1}^{3}\frac{\partial}{\partial y_k}\,\mathcal{P}_{q\star}(f a_{kj\star}^{(q)}\partial_j a_q)\Big]. \qquad (3.66)$$

Therefore from Theorem 3.5 immediately follows

Theorem 3.6. *The following operators are continuous*

$$\mathcal{R}_q : \widetilde{H}^s(\Omega_q) \longrightarrow H^s(\Omega_q), \;\; s\in\mathbb{R}, \;\; \chi\in X^1, \qquad (3.67)$$

$$: H^s(\Omega_q) \longrightarrow H^{k-\frac12-\varepsilon}(\Omega_q), \;\; \frac12 \leq s, \;\; \chi\in X^k, \;\; k=2,3, \qquad (3.68)$$

where ε is an arbitrarily small positive number.

In view of compactness of the imbedding $H^s(\Omega_q) \subset H^t(\Omega_q)$ for $s > t$ and bounded Ω_q from Theorem 3.6 we obtain the following statement.

Lemma 3.7. *The operators*

$$\mathcal{R}_q : H^1(\Omega_q) \longrightarrow H^t(\Omega_q), \;\; t < \frac32, \;\; \chi\in X^2, \qquad (3.69)$$

$$\gamma_q\mathcal{R}_q : H^1(\Omega_q) \longrightarrow H^{t-\frac12}(\partial\Omega_q), \;\; t < \frac32, \;\; \chi\in X^2, \qquad (3.70)$$

$$T_q\mathcal{R}_q : H^1(\Omega_q) \longrightarrow H^{t-\frac12}(\partial\Omega_q), \;\; t < \frac32, \;\; \chi\in X^3, \qquad (3.71)$$

are compact.

Now we study the mapping properties and jump relations of the localized layer potentials.

First of all let us note that for the single layer potential we have the following representation (cf. [7])

$$V_{S\star}^{(q)}\psi(y) = -\big\langle \gamma_S P_{q\star}(\cdot - y), \psi\big\rangle_S = -\big\langle P_{q\star}(\cdot - y), \gamma_S^*\psi\big\rangle_{\mathbb{R}^3} = \\ = -\big[P_{q\star} * (\gamma_S^*\psi)\big](y) = -\mathbf{P}_{q\star}(\gamma_S^*\psi)(y), \qquad (3.72)$$

where $*$ denotes the convolution operator. The operator γ_S^* is adjoint to the trace operator $\gamma_S : H^t(\mathbb{R}^3) \longrightarrow H^{t-\frac12}(S)$, $t > 1/2$, i.e., is defined by the relation

$$\langle\gamma_{_S}^*\psi, h\rangle := \langle\psi, \gamma_{_S} h\rangle_{_S} \ \text{ for all } \ h \in H^t(\mathbb{R}^3), \ \ \psi \in H^{\frac{1}{2}-t}(S), \ \ t > \frac{1}{2}\,, \tag{3.73}$$

and thus the operator

$$\gamma_{_S}^* : H^{\frac{1}{2}-t}(S) \longrightarrow H^{-t}(\mathbb{R}^3), \ \ t > 1/2 \tag{3.74}$$

is continuous. Since $\gamma_{_S} h = 0$ for any $h \in C^\infty_{comp}(\mathbb{R}^3\backslash S)$, then $\operatorname{supp}\gamma_{_S}^*\psi \in S$, i.e. in fact the operator

$$\gamma_{_S}^* : H^{\frac{1}{2}-t}(S) \longrightarrow H_S^{-t} := \big\{f \in H^{-t}(\mathbb{R}^3) : \ \operatorname{supp} f \in S\big\} \tag{3.75}$$

is also continuous for $t > 1/2$.

Quite analogously, for the double layer potential we have the following representation

$$\begin{aligned} W_{S\star}^{(q)}\varphi(y) = -\big\langle T_{q\star S} P_{q\star}(\cdot - y), \varphi\big\rangle_S = -\big\langle P_{q\star}(\cdot - y), T^*_{q\star S}\varphi\big\rangle_{\mathbb{R}^3} = \\ = -\big[P_{q\star} * T^*_{q\star S}\varphi\big](y) = -\mathbf{P}_{q\star}[T^*_{q\star S}\varphi](y). \end{aligned} \tag{3.76}$$

Here $T_{q\star S} = a^{(q)}_{kj\star} n^{(q)}_k(x)\gamma_{_S}\partial_{x_j} : H^t(\mathbb{R}^3) \longrightarrow H^{t-\frac{3}{2}}(S)$ is the classical (defined in terms of the trace) co-normal derivative operator on S that is continuous for $t > \frac{3}{2}$ (for the infinitely smooth S), while $T^*_{q\star S}$ is the operator adjoint to it, i.e., defined by the relation

$$\langle T^*_{q\star S}\varphi, h\rangle_{\mathbb{R}^3} := \langle\varphi, T_{q\star S} h\rangle_{_S} \ \text{ for any } \ h \in H^t(\mathbb{R}^3), \ \ \varphi \in H^{\frac{3}{2}-t}(S), \tag{3.77}$$

and thus the operator

$$T^*_{q\star S} : H^{\frac{3}{2}-t}(S) \longrightarrow H^{-t}(\mathbb{R}^3), \ \ t > \frac{3}{2}, \tag{3.78}$$

is continuous. Since $T_{q\star S} h = 0$ for any $h \in C^\infty_{comp}(\mathbb{R}^3\backslash S)$, then $\operatorname{supp} T^*_{q\star S}\varphi \in S$, i.e. in fact the operator

$$T^*_{q\star S} : H^{\frac{3}{2}-t}(S) \longrightarrow H_S^{-t} \tag{3.79}$$

is also continuous for $t > 3/2$.

Theorem 3.8. *If $\chi \in X^k$, $k = 2, 3$, then the following operators are continuous*

$$V^{(q)}_{S\star} : H^s(S) \longrightarrow H^{s+\frac{3}{2}}(\Omega^S) \ \textit{ for } \ s < k-1, \tag{3.80}$$

$$A_{q\star}V^{(q)}_{S\star} : H^s(S) \longrightarrow H^{s+k-\frac{3}{2}}(\Omega^S) \ \textit{ for } \ s < 0, \tag{3.81}$$

$$A_{q\star}V^{(q)}_{S\star} : H^s(S) \longrightarrow H^{-\epsilon+k-\frac{3}{2}}(\Omega^S) \ \textit{ for } \ s \geq 0, \ \ \forall\,\epsilon > 0, \tag{3.82}$$

$$W^{(q)}_{S\star} : H^s(S) \longrightarrow H^{s+\frac{1}{2}}(\Omega^S) \ \textit{ for } \ s < k-1, \tag{3.83}$$

$$A_{q\star}W^{(q)}_{S\star} : H^s(S) \longrightarrow H^{s+k-\frac{5}{2}}(\Omega^S) \ \textit{ for } \ s < 0, \tag{3.84}$$

$$A_{q\star}W^{(q)}_{S\star} : H^s(S) \longrightarrow H^{-\epsilon+k-\frac{5}{2}}(\Omega^S) \ \textit{ for } \ s \geq 0, \ \ \forall\,\epsilon > 0, \tag{3.85}$$

where Ω^S is an interior or exterior domain bounded by S.

Proof. For $\chi \in X^k$, $k = 2,3$, by Lemma 3.1 we have $\widetilde{P}_{q\star} \in C(\mathbb{R}^3)$ and in view of (3.58) we have

$$\widetilde{P}_{q\star}(\xi) = -|\zeta|^{-2} + \widetilde{Q}_q(\xi), \tag{3.86}$$

where $\zeta = \mathbf{d}_{q\star}^{-1}\xi \in \mathbb{R}^3 \setminus \{0\}$, and $\widetilde{Q}_q(\xi)$ is defined in (3.59).

Note that the symbol of the localized operator $\mathbf{P}_{q\star}$ is of neither classical nor rational type, in general. Therefore we can not apply directly the well known theorems for pseudodifferential operators with rational type symbols (see, e.g. [4], [13], [25]).

However, due to (3.72), ansatz (3.86) gives us possibility to represent the localized single layer potential $V_{S\star}^{(q)}(\psi)$ as

$$V_{S\star}^{(q)}(\psi) = V_{S1\star}^{(q)}(\psi) + \mathbf{Q}_q\gamma_{_S}^*\psi, \tag{3.87}$$

where $V_{_S}^{(q1\star)}(\psi)$ is the non-localized single layer potential constructed by the fundamental solution $P_{q1\star}(x-y)$,

$$V_{S1\star}^{(q)}(\psi) = -\int\limits_S P_{q1\star}(x-y)\psi(x)\,dS = -\mathbf{P}_{q1\star}\gamma_{_S}^*\psi, \tag{3.88}$$

where the symbol of the operator $\mathbf{P}_{q1\star}$ is $-|\zeta|^{-2}$, while $\mathbf{Q}_q$ is pseudodifferential operator with the symbol $\widetilde{Q}_q$.

The principal homogeneous symbol of the pseudodifferential operator $\mathbf{P}_{q1\star}$ is rational function in ξ, and due to equality (3.88) and [4, Ch. 5, Theorem 2.4] (see also [13, Theorem 8.5.8]) we have

$$\mu V_{S1\star}^{(q)} : H^s(S) \longrightarrow H^{s+\frac{3}{2}}(\Omega^S) \;\text{ for }\; s \in \mathbb{R},\;\; \forall\, \mu \in C_{comp}^\infty(\overline{\Omega}^S). \tag{3.89}$$

On the other hand, the asymptotic relation (3.59) and mapping property (3.74) imply continuity of the mapping

$$\mu\mathbf{Q}_q\gamma_{_S}^* : H^s(S) \longrightarrow H^{s+k+\frac{1}{2}}(\mathbb{R}^3) \;\text{ for }\; s < 0,\;\; \forall\, \mu \in C_{comp}^\infty(\mathbb{R}^3), \tag{3.90}$$

and thus also of the mapping

$$\mu\mathbf{Q}_q\gamma_{_S}^* : H^s(S) \longrightarrow H^{k+\frac{1}{2}-\epsilon}(\mathbb{R}^3) \;\text{ for }\; s \geq 0,\;\; \forall\, \mu \in C_{comp}^\infty(\mathbb{R}^3) \tag{3.91}$$

for $k = 2,3$ and $\forall\, \epsilon > 0$.

Let first Ω^S be a bounded domain. Then (3.80) follows from (3.87) by (3.89), (3.90) and (3.91). Since $A_{q\star}V_{S1\star}^{(q)}\psi = 0$ in Ω^S, we have, $A_{q\star}V_{S\star}^{(q)}\psi = A_{q\star}\mathbf{Q}_q\gamma_{_S}^*\psi$ in Ω^S, which by (3.90) and (3.91) also implies (3.81) and (3.82).

Let now Ω^S be an unbounded domain. Let $\lambda \in C_{comp}^\infty(\mathbb{R}^3)$ be such that $\lambda(0) = 1$ and represent $\chi = \chi_0 + \chi_\infty$, where $\chi_0 = \lambda\chi$, $\chi_\infty = (1-\lambda)\chi$. Then evidently $V_{S\star}^{(q)}(\psi)$ is represented in terms of the potentials with the localizing functions χ_0 and χ_∞, respectively,

$$V_{S\star}^{(q)}(\psi) = V_{S\chi_0\star}^{(q)}(\psi) + V_{S\chi_\infty\star}^{(q)}(\psi).$$

Let us analyze the potential $V_{S\chi_0\star}^{(q)}(\psi)$ first. Follow the same arguments as above, we split it in two parts as in (3.87) and arrive at the continuity of the mappings similar to (3.89)–(3.91) for them. Due to the compact support of λ and the compactness of the surface S in $\mathbb{R}^3$, the support of $V^{(q)}_{S\chi_0\star}(\psi)$ is also compact in $\mathbb{R}^3$ and does not depend on ψ. This means that for μ such that $\mu = 1$ in the support of $V^{(q)}_{S\chi_0\star}(\psi)$, we have $\mu V^{(q)}_{S\chi_0\star} = V^{(q)}_{S\chi_0\star}$, that is, μ can be dropped in the mappings similar to (3.89)–(3.91) for them. This implies the counterparts of mappings (3.80)–(3.82) for $V^{(q)}_{S\chi_0\star}$ in unbounded domains Ω^S.

Let us now analyze the potential $V^{(q)}_{S\chi_\infty\star}(\psi)$. Since $\chi_\infty(0) = 0$ the term with $m = 0$ in the sum in the representation (3.36) for the symbol $\widetilde{P}_{q\chi_\infty\star}$ of the corresponding volume potential $\mathbf{P}_{q\chi_\infty\star}$ vanishes, and we have the estimate

$$|\widetilde{P}_{q\chi_\infty\star}(\xi)| \le c(1+|\xi|^2)^{-\frac{k+1}{2}} \ \text{ for all } \ \xi \in \mathbb{R}^3 \ \text{ if } \ \chi \in X^k, \ \ k = 2,3.$$

This implies continuity of the mapping

$$V^{(q)}_{S\chi_\infty\star} = \mathbf{P}_{q\chi_\infty\star}\gamma^*_S : H^s(S) \longrightarrow H^{s+k+\frac{1}{2}}(\mathbb{R}^3) \ \text{ for } \ s<0, \ \ k=2,3,$$

and thus also of the mapping

$$V^{(q)}_{S\chi_\infty\star} = \mathbf{P}_{q\chi_\infty\star}\gamma^*_S : H^s(S) \longrightarrow H^{k+\frac{1}{2}-\epsilon}(\mathbb{R}^3) \ \text{ for } \ s \ge 0, \ \ k=2,3, \ \ \forall\, \epsilon>0,$$

which give the counterparts of mappings (3.80)–(3.82) for $V^{(q)}_{S\chi_\infty\star}$ and thus mappings (3.80)–(3.82) for $V^{(q\star)}_{S}$ in unbounded domains Ω^S.

To show the mapping properties (3.83)–(3.85), we rewrite (3.23) in the form

$$\begin{aligned} W^{(q)}_{S\star} g(y) &= \int\limits_S \big[T_{q\star}(x,\partial_y)P_{q\star}(x-y)\big]g(x)\,dS_x = \\ &= \sum_{k,j}^{3} a^{(q)}_{kj\star}\frac{\partial}{\partial y_j}\int\limits_S P_{q\star}(x-y)[n^{(q)}_k(x)g(x)]\,dS_x = \\ &= \sum_{k,j}^{3} a^{(q)}_{kj\star}\frac{\partial}{\partial y_j}V^{(q)}_{S\star}\big(n^{(q)}_k g\big), \ \ g \in H^{s+1}(S). \end{aligned} \tag{3.92}$$

Whence (3.83)–(3.85) follow from (3.80)–(3.82). □

From Theorem 3.8 we have the following assertion.

Theorem 3.9. *The localized single and double layer potentials possess the following mapping properties*

$$V^{(q)} : H^{-\frac{1}{2}}(\partial\Omega_q) \longrightarrow H^{1,0}(\Omega_q; A_q), \ \ \chi \in X^2, \tag{3.93}$$

$$W^{(q)} : H^{\frac{1}{2}}(\partial\Omega_q) \longrightarrow H^{1,0}(\Omega_q; A_q), \ \ \chi \in X^3. \tag{3.94}$$

Moreover, the operators

$$r_{S_e}\gamma_2 V^{(2)}_{S_i} : H^{-\frac{1}{2}}(S_i) \longrightarrow H^{\frac{1}{2}}(S_e), \ \ \chi \in X^2, \tag{3.95}$$

$$r_{S_e} T_2 V^{(2)}_{S_i} : H^{-\frac{1}{2}}(S_i) \longrightarrow H^{-\frac{1}{2}}(S_e), \;\; \chi \in X^2, \tag{3.96}$$

$$r_{S_e} \gamma_2 W^{(2)}_{S_i} : H^{-\frac{1}{2}}(S_i) \longrightarrow H^{\frac{1}{2}}(S_e), \;\; \chi \in X^2, \tag{3.97}$$

$$r_{S_e} T_q W^{(2)}_{S_i} : H^{-\frac{1}{2}}(S_i) \longrightarrow H^{-\frac{1}{2}}(S_e), \;\; \chi \in X^2, \tag{3.98}$$

are compact.

Proof. Mappings (3.93) and (3.94) immediately follow from Theorem 3.8 and the relations (3.31) and (3.32) . Therefore the co-normal derivative $T_q^{\pm} V^{(q)} g$ of the localized single layer potential with $g \in H^{-\frac{1}{2}}(\partial\Omega_q)$ and $\chi \in X^2$ is well defined, as well as the co-normal derivative of the localized double layer potential,

$$T_q^{\pm} W^{(q)} h =: \mathcal{L}^{(q)\pm} h, \;\; h \in H^{\frac{1}{2}}(\partial\Omega_q), \;\; \chi \in X^3. \tag{3.99}$$

Compactness of the operators (3.95)–(3.98) is evident since the surfaces S_i and S_e are disjoint. $\square$

By the same arguments as in [7, Theorem 5.13] one can easily show also the following jump relations for localized layer potentials.

Theorem 3.10. *Let $g \in H^{-\frac{1}{2}}(\partial\Omega_q)$ and $h \in H^{\frac{1}{2}}(\partial\Omega_q)$. Then*

$$\gamma_q^+ V^{(q)} g = \gamma_q^- V^{(q)} g = \mathcal{V}^{(q)} g, \;\; \chi \in X^2, \tag{3.100}$$

$$T_q^{\pm} V^{(q)} g = \pm \frac{1}{2}\, g + \mathcal{W}'^{(q)} g, \;\; \chi \in X^2, \tag{3.101}$$

$$\gamma_q^{\pm} W^{(q)} h = \mp \frac{1}{2}\, h + \mathcal{W}^{(q)} h, \;\; \chi \in X^3, \tag{3.102}$$

$$T_q^+ W^{(q)} h - T_q^- W^{(q)} h \equiv \mathcal{L}^{+(q)} h - \mathcal{L}^{-(q)} h = \quad (T_{q\star} a_q) h, \;\; \chi \in X^3. \tag{3.103}$$

In particular, for $a_q = 1$ and $S = \partial\Omega_q$ the following equalities hold

$$T^+_{q\star} W^{(q)}_{S\star} h = T^-_{q\star} W^{(q)}_{S\star} h =: \mathcal{L}^{(q)}_{S\star} h, \;\; \chi \in X^3, \;\; q = 1, 2. \tag{3.104}$$

The following statement is implied by Theorems 3.10 and 3.9, and the relations(3.18).

Theorem 3.11. *The following boundary operators are continuous,*

$$\mathcal{V}^{(q)}_S : H^{-\frac{1}{2}}(S) \longrightarrow H^{\frac{1}{2}}(S), \;\; \chi \in X^2, \tag{3.105}$$

$$\mathcal{W}'^{(q)}_S : H^{-\frac{1}{2}}(S) \longrightarrow H^{-\frac{1}{2}}(S), \;\; \chi \in X^2, \tag{3.106}$$

$$\mathcal{W}^{(q)}_S : H^{\frac{1}{2}}(S) \longrightarrow H^{\frac{1}{2}}(S), \;\; \chi \in X^3, \tag{3.107}$$

$$\mathcal{L}^{(q)\pm}_S : H^{\frac{1}{2}}(S) \longrightarrow H^{-\frac{1}{2}}(S), \;\; \chi \in X^3. \tag{3.108}$$

Moreover, the operators (3.106) *and* (3.107) *are compact.*

Proof. The continuity of the operators (3.105)–(3.108) follows from the mapping properties (3.93)–(3.94). On the other hand, from the relations (3.18) it follows that the kernels of the integral operators $\mathcal{W}'^{(q)}_S$ and $\mathcal{W}^{(q)}_S$ are weakly singular of type $\mathcal{O}(|x-y|^{-2+\alpha})$. Therefore, $\mathcal{W}'^{(q)}_S$ and $\mathcal{W}^{(q)}_S$ are pseudodifferential operators on S of order $-\alpha < 0$ and possess the following mapping properties

$$\mathcal{W}_S'^{(q)} : H^{-\frac{1}{2}}(S) \longrightarrow H^{-\frac{1}{2}+\alpha}(S), \ \ \chi \in X^2,$$
$$\mathcal{W}_S^{(q)} : H^{\frac{1}{2}}(S) \longrightarrow H^{\frac{1}{2}+\alpha}(S), \ \ \chi \in X^3,$$

implying the compactness of the operators (3.106) and (3.107) due to the Rellich compact imbedding theorem. □

Taking $v(x) := P_q(x-y,y)$ and $u = u_q \in H^{1,0}(\Omega_q; A_q)$ in the second Green identity (2.8), by the standard limiting procedure (see, e.g., [23]), we obtain the following third Green identity based on the localized parametrix,

$$u_q + \mathcal{R}_q u_q - V^{(q)} T_q u_q + W^{(q)} \gamma_q u_q = \mathcal{P}_q A_q u_q \ \text{ in } \ \Omega_q. \tag{3.109}$$

Recall that for layer potentials we drop the subindex S when $S = \partial\Omega_q$.

Taking in mind the properties of the localized potentials, the trace and co-normal derivative of (3.109) have the following form,

$$\frac{1}{2}\gamma_q u_q + \gamma_q \mathcal{R}_q u_q - \mathcal{V}^{(q)} T_q u_q + \mathcal{W}^{(q)} \gamma_q u_q = \gamma_q \mathcal{P}_q A_q u_q \ \text{ on } \ \partial\Omega_q, \tag{3.110}$$
$$\frac{1}{2} T_q u_q + T_q \mathcal{R}_q u_q - \mathcal{W}'^{(q)} T_q u_q + \mathcal{L}^{(q)} \gamma_q u_q = T_q \mathcal{P}_q A_q u_q \ \text{ on } \ \partial\Omega_q. \tag{3.111}$$

Recall that $\mathcal{L}_S^{(q)} := \mathcal{L}_S^{(q)+} \neq \mathcal{L}_S^{(q)-}$ if a_q is not a constant function (see Theorem 3.10).

With the help of these relations we will construct various types of localized boundary domain integral equation systems for the above formulated Dirichlet and mixed type transmission BVPs with and without crack.

4. Some Injectivity Results

Before formulating the boundary-domain integral equations, we present in this section some auxiliary lemmata which play a crucial role in our analysis.

Lemma 4.1. *If $\chi \in X_*^k$, $k \geq 1$, and $s \geq -1$, then the operator*

$$-\mathbf{P}_{q\star} : H^s(\mathbb{R}^3) \longrightarrow H^{-s}(\mathbb{R}^3), \ \ q = 1, 2, \tag{4.1}$$

is positive, i.e.,

$$-\langle \mathbf{P}_{q\star} g, \overline{g} \rangle_{\mathbb{R}^3} > 0 \ \ \forall\, g \in H^s(\mathbb{R}^3), \ \ g \neq 0,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ denotes the duality brackets between the spaces $H^{-s}(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3)$.

Proof. The continuity of operator (4.1) is implied by Theorem 3.2. For any $g \in H^s(\mathbb{R}^3)$, $s \geq -1$, we have,

$$\langle \mathbf{P}_{q\star} g, \overline{g} \rangle_{\mathbb{R}^3} = \langle \mathcal{F}^{-1}[\widetilde{P}_{q\star} \widetilde{g}], \overline{g} \rangle_{\mathbb{R}^3} =$$

$$= (2\pi)^{-3}\langle \widetilde{P}_{q\star}\widetilde{g}, \overline{\widetilde{g}}\rangle_{\mathbb{R}^3} = (2\pi)^{-3}\int\limits_{\mathbb{R}^3} \widetilde{P}_{q\star}(\xi)|\widetilde{g}(\xi)|^2\, d\xi. \quad (4.2)$$

By Lemma 3.1(ii) $\widetilde{P}_{q\star}(\xi) < 0$ for a.e. $\xi \in \mathbb{R}^3$. Hence the conclusion. □

Throughout the rest of this section and in the main statements further on we assume that the following relation holds on S_i

$$a_2(x) = \varkappa a_1(x) \ \text{ for } \ x \in S_i, \ \ \varkappa = const > 0. \quad (4.3)$$

Lemma 4.2. *Let* $\chi \in X^3_{1*}$, $G_q \in H^0(\Omega_q)$, $g_{i1} \in H^{-\frac{1}{2}}(S_i)$, $g_{i2} \in H^{\frac{1}{2}}(S_i)$, $g_e \in H^{-\frac{1}{2}}(S_e)$ *and condition* (4.3) *hold. Further let*

$$V^{(1)}_{S_i\star}(g_{i1}) + W^{(1)}_{S_i\star}(a_1 g_{i2}) + \mathcal{P}_{1\star}(G_1) = 0 \ \text{ in } \ \Omega_1, \quad (4.4)$$

$$V^{(2)}_{S_i\star}(g_{i1}) - W^{(2)}_{S_i\star}(a_2 g_{i2}) + V^{(2)}_{S_e\star}(g_e) + \mathcal{P}_{2\star}(G_2) = 0 \ \text{ in } \ \Omega_2. \quad (4.5)$$

Then

$$G_q = 0 \ in\ \Omega_q, \ q = 1,2, \ \ g_{i1} = 0, \ g_{i2} = 0 \ on\ S_i, \ \ and \ \ g_e = 0 \ on\ S_e. \quad (4.6)$$

Proof. We set

$$U_1 := V^{(1)}_{S_i\star}(g_{i1}) + W^{(1)}_{S_i\star}(a_1 g_{i2}) + \mathcal{P}_{1\star}(G_1) \ \text{ in } \ \mathbb{R}^3 \setminus \partial\Omega_1, \quad (4.7)$$

$$U_2 := V^{(2)}_{S_i\star}(g_{i1}) - W^{(2)}_{S_i\star}(a_2 g_{i2}) + V^{(2)}_{S_e\star}(g_e) + \mathcal{P}_{2\star}(G_2) \ \text{ in } \ \mathbb{R}^3 \setminus \partial\Omega_2. \quad (4.8)$$

Due to (4.4) and (4.5),

$$U_q = 0 \ \text{ in } \ \Omega_q, \ \ q = 1,2. \quad (4.9)$$

In view of the restrictions on the density functions G_q, g_{iq}, $q = 1,2$, and g_e, and on the localizing function χ and due to mapping properties (3.43), (3.93) and (3.94) we have

$$U_q \in H^{1,0}(\mathbb{R}^3 \setminus \partial\Omega_q; A_{q\star}). \quad (4.10)$$

Then we can write the following Green's formulas

$$\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_1} (A_{1\star}U_1)U_1\, dx + \int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_1} E_{q\star}(U_1,U_1)\, dx = -\langle T^-_{1\star}U_1, \gamma^-_1 U_1\rangle_{S_i}, \quad (4.11)$$

$$\int\limits_{\Omega_1} (A_{2\star}U_2)U_2\, dx + \int\limits_{\Omega_1} E_{2\star}(U_2,U_2)\, dx = -\langle T^-_{2\star}U_2, \gamma^-_2 U_2\rangle_{S_i}, \quad (4.12)$$

$$\int\limits_{\mathbb{R}^3\setminus(\overline{\Omega}_1\cup\overline{\Omega}_2)} (A_{2\star}U_2)U_2\, dx + \int\limits_{\mathbb{R}^3\setminus(\overline{\Omega}_1\cup\overline{\Omega}_2)} E_{2\star}(U_2,U_2)\, dx = -\langle T^-_{2\star}U_2, \gamma^-_2 U_2\rangle_{S_e}, \quad (4.13)$$

where

$$E_{q\star}(U_q,U_q) := \sum_{k,j=1}^{3} a^{(q)}_{kj\star}\partial_k U_q \partial_j U_q \geq c|\nabla U_q|^2, \ \ q = 1,2, \quad (4.14)$$

with some positive constant $c > 0$ due to the positive definiteness of the matrix $\mathbf{a}_{q\star} = [a^{(q)}_{kj\star}]_{3\times 3}$.

With the help of the jump relations and the mapping properties of the localized layer potentials (3.100)–(3.103) we get

$$\begin{gathered}\gamma_1^+U_1-\gamma_1^-U_1=-a_1g_{i2},\quad \gamma_2^+U_2-\gamma_2^-U_2=a_2g_{i2}\ \text{ on }\ S_i,\\ T_{1\star}^+U_1-T_{1\star}^-U_1=T_{2\star}^+U_2-T_{2\star}^-U_2=g_{i1}\ \text{ on }\ S_i,\\ \gamma_2^+U_2=\gamma_2^-U_2=0,\ \ T_{2\star}^+U_2-T_{2\star}^-U_2^-=g_e\ \text{ on }\ S_e.\end{gathered}\tag{4.15}$$

Therefore, from (4.11)–(4.13) with the help of (4.3), (4.9) and (4.15) we derive

$$\varkappa\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_1}\big[(A_{1\star}U_1)U_1+E_{1\star}(U_1,U_1)\big]\,dx+ \\ +\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_2}\big[(A_{2\star}U_2)U_2+E_{2\star}(U_2,U_2)\big]\,dx=0.\tag{4.16}$$

Further we proceed as follows. Denote by $\overset{\circ}{G}_q:=\ell_{0q}G_q\in\widetilde{H}^0(\Omega_q)$ the extensions of the functions G_q onto the whole of $\mathbb{R}^3$ by zero. Then clearly $\mathcal{P}_{q\star}G_q=\mathbf{P}_{q\star}\overset{\circ}{G}_q$ and in view of formulas (3.72), (3.76) we can rewrite (4.7) and (4.8) as

$$U_q=\mathbf{P}_{q\star}F_q\ \text{ in }\ \mathbb{R}^3,\quad q=1,2,\tag{4.17}$$

in the distributional sense, where the distributions F_1 and F_2 on $\mathbb{R}^3$ read as

$$\begin{aligned}F_1&=\overset{\circ}{G}_1-\gamma_{S_i}^*g_{i1}-T_{q\star S_i}^*(a_1g_{i2}),\\ F_2&=\overset{\circ}{G}_2-\gamma_{S_i}^*g_{i1}+T_{q\star S_i}^*(a_2g_{i2})-\gamma_{S_e}^*g_e,\end{aligned}\tag{4.18}$$

and thus $F_q\in\widetilde{H}^{-2}(\Omega_q)$ by (3.75) and (3.79). Whence in view of (3.20) we have

$$A_{q\star}U_q=F_q+\mathbf{R}_{q\star}F_q=\mathbf{R}_{q\star}F_q\ \text{ in }\ \mathbb{R}^3\setminus\overline{\Omega}_q,\tag{4.19}$$

and $\mathbf{R}_{q\star}F_q\in H^0(\mathbb{R}^3)$ by Theorem 3.4. Consequently, from (4.16) we derive

$$\sum_{q=1}^{2}\varkappa_q\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_q}\big[(\mathbf{R}_{q\star}F_q)(\mathbf{P}_{q\star}F_q)+E_{q\star}(U_q,U_q)\big]\,dx=0,\tag{4.20}$$

where $\varkappa_1=\varkappa$ and $\varkappa_2=1$.

Keeping in mind that $\mathbf{P}_{q\star}F_q\in H^0(\mathbb{R}^3)$ and $\mathbf{P}_{q\star}F_q=U_q=0$ in Ω_q, we can extend the integration to the whole space $\mathbb{R}^3$ and apply Parseval's formula to obtain

$$\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_q}(\mathbf{R}_{q\star}F_q)(\mathbf{P}_{q\star}F_q)\,dx=\int\limits_{\mathbb{R}^3}(\mathbf{R}_{q\star}F_q)(\mathbf{P}_{q\star}F_q)\,dx= \\ =\int\limits_{\mathbb{R}^3}\widetilde{R}_{q\star}\overline{\widetilde{P}}_{q\star}|\widetilde{F}_q|^2\,d\xi\geq 0\tag{4.21}$$

since $\widetilde{P}_{q\star}(\xi) \le 0$, $\widetilde{R}_{q\star}(\xi) \le 0$ by Lemma 3.1(ii) and Theorem 3.4(ii). Then (4.20) and (4.14) imply $\nabla U_q = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}_q$.

Consequently, $U_1 = C_1$ in $\mathbb{R}^3 \setminus \overline{\Omega}_1$, $U_2 = C_2$ in $\mathbb{R}^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$, and $U_2 = C_3$ in Ω_1, where C_j, $j = 1,2,3$, are arbitrary constants. Since $U_q \in H^1\big(\mathbb{R}^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)\big)$, we get $C_1 = C_2 = 0$. Then with the help of the first two equalities in (4.15) we conclude that $C_3 = r_{_{S_i}} \gamma_2^- U_2 = 0$. Thus $U_q = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}_q$ and in view of (4.17) we have $U_q = 0$, $q = 1,2$, in $\mathbb{R}^3$.

Now taking jumps of traces and co-normal derivatives of (4.7) and (4.8) on $\partial\Omega_1$ and $\partial\Omega_2$, respectively, gives $g_{i1} = 0$ and $g_{i2} = 0$ on S_i, and $g_e = 0$ on S_e (see (4.15)). Finally Lemma 4.1 implies $\overset{\circ}{G}_q = 0$ in $\mathbb{R}^3$. □

Lemma 4.3. *Let* $\chi \in X^3_{1*}$, $G_q \in H^0(\Omega_q)$, $g_{i1} \in H^{-\frac{1}{2}}(S_i)$, $g_{i2} \in H^{\frac{1}{2}}(S_i)$, $g_e \in H^{\frac{1}{2}}(S_e)$ *and condition* (4.3) *hold. Further let*

$$V^{(1)}_{S_i\star}(g_{i1}) + W^{(1)}_{S_i\star}(a_1 g_{i2}) + \mathcal{P}_{1\star}(G_1) = 0 \ \text{ in } \ \Omega_1, \tag{4.22}$$

$$V^{(2)}_{S_i\star}(g_{i1}) - W^{(2)}_{S_i\star}(a_2 g_{i2}) + W^{(2)}_{S_e\star}(g_e) + \mathcal{P}_{2\star}(G_2) = 0 \ \text{ in } \ \Omega_2. \tag{4.23}$$

Then

$$G_q = 0 \ \textit{in}\ \Omega_q,\ q = 1,2,\ \ g_{i1} = 0,\ \ g_{i2} = 0 \ \textit{on}\ S_i,\ \ \textit{and}\ \ g_e = 0 \ \textit{on}\ S_e. \tag{4.24}$$

Proof. As in the proof of Lemma 4.2 here we set

$$U_1 := V^{(1)}_{S_i\star}(g_{i1}) + W^{(1)}_{S_i\star}(a_1 g_{i2}) + \mathcal{P}_{1\star}(G_1) \ \text{ in } \ \mathbb{R}^3 \setminus S_i, \tag{4.25}$$

$$U_2 := V^{(2)}_{S_i\star}(g_{i1}) - W^{(2)}_{S_i\star}(a_2 g_{i2}) + W^{(2)}_{S_e\star}(g_e) + \mathcal{P}_{2\star}(G_2) \text{ in } \mathbb{R}^3 \setminus (S_i \cup S_e). \tag{4.26}$$

Again, by the assumptions stated in the lemma and the mapping properties of the localized volume and surface potentials we have

$$U_q \in H^{1,0}(\mathbb{R}^3 \setminus \partial\Omega_q; A_{q\star}), \tag{4.27}$$

and we can write Green's formulas (4.11)–(4.13). By relations

$$\begin{gathered}
\gamma_1^+ U_1 - \gamma_1^- U_1 = a_1 g_{i2}, \quad \gamma_2^+ U_2 - \gamma_2^- U_2 = a_2 g_{i2} \ \text{ on } \ S_i, \\
T^+_{1\star} U_1 - T^-_{1\star} U_1 = T^+_{2\star} U_2 - T^-_{2\star} U_2 = g_{i1} \ \text{ on } \ S_i, \\
\gamma_2^+ U_2 - \gamma_2^- U_2 = -g_e, \quad T^+_{2\star} U_2 = T^-_{2\star} U_2 = 0 \ \text{ on } \ S_e,
\end{gathered} \tag{4.28}$$

and taking into account that $U_q = 0$ in Ω_q along with the relation (4.3), we arrive at the formula (4.16). By the word for word arguments from the proof of Lemma 4.2 we complete the proof. □

Lemma 4.4. *Let* $\chi \in X^3_{1*}$, $G_q \in H^0(\Omega_q)$, $g_{i1} \in H^{-\frac{1}{2}}(S_i)$, $g_{i2} \in H^{\frac{1}{2}}(S_i)$, $g_{eD} \in \widetilde{H}^{-\frac{1}{2}}(S_{eD})$, $g_{eN} \in \widetilde{H}^{\frac{1}{2}}(S_{eN})$ *and condition* (4.3) *hold. Further let*

$$V^{(1)}_{S_i\star}(g_{i1}) + W^{(1)}_{S_i\star}(a_1 g_{i2}) + \mathcal{P}_{1\star}(G_1) = 0 \text{ in } \Omega_1, \tag{4.29}$$

$$V^{(2)}_{S_i\star}(g_{i1}) - W^{(2)}_{S_i\star}(a_2 g_{i2}) + V^{(2)}_{S_e\star}(g_{eD}) + W^{(2)}_{S_e\star}(g_{eN}) + \mathcal{P}_{2\star}(G_2) = 0 \text{ in } \Omega_2. \tag{4.30}$$

Then $G_q = 0$ *in* Ω_q, $q = 1,2$, $g_{i1} = 0$ *and* $g_{i2} = 0$ *on* S_i, $g_{eD} = 0$ *and* $g_{eN} = 0$ *on* S_e.

Proof. As in the proof of Lemma 4.2 here we set

$$U_1 := V^{(1)}_{S_i\star}(g_{i1}) + W^{(1)}_{S_i\star}(a_1 g_{i2}) + \mathcal{P}_{1\star}(G_1) \ \text{ in } \ \mathbb{R}^3 \setminus S_i, \tag{4.31}$$

$$U_2 := V^{(2)}_{S_i\star}(g_{i1}) - W^{(2)}_{S_i\star}(a_2 g_{i2}) + V^{(2)}_{S_e\star}(g_{eD}) + W^{(2)}_{S_e\star}(g_{eN}) + \\ + \mathcal{P}_{2\star}(G_2) \ \text{ in } \ \mathbb{R}^3 \setminus (S_i \cup S_e). \tag{4.32}$$

Again, in view of the assumptions stated in the lemma and with the help of the mapping properties of the localized volume and surface potentials we have

$$U_q \in H^{1,0}(\mathbb{R}^3 \setminus \partial\Omega_q; A_{q\star}), \tag{4.33}$$

and we can write Green's formulas (4.11)–(4.13). By relations

$$\begin{gathered}
\gamma_1^+ U_1 - \gamma_1^- U_1 = a_1 g_{i2}, \ \ \gamma_2^+ U_2 - \gamma_2^- U_2 = a_2 g_{i2} \ \text{ on } \ S_i, \\
T^+_{1\star} U_1 - T^-_{1\star} U_1 = T^+_{2\star} U_2 - T^-_{2\star} U_2 = g_{i1} \ \text{ on } \ S_i, \\
\gamma_2^+ U_2 - \gamma_2^- U_2 = -g_{eN}, \ \ T^+_{2\star} U_2 - T^-_{2\star} U_2 = g_{eD} \ \text{ on } \ S_e, \\
r_{S_{eD}} \gamma_2^+ U_2 = r_{S_{eD}} \gamma_2^- U_2 = 0 \ \text{ on } \ S_{eD}, \\
r_{S_{eN}} T^+_{2\star} U_2 = r_{S_{eN}} T^-_{2\star} U_2 = 0 \ \text{ on } \ S_{eN},
\end{gathered} \tag{4.34}$$

and taking into account that $U_q = 0$ in Ω_q along with the relation (4.3), we easily arrive at the formula (4.16). By the word for word arguments applied in the proof of Lemma 4.2 we complete the proof. □

Lemma 4.5. *Let* $\chi \in X^3_{1*}$, *condition* (4.3) *hold and*

$$\begin{gathered}
G_q \in H^0(\Omega_q), \ \ g_{i1} \in \widetilde{H}^{-\frac{1}{2}}(S_i^{(t)}), \ \ g_{i2}, g_{i3} \in H^{\frac{1}{2}}(S_i), \\
g_{i2} - g_{i3} \in \widetilde{H}^{\frac{1}{2}}(S_i^{(c)}), \ \ g_e \in H^{-\frac{1}{2}}(S_e).
\end{gathered}$$

Further let

$$V^{(1)}_{S_i\star}(g_{i1}) + W^{(1)}_{S_i\star}(a_1 g_{i2}) + \mathcal{P}_{1\star}(G_1) = 0 \ \text{ in } \ \Omega_1, \tag{4.35}$$

$$-V^{(2)}_{S_i\star}(g_{i1}) + W^{(2)}_{S_i\star}(a_2 g_{i3}) + V^{(2)}_{S_e\star}(g_e) + \mathcal{P}_{2\star}(G_2) = 0 \ \text{ in } \ \Omega_2. \tag{4.36}$$

Then $g_{i1} = g_{i2} = g_{i3} = 0$ *on* S_i, $g_e = 0$ *on* S_e *and* $G_q = 0$ *in* Ω_q, $q = 1, 2$.

Proof. Introduce the functions

$$U_1 := V^{(1)}_{S_i\star}(g_{i1}) + W^{(1)}_{S_i\star}(a_1 g_{i2}) + \mathcal{P}_{1\star}(G_1) \ \text{ in } \ \mathbb{R}^3 \setminus S_1, \tag{4.37}$$

$$U_2 := -V^{(2)}_{S_i\star}(g_{i1}) + W^{(2)}_{S_i\star}(a_2 g_{i3}) + V^{(2)}_{S_e\star}(g_e) + \mathcal{P}_{2\star}(G_2) \text{ in } \mathbb{R}^3 \setminus (S_i \cup S_e). \tag{4.38}$$

Clearly $U_q = 0$ in Ω_q, $q = 1, 2$. Denote again by $\overset{\circ}{G}_q := \ell_{0q} G_q \in \widetilde{H}^0(\Omega_q)$ the extensions of the functions G_q by zero on the whole of $\mathbb{R}^3$. Then $\mathcal{P}_{q\star} G_q = \mathbf{P}_{q\star} \overset{\circ}{G}_q$, $q = 1, 2$.

In view of the assumptions stated in the lemma and with the help of the mapping properties of the localized volume and surface potentials we have

$$U_q \in H^{1,0}(\mathbb{R}^3 \setminus \partial\Omega_q; A_{q\star}), \quad q = 1, 2. \tag{4.39}$$

Therefore we can write Green's formulas (4.11)–(4.13). Note that with the help of the jump relations of the localized layer potentials we get

$$\begin{gathered}\gamma_1^+U_1-\gamma_1^-U_1=-a_1g_{i2},\quad \gamma_2^+U_2-\gamma_2^-U_2=-a_2g_{i3}\ \text{ on }\ S_i,\\ T_{1\star}^+U_1-T_{1\star}^-U_1=g_{i1},\quad T_{2\star}^+U_2-T_{2\star}^-U_2=-g_{i1}\ \text{ on }\ S_i,\\ \gamma_2^+U_2=\gamma_2^-U_2=0,\quad T_{2\star}^+U_2-T_{2\star}^-U_2^-=g_e\ \text{ on }\ S_e.\end{gathered}\tag{4.40}$$

Thus from (4.11)–(4.13) due to the lemma hypotheses and (4.3), we derive

$$\begin{gathered}\varkappa\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_1}\big[(A_{1\star}U_1)U_1+E_{1\star}(U_1,U_1)\big]\,dx+\\ +\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_2}\big[(A_{2\star}U_2)U_2+E_{2\star}(U_2,U_2)\big]\,dx=\\ =-\langle g_{i1},\varkappa a_1g_{i2}\rangle_{S_i}+\langle g_{i1},a_2g_{i3}\rangle_{S_i}=0.\end{gathered}\tag{4.41}$$

Now, applying the same arguments as in the proof of Lemma 4.2 we conclude $U_1=C_1$ in $\mathbb{R}^3\setminus\overline{\Omega}_1$, $U_2=C_2$ in $\mathbb{R}^3\setminus(\overline{\Omega}_1\cup\overline{\Omega}_2)$, and $U_2=C_3$ in Ω_1, where C_j, $j=1,2,3$, are arbitrary constants. Since $U_q\in H^1\big(\mathbb{R}^3\setminus(\overline{\Omega}_1\cup\overline{\Omega}_2)\big)$, we get $C_1=C_2=0$, implying $U_1=0$ in $\mathbb{R}^3$ and $U_2=0$ in $\mathbb{R}^3\setminus\overline{\Omega}_1$. Consequently, $g_{i2}=0$ on S_i.

Further, since $g_{i2}-g_{i3}\in\widetilde{H}^{\frac12}(S_i^{(c)})$, from the second equation in (4.40) we derive

$$r_{S_i^{(t)}}\big(\gamma_2^-U_2\big)=r_{S_i^{(t)}}(a_2g_{i3})=r_{S_i^{(t)}}(a_2g_{i2})=0\ \text{ on }\ S_i^{(t)}.$$

Then it follows that $C_3=r_{S_i}\gamma_2^-U_2=0$. Thus $U_q=0$ in $\mathbb{R}^3$, $q=1,2$, and the relations (4.40) and Lemma 4.1 complete the proof. □

In view of formulas (3.31)–(3.33) the above lemmata lead to the following corollaries.

Corollary 4.6. *Let $\chi\in X^3_{1*}$, $G_q\in H^0(\Omega_q)$, $g_{i1}\in H^{-\frac12}(S_i)$, $g_{i2}\in H^{\frac12}(S_i)$, $g_e\in H^{-\frac12}(S_e)$ and condition (4.3) hold. Further let*

$$V_{S_i}^{(1)}(g_{i1})+W_{S_i}^{(1)}(g_{i2})+\mathcal{P}_1(G_1)=0\ \text{ in }\ \Omega_1,\tag{4.42}$$

$$V_{S_i}^{(2)}(g_{i1})-W_{S_i}^{(2)}(g_{i2})+V_{S_e}^{(2)}(g_e)+\mathcal{P}_2(G_2)=0\ \text{ in }\ \Omega_2.\tag{4.43}$$

Then $g_{i1}=0$, $g_{i2}=0$ on S_i, $g_e=0$ on S_e and $G_q=0$ in Ω_q, $q=1,2$.

Corollary 4.7. *Let $\chi\in X^3_{1*}$, $G_q\in H^0(\Omega_q)$, $g_{i1}\in H^{-\frac12}(S_i)$, $g_{i2}\in H^{\frac12}(S_i)$, $g_e\in H^{\frac12}(S_e)$ and condition (4.3) hold. Further let*

$$V_{S_i}^{(1)}(g_{i1})+W_{S_i}^{(1)}(g_{i2})+\mathcal{P}_1(G_1)=0\ \text{ in }\ \Omega_1,\tag{4.44}$$

$$V_{S_i}^{(2)}(g_{i1})-W_{S_i}^{(2)}(g_{i2})+W_{S_e}^{(2)}(g_e)+\mathcal{P}_2(G_2)=0\ \text{ in }\ \Omega_2.\tag{4.45}$$

Then $g_{i1}=0$, $g_{i2}=0$ on S_i, $g_e=0$ on S_e and $G_q=0$ in Ω_q, $q=1,2$.

Corollary 4.8. *Let* $\chi \in X^3_{1*}$, $G_q \in H^0(\Omega_q)$, $g_{i1} \in H^{-\frac{1}{2}}(S_i)$, $g_{i2} \in H^{\frac{1}{2}}(S_i)$, $g_{eD} \in \widetilde{H}^{-\frac{1}{2}}(S_{eD})$, $g_{eN} \in \widetilde{H}^{\frac{1}{2}}(S_{eN})$, *and condition* (4.3) *hold. Further let*

$$V^{(1)}_{S_i}(g_{i1}) + W^{(1)}_{S_i}(g_{i2}) + \mathcal{P}_1(G_1) = 0 \text{ in } \Omega_1, \quad (4.46)$$

$$V^{(2)}_{S_i}(g_{i1}) - W^{(2)}_{S_i}(g_{i2}) + V^{(2)}_{S_e}(g_{eD}) + W^{(2)}_{S_e}(g_{eN}) + \mathcal{P}_2(G_2) = 0 \text{ in } \Omega_2. \quad (4.47)$$

Then $g_{i1} = 0$ *and* $g_{i2} = 0$ *on* S_i, $g_{eD} = 0$, $g_{eN} = 0$ *on* S_e *and* $G_q = 0$ *in* Ω_q, $q = 1, 2$.

Corollary 4.9. *Let* $\chi \in X^3_{1*}$,

$$G_q \in H^0(\Omega_q), \quad g_{i1} \in \widetilde{H}^{-\frac{1}{2}}(S^{(t)}_i), \quad g_{i2}, g_{i3} \in H^{\frac{1}{2}}(S_i),$$
$$g_{i2} - g_{i3} \in \widetilde{H}^{\frac{1}{2}}(S^{(c)}_i), \quad g_e \in H^{-\frac{1}{2}}(S_e).$$

and condition (4.3) *hold. Further let*

$$V^{(1)}_{S_i}(g_{i1}) + W^{(1)}_{S_i}(g_{i2}) + \mathcal{P}_1(G_1) = 0 \text{ in } \Omega_1, \quad (4.48)$$

$$-V^{(2)}_{S_i}(g_{i1}) + W^{(2)}_{S_i}(g_{i3}) + V^{(2)}_{S_e}(g_e) + \mathcal{P}_2(G_2) = 0 \text{ in } \Omega_2. \quad (4.49)$$

Then $g_{i1} = g_{i2} = g_{i3} = 0$ *on* S_i, $g_e = 0$ *on* S_e *and* $G_q = 0$ *in* Ω_q, $q = 1, 2$.

5. LBDIE Systems for the Transmission-Dirichlet Problem

Let a pair $(u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$ be a solution to the transmission Dirichlet problem (2.9)–(2.12), i.e., Problem (TD). Assume that the problem right hand sides satisfy the imbeddings

$$\varphi_{0i} \in H^{\frac{1}{2}}(S_i), \quad \psi_{0i} \in H^{-\frac{1}{2}}(S_i), \quad \varphi_{0e} \in H^{\frac{1}{2}}(S_e), \quad f_q \in H^0(\Omega_q), \quad q = 1, 2. \quad (5.1)$$

Let us introduce the following combinations of the unknown boundary functions

$$\psi_i = \frac{1}{2}(T_1 u_1 - T_2 u_2), \quad \varphi_i = \frac{1}{2}(\gamma_1 u_1 + \gamma_2 u_2), \quad \psi_e = T_2 u_2. \quad (5.2)$$

Then evidently $\psi_i \in H^{-\frac{1}{2}}(S_i)$, $\varphi_i \in H^{\frac{1}{2}}(S_i)$, $\psi_e \in H^{-\frac{1}{2}}(S_e)$.

5.1. **LBDIE system (TD1).**

Let us introduce the vector function

$$U^{(TD)} := (u_1, u_2, \psi_i, \varphi_i, \psi_e) \in \mathbb{H}^{(TD)}, \quad (5.3)$$

where

$$\mathbb{H}^{(TD)} :=$$
$$:= H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{-\frac{1}{2}}(S_i) \times H^{\frac{1}{2}}(S_i) \times H^{-\frac{1}{2}}(S_e), \quad (5.4)$$

and assume formally that the components of $U^{(TD)}$ are unrelated to each other (i.e., segregated).

Further, let us employ the third Green identities (3.109) in Ω_1 and Ω_2, difference of their traces (3.110) and sum of their co-normal derivatives (3.111) on S_i, and also the trace (3.110) on S_e.

Then after substituting transmission and boundary conditions (2.10)–(2.12) and notations (5.2) we arrive at the following system of direct segregated LBDIEs **(TD1)** for the components of the vector function $U^{(TD)} = (u_1, u_2, \psi_i, \varphi_i, \psi_e)$,

$$u_1 + \mathcal{R}_1 u_1 - V^{(1)}_{S_i}\psi_i + W^{(1)}_{S_i}\varphi_i = F^{(TD)}_1 \ \text{ in } \ \Omega_1, \tag{5.5}$$

$$u_2 + \mathcal{R}_2 u_2 + V^{(2)}_{S_i}\psi_i + W^{(2)}_{S_i}\varphi_i - V^{(2)}_{S_e}\psi_e = F^{(TD)}_2 \ \text{ in } \ \Omega_2, \tag{5.6}$$

$$\begin{aligned}\gamma_1\mathcal{R}_1 u_1 - \gamma_2\mathcal{R}_2 u_2 - (\mathcal{V}^{(1)}_{S_i} + \mathcal{V}^{(2)}_{S_i})\psi_i + (\mathcal{W}^{(1)}_{S_i} - \mathcal{W}^{(2)}_{S_i})\varphi_i + \gamma_2 V^{(2)}_{S_e}\psi_e =\\ = \gamma_1 F^{(TD)}_1 - \gamma_2 F^{(TD)}_2 - \varphi_{0i} \ \text{ on } \ S_i,\end{aligned} \tag{5.7}$$

$$\begin{aligned}T_1\mathcal{R}_1 u_1 + T_2\mathcal{R}_2 u_2 - (\mathcal{W}'^{(1)}_{S_i} - \mathcal{W}'^{(2)}_{S_i})\psi_i + (\mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i})\varphi_i - T_2 V^{(2)}_{S_e}\psi_e =\\ = T_1 F^{(TD)}_1 + T_2 F^{(TD)}_2 - \psi_{0i} \ \text{ on } \ S_i,\end{aligned} \tag{5.8}$$

$$\gamma_2\mathcal{R}_2 u_2 + \gamma_2 V^{(2)}_{S_i}\psi_i + \gamma_2 W^{(2)}_{S_i}\varphi_i - \mathcal{V}^{(2)}_{S_e}\psi_e = \gamma_2 F^{(TD)}_2 - \varphi_{0e} \ \text{ on } \ S_e, \tag{5.9}$$

where

$$F^{(TD)}_1 = \mathcal{P}_1 f_1 + \frac{1}{2}V^{(1)}_{S_i}\psi_{0i} - \frac{1}{2}W^{(1)}_{S_i}\varphi_{0i}, \tag{5.10}$$

$$F^{(TD)}_2 = \mathcal{P}_2 f_2 + \frac{1}{2}V^{(2)}_{S_i}\psi_{0i} + \frac{1}{2}W^{(2)}_{S_i}\varphi_{0i} - W^{(2)}_{S_e}\varphi_{0e}. \tag{5.11}$$

If we introduce the notation

$$\begin{aligned}&\mathcal{K}^{(TD1)} = [\mathcal{K}^{(TD1)}_{kj}]_{5\times 5} := \operatorname{diag}(r_{\Omega_1}, r_{\Omega_2}, r_{S_i}, r_{S_i}, r_{S_e})\times\\ &\times\begin{bmatrix} I+\mathcal{R}_1 & 0 & -V^{(1)}_{S_i} & W^{(1)}_{S_i} & 0\\ 0 & I+\mathcal{R}_2 & V^{(2)}_{S_i} & W^{(2)}_{S_i} & -V^{(2)}_{S_e}\\ \gamma_1\mathcal{R}_1 & -\gamma_2\mathcal{R}_2 & -\mathcal{V}^{(1)}_{S_i}-\mathcal{V}^{(2)}_{S_i} & \mathcal{W}^{(1)}_{S_i}-\mathcal{W}^{(2)}_{S_i} & \gamma_2 V^{(2)}_{S_e}\\ T_1\mathcal{R}_1 & T_2\mathcal{R}_2 & -\mathcal{W}'^{(1)}_{S_i}+\mathcal{W}'^{(2)}_{S_i} & \mathcal{L}^{(1)}_{S_i}+\mathcal{L}^{(2)}_{S_i} & -T_2 V^{(2)}_{S_e}\\ 0 & \gamma_2\mathcal{R}_2 & \gamma_2 V^{(2)}_{S_i} & \gamma_2 W^{(2)}_{S_i} & -\mathcal{V}^{(2)}_{S_e}\end{bmatrix}\end{aligned} \tag{5.12}$$

the LBDIEs system (5.5)–(5.9) can be rewritten as

$$\mathcal{K}^{(TD1)} U^{(TD)} = \mathcal{F}^{(TD1)}, \tag{5.13}$$

where $U^{(TD)} \in \mathbb{H}^{(TD)}$ is the unknown vector, while $\mathcal{F}^{(TD1)} \in \mathbb{F}^{(TD1)}$ is the known vector generated by the right hand side functions in (5.5)–(5.9) and

$$\mathbb{F}^{(TD1)} := H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{\frac{1}{2}}(S_i) \times H^{-\frac{1}{2}}(S_i) \times H^{\frac{1}{2}}(S_e).$$

5.2. **LBDIE system (TD2).** Alternatively, let us employ the third Green identities (3.109) in Ω_1 and Ω_2, difference of their co-normal derivatives (3.111) on S_i and sum of their traces (3.110), and also the co-normal derivative (3.111) on S_e. Then after substituting transmission and boundary conditions (2.10)–(2.12) and notations (5.2) we arrive at the following system of direct segregated LBDIEs **(TD2)** of the second kind for the components of the vector function $U^{(TD)} = (u_1, u_2, \psi_i, \varphi_i, \psi_e) \in \mathbb{H}^{(TD)}$,

$$u_1 + \mathcal{R}_1 u_1 - V^{(1)}_{S_i}\psi_i + W^{(1)}_{S_i}\varphi_i = F^{(TD)}_1 \ \text{ in } \ \Omega_1, \tag{5.14}$$

$$u_2 + \mathcal{R}_2 u_2 + V^{(2)}_{S_i}\psi_i + W^{(2)}_{S_i}\varphi_i - V^{(2)}_{S_e}\psi_e = F^{(TD)}_2 \ \text{ in } \ \Omega_2, \tag{5.15}$$

$$\psi_i + T_1\mathcal{R}_1 u_1 - T_2\mathcal{R}_2 u_2 - (\mathcal{W}'^{(1)}_{S_i} + \mathcal{W}'^{(2)}_{S_i})\psi_i + (\mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i})\varphi_i + T_2 V^{(2)}_{S_e}\psi_e = \\ = T_1 F^{(TD)}_1 - T_2 F^{(TD)}_2 \ \text{ on } \ S_i, \tag{5.16}$$

$$\varphi_i + \gamma_1\mathcal{R}_1 u_1 + \gamma_2\mathcal{R}_2 u_2 - (\mathcal{V}^{(1)}_{S_i} - \mathcal{V}^{(2)}_{S_i})\psi_i + (\mathcal{W}^{(1)}_{S_i} + \mathcal{W}^{(2)}_{S_i})\varphi_i - \gamma_2 V^{(2)}_{S_e}\psi_e = \\ = \gamma_1 F^{(TD)}_1 + \gamma_2 F^{(TD)}_2 \ \text{ on } \ S_i, \tag{5.17}$$

$$\frac{1}{2}\,\psi_e + T_2\mathcal{R}_2 u + T_2 V^{(2)}_{S_i}\psi_i + T_2 W^{(2)}_{S_i}\varphi_i - \mathcal{W}'^{(2)}_{S_e}\psi_e = T_2 F^{(TD)}_2 \ \text{ on } \ S_e, \tag{5.18}$$

where $F^{(TD)}_1$, $F^{(TD)}_2$ are given by (5.10), (5.11).

If we introduce the notations

$$\mathcal{K}^{(TD2)} = [\mathcal{K}^{(TD2)}_{kj}]_{5\times 5} := \operatorname{diag}(r_{\Omega_1}, r_{\Omega_2}, r_{S_i}, r_{S_i}, r_{S_e})\times \\ \times \begin{bmatrix} I+\mathcal{R}_1 & 0 & -V^{(1)}_{S_i} & W^{(1)}_{S_i} & 0 \\ 0 & I+\mathcal{R}_2 & V^{(2)}_{S_i} & W^{(2)}_{S_i} & -V^{(2)}_{S_e} \\ T_1\mathcal{R}_1 & -T_2\mathcal{R}_2 & I-\mathcal{W}'^{(1)}_{S_i}-\mathcal{W}'^{(2)}_{S_i} & \mathcal{L}^{(1)}_{S_i}-\mathcal{L}^{(2)}_{S_i} & +T_2V^{(2)}_{S_e} \\ \gamma_1\mathcal{R}_1 & \gamma_2\mathcal{R}_2 & -\mathcal{V}^{(1)}_{S_i}+\mathcal{V}^{(2)}_{S_i} & I+\mathcal{W}^{(1)}_{S_i}+\mathcal{W}^{(2)}_{S_i} & -\gamma_2V^{(2)}_{S_e} \\ 0 & T_2\mathcal{R}_2 & T_2V^{(2)}_{S_i} & T_2W^{(2)}_{S_i} & \frac{1}{2}\,I-\mathcal{W}'^{(2)}_{S_e} \end{bmatrix}, \tag{5.19}$$

the LBDIEs system (5.14)–(5.18) can be rewritten as

$$\mathcal{K}^{(TD2)}U^{(TD)} = \mathcal{F}^{(TD2)}, \tag{5.20}$$

where $U^{(TD)} \in \mathbb{H}^{(TD)}$ is the unknown vector, while $\mathcal{F}^{(TD2)} \in \mathbb{F}^{(TD2)}$ is the known vector generated by the right hand side functions in (5.14)–(5.18) and

$$\mathbb{F}^{(TD2)} := H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{-\frac{1}{2}}(S_i) \times H^{\frac{1}{2}}(S_i) \times H^{-\frac{1}{2}}(S_e).$$

5.3. **Main theorems for LBDIE systems (TD1) and (TD2).** There holds the following equivalence theorem.

Theorem 5.1. *Let conditions* (5.1) *hold and* $\chi \in X^3_{1*}$.

(i) *If a pair* $(u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$ *solves the Problem* (TD), *then the vector* $U^{(TD)} \in \mathbb{H}^{(TD)}$ *given by* (5.3), *where* ψ_i, φ_i *and* ψ_e *are defined by* (5.2), *solves both* LBDIE *systems* **(TD1)** *and* **(TD2)**.

(ii) *Vice versa, if a vector* $U^{(TD)} \in \mathbb{H}^{(TD)}$ *solves LBDIE system* **(TD1)** *or* LBDIE *system* **(TD2)** *and condition* (4.3) *holds, then* $(u_1, u_2) \in H^{(1,0)}(\Omega_1; A_1) \times H^{(1,0)}(\Omega_2; A_2)$ *solves Problem (TD) and relations* (5.2) *hold.*

Proof. Claim (i) immediately follows from the deduction of **(TD1)** and **(TD2)**.

Now, let a vector $U^{(TD)} \in \mathbb{H}^{(TD)}$ solves LBDIE system **(TD1)**. Subtracting from equation (5.7) the trace γ_1 of equation (5.5) and adding the trace γ_2 of equation (5.6), we prove (2.10). Similarly, subtracting from equation (5.8) the co-normal derivative T_1 of equation (5.5) and the co-normal derivative T_2 of equation (5.6), we prove (2.11). At last, subtracting from equation (5.9) the trace γ_2 of equation (5.6), we prove (2.12). That is, the transmission conditions on S_i and the Dirichlet boundary condition on S_e are fulfilled.

It remains to show that u_q solve differential equations (2.9) and that the conditions (5.2) hold true. Due to the embedding $U^{(TD)} \in \mathbb{H}^{(TD)}$, the third Green identities (3.109) hold. Comparing these identities with the first two equations of the LBDIE system, (5.5) and (5.6), and taking into account transmission conditions (2.10)–(2.11) and the Dirichlet boundary condition (2.12) already proved, we arrive at the relations

$$V^{(1)}_{S_i}\Big(\frac{T_1u_1 - T_2u_2}{2} - \psi_i\Big) + W^{(1)}_{S_i}\Big(\varphi_i - \frac{\gamma_1u_1 + \gamma_2u_2}{2}\Big) = \\ = \mathcal{P}_1(f_1 - A_1u_1) \ \text{ in } \ \Omega_1,$$

$$V^{(2)}_{S_i}\Big(\frac{T_1u_1 - T_2u_2}{2} - \psi_i\Big) - W^{(2)}_{S_i}\Big(\varphi_i - \frac{\gamma_1u_1 + \gamma_2u_2}{2}\Big) + V^{(2)}_{S_e}(\psi_e - T_2u_2) = \\ = \mathcal{P}_2(A_2u_2 - f_2) \ \text{ in } \ \Omega_2.$$

Whence by Corollary 4.6 we conclude that conditions (5.2) are satisfied and

$$A_1u_1 - f_1 = 0 \ \text{ in } \ \Omega_1, \quad A_2u_2 - f_2 = 0 \ \text{ in } \ \Omega_2. \tag{5.21}$$

This completes the proof of item (ii) for LBDIE system **(TD1)**.

Let now a vector $U^{(TD)} \in \mathbb{H}^{(TD)}$ solve LBDIE system **(TD2)**. Subtracting from equation (5.2) the co-normal derivative T_1 of equation (5.14) and adding the co-normal derivative T_2 of equation (5.15), we prove the first relation in (5.2). Similarly, subtracting from equation (5.2) the trace γ_1 of equation (5.14) and the trace γ_2 of equation (5.15), we prove the second relation in (5.2). At last, subtracting from equation (5.11) the co-normal derivative T_2 of equation (5.15), we prove the third relation in (5.2).

It remains to show that u_q solve differential equations (2.9) and that the transmission conditions on S_i and the Dirichlet boundary condition on S_e are fulfilled. Due to the embedding $U^{(TD)} \in \mathbb{H}^{(TD)}$, the third Green identities (3.109) hold. Comparing these identities with the first two equations of the LBDIEs system, (5.5) and (5.6), and taking into account relations (5.2) already proved, we arrive at the relations

$$\frac{1}{2}V^{(1)}_{S_i}\,(T_1u_1 + T_2u_2 - \psi_{0i}) + \frac{1}{2}W^{(1)}_{S_i}\,(\varphi_{0i} - \gamma_1u_1 + \gamma_2u_2) = \\ = \mathcal{P}_1(f_1 - A_1u_1) \quad \text{in} \quad \Omega_1,$$

$$\frac{1}{2}\,V^{(2)}_{S_i}\,(T_1u_1 + T_2u_2 - \psi_{0i}) - \frac{1}{2}\,W^{(2)}_{S_i}\,(\varphi_{0i} - \gamma_1u_1 + \gamma_2u_2) +$$

$$+W^{(2)}_{S_e}(\varphi_{0e} - \gamma_2 u_2) = \mathcal{P}_2(f_2 - A_2 u_2) \quad \text{in} \quad \Omega_2.$$

Whence by Corollary 4.7 we conclude that the transmission conditions on S_i and the Dirichlet boundary condition on S_e are satisfied and

$$A_1 u_1 - f_1 = 0 \ \text{ in } \ \Omega_1, \quad A_2 u_2 - f_2 = 0 \ \text{ in } \ \Omega_2. \tag{5.22}$$

This completes the proof of item (ii) for LBDIE system **(TD2)**. □

Due to this equivalence theorem we conclude that the LBDIE system (5.5)–(5.9) with the special right hand side functions which belong to the space $\mathbb{F}^{(TD1)}$ is uniquely solvable in the space $\mathbb{H}^{(TD)}$ defined by (5.4). In particular, the corresponding homogeneous LBDIE system possesses only the trivial solution. By the way, one can easily observe that the right hand side in LBDIE system (5.5)–(5.9) vanishes if $f_q = 0$ in Ω_q, $q = 1, 2$, $\varphi_{0i} = 0$ and ψ_{0i}=0 on S_i, and $\varphi_{0e} = 0$ on S_e.

Our next aim is to establish the invertibility of the matrix operator generated by the left hand side expressions in the LBDIE system (5.5)–(5.9) in two sets of function spaces

$$\mathcal{K}^{(TD1)} : \mathbb{H}^{(TD)} \longrightarrow \mathbb{F}^{(TD1)}, \tag{5.23}$$

$$: \mathbb{X}^{(TD)} \longrightarrow \mathbb{Y}^{(TD1)}, \tag{5.24}$$

where we introduced the following notations for the wider function spaces,

$$\mathbb{X}^{(TD)} := H^1(\Omega_1) \times H^1(\Omega_2) \times H^{-\frac{1}{2}}(S_i) \times H^{\frac{1}{2}}(S_i) \times H^{-\frac{1}{2}}(S_e), \tag{5.25}$$

$$\mathbb{Y}^{(TD1)} := H^1(\Omega_1) \times H^1(\Omega_2) \times H^{\frac{1}{2}}(S_i) \times H^{-\frac{1}{2}}(S_i) \times H^{\frac{1}{2}}(S_e). \tag{5.26}$$

Evidently $\mathbb{H}^{(TD)} \subset \mathbb{X}^{(TD)}$ and $\mathbb{F}^{(TD1)} \subset \mathbb{Y}^{(TD1)}$. Due to Theorems 3.6, 3.9 and 3.11 the operators (5.25) and (5.26) are bounded.

Theorem 5.2. *Let $\chi \in X^3_{1*}$ and condition* (4.3) *hold. Then the operators* (5.23) *and* (5.24) *are invertible.*

Proof. We can easily see that the upper triangular matrix operator

$$\mathcal{K}_0^{(TD1)} := \begin{bmatrix} I & 0 & -r_{\Omega_1} V^{(1)}_{S_i} & r_{\Omega_1} W^{(1)}_{S_i} & 0 \\ 0 & I & r_{\Omega_2} V^{(2)}_{S_i} & r_{\Omega_2} W^{(2)}_{S_i} & -r_{\Omega_2} V^{(2)}_{S_e} \\ 0 & 0 & -\mathcal{V}^{(1)}_{S_i} - \mathcal{V}^{(2)}_{S_i} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} & 0 \\ 0 & 0 & 0 & 0 & -\mathcal{V}^{(2)}_{S_e} \end{bmatrix} \tag{5.27}$$

possesses the same mapping properties as the operator $\mathcal{K}^{(TD1)}$,

$$\mathcal{K}_0^{(TD1)} : \mathbb{X}^{(TD)} \longrightarrow \mathbb{Y}^{(TD1)}, \tag{5.28}$$

and by Lemma 3.7 and Theorems 3.9 and 3.11 the operator (5.28) is a compact perturbation of the operator (5.24).

On the other hand, for $q=1,2$ the operators (3.105) are strongly elliptic pseudodifferential operators of order -1 with strictly positive principal homogenous symbol $\sigma_{_{\mathcal{V}^{(q)}}}(y,\xi')$, while (3.108) are strongly elliptic pseudodifferential operators of order $+1$ with strictly negative principal homogenous symbol $\sigma_{_{\mathcal{L}^{(q)}}}(y,\xi')$ for $\xi'\in\mathbb{R}^2\setminus\{0\}$ and $y\in\partial\Omega_q$ (see formulas (B.8) and (B.9) in Appendix B). Therefore by standard arguments it can be shown that the operators on the main diagonal in (5.27) are Fredholm of zero index in the appropriate function spaces (see, e.g. [1]). Thus operator (5.24) is also Fredholm with zero index.

It remains to show that the null space of operator (5.24) is trivial. We proceed as follows. Let $U^{(TD)}\in\mathbb{X}^{(TD)}$ be a solution to the homogeneous system of equations $\mathcal{K}^{(TD1)}U^{(TD)}=0$. Then due Theorems 3.6 and 3.9 we see from the first two equations of the system that $U^{(TD)}\in\mathbb{H}^{(TD)}$ and by the equivalence Theorem 5.1 we conclude $U^{(TD)}=0$. Thus the kernel of the operator (5.24) is trivial and consequently (5.24) is invertible.

To prove invertibility of operator (5.23), we remark that for any $\mathcal{F}^{(TD)}\in\mathbb{F}^{(TD1)}$ a unique solution $U^{(TD)}\in\mathbb{X}^{(TD)}$ of equation (5.13) is delivered by the inverse to the operator (5.24). On the other hand, since $\mathcal{F}^{(TD)}\in\mathbb{F}^{(TD1)}$, the first two lines of the matrix operator $\mathcal{K}^{(TD)}$ imply that in fact $U^{(TD)}\in\mathbb{H}^{(TD)}$ and the mapping $\mathbb{F}^{(TD1)}\longrightarrow\mathbb{H}^{(TD)}$ delivered by the inverse to the operator (5.24) is continuous, i.e., this operator is inverse to operator (5.23). $\square$

6. The Transmission Mixed Problem (TM)

Let us consider the mixed type transmission problems (2.9), (2.10), (2.11), (2.14), (2.15), with the right hand sides

$$\begin{gathered}\varphi_{0i}\in H^{\frac12}(S_i),\quad \psi_{0i}\in H^{-\frac12}(S_i),\\ \varphi_{0e}^{(M)}\in H^{\frac12}(S_{eD}),\quad \psi_{0e}^{(M)}\in H^{-\frac12}(S_{eN}),\quad f_q\in H^0(\Omega_q),\ \ q=1,2.\end{gathered}\tag{6.1}$$

Let us denote by $\Phi_{0e}\in H^{\frac12}(S_e)$ and $\Psi_{0e}\in H^{-\frac12}(S_e)$ some fixed extensions of the boundary functions $\varphi_{0e}^{(M)}$ and $\psi_{0e}^{(M)}$ from S_{eD} and S_{eN}, respectively, onto the whole surface S_e, preserving the space. Then $r_{_{S_{eD}}}\Phi_{0e}=\varphi_{0e}^{(M)}$, $r_{_{S_{eN}}}\Psi_{0e}=\psi_{0e}^{(M)}$.

Any other extensions $\Phi\in H^{\frac12}(S_e)$ and $\Psi\in H^{-\frac12}(S_e)$ can be evidently represented then in the form

$$\Phi=\Phi_{0e}+\varphi_e,\quad \varphi_e\in\widetilde{H}^{\frac12}(S_{eN});\ \ \Psi=\Psi_{0e}+\psi_e,\quad \psi_e\in\widetilde{H}^{-\frac12}(S_{eD}).$$

Similar to (5.2) for the Problem (TD), let us introduce the following combinations of the unknown boundary functions

$$\begin{gathered}\psi_i=\frac12\,(T_1u_1-T_2u_2)\in H^{-\frac12}(S_i),\quad \varphi_i=\frac12\,(\gamma_1u_1+\gamma_2u_2)\in H^{\frac12}(S_i),\\ \psi_e=T_2u_2-\Psi_{0e}\in\widetilde{H}^{-\frac12}(S_{eD}),\quad \varphi_e=\gamma_2u_2-\Phi_{0e}\in\widetilde{H}^{\frac12}(S_{eN}).\end{gathered}\tag{6.2}$$

Further, let us set

$$U^{(TM)} := (u_1, u_2, \psi_i, \varphi_i, \psi_e, \varphi_e) \in \mathbb{H}^{(TM)}, \tag{6.3}$$

where

$$\mathbb{H}^{(TM)} := H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{-\frac{1}{2}}(S_i) \times \\ \times H^{\frac{1}{2}}(S_i) \times \widetilde{H}^{-\frac{1}{2}}(S_{eD}) \times \widetilde{H}^{\frac{1}{2}}(S_{eN}) \tag{6.4}$$

and we assume again that the components of the vector $U^{(TM)}$ are formally unrelated.

Let us employ the third Green identities (3.109) in Ω_1 and Ω_2, difference of their traces (3.110) and sum of their co-normal derivatives (3.111) on S_i, and also the trace (3.110) on S_{eD} and the co-normal derivative (3.111) on S_{eN}. Then after substituting transmission conditions (2.10)–(2.11) and mixed boundary conditions (2.14)–(2.15) along with notations (6.2), we arrive at the following system of direct segregated LBDIEs for the components of the vector $U^{(TM)}$,

$$u_1 + \mathcal{R}_1 u_1 - V_{S_i}^{(1)}\psi_i + W_{S_i}^{(1)}\varphi_i = F_1^{(TM)} \ \text{ in } \ \Omega_1, \tag{6.5}$$

$$u_2 + \mathcal{R}_2 u_2 + V_{S_i}^{(2)}\psi_i + W_{S_i}^{(2)}\varphi_i - V_{S_e}^{(2)}\psi_e + W_{S_e}^{(2)}\varphi_e = F_2^{(TM)} \ \text{ in } \ \Omega_2, \tag{6.6}$$

$$\gamma_1\mathcal{R}_1 u_1 - \gamma_2\mathcal{R}_2 u_2 - (\mathcal{V}_{S_i}^{(1)} + \mathcal{V}_{S_i}^{(2)})\psi_i + (\mathcal{W}_{S_i}^{(1)} - \mathcal{W}_{S_i}^{(2)})\varphi_i + \\ +\gamma_2 V_{S_e}^{(2)}\psi_e - \gamma_2 W_{S_e}^{(2)}\varphi_e = \gamma_1 F_1^{(TM)} - \gamma_2 F_2^{(TM)} - \varphi_{0i} \ \text{ on } \ S_i, \tag{6.7}$$

$$T_1\mathcal{R}_1 u_1 + T_2\mathcal{R}_2 u_2 - (\mathcal{W}_{S_i}^{\prime(1)} - \mathcal{W}_{S_i}^{\prime(2)})\psi_i + (\mathcal{L}_{S_i}^{(1)} + \mathcal{L}_{S_i}^{(2)})\varphi_i - \\ -T_2 V_{S_e}^{(2)}\psi_e + T_2 W_{S_e}^{(2)}\varphi_e = T_1 F_1^{(TM)} + T_2 F_2^{(TD)} - \psi_{0i} \ \text{ on } \ S_i, \tag{6.8}$$

$$\gamma_2\mathcal{R}_2 u + \gamma_2 V_{S_i}^{(2)}\psi_i + \gamma_2 W_{S_i}^{(2)}\varphi_i - \mathcal{V}_{S_e}^{(2)}\psi_e + \mathcal{W}_{S_e}^{(2)}\varphi_e = \\ = \gamma_2 F_2^{(TM)} - \varphi_{0e} \ \text{ on } \ S_{eD}, \tag{6.9}$$

$$T_2\mathcal{R}_2 u + T_2 V_{S_i}^{(2)}\psi_i + T_2 W_{S_i}^{(2)}\varphi_i - \mathcal{W}_{S_e}^{\prime(2)}\psi_e + \mathcal{L}_{S_e}^{(2)}\varphi_e = \\ = T_2 F_2^{(TM)} - \psi_{0e} \ \text{ on } \ S_{eN}, \tag{6.10}$$

where

$$F_1^{(TM)} = \mathcal{P}_1 f_1 + \frac{1}{2}\, V_{S_i}^{(1)}\psi_{0i} - \frac{1}{2}\, W_{S_i}^{(1)}\varphi_{0i}, \tag{6.11}$$

$$F_2^{(TM)} = \mathcal{P}_2 f_2 + \frac{1}{2}\, V_{S_i}^{(2)}\psi_{0i} + \frac{1}{2}\, W_{S_i}^{(2)}\varphi_{0i} + V_{S_e}^{(2)}\Psi_{0e} - W_{S_e}^{(2)}\Phi_{0e}. \tag{6.12}$$

As in the case of the problem (TD), we have here the following equivalence theorem.

Theorem 6.1. *Let* $\chi \in X_{1*}^3$ *and conditions* (6.1) *hold. Further, let* $\Phi_{0e} \in H^{\frac{1}{2}}(S_e)$ *and* $\Psi_{0e} \in H^{-\frac{1}{2}}(S_e)$ *be some fixed extensions of the boundary functions* $\varphi_{0e}^{(M)}$ *and* $\psi_{0e}^{(N)}$ *from* S_{eD} *and* S_{eN}*, respectively, onto the whole surface* S_e*.*

(i) *If a pair* $(u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$ *solves the transmission mixed problem (TM), then the vector* $U^{(TM)} \in \mathbb{H}^{(TM)}$ *given by* (6.3)*, where* ψ_i*,* φ_i*,* ψ_e *and* φ_e *are defined by* (6.2)*, solves LBDIE system* (6.5)*–*(6.12)*.*

(i) *Vice versa, if a vector* $U^{(TM)} \in \mathbb{H}^{(TM)}$ *solves the LBDIE system* (6.5)*–*(6.12) *and condition* (4.3) *holds, then the pair* (u_1, u_2) *solves the Problem (TM) and relations* (6.2) *hold.*

Proof. The claim (i) immediately follows from the deduction of (6.5)–(6.12).

Now, let a vector $U^{(TM)}$ solve the LBDIE system (6.5)–(6.12). Subtracting from equation (6.7) the trace γ_1 of equation (6.5) and adding the trace γ_2 of equation (6.6), we prove (2.10). Similarly, subtracting from equation (6.8) the co-normal derivative T_1 of equation (6.5) and the co-normal derivative T_2 of equation (6.6), we prove (2.11). Subtracting from equation (6.9) the trace γ_2 of equation (6.6), we prove (2.14). Similarly, subtracting from equation (6.10) the co-normal derivative T_2 of equation (6.6), we prove (2.15). That is, the transmission conditions on S_i and the mixed boundary conditions on S_e are fulfilled.

It remains to show that equations (2.9) and the relations (6.2) hold true. Due to the embedding $U^{(TM)} \in \mathbb{H}^{(TM)}$, the third Green identities (3.109) hold. Comparing these identities with the first two equations of the LBDIE system, (6.5) and (6.6), and taking into account transmission conditions (2.10)–(2.11) and mixed boundary conditions (2.14)–(2.15), already proved, we arrive at the relations

$$V^{(1)}_{S_i}\Big(\frac{T_1u_1 - T_2u_2}{2} - \psi_i\Big) + W^{(1)}_{S_i}\Big(\varphi_i - \frac{\gamma_1u_1 + \gamma_2u_2}{2}\Big) = \\ = \mathcal{P}_1(f_1 - A_1u_1) \ \text{ in } \ \Omega_1,$$

$$V^{(2)}_{S_i}\Big(\frac{T_1u_1 - T_2u_2}{2} - \psi_i\Big) - W^{(2)}_{S_i}\Big(\varphi_i - \frac{\gamma_1u_1 + \gamma_2u_2}{2}\Big) + \\ + V^{(2)}_{S_e}(-T_2u_2 + \psi_e + \Psi_{0e}) + W^{(2)}_{S_e}(\gamma_2u_2 - \varphi_e - \Phi_{0e}) = \mathcal{P}_2(A_2u_2 - f_2) \ \text{ in } \ \Omega_2.$$

Whence by Corollary 4.8 we conclude that (2.9) and (6.2) are satisfied. □

Denote by $\mathcal{K}^{(TM)}$ the localized boundary-domain 6×6 matrix integral operator generated by the left hand side expressions in (6.5)—(6.10),

$$\mathcal{K}^{(TM)} = [\mathcal{K}^{(TM)}_{kj}]_{6\times 6} := \operatorname{diag}(r_{\Omega_1}, r_{\Omega_2}, r_{S_i}, r_{S_i}, r_{S_{eD}}, r_{S_{eN}}) \times \\ \times \begin{bmatrix} I+\mathcal{R}_1 & 0 & -V^{(1)}_{S_i} & W^{(1)}_{S_i} & 0 & 0 \\ 0 & I+\mathcal{R}_2 & V^{(2)}_{S_i} & W^{(2)}_{S_i} & -V^{(2)}_{S_e} & W^{(2)}_{S_e} \\ \gamma_1\mathcal{R}_1 & -\gamma_2\mathcal{R}_2 & -\mathcal{V}^{(1)}_{S_i} - \mathcal{V}^{(2)}_{S_i} & \mathcal{W}^{(1)}_{S_i} - \mathcal{W}^{(2)}_{S_i} & \gamma_2 V^{(2)}_{S_e} & -\gamma_2 W^{(2)}_{S_e} \\ T_1\mathcal{R}_1 & T_2^-\mathcal{R}_2 & \mathcal{W}'^{(2)}_{S_i} - \mathcal{W}'^{(1)}_{S_i} & \mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} & -T_2V^{(2)}_{S_e} & T_2W^{(2)}_{S_e} \\ 0 & \gamma_2\mathcal{R}_2 & \gamma_2V^{(2)}_{S_i} & \gamma_2W^{(2)}_{S_i} & -\mathcal{V}^{(2)}_{S_e} & \mathcal{W}^{(2)}_{S_e} \\ 0 & T_2\mathcal{R}_2 & T_2V^{(2)}_{S_i} & T_2W^{(2)}_{S_i} & -\mathcal{W}'^{(2)}_{S_e} & \mathcal{L}^{(2)}_{S_e} \end{bmatrix} \tag{6.13}$$

and set

$$\mathbb{F}^{(TM)} := H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{\frac{1}{2}}(S_i)\times \\ \times H^{-\frac{1}{2}}(S_i) \times H^{\frac{1}{2}}(S_{eD}) \times H^{-\frac{1}{2}}(S_{eN}). \quad (6.14)$$

Then the LBDIEs system (6.5)–(6.10) can be written in matrix form as

$$\mathcal{K}^{(TM)} U^{(TM)} = \mathcal{F}^{(TM)}, \quad (6.15)$$

where $U^{(TM)}$ is the unknown vector function (6.3), while $\mathcal{F}^{(TM)} \in \mathbb{F}^{(TM)}$ is the known vector function compiled by the right hand side functions in (6.5)–(6.12).

From Theorem 6.1 it follows that LBDIE system (6.5)–(6.10), i.e., equation (6.15) is uniquely solvable in the space $\mathbb{H}^{(TM)}$ for the special right hand side vector-function (see the right hand side functions in (6.5)–(6.12)) which belong to the space $\mathbb{F}^{(TM)}$ defined by (6.14). One can easily observe that the right hand side expressions in LBDIE system (6.5)–(6.10) vanish if $f_q = 0$ in Ω_q, $q = 1, 2$, $f_1 = 0$ and $\psi_{0i} = 0$ on S_i, $\Phi_{0e} = 0$ and $\Psi_{0e} = 0$ on S_e.

Now we establish that actually equation (6.15) is uniquely solvable in two sets of spaces. To this end let us consider the operators

$$\mathcal{K}^{(TM)} : \mathbb{H}^{(TM)} \longrightarrow \mathbb{F}^{(TM)}, \quad (6.16)$$

$$: \mathbb{X}^{(TM)} \longrightarrow \mathbb{Y}^{(TM)}, \quad (6.17)$$

where

$$\mathbb{X}^{(TM)} := H^1(\Omega_1)\times H^1(\Omega_2)\times H^{-\frac{1}{2}}(S_i)\times H^{\frac{1}{2}}(S_i)\times \widetilde{H}^{-\frac{1}{2}}(S_{eD})\times \widetilde{H}^{\frac{1}{2}}(S_{eN}), \quad (6.18)$$

$$\mathbb{Y}^{(TM)} := H^1(\Omega_1)\times H^1(\Omega_2)\times H^{\frac{1}{2}}(S_i)\times H^{-\frac{1}{2}}(S_i)\times H^{\frac{1}{2}}(S_{eD})\times H^{-\frac{1}{2}}(S_{eN}). \quad (6.19)$$

As follows from the mapping properties of the potentials (see Theorem 3.6, 3.9 and 3.11), the operators (6.16) and (6.17) are bounded. Further we show that the operator (6.17) is Fredholm with zero index and thus (6.17) and consequently (6.16) are invertible.

Consider the upper triangular operator

$$\mathcal{K}_0^{(TM)} := \\ := \begin{bmatrix} I & 0 & -r_{\Omega_1} V^{(1)}_{S_i} & r_{\Omega_1} W^{(1)}_{S_i} & 0 & 0 \\ 0 & I & r_{\Omega_2} V^{(2)}_{S_i} & r_{\Omega_2} W^{(2)}_{S_i} & -r_{\Omega_2} V^{(2)}_{S_e} & r_{\Omega_2} W^{(2)}_{S_e} \\ 0 & 0 & -\mathcal{V}^{(1)}_{S_i} - \mathcal{V}^{(2)}_{S_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} & 0 & 0 \\ 0 & 0 & 0 & 0 & -r_{S_{eD}} \mathcal{V}^{(2)}_{S_e} & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{S_{eN}} \mathcal{L}^{(2)}_{S_e} \end{bmatrix}. \quad (6.20)$$

It is easy to see that, on the one hand, the operator

$$\mathcal{K}_0^{(TM)} : \mathbb{X}^{(TM)} \longrightarrow \mathbb{Y}^{(TM)}, \quad (6.21)$$

is bounded, while due to Lemma 3.7 and Theorems 3.9 and 3.11,

$$\mathcal{K}^{(TM)} - \mathcal{K}_0^{(TM)} : \mathbb{X}^{(TM)} \longrightarrow \mathbb{Y}^{(TM)} \tag{6.22}$$

is a compact operator.

On the other hand, as it has been mentioned in the proof of Theorem 5.2, the third and forth operators in the main diagonal

$$-[\mathcal{V}_{S_i}^{(1)} + \mathcal{V}_{S_i}^{(2)}] : H^{-\frac{1}{2}}(S_i) \longrightarrow H^{\frac{1}{2}}(S_i), \tag{6.23}$$

$$\mathcal{L}_{S_i}^{(1)} + \mathcal{L}_{S_i}^{(2)} : H^{\frac{1}{2}}(S_i) \longrightarrow H^{-\frac{1}{2}}(S_i), \tag{6.24}$$

are Fredholm with zero index.

Moreover, applying the results of the theory of strongly elliptic pseudo-differential equations on manifolds with boundary (see, e.g., [3, Theorem 3.5], [6, Lemma 3.4]) we conclude that the operators on the main diagonal

$$r_{_{S_{eD}}}\mathcal{V}_{S_e}^{(2)} : \widetilde{H}^{-\frac{1}{2}}(S_{eD}) \longrightarrow H^{\frac{1}{2}}(S_{eD}), \tag{6.25}$$

$$r_{_{S_{eN}}}\mathcal{L}_{S_e}^{(2)} : \widetilde{H}^{\frac{1}{2}}(S_{eN}) \longrightarrow H^{-\frac{1}{2}}(S_{eN}), \tag{6.26}$$

are Fredholm with zero index.

Therefore, (6.21) and consequently (6.17) is a Fredholm operator with zero index. It remains to show that the null space of operator (6.17) is trivial. Let $U^{(TM)} \in \mathbb{X}^{(TM)}$ be a solution to the homogeneous equation $\mathcal{K}^{(TM)}U^{(TM)} = 0$. Then due to the first two lines of the matrix equation and mapping properties (3.68), (3.93) and (3.94) we see that $U^{(TM)} \in \mathbb{H}^{(TM)}$ and by the equivalence Theorem 6.1 we conclude $U^{(TM)} = 0$ due to the uniqueness theorem for the problem (TM) in the space $\mathbb{H}^{(TM)}$. Thus the operator (6.17) is invertible.

To prove invertibility of operator (6.16), we remark that for any $\mathcal{F}^{(TM)} \in \mathbb{F}^{(TM)}$ a unique solution $U^{(TM)} \in \mathbb{X}^{(TM)}$ of equation (6.15) is delivered by the inverse to the operator (6.17). On the other hand, since $\mathcal{F}^{(TM)} \in \mathbb{F}^{(TM)}$, the first two lines of the matrix operator $\mathcal{K}^{(TM)}$ imply that in fact $U^{(TM)} \in \mathbb{H}^{(TM)}$ and the mapping $\mathbb{F}^{(TM)} \longrightarrow \mathbb{H}^{(TM)}$ delivered by the inverse to the operator (6.17) is continuous, i.e., this operator gives inverse to operator (6.16) as well.

Now we can summarize the results obtained above as the following

Theorem 6.2. *Let $\chi \in X^3_{1*}$ and condition* (4.3) *hold. Then the operators* (6.16) *and* (6.17) *are invertible.*

7. Crack Type Transmission Dirichlet Problem (CTD)

Let a pair $(u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$ be a solution to the problem (CTD) with the interface crack-transmission conditions (2.17)–(2.20) on S_i and the Dirichlet type boundary condition (2.12) on the exterior boundary S_e, i.e.,

$$A_q(x,\partial)u_q = f_q \ \text{ in } \ \Omega_q, \ \ q = 1, 2, \tag{7.1}$$

$$\gamma_1 u_1 - \gamma_2 u_2 = \varphi_{0i}^{(t)} \ \text{ on } \ S_i^{(t)}, \tag{7.2}$$

$$T_1 u_1 + T_2 u_2 = \psi_{0i}^{(t)} \ \text{ on } \ S_i^{(t)}, \tag{7.3}$$

$$T_1 u_1 = \psi'_{0i} \ \text{ on } \ S_i^{(c)}, \tag{7.4}$$

$$T_2 u_2 = \psi''_{0i} \ \text{ on } \ S_i^{(c)}, \tag{7.5}$$

$$\gamma_2 u_2 = \varphi_{0e} \ \text{ on } \ S_e. \tag{7.6}$$

Let ψ_{0i} be defined by (2.22). We assume that the conditions (2.21)–(2.23) are satisfied along with the conditions (2.16) for the function φ_{0e} and f_q, $q = 1, 2$.

Denote by $\Psi_{0i} \in H^{-\frac{1}{2}}(S_i)$ some fixed extension of the function $\psi'_{0i} - \psi''_{0i}$ from $S_i^{(c)}$ onto the whole of S_i preserving the function space. Analogously, let $\Phi_{0i} \in H^{\frac{1}{2}}(S_i)$ be some fixed extension of the function $\varphi_{0i}^{(t)}$ from $S_i^{(t)}$ onto the whole of S_i preserving the function space. Then we can write the following relations on S_i

$$T_1 u_1 = \frac{1}{2}\,[T_1 u_1 + T_2 u_2] + \frac{1}{2}\,[T_1 u_1 - T_2 u_2] = \frac{1}{2}\,\psi_{0i} + \frac{1}{2}\,\Psi_{0i} + \widetilde{\psi}_i, \tag{7.7}$$

$$T_2 u_2 = \frac{1}{2}\,[T_1 u_1 + T_2 u_2] - \frac{1}{2}\,[T_1 u_1 - T_2 u_2] = \frac{1}{2}\,\psi_{0i} - \frac{1}{2}\,\Psi_{0i} - \widetilde{\psi}_i, \tag{7.8}$$

$$\gamma_1 u_1 = \frac{1}{2}\,[\gamma_1 u_1 + \gamma_2 u_2] + \frac{1}{2}\,[\gamma_1 u_1 - \gamma_2 u_2] = \frac{1}{2}\,\Phi_{0i} + \varphi_i + \widetilde{\varphi}_i, \tag{7.9}$$

$$\gamma_2 u_2 = \frac{1}{2}\,[\gamma_1 u_1 + \gamma_2 u_2] - \frac{1}{2}\,[\gamma_1 u_1 - \gamma_2 u_2] = -\frac{1}{2}\,\Phi_{0i} + \varphi_i - \widetilde{\varphi}_i, \tag{7.10}$$

where

$$\widetilde{\psi}_i := \frac{1}{2}\,[T_1 u_1 - T_2 u_2] - \frac{1}{2}\,\Psi_{0i} \in \widetilde{H}^{-1/2}(S_i^{(t)}), \tag{7.11}$$

$$\varphi_i := \frac{1}{2}\,[\gamma_1 u_1 + \gamma_2 u_2] \in H^{1/2}(S_i), \tag{7.12}$$

$$\widetilde{\varphi}_i := \frac{1}{2}\,[\gamma_1 u_1 - \gamma_2 u_2] - \frac{1}{2}\,\Phi_{0i} \in \widetilde{H}^{1/2}(S_i^{(c)}), \tag{7.13}$$

are unknown functions. Let us introduce one more unknown function defined on S_e

$$\psi_e := T_2 u_2 \in H^{-1/2}(S_e), \tag{7.14}$$

and denote

$$U^{(CTD)} = (u_1, u_2, \widetilde{\psi}_i, \varphi_i, \widetilde{\varphi}_i, \psi_e) \in \mathbb{H}^{(TD)}, \tag{7.15}$$

$$\begin{aligned} \mathbb{H}^{(CTD)} := {} & H^{1,0}(\Omega_1; L_1) \times H^{1,0}(\Omega_2; L_2) \times \widetilde{H}^{-\frac{1}{2}}(S_i^{(t)}) \times H^{\frac{1}{2}}(S_i) \times \\ & \times \widetilde{H}^{\frac{1}{2}}(S_i^{(c)}) \times H^{-\frac{1}{2}}(S_e), \end{aligned} \tag{7.16}$$

We choose equations (3.109) in Ω_1 and Ω_2, difference of equations (3.110) for $q = 1$ and $q = 2$ on $S_i^{(t)}$, sum of equations (3.111) for $q = 1$ and $q = 2$ on the whole of S_i, difference of equations (3.111) for $q = 1$ and $q = 2$ on $S_i^{(c)}$ and equation (3.111) for $q = 2$ on S_e. Then after substituting there the notation (7.7)–(7.10) and (7.14) and taking into consideration the relations (7.1)–(7.6), we arrive at the following system of direct segregated LBDIEs for the components of the vector $U^{(CTD)} = (u_1, u_2, \widetilde{\psi}_i, \varphi_i, \widetilde{\varphi}_i, \psi_e)$,

$$u_1 + \mathcal{R}_1 u_1 - V^{(1)}_{S_i}\widetilde{\psi}_i + W^{(1)}_{S_i}\varphi_i + W^{(1)}_{S_i}\widetilde{\varphi}_i = F_1^{(CTD)} \text{ in } \Omega_1, \tag{7.17}$$

$$u_2 + \mathcal{R}_2 u_2 + V^{(2)}_{S_i}\widetilde{\psi}_i + W^{(2)}_{S_i}\varphi_i - W^{(2)}_{S_i}\widetilde{\varphi}_i - V^{(2)}_{S_e}\psi_e = F_2^{(CTD)} \text{ in } \Omega_2, \tag{7.18}$$

$$\begin{aligned}\gamma_1\mathcal{R}_1 u_1 - \gamma_2\mathcal{R}_2 u_2 - [\mathcal{V}^{(1)}_{S_i} + \mathcal{V}^{(2)}_{S_i}]\widetilde{\psi}_i + [\mathcal{W}^{(1)}_{S_i} - \mathcal{W}^{(2)}_{S_i}]\varphi_i + [\mathcal{W}^{(1)}_{S_i} + \mathcal{W}^{(2)}_{S_i}]\widetilde{\varphi}_i +\\ + \gamma_2 V^{(2)}_{S_e}\psi_e = \gamma_1 F_1^{(CTD)} - \gamma_2 F_2^{(CTD)} - \Phi_{0i} \text{ on } S_i^{(t)},\end{aligned} \tag{7.19}$$

$$\begin{aligned}T_1\mathcal{R}_1 u_1 + T_2\mathcal{R}_2 u_2 - [\mathcal{W}'^{(1)}_{S_i} - \mathcal{W}'^{(2)}_{S_i}]\widetilde{\psi}_i + [\mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i}]\varphi_i + [\mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i}]\widetilde{\varphi}_i -\\ - T_2 V^{(2)}_{S_e}\psi_e = T_1 F_1^{(CTD)} + T_2 F_2^{(CTD)} - \psi_{0i} \text{ on } S_i,\end{aligned} \tag{7.20}$$

$$\begin{aligned}T_1\mathcal{R}_1 u_1 - T_2\mathcal{R}_2 u_2 - [\mathcal{W}'^{(1)}_{S_i} + \mathcal{W}'^{(2)}_{S_i}]\widetilde{\psi}_i + [\mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i}]\varphi_i + [\mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i}]\widetilde{\varphi}_i +\\ + T_2 V^{(2)}_{S_e}\psi_e = T_1 F_1^{(CTD)} - T_2 F_2^{(CTD)} - \Psi_{0i} \text{ on } S_i^{(c)},\end{aligned} \tag{7.21}$$

$$\begin{aligned}\gamma_2\mathcal{R}_2 u + \gamma_2 V^{(2)}_{S_i}\widetilde{\psi}_i + \gamma_2 W^{(2)}_{S_i}\varphi_i - \gamma_2 W^{(2)}_{S_i}\widetilde{\varphi}_i - \mathcal{V}^{(2)}_{S_e}\psi_e =\\ = \gamma_2 F_2^{(TM)} - \varphi_{0e} \text{ on } S_e,\end{aligned} \tag{7.22}$$

where

$$F_1^{(CTD)} = \mathcal{P}_1 f_1 + \frac{1}{2}V^{(1)}_{S_i}\psi_{0i} + \frac{1}{2}V^{(1)}_{S_i}\Psi_{0i} - \frac{1}{2}W^{(1)}_{S_i}\Phi_{0i} \text{ in } \Omega_1, \tag{7.23}$$

$$F_2^{(CTD)} = \mathcal{P}_2 f_2 + \frac{1}{2}V^{(2)}_{S_i}\psi_{0i} - \frac{1}{2}V^{(2)}_{S_i}\Psi_{0i} + \frac{1}{2}W^{(2)}_{S_i}\Phi_{0i} - W^{(2)}_{S_e}\varphi_{0e} \text{ in } \Omega_2. \tag{7.24}$$

There holds the following equivalence theorem.

Theorem 7.1. *Let $\chi \in X^3_{1*}$, conditions* (2.21)–(2.23) *be satisfied along with the conditions* (2.16) *for the functions φ_{0e} and f_q, $q = 1, 2$, ψ_{0i} be defined by* (2.22)*, and Ψ_{0i}, Φ_{0i} and Φ_{0i} be the above introduced extended functions.*

(i) *If a pair $(u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$ solves the interface crack problem (CTD), then the vector $(u_1, u_2, \widetilde{\psi}_i, \varphi_i, \widetilde{\varphi}_i, \psi_e)$, where $\widetilde{\psi}_i$, φ_i, $\widetilde{\varphi}_i$ and ψ_e are defined by relations* (7.11)–(7.14)*, solves LBDIE system* (7.17)–(7.22).

(ii) *Vice versa, if a vector $(u_1, u_2, \widetilde{\psi}_i, \varphi_i, \widetilde{\varphi}_i, \psi_e) \in \mathbb{H}^{(TD)}$ solves LBDIE system* (7.17)–(7.22) *and condition* (4.3) *holds, then the pair (u_1, u_2) solves the problem (CTD) and relations* (7.7)–(7.14) *hold true.*

Proof. The proof of the claim (i) immediately follows from the deduction of system (7.17)–(7.22).

Now, let the vector (7.15) solve LBDIE system (7.17)–(7.22). One can easily verify that the boundary-transmission and crack conditions (7.2)–(7.6) are satisfied. To this end one needs, similar to the proof of Theorem 6.1, to take the traces and co-normal derivatives of the first two equations (7.17) and (7.18) and compare them with the last four equations (7.19)–(7.22).

It remains to show that u_1 and u_2 solve the differential equations (7.1) and that the relations (7.7)–(7.14) hold true. Due to the embedding (7.16), we can write the third Green identities (3.109). Comparing these equalities with the first two equations of the LBDIE system, (7.17) and (7.18), and keeping in mind that for the functions u_1 and u_2 the boundary-transmission conditions (7.2)–(7.6) are already proved, we arrive at the relations

$$V^{(1)}_{S_i}(g_{i1}) + W^{(1)}_{S_i}(g_{i2}) + \mathcal{P}_1(G_1) = 0 \ \text{ in } \ \Omega_1, \tag{7.25}$$

$$V^{(2)}_{S_i}(g'_{i1}) + W^{(2)}_{S_i}(g_{i4}) + V^{(2)}_{S_e}(g_e) + \mathcal{P}_2(G_2) = 0 \ \text{ in } \ \Omega_2, \tag{7.26}$$

where

$$\begin{gathered} G_1 := A_1 u_1 - f_1 \ \text{ in } \ \Omega_1, \quad G_2 := A_2 u_2 - f_2 \ \text{ in } \ \Omega_2, \\ g_{i1} := T_1 u_1 - \widetilde{\psi}_i - \frac{1}{2}\,\psi_{0i} - \frac{1}{2}\,\Psi_{0i} \ \text{ on } \ S_i, \\ g_{i2} := \varphi_i + \widetilde{\varphi}_i + \frac{1}{2}\,\Phi_{0i} - \gamma_1 u_1 \ \text{ on } \ S_i, \\ g'_{i1} := T_2 u_2 + \widetilde{\psi}_i - \frac{1}{2}\,\psi_{0i} + \frac{1}{2}\,\Psi_{0i} \ \text{ on } \ S_i, \\ g_{i3} := \varphi_i - \widetilde{\varphi}_i - \frac{1}{2}\,\Phi_{0i} - \gamma_2 u_2 \ \text{ on } \ S_i, \\ g_e := T_2 u_2 - \psi_e \ \text{ on } \ S_e. \end{gathered} \tag{7.27}$$

Due to the boundary-transmission conditions (7.2)–(7.6) and equalities (2.22) we obtain,

$$g_{i1} = -g'_{i1} \in \widetilde{H}^{-\frac{1}{2}}(S_i^{(t)}), \ \ g_{i2} - g_{i3} \in \widetilde{H}^{\frac{1}{2}}(S_i^{(c)}), \ \ g_e \in H^{-\frac{1}{2}}(S_e). \tag{7.28}$$

Therefore by Corollary 4.9 we have $g_{i1} = g'_{i1} = g_{i2} = g_{i3} = 0$ on S_i, $g_e = 0$ on S_e and $G_q = 0$ in Ω_q, $q = 1, 2$, which completes the proof. □

Due to this equivalence theorem we conclude that the LBDIEs system (7.17)–(7.22) with the special right hand side functions which belong to the space

$$\begin{aligned} \mathbb{F}^{(CTD)} := H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2) \times H^{\frac{1}{2}}(S_i^{(t)}) \times \\ \times H^{-\frac{1}{2}}(S_i) \times H^{-\frac{1}{2}}(S_i^{(c)}) \times H^{\frac{1}{2}}(S_e) \end{aligned} \tag{7.29}$$

is uniquely solvable in the space $\mathbb{H}^{(CTD)}$ defined in (7.16). In particular, the corresponding homogeneous LBDIEs system possesses only the trivial solution. By the way, one can easily observe that the right hand side expressions in LBDIEs system (7.17)–(7.22) vanish if and only if $f_q = 0$ in Ω_q, $q = 1, 2$, $\varphi_{0i} = \psi_{0i} = 0$ on $S_i^{(t)}$, $\psi'_{0i} = \psi''_{0i} = 0$ on $S_i^{(c)}$ and $\varphi_{0e} = 0$ on S_2.

Our next aim is to establish that the matrix operator $\mathcal{K}^{(CTD)}$ generated by the left hand side expressions in the LBDIEs system (7.17)–(7.22) is invertible in two sets spaces. We have

$$\mathcal{K}^{(CTD)}=[\mathcal{K}^{(CTD)}_{kj}]_{6\times 6}:=\operatorname{diag}(r_{\Omega_1},r_{\Omega_2},r_{S_i^{(t)}},r_{S_i},r_{S_i^{(c)}},r_{S_e})\times$$
$$\times\begin{bmatrix} I+\mathcal{R}_1 & 0 & -V^{(1)}_{S_i} & W^{(1)}_{S_i} & W^{(1)}_{S_i} & 0\\ 0 & I+\mathcal{R}_2 & V^{(2)}_{S_i} & W^{(2)}_{S_i} & -W^{(2)}_{S_i} & -V^{(2)}_{S_e}\\ \gamma_1\mathcal{R}_1 & -\gamma_2\mathcal{R}_2 & -\mathcal{V}^{(1)}_{S_i}-\mathcal{V}^{(2)}_{S_i} & \mathcal{W}^{(1)}_{S_i}-\mathcal{W}^{(2)}_{S_i} & \mathcal{W}^{(1)}_{S_i}+\mathcal{W}^{(2)}_{S_i} & \gamma_2 V^{(2)}_{S_e}\\ T_1\mathcal{R}_1 & T_2\mathcal{R}_2 & -\mathcal{W}'^{(1)}_{S_i}+\mathcal{W}'^{(2)}_{S_i} & \mathcal{L}^{(1)}_{S_i}+\mathcal{L}^{(2)}_{S_i} & \mathcal{L}^{(1)}_{S_i}-\mathcal{L}^{(2)}_{S_i} & -T_2V^{(2)}_{S_e}\\ T_1\mathcal{R}_1 & -T_2\mathcal{R}_2 & -\mathcal{W}'^{(1)}_{S_i}-\mathcal{W}'^{(2)}_{S_i} & \mathcal{L}^{(1)}_{S_i}-\mathcal{L}^{(2)}_{S_i} & \mathcal{L}^{(1)}_{S_i}+\mathcal{L}^{(2)}_{S_i} & T_2V^{(2)}_{S_e}\\ 0 & \gamma_2\mathcal{R}_2 & \gamma_2 V^{(2)}_{S_i} & \gamma_2 W^{(2)}_{S_i} & -\gamma_2 W^{(2)}_{S_i} & -\mathcal{V}^{(2)}_{S_e} \end{bmatrix}. \tag{7.30}$$

Introduce the function spaces

$$\mathbb{X}^{(CTD)}:=H^1(\Omega_1)\times H^1(\Omega_2)\times\widetilde{H}^{-\frac12}(S_1^{(t)})\times H^{\frac12}(S_1)\times$$
$$\times\widetilde{H}^{\frac12}(S_1^{(c)})\times H^{-\frac12}(S_2), \tag{7.31}$$
$$\mathbb{Y}^{(CTD)}:=H^1(\Omega_1)\times H^1(\Omega_2)\times H^{\frac12}(S_1^{(t)})\times H^{-\frac12}(S_1)\times$$
$$\times H^{-\frac12}(S_1^{(c)})\times H^{\frac12}(S_2). \tag{7.32}$$

By virtue of Theorems 3.9 and 3.11 we see that the operator $\mathcal{K}^{(CTD)}$ has the following mapping property

$$\mathcal{K}^{(CTD)}:\mathbb{H}^{(CTD)}\longrightarrow\mathbb{F}^{(CTD)}, \tag{7.33}$$
$$:\mathbb{X}^{(CTD)}\longrightarrow\mathbb{Y}^{(CTD)}. \tag{7.34}$$

Theorem 7.2. *Let $\chi\in X^3_{1*}$ and condition* (4.3) *hold. Then operators* (7.33) *and* (7.34) *are invertible.*

Proof. Due to compactness of the operators from Lemma 3.7 and Theorems 3.9 and 3.11, the upper block-triangular matrix operator

$$\mathcal{K}^{(CTD)}_0:=\operatorname{diag}(r_{\Omega_1},r_{\Omega_2},r_{S_i^{(t)}},r_{S_i},r_{S_i^{(c)}},r_{S_e})\times$$
$$\times\begin{bmatrix} I & 0 & -V^{(1)}_{S_i} & W^{(1)}_{S_i} & W^{(1)}_{S_i} & 0\\ 0 & I & V^{(2)}_{S_i} & W^{(2)}_{S_i} & -W^{(2)}_{S_i} & -V^{(2)}_{S_e}\\ 0 & 0 & -\mathcal{V}^{(1)}_{S_i}+\mathcal{V}^{(2)}_{S_i} & 0 & 0 & 0\\ 0 & 0 & 0 & [\mathcal{L}^{(1)}_{S_i}+\mathcal{L}^{(2)}_{S_i}] & \mathcal{L}^{(1)}_{S_i}-\mathcal{L}^{(2)}_{S_i} & 0\\ 0 & 0 & 0 & [\mathcal{L}^{(1)}_{S_i}-\mathcal{L}^{(2)}_{S_i}] & \mathcal{L}^{(1)}_{S_i}+\mathcal{L}^{(2)}_{S_i} & 0\\ 0 & 0 & 0 & 0 & 0 & -\mathcal{V}^{(2)}_{S_e} \end{bmatrix}$$

is a compact perturbation of the operator (7.34) and possesses the same mapping property,

$$\mathcal{K}^{(CTD)}_0:\mathbb{X}^{(CTD)}\longrightarrow\mathbb{Y}^{(CTD)}. \tag{7.35}$$

Our goal is to show that the operator (7.35) is Fredholm with zero index. To this end, let us note that the operator (3.105) is a strongly elliptic pseudodifferential operator of order -1 with strictly positive principal homogenous symbol, while (3.108) is a strongly elliptic pseudodifferential operator of order $+1$ with strictly negative principal homogenous symbol. This can be shown by a standard approach since the principal homogeneous symbols of the localized operators and the corresponding non-localized ones coincide (cf. [7], [13]).

Therefore, applying the theory of pseudodifferential equations on manifolds with and/or without boundary ([11], [26]) one can show that the third and sixth operators in the main diagonal of $\mathcal{K}_0^{(CTD)}$

$$r_{S_i^{(t)}}[\mathcal{V}^{(1)}_{S_i}+\mathcal{V}^{(2)}_{S_i}] : \widetilde{H}^{-\frac{1}{2}}(S_i^{(t)}) \longrightarrow H^{-\frac{1}{2}}(S_i^{(t)}),$$
$$\mathcal{V}^{(2)}_{S_e} : H^{-\frac{1}{2}}(S_e) \longrightarrow H^{\frac{1}{2}}(S_e)$$

are Fredholm with zero index.

Now let us consider the following 2×2 matrix operator block which stands in the main diagonal of the upper block-triangular matrix operator $\mathcal{K}_0^{(CTD)}$

$$\mathcal{L} := \begin{bmatrix} \mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} & \mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i} \\ r_{S_i^{(c)}}[\mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i}] & r_{S_i^{(c)}}[\mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i}] \end{bmatrix}. \tag{7.36}$$

Clearly,

$$\mathcal{L} : H^{\frac{1}{2}}(S_i) \times \widetilde{H}^{\frac{1}{2}}(S_i^{(c)}) \longrightarrow H^{-\frac{1}{2}}(S_i) \times H^{-\frac{1}{2}}(S_i^{(c)}) \tag{7.37}$$

is continuous. Denote by $\sigma^{(q)}(y,\xi')$, $y \in S_i$, $\xi' \in \mathbb{R}^2$, the principal homogeneous symbol of the operator $\mathcal{L}^{(q)}_{S_i}$, $q = 1, 2$ (see formula (B.9)). As it is shown in Appendix B, $\sigma^{(q)}(y,\xi')$ is a homogeneous function in ξ' of order 1 and $\sigma^{(q)}(y,\xi') < 0$ for all $\xi' \in \mathbb{R}^2 \setminus \{0\}$ and for all $y \in S_i$.

Therefore there is a compact operator $\mathcal{C} : H^{\frac{1}{2}}(S_i) \longrightarrow H^{-\frac{1}{2}}(S_i)$ such that

$$\mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} + \mathcal{C} : H^{\frac{1}{2}}(S_i) \longrightarrow H^{-\frac{1}{2}}(S_i) \tag{7.38}$$

is invertible. Denote the inverse operator by $[\mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} + \mathcal{C}]^{-1}$.
Further, let us introduce a compact perturbation of the operator $\mathcal{L}$ in (7.36)–(7.37) defined by the relation

$$\widetilde{\mathcal{L}} := \begin{bmatrix} \mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} + \mathcal{C} & \mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i} \\ r_{S_i^{(c)}}[\mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i}] & r_{S_i^{(c)}}[\mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i}] \end{bmatrix}. \tag{7.39}$$

It is easy to check that $\widetilde{\mathcal{L}}$ can be represented as the composition of two operators

$$\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}_1 \widetilde{\mathcal{L}}_2,$$

where

$$\widetilde{\mathcal{L}}_1 := \begin{bmatrix} 0 & \mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} + \mathcal{C} \\ r_{S_i^{(c)}}\mathcal{N}_{S_i} & r_{S_i^{(c)}}[\mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i}] \end{bmatrix} \tag{7.40}$$

with

$$\mathcal{N}_{S_i} := \mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} - [\mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i}]\,[\mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} + \mathcal{C}]^{-1}[\mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i}] \tag{7.41}$$

and

$$\widetilde{\mathcal{L}}_2 := \begin{bmatrix} 0 & I \\ I & [\mathcal{L}^{(1)}_{S_i} + \mathcal{L}^{(2)}_{S_i} + \mathcal{C}]^{-1}[\mathcal{L}^{(1)}_{S_i} - \mathcal{L}^{(2)}_{S_i}] \end{bmatrix}. \tag{7.42}$$

Note that the operator

$$\widetilde{\mathcal{L}}_2 : H^{\frac{1}{2}}(S_i) \times \widetilde{H}^{\frac{1}{2}}(S_i^{(c)}) \longrightarrow \widetilde{H}^{\frac{1}{2}}(S_i) \times H^{\frac{1}{2}}(S_i), \tag{7.43}$$

is invertible, while the operator

$$\widetilde{\mathcal{L}}_1 : \widetilde{H}^{\frac{1}{2}}(S_i^{(c)}) \times H^{\frac{1}{2}}(S_i) \longrightarrow H^{-\frac{1}{2}}(S_i) \times H^{-\frac{1}{2}}(S_i^{(c)}) \tag{7.44}$$

is bounded. Due to the triangular structure of the operator $\widetilde{\mathcal{L}}_1$ in (7.40) and in view of invertibility of the operator (7.38) we see that (7.44) is Fredholm with zero index if the pseudodifferential operator

$$r_{S_i^{(c)}} \mathcal{N}_{S_i} : \widetilde{H}^{\frac{1}{2}}(S_i^{(c)}) \longrightarrow H^{-\frac{1}{2}}(S_i^{(c)}) \tag{7.45}$$

is Fredholm with zero index. Taking into consideration that $\sigma^{(q)}(y, \xi') < 0$ for all $\xi' \in \mathbb{R}^2 \setminus \{0\}$ and for all $y \in S_i$, we deduce that the principal homogeneous symbol $\sigma_{\mathcal{N}}(y, \xi')$ of the operator $\mathcal{N}_{S_i}$ is strictly negative,

$$\sigma_{\mathcal{N}}(y, \xi') = \sigma^{(1)}(y, \xi') + \sigma^{(2)}(y, \xi') - \frac{[\sigma^{(1)}(y, \xi') - \sigma^{(2)}(y, \xi')]^2}{\sigma^{(1)}(y, \xi') + \sigma^{(2)}(y, \xi')} =$$

$$= \frac{4\sigma^{(1)}(y, \xi')\sigma^{(2)}(y, \xi')}{\sigma^{(1)}(y, \xi') + \sigma^{(2)}(y, \xi')} < 0$$

for all $\xi' \in \mathbb{R}^2 \setminus \{0\}$ and for all $y \in S_i$.

Therefore the pseudodifferential operator (7.45) and, consequently, (7.44) and (7.39) are Fredholm with zero index ([11], [26]). The operator (7.37) possesses the same property, since $\mathcal{L} - \widetilde{\mathcal{L}}$ is compact. This implies that the operator (7.35) is Fredholm with zero index and since

$$\mathcal{K}^{(CTD)} - \mathcal{K}_0^{(CTD)} : \mathbb{X}^{(CTD)} \longrightarrow \mathbb{Y}^{(CTD)}$$

is compact, the operator (7.34) is Fredholm with zero index as well.

It remains to show that the null space of the operator (7.34) is trivial. Let $U_0 \in \mathbb{X}^{(CTD)}$ be a solution to the homogeneous equation $\mathcal{K}^{(CTD)} U_0 = 0$. From equations (7.17) and (7.18) with zero right hand sides due to the mapping properties (3.68), (3.93) and (3.94) we then see that $U_0 \in \mathbb{H}^{(CTD)}$. By the equivalence Theorem 7.1 and the uniqueness Theorem 2.1 then it follows that $U_0 = 0$. Thus the kernel of the operator (7.34) is trivial and consequently it is invertible.

To prove invertibility of operator (7.33), we remark that for any $\mathcal{F}^{(CTD)} \in \mathbb{F}^{(CTD)}$ a unique solution $U^{(CTD)} \in \mathbb{X}^{(CTD)}$ of equation

$$\mathcal{K}^{(CTD)} U^{(CTD)} = \mathcal{F}^{(CTD)}, \tag{7.46}$$

is delivered by the inverse to the operator (7.34). On the other hand, since $\mathcal{F}^{(CTD)} \in \mathbb{F}^{(CTD)}$, the first two lines of the matrix operator $\mathcal{K}^{(CTD)}$ imply that in fact $U^{(CTD)} \in \mathbb{H}^{(CTD)}$ and the mapping $\mathbb{F}^{(CTD)} \longrightarrow \mathbb{H}^{(CTD)}$ delivered by the inverse to the operator (7.34) is continuous, i.e., this operator gives inverse to operator (7.33) as well. □

8. Appendix A: Classes of Localizing Functions

Let us introduce the classes for *localizing functions.*

Definition A.1.

(i) We say $\chi \in X^k$ for integer $k \geq 0$ if

$$\chi(x) = \breve{\chi}(|x|), \quad \breve{\chi} \in W_1^k(0,\infty), \quad \varrho\breve{\chi}(\varrho) \in L_1(0,\infty). \tag{A.1}$$

(ii) We say $\chi \in X_*^k$ for $k \geq 1$ if $\chi \in X^k$, $\chi(0) = 1$ and

$$\sigma_\chi(\omega) > 0 \ \text{ for a.e. } \ \omega \in \mathbb{R}, \tag{A.2}$$

where

$$\sigma_\chi(\omega) := \begin{cases} \dfrac{1}{\omega}\,\widehat{\chi}_s(\omega) & \text{for } \ \omega \in \mathbb{R} \setminus \{0\}, \\ \displaystyle\int_0^\infty \varrho\breve{\chi}(\varrho)\,d\varrho & \text{for } \ \omega = 0, \end{cases} \tag{A.3}$$

and $\widehat{\chi}_s(\omega)$ denotes the sine-transform of the function $\breve{\chi}$,

$$\widehat{\chi}_s(\omega) := \int_0^\infty \breve{\chi}(\varrho)\, \sin(\varrho\omega)\,d\varrho. \tag{A.4}$$

(iii) We say $\chi \in X_{1*}^k$ for $k \geq 1$ if $\chi \in X_*^k$ and

$$\omega\widehat{\chi}_s(\omega) \leq 1 \ \ \forall\, \omega \in \mathbb{R}. \tag{A.5}$$

Note that if $\breve{\chi}$ has a compact support, then the third condition in (A.1) is evidently satisfied. If $\breve{\chi} \in W^k(0,\infty)$, $k \geq 1$, then $\breve{\chi}$ is continuous due to the Sobolev embedding theorem, and $\chi(0) = \breve{\chi}(0)$ is well defined as the trace of $\breve{\chi}$. Evidently, we have the following embeddings, $X^{k_1} \subset X^{k_2}$ and $X_*^{k_1} \subset X_*^{k_2}$, $X_{1*}^{k_1} \subset X_{1*}^{k_2}$ for $k_1 > k_2$.

The class X_*^k is defined in terms of the sine-transform. Since the classes X_+^k and X_{1+}^k introduced in [7] are subsets of the corresponding classes X_*^k and X_{1*}^k, the following lemma implied by [7, Lemma 3.2] gives an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class.

Lemma A.2. *If $\chi \in X^k$, $k \geq 1$, $\breve{\chi}(0) = 1$, $\breve{\chi}(\varrho) \geq 0$ for all $\varrho \in (0,\infty)$, and $\breve{\chi}$ is a non-increasing function on $[0,+\infty)$, then $\chi \in X_*^k$.*

The following examples for χ are presented in [7],

$$\chi_1(x) = \begin{cases} \Big[1 - \dfrac{|x|}{\varepsilon}\Big]^k & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \tag{A.6}$$

$$\chi_2(x) = \begin{cases} \exp\Big[\dfrac{|x|^2}{|x|^2 - \varepsilon^2}\Big] & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \tag{A.7}$$

One can observe that $\chi_1 \in X_*^k$, while $\chi_2 \in X_*^\infty$ due to Lemma A.2 and for them the inequality (A.2) holds for all $\omega \in \mathbb{R}$. Moreover, $\chi_1 \in X_{1*}^k$ for $k = 2$ and $k = 3$. For details and further examples see [7].

9. Appendix B: Calculation of Symbols of Boundary Operators

Here we calculate the principal homogeneous symbols $\sigma_{\mathcal{V}^{(q)}}(y, \xi')$ and $\sigma_{\mathcal{L}^{(q)}}(y, \xi')$ of the boundary pseudodifferential operators $\mathcal{V}^{(q)}$ and $\mathcal{L}^{(q)}$, $q = 1, 2$, defined by formulas (3.14) and (3.17). Without loss of generality, we assume that the point $y \in \partial\Omega_q$ is the origin of some local co-ordinate system with the third co-ordinate axis coinciding with the outward unit normal vector $n^{(q)}(y)$. Due to the local principal technique (see, e.g. [11]), instead of Ω_q, actually, we can consider the half-space $\mathbb{R}^3_- := \{x \in \mathbb{R}^3 : x_3 < 0\}$ with the outward unit normal vector $n^{(q)}(y) = (0, 0, 1)$ to the boundary $\partial\mathbb{R}^3_-$.

First we rewrite the fundamental solution (Levi function) of the operator $A_q(y, \partial_x) = a_q(y) A_{q\star}(\partial_x)$ (see (2.1) and (3.1)) in the following form

$$\begin{aligned} P_{q1}(x, y) = a_q^{-1}(y) P_{q1\star}(x, y) = a_q^{-1}(y) \mathfrak{F}^{-1}_{\xi \to x}[A^{-1}_{q*}(-i\xi)] = \\ = a_q^{-1}(y) \mathfrak{F}^{-1}_{\xi' \to x'}\Big[\pm \frac{1}{2\pi} \int\limits_{l^\pm} A^{-1}_{q\star}(-i\xi', -i\tau) e^{-i\tau x_3}\, d\tau \Big], \end{aligned} \tag{B.1}$$

where $P_{q1\star}(x, y)$ is defined by (3.2), the sign "+" corresponds to the case $x_3 < 0$, while the sign "$-$" corresponds to the case $x_3 > 0$. Here we use the notation: $x' = (x_1, x_2)$, $x = (x', x_3)$, $\xi' = (\xi_1, \xi_2)$, $\xi = (\xi', \xi_3)$, $l^+ (l^-)$ is a closed contour orientated counterclockwise and enclosing all the roots of the polynomial $A_{q\star}(-i\xi', -i\tau)$ with respect to the variable τ in the half-plane $\operatorname{Im}\tau > 0$ ($\operatorname{Im}\tau < 0$).

Note that due to formulas (3.1) and (3.26)

$$A_{q\star}(\xi', \tau) = a^{(q)}_{33\star}\tau^2 + 2\tau \sum_{k=1}^{2} a^{(q)}_{k3\star}\xi_k + \sum_{k,j=1}^{2} a^{(q)}_{kj\star}\xi_k \xi_j, \tag{B.2}$$

$$T_{q\star}(\xi', \tau) = a^{(q)}_{33\star}\tau + \sum_{k=1}^{2} a^{(q)}_{k3\star}\xi_k, \tag{B.3}$$

since $n^{(q)} = (0, 0, 1)$.

Denote by τ_q^+ and τ_q^- the zeros of the polynomial $A_{q\star}(\xi', \tau)$ with positive and negative imaginary parts respectively,

$$\tau_q^{\pm}(\xi') = \tau_{q1}(\xi') \pm i\tau_{q2}(\xi'), \quad \tau_{q2}(\xi') > 0, \tag{B.4}$$

where

$$\tau_{q1}(\xi') = -[a_{33\star}^{(q)}]^{-1} \sum_{k=1}^{2} a_{k3\star}^{(q)} \xi_k, \tag{B.5}$$

$$\tau_{q2}(\xi') = [a_{33\star}^{(q)}]^{-1} \Big[a_{33\star}^{(q)} \sum_{k,j=1}^{2} a_{kj\star}^{(q)} \xi_k \xi_j - \Big(\sum_{k=1}^{2} a_{k3\star}^{(q)} \xi_k \Big)^2 \Big]^{1/2} > 0 \tag{B.6}$$

for all $\xi' \in \mathbb{R}^2 \setminus \{0\}$.

The latter inequality follows from the positive definiteness of the matrix $[a_{kj\star}^{(q)}]_{3\times 3}$.

Now, in view of the representation (B.1) and formula (3.14), we get the following expression for the principal homogeneous symbol of the operator $\mathcal{V}^{(q)}$:

$$\sigma_{_{\mathcal{V}^{(q)}}}(y, \xi') = -\frac{1}{2\pi a_q(y)} \int_{l^+} A_{q\star}^{-1}(-i\xi', -i\tau)\, d\tau = \frac{1}{2\pi a_q(y)} \int_{l^+} \frac{d\tau}{A_{q\star}(\xi', \tau)} \tag{B.7}$$

and with the help of the residue theorem finally we deduce

$$\begin{aligned}\sigma_{_{\mathcal{V}^{(q)}}}(y, \xi') &= \frac{i}{2a_q(y)} \, \frac{1}{a_{33\star}^{(q)} \tau_q^+ + \sum\limits_{k=1}^{2} a_{k3\star}^{(q)} \xi_k} = \\ &= \frac{1}{2a_{33\star}^{(q)} a_q(y) \tau_{q2}(\xi')} > 0 \ \text{ for all } \ \xi' \in \mathbb{R}^2 \setminus \{0\}. \end{aligned} \tag{B.8}$$

Quite similarly, for the principal homogeneous symbol of the boundary pseudodifferential operator $\mathcal{L}^{(q)}$ with the help of (3.17) and (B.1) we get:

$$\begin{aligned}\sigma^{(q)}(y, \xi') &\equiv \sigma_{_{\mathcal{L}^{(q)}}}(y, \xi') = -\frac{1}{2\pi} \int_{l^+} \frac{T_q(y, -i\xi', -i\tau) T_q(y, i\xi', i\tau)}{A_q(-i\xi', -i\tau)}\, d\tau = \\ &= \frac{1}{2\pi} \int_{l^+} \frac{[T_q(y, \xi', \tau)]^2}{A_q(\xi', \tau)}\, d\tau = \frac{1}{2\pi} \int_{l^+} \frac{a_q^2(y) [T_{q\star}(y, \xi', \tau)]^2}{a_q(y) A_{q\star}(\xi', \tau)}\, d\tau = \\ &= \frac{i a_q(y)}{2} \Big[a_{33\star}^{(q)} \tau_q^+ + \sum_{k=1}^{2} a_{k3\star}^{(q)} \xi_k \Big] = -\frac{1}{2} a_{33\star}^{(q)} a_q(y) \tau_{q2}(\xi') < 0 \end{aligned} \tag{B.9}$$

for all $\xi' \in \mathbb{R}^2 \setminus \{0\}$.

Concluding Remarks

Four *segregated* direct localized boundary-domain integral equation systems for several transmission problems for a scalar linear divergence PDE with *matrix variable* coefficients of a special form were formulated and analyzed in the paper. They give some representative samples of different LBDIE systems that can be formulated and analyzed for such problems. The first two LBDIE systems, (TD1) and (TD2) are associated with the transmission-Dirichlet problem, where the boundary equations of the system (TD1) are of the first kind, while all the equations of the system (TD2) are of the second kind. The last two LBDIE systems are associated with the transmission-mixed problem and with the transmission-Dirichlet problem with the interface crack on a part of the interface. The boundary equations of the both these LBDIE systems are of the first kind.

Equivalence of the LBDIEs to the original variable-coefficient transmission-boundary-crack problems was proved in the case when right-hand side of the PDE is from $L_2(\Omega_q)$, and the Dirichlet and the Neumann data from the spaces $H^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}$, respectively, on the corresponding parts of the boundary. The invertibility of the operators for the LBDIE systems (TD1), (TM) and (CTD) was proved in the corresponding Sobolev spaces, employing the technique of pseudodifferential operators on manifolds. The main theorems for LBDIEs were proved under condition $\chi \in X^3_{1*}$ on the localizing function, which is more relaxed then the condition $\chi \in X^3_{1+}$ from [7]. Condition (4.3) that the ratio of the coefficients on the interface should be constant appeared to be essential in the proof. A special consideration is needed to relax the latter condition.

Quite similarly the problems (TN), (CTN) and (CTM) can be reduced to the corresponding LBDIE systems which can be analyzed by the analogous arguments. By the same approach, the corresponding LBDIDE systems for unbounded domains can be analyzed as well. The approach can be extended also to more general PDEs and to systems of PDEs, while smoothness of the variable coefficients and the boundary can be essentially relaxed, and the PDE right hand side can be considered in more general spaces, c.f. [18, 19].

This study can serve as a basis for rigorous analysis of numerical, especially mesh-less methods for the LBDIEs that after discretization lead to sparsely populated systems of linear algebraic equations attractive for numerical computations (see e.g. [17, 21] for algorithm and implementation).

Acknowledgements

This research was supported by the EPSRC grant No. EP/H020497/1: "Mathematical analysis of Localized Boundary-Domain Integral Equations for Variable-Coefficient Boundary Value Problems" and partly by the Georgian Technical University grant in the case of the third author.

References

1. M. S. Agranovich, Elliptic singular integro-differential operators. (Russian) *Usp. Mat. Nauk* **20** (1965), No. 5(125), 3–120.

2. L. Boutet de Monvel, Boundary problems for pseudo-differential operators. *Acta Math.* **126** (1971), No. 1-2, 11–51.
3. T. Buchukuri, O. Chkadua, and D. Natroshvili, Mixed boundary value problems of thermopiezoelectricity for solids with interior cracks. *Integral Equations Operator Theory* **64** (2009), No. 4, 495–537.
4. J. Chazarain and A. Piriou, Introduction to the theory of linear partial differential equations. Translated from the French. Studies in Mathematics and its Applications, 14. *North-Holland Publishing Co., Amsterdam–New York*, 1982.
5. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. I. Equivalence and invertibility. *J. Integral Equations Appl.* **21** (2009), No. 4, 499–543. 499-542.
6. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. II. Solution regularity and asymptotics. *J. Integral Equations Appl.* **22** (2010), No. 1, 19–37.
7. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Analysis of some localized boundary-domain integral equations. *J. Integral Equations Appl.* **21** (2009), No. 3, 405–445.
8. O. Chkadua, S. E. Mikhailov, and D. Natroshvili, Analysis of segregated boundary-domain integral equations for variable-coefficient problems with cracks. *Numer. Methods Partial Differ. Equations* 27 (2011), No. 1, 121–140.
9. M. Costabel, Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.* **19** (1988), No. 3, 613–626.
10. R. Dautray and J.-L. Lions, Mathematical analysis and numerical methods for science and technology. Vol. 4. Integral equations and numerical methods. With the collaboration of Michel Artola, Philippe Benilan, Michel Bernadou, Michel Cessenat, Jean-Claude Nedelec, Jacques Planchard and Bruno Scheurer. Translated from the French by John C. Amson. *Springer-Verlag, Berlin*, 1990.
11. G. Eskin, Boundary value problems for elliptic pseudodifferential equations. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, 52. *American Mathematical Society, Providence, R.I.*, 1981.
12. G. Fichera, Boundary value problems of elasticity with unilateral constraints. In: Handbuch der Physik, Bd. VIa/2: Mechanics of Solid II (Truesdell C., ed.), *Springer-Verlag, Berlin*, 1972, 391-424.
13. G. C. Hsiao and W. L. Wendland, Boundary integral equations. Applied Mathematical Sciences, 164. *Springer-Verlag, Berlin*, 2008.
14. A. M. Khludnev and V. A. Kovtunenko, Analysis of cracks in solids. *WIT Press, Southampton–Boston*, 2000.
15. J.-L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications. Vol. I. Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 181. *Springer-Verlag, New York-Heidelberg*, 1972.
16. W. McLean,Strongly elliptic systems and boundary integral equations. *Cambridge University Press, Cambridge*, 2000.
17. S. E. Mikhailov, Localized boundary-domain integral formulations for problems with variable coefficients. *Eng. Anal. Bound. Elem.* **26** (2002), No. 8, 681–690.
18. S. E. Mikhailov, Analysis of extended boundary-domain integral and integro-differential equations of some variable-coefficient BVP. In: Ke Chen (Edr.), *Advances in Boundary Integral Methods – Proceedings of the 5th UK Conference on Boundary Integral Methods*, *Liverpool, UK. University of Liverpool Publ.*, 2005, pp. 106–125.
19. S. E. Mikhailov,Analysis of united boundary-domain integro-differential and integral equations for a mixed BVP with variable coefficient. *Math. Methods Appl. Sci.* **29** (2006), No. 6, 715–739.
20. S. E. Mikhailov, Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains. *J. Math. Anal. Appl.* (2010), doi:10.1016/j.jmaa.2010.12.027.

21. S. E. Mikhailov and I. S. Nakhova, Mesh-based numerical implementation of the localized boundary-domain integral-equation method to a variable-coefficient Neumann problem. *J. Engrg. Math.* **51** (2005), No. 3, 251–259.
22. S. G. Mikhlin, Equtions of mathematical physics. (Russian) *Nauka, Moscow*, 1968.
23. C. Miranda, Partial differential equations of elliptic type. Second revised edition. Translated from the Italian by Zane C. Motteler. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2. *Springer-Verlag, New York–Berlin*, 1970.
24. A. Pomp, The boundary-domain integral method for elliptic systems. With an application to shells. Lecture Notes in Mathematics, 1683. *Springer-Verlag, Berlin*, 1998.
25. R. T. Seeley, Singular integrals and boundary value problems. *Amer. J. Math.* **88** (1966), 781–809.
26. E. Shargorodsky, An L_p-analogue of the Vishik–Eskin theory. *Mem. Differential Equations Math. Phys.* **2** (1994), 41–146.
27. J. Sladek, V. Sladek, and S. N. Atluri, Local boundary integral equation (LBIE) method for solving problems of elasticity with nonhomogeneous material properties. *Comput. Mech.* **24** (2000), No. 6, 456-462.
28. A. E. Taigbenu, The Green element method. *Kluwer Academic Publishers, Dordrecht,* 1999.
29. V. S. Vladimirov, Generalized functions in mathematical physics. (Russian) Current Problems in Physics and Technology Series. *Izdat. "Nauka", Moscow*, 1976.
30. T. Zhu, J.-D. Zhang, and S. N. Atluri, A local boundary integral equation (LBIE) method in computational mechanics, and a meshless discretization approach. *Comput. Mech.* **21** (1998), No. 3, 223–235.
31. T. Zhu, J.-D. Zhang, and S. N. Atluri, A meshless numerical method based on the local boundary integral equation (LBIE) to solve linear and non-linear boundary value problems. *Eng. Anal. Bound. Elem.* **23** (1999), No. 5-6, 375–389.

(Received 13.10.2010)

Authors' address:

O. Chkadua
Andrea Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: chkadua@rmi.ge

S. E. Mikhailov
Department of Mathematics
Brunel University West London
Uxbridge, UB8 3PH
UK
E-mail: Sergey.Mikhailov@brunel.ac.uk

D. Natroshvili
Department of Mathematics, Georgian Technical University
77, M. Kostava St., Tbilisi 0175
Georgia
E-mail: natrosh@hotmail.com