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On some mixed-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces

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In memory of Professor Gabriela Kohr, with affection and deep respect

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A B S T R A C T

The main purpose of this paper is the analysis of mixed-transmission problems for the anisotropic Stokes system in a compressible framework and in bounded Lipschitz domains with transversal Lipschitz interfaces in $\mathbb{R}^n$, $n \geq 2$. Mixed problems and mixed-transmission problems for the anisotropic Navier-Stokes system in dimension $n \in \{2, 3\}$ are also considered. The anisotropy is introduced by an L^∞-viscosity tensor coefficient, which satisfies an ellipticity condition in terms of symmetric matrices in $\mathbb{R}^{n\times n}$ with null matrix traces. In the first part we use a variational approach to show the well-posedness of the analyzed linear problems for the Stokes system in L^2-based Sobolev spaces. In the second part we show the existence and uniqueness of a weak solution of the mixed problem for the anisotropic compressible Navier-Stokes system with small data in L^2-based Sobolev spaces in a bounded Lipschitz domain in $\mathbb{R}^n$, $n \in \{2, 3\}$. A mixed-transmission problem for the Navier-Stokes system in a Lipschitz domain with a transversal Lipschitz interface is also considered.

1. Introduction: Anisotropic Stokes system with elliptic L^∞ viscosity tensor coefficient

Let $n \geq 2$ and Ω be an open set in $\mathbb{R}^n$. Throughout our paper we use the notation ∂_α for the first order partial derivative $\dfrac{\partial}{\partial x_\alpha}$, $\alpha = 1, \ldots, n$, as well as the Einstein summation rule on repeated indices.

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Let $\mathfrak{L}$ be a second order divergence form differential operator

$$\mathfrak{L}\mathbf{u} := \partial_\alpha\left(A^{\alpha\beta}\partial_\beta\mathbf{u}\right), \tag{1.1}$$

such that the tensor coefficient $\mathbb{A}$ consists of $n\times n$ matrix valued functions $A^{\alpha\beta}$ with bounded, measurable, real-valued entries $a_{ij}^{\alpha\beta}$, that is,

$$\mathbb{A}=\left(A^{\alpha\beta}\right)_{1\le\alpha,\beta\le n},\ A^{\alpha\beta}=\left(a_{ij}^{\alpha\beta}\right)_{1\le i,j\le n},\ a_{ij}^{\alpha\beta}\in L^\infty(\Omega),\ 1\le i,j,\alpha,\beta\le n. \tag{1.2}$$

Assume that the following symmetry conditions hold

$$a_{ij}^{\alpha\beta}(x)=a_{\alpha j}^{i\beta}(x)=a_{i\beta}^{\alpha j}(x),\quad x\in\Omega \tag{1.3}$$

(see also [43, (3.1), (3.3)]). In addition, assume that the tensor coefficient $\mathbb{A}$ satisfies the ellipticity condition only in terms of all symmetric matrices in $\mathbb{R}^{n\times n}$, with zero matrix trace. Thus, there is a constant $C_\mathbb{A}>0$ such that, for almost all $x\in\Omega$,

$$a_{ij}^{\alpha\beta}(x)\xi_{i\alpha}\xi_{j\beta}\ge C_\mathbb{A}^{-1}|\boldsymbol{\xi}|^2,\ \forall\,\boldsymbol{\xi}=(\xi_{i\alpha})_{i,\alpha=1,\dots,n}\in\mathbb{R}^{n\times n}\ \text{ such that }\ \boldsymbol{\xi}=\boldsymbol{\xi}^\top\ \text{ and }\ \sum_{i=1}^n\xi_{ii}=0, \tag{1.4}$$

where $|\boldsymbol{\xi}|^2=\xi_{i\alpha}\xi_{i\alpha}$, and the superscript $\top$ refers to the transpose of a matrix (see also [26]). The tensor coefficient $\mathbb{A}$ is endowed with the norm

$$\|\mathbb{A}\|:=\max\left\{\|a_{ij}^{\alpha\beta}\|_{L^\infty(\Omega)}: i,j,\alpha,\beta=1\dots,n\right\}. \tag{1.5}$$

Let $\mathbf{u}$ and π be unknown vector and scalar fields. Let us assume that $\mathbf{f}$ is a given vector field and g is a given scalar field defined in Ω. Then the equations

$$\boldsymbol{\mathcal{L}}(\mathbf{u},\pi):=\mathfrak{L}\mathbf{u}-\nabla\pi=\mathbf{f},\ \operatorname{div}\,\mathbf{u}=g\ \text{in}\ \Omega \tag{1.6}$$

determine the *anisotropic Stokes system with variable viscosity tensor coefficient* $\mathbb{A}=\left(A^{\alpha\beta}\right)_{1\le\alpha,\beta\le n}$ *in a compressible framework.*

Relation (1.1) and conditions (1.3) show that the Stokes operator $\boldsymbol{\mathcal{L}}$ can be written in any of the alternative forms

$$\boldsymbol{\mathcal{L}}(\mathbf{u},\pi)=\partial_\alpha\left(A^{\alpha\beta}\partial_\beta\mathbf{u}\right)-\nabla\pi,\quad\left(\boldsymbol{\mathcal{L}}(\mathbf{u},\pi)\right)_i=\partial_\alpha\left(a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u})\right)-\partial_i\pi,\ i=1,\dots,n, \tag{1.7}$$

where $\mathbf{u}=(u_1,\dots,u_n)^\top$ and $E_{j\beta}(\mathbf{u}):=\frac{1}{2}(\partial_j u_\beta+\partial_\beta u_j)$ are the entries of the symmetric part $\mathbb{E}(\mathbf{u})$ of $\nabla\mathbf{u}$ that is the gradient of $\mathbf{u}$.

The *anisotropic Navier-Stokes system in a compressible framework with variable viscosity tensor coefficient* $\mathbb{A}=\left(A^{\alpha\beta}\right)_{1\le\alpha,\beta\le n}$ is given by the following equations

$$\boldsymbol{\mathcal{L}}(\mathbf{u},\pi)-(\mathbf{u}\cdot\nabla)\mathbf{u}=\mathbf{f},\quad\operatorname{div}\mathbf{u}=g\ \text{in}\ \Omega. \tag{1.8}$$

The anisotropic Stokes and Navier-Stokes systems in the incompressible case are given by the equations of (1.6) and (1.8), respectively, with $\operatorname{div}\mathbf{u}=0$.

In the isotropic case, the tensor $\mathbb{A}$ in (1.2) has the entries

$$a_{ij}^{\alpha\beta}(x)=\lambda(x)\delta_{i\alpha}\delta_{j\beta}+\mu(x)\left(\delta_{\alpha j}\delta_{\beta i}+\delta_{\alpha\beta}\delta_{ij}\right),\ 1\le i,j,\alpha,\beta\le n, \tag{1.9}$$

where $\lambda, \mu \in L^\infty(\Omega)$, and $c_\mu^{-1} \le \mu(x) \le c_\mu$ for a.e. $x \in \Omega$, with some constant $c_\mu > 0$. This implies that condition (1.4) is satisfied (cf., e.g., Appendix III, Part I, Section 1 in [46]; see also [26]).

The anisotropic Stokes and Navier-Stokes systems play a main role in various applications related to the flow of immiscible fluids, liquid crystals, and flows of non-homogeneous fluids with variable anisotropic viscosity tensors depending on physical properties of the fluids (cf., e.g., [12], [15], [16], [34, Chapter 3]).

The boundary value problems for the (isotropic) Stokes and Navier-Stokes systems involving mixed conditions have been intensively analyzed by using various mathematical tools, such as variational methods and layer potential theoretic methods (see, e.g., [8,13,20,17,44–46] and the references therein) due to their applications in mathematical physics and engineering. Brown, Mitrea, Mitrea, and Wright [8] obtained the well-posedness of the mixed problem for the Stokes system with constant coefficients in a class of Lipschitz domains in $\mathbb{R}^n$, $n \ge 3$, by using a layer potential approach that reduces the mixed problem to a boundary integral equation. Cocquet, Rakotobe, Ramalingom, and Bastide [13] developed a variational analysis and a finite element approximation of the Darcy-Brinkman-Forchheimer model for porous media with mixed boundary conditions. (The Darcy-Brinkman-Forchheimer equation is a perturbation of the Navier-Stokes equation with a compact operator.) Ebmeyer and Frehse [17] used a variational approach in the analysis of constant coefficient steady Navier-Stokes equations with mixed boundary conditions (involving Dirichlet and Navier-type conditions) in three-dimensional polyhedral domains and a class of Lipschitzian domains. Ott, Kim, and Brown [44] constructed the Green function for the mixed Dirichlet-Neumann boundary value problem for the Stokes system in a two-dimensional Lipschitz domain. Recently, Amrouche and Boussetouan [2] have proved existence, uniqueness and regularity of some vector potentials, associated with a divergence-free vector field satisfying mixed boundary conditions. These results have been used to obtain weak and strong solutions for a mixed boundary problem for the Stokes system with a pressure condition on some part of the boundary and Navier-type boundary condition on the remaining part.

Variational approaches have been also used in the analysis of many other elliptic boundary valued problems. By using such an approach, Angot [4,5] obtained the well-posedness of some Stokes/Brinkman problems with constant isotropic viscosity and a family of embedded jump conditions on an immersed (transversal) interface with weak regularity assumptions. The authors in [23] used a layer potential analysis and the Leray-Schauder fixed point theorem in order to show existence results for a nonlinear Neumann-transmission problem for the constant coefficient Stokes and Brinkman systems in L^p, Sobolev, and Besov spaces. Regularity results for the Stokes system with measurable coefficients in one direction have been obtained by Dong and Kim [15] by using a variational technique (see also [12]). Brewster et al. [7] developed a variational approach to prove the well-posedness of mixed boundary problems for higher order divergence-form elliptic equations with L^∞ coefficients in locally (ϵ, δ)-domains and in Besov and Bessel potential spaces. Mazzucato and Nistor [36] obtained the well-posedness and regularity in weighted Sobolev spaces for the anisotropic linear elasticity equations with mixed conditions on polyhedral domains.

An alternative integral approach using explicit parametrix-based integral potentials, which reduces boundary value problems for the Stokes system with variable coefficients, as well as other variable-coefficient elliptic partial differential equations, to boundary-domain integral equations has been developed in [9–11,19,40] (see also the references therein).

The authors in [25] developed a variational analysis in a pseudostress setting for transmission problems with internal interfaces in weighted Sobolev spaces for the anisotropic Stokes and Navier-Stokes systems with an L^∞ strongly elliptic coefficient tensor (see also [15]), by using the strong ellipticity condition in terms of all matrices in $\mathbb{R}^{n\times n}$ (see also [29,30] for boundary value problems for the Stokes and Navier-Stokes systems with L^∞ coefficients in Lipschitz domains on compact Riemannian manifolds, and [24] in the case of smooth coefficients in the same setting). The authors in [27] and [26] extended their variational analysis to other transmission and exterior boundary problems with internal interfaces for the anisotropic Stokes and Navier-Stokes systems by assuming that the corresponding L^∞ viscosity tensor coefficient satisfies a weaker ellipticity condition in terms of only symmetric matrices in $\mathbb{R}^{n\times n}$ with zero traces, that is, the ellipticity condition (1.4), which is indeed weaker than that employed in [25] and [15]. Non-homogeneous Dirichlet-transmission problems for the anisotropic Stokes and Navier-Stokes systems in a bounded Lipschitz domain in $\mathbb{R}^n$ (in the case of the nonlinear problems it is assumed that $n = 2, 3$) with a transversal Lipschitz interface have been investigated in [28] by imposing the ellipticity condition (1.4). The authors have used a variational approach and the Leray-Schauder theorem in order to show the existence of a weak solution for the nonlinear Dirichlet-transmission problem.

In this paper we obtain well-posedness results in L^2-based Sobolev spaces for mixed and mixed-transmission problems for a compressible anisotropic Stokes system in a bounded Lipschitz domain of $\mathbb{R}^n$, $n \geq 2$, with an internal Lipschitz interface that intersects transversally the boundary of the domain. We show also the existence and uniqueness of a weak solution for nonlinear mixed and mixed-transmission problems for the anisotropic compressible Navier-Stokes system with small data in L^2-based Sobolev spaces in a bounded Lipschitz domain with the same geometry as in the linear case, but with $n = 2, 3$. The proof of the well-posedness in the nonlinear case is based on the well-posedness of the linear mixed or mixed-transmission problems for the anisotropic Stokes system and on the Banach fixed point theorem. We assume that the L^∞ viscosity tensor coefficient satisfies the ellipticity condition (1.4).

The anisotropic Stokes and Navier-Stokes problems of mixed-transmission type considered below and involving mixed and transmission conditions may describe various physical phenomena, like lubrication and blood flows (cf. [2] and the references therein), multiphase flows of immiscible fluids with variable anisotropic viscosity tensors and variable compressibility (see, e.g., [16], [34, Chapter 3]).

2. Preliminary results

Given a Banach space $\mathcal{X}$, its topological dual is denoted by $\mathcal{X}'$, and the notation $\langle \cdot, \cdot \rangle_X$ means the duality pairing of two dual spaces defined on a set $X \subseteq \mathbb{R}^n$.

2.1. Sobolev spaces on Lipschitz domains in $\mathbb{R}^n$

Let $n \geq 2$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with connected boundary $\partial\Omega$. Let $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω, equipped with the inductive limit topology. Let $\mathcal{D}'(\Omega)$ denote the corresponding space of distributions on Ω, i.e., the dual of the space $\mathcal{D}(\Omega)$. Let $L^2(\Omega)$ be the Lebesgue space of square-integrable functions on Ω, and $L^\infty(\Omega)$ be the space of (equivalence classes of) essentially bounded measurable functions on Ω. Let also

$$L_0^2(\Omega) := \{f \in L^2(\Omega) : \langle f, 1 \rangle_\Omega = 0\}. \tag{2.1}$$

The dual of $L_0^2(\Omega)$ is the space $L^2(\Omega)/\mathbb{R}$. The Sobolev space $H^1(\Omega)$ is defined as

$$H^1(\Omega) := \big\{f \in L^2(\Omega) : \nabla f \in L^2(\Omega)^n\big\}, \tag{2.2}$$

and is endowed with the norm

$$\|f\|_{H^1(\Omega)}^2 = \|f\|_{L^2(\Omega)}^2 + \|\nabla f\|_{L^2(\Omega)^n}^2 . \tag{2.3}$$

The space $\widetilde{H}^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\mathbb{R}^n)$, and can be also described as

$$\widetilde{H}^1(\Omega) := \big\{\widetilde{f} \in H^1(\mathbb{R}^n) : \operatorname{supp} \widetilde{f} \subseteq \overline{\Omega}\big\}, \tag{2.4}$$

where $\operatorname{supp} f := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$. The dual of $\widetilde{H}^1(\Omega)$ is the space $H^{-1}(\Omega)$. Then the following equivalent characterization of the spaces $H^{\pm 1}(\Omega)$ holds

$$H^{\pm 1}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists\, F \in H^{\pm 1}(\mathbb{R}^n) \text{ such that } F|_\Omega = f\}, \tag{2.5}$$

where $|_X = r_X$ is the restriction operator of functions or distributions to a set X of $\mathbb{R}^n$.

The closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ is denoted by $\mathring{H}^1(\Omega)$ and can be equivalently described as the space of all functions in $H^1(\Omega)$ with null traces on $\partial\Omega$, that is,

$$\mathring{H}^1(\Omega) := \{f \in H^1(\Omega) : \gamma_\Omega f = 0 \text{ on } \partial\Omega\}, \tag{2.6}$$

where $\gamma_\Omega : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is the trace operator. Recall that this is a linear, bounded and surjective operator (cf. [14], [38, Lemma 2.6], [42, Theorem 2.5.2]). We will use the same notation γ_Ω for the trace operator acting on vector-valued functions.

Note that the spaces $\widetilde{H}^1(\Omega)$ and $\mathring{H}^1(\Omega)$ can be identified isomorphically (see, e.g., [37, Theorem 3.33]). The dual of $H^1(\Omega)$ is denoted by $\widetilde{H}^{-1}(\Omega)$, and is a space of distributions. (Note that $\widetilde{H}^{-1}(\mathbb{R}^n) = H^{-1}(\mathbb{R}^n)$.) Moreover, the following spaces can be isomorphically identified (cf., e.g., [37, Theorem 3.14])

$$\left(H^1(\Omega)\right)' = \widetilde{H}^{-1}(\Omega), \quad H^{-1}(\Omega) = \left(\widetilde{H}^1(\Omega)\right)'. \tag{2.7}$$

Let $s \in (0,1)$. Then the boundary Sobolev space $H^s(\partial\Omega)$ is defined by

$$H^s(\partial\Omega) := \left\{ f \in L^2(\partial\Omega) : \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(\mathbf{x}) - f(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n-1+2s}} d\sigma_{\mathbf{x}} d\sigma_{\mathbf{y}} < \infty \right\}, \tag{2.8}$$

where $\sigma_{\mathbf{y}}$ is the surface measure on $\partial\Omega$ (see, e.g., [42, Proposition 2.5.1]). The dual of $H^s(\partial\Omega)$ is the space $H^{-s}(\partial\Omega)$, and $H^0(\partial\Omega) = L^2(\partial\Omega)$.

By $H^1(\Omega)^n$, $\widetilde{H}^1(\Omega)^n$, $H^s(\partial\Omega)^n$ we denote the spaces of vector-valued functions whose components belong to the spaces $H^1(\Omega)$, $\widetilde{H}^1(\Omega)$, and $H^s(\partial\Omega)$, respectively. For further properties of Sobolev spaces we refer the reader to [22,37,42].

We will need the following well known result (see, e.g., [32, Lemma 2.5], [6], [3, Theorem 3.1]), for which we will provide several generalizations further on.

Proposition 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, with connected boundary. Then the divergence operator* $\operatorname{div} : \mathring{H}^1(\Omega)^n \to L^2_0(\Omega)$ *is bounded, linear and surjective. It has a bounded, linear right inverse $\mathcal{R}_\Omega : L^2_0(\Omega) \to \mathring{H}^1(\Omega)^n$. Thus, there exists a constant $C = C(\Omega, n) > 0$ such that*

$$\operatorname{div}(\mathcal{R}_\Omega f) = f, \ \|\mathcal{R}_\Omega f\|_{H^1(\Omega)^n} \leq C\|f\|_{L^2(\Omega)}, \ \forall\, f \in L^2_0(\Omega). \tag{2.9}$$

2.2. Sobolev spaces on bounded domains with partially vanishing traces

Let $\Omega_0 \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain with connected boundary Γ_0. Let D and N be relatively open subsets of Γ_0, such that D has positive $(n-1)$-Hausdorff measure, $D \cap N = \emptyset$, $\overline{D} \cup \overline{N} = \Gamma_0$, and $\overline{D} \cap \overline{N} = \Sigma_1$, where Σ_1 is an $(n-2)$-dimensional closed Lipschitz submanifold of Γ_0.

We need the following space defined on the Lipschitz domains Ω_0

$$C^\infty_D(\Omega_0)^n := \left\{\boldsymbol{\varphi}|_{\Omega_0} : \boldsymbol{\varphi} \in C^\infty(\mathbb{R}^n)^n, \operatorname{supp}(\boldsymbol{\varphi}) \cap \overline{D} = \emptyset\right\}, \tag{2.10}$$

and let $H^1_D(\Omega_0)^n$ be the closure of $C^\infty_D(\Omega_0)^n$ in $H^1(\Omega_0)^n$. The space $H^1_D(\Omega_0)^n$ can be equivalently characterized as

$$H^1_D(\Omega_0)^n = \left\{\mathbf{v} \in H^1(\Omega_0)^n : \left(\gamma_{\Omega_0}\mathbf{v}\right)|_D = \mathbf{0}\right\} \tag{2.11}$$

(cf. [7, Corollary 3.11], [21, Definition 2.4]). Let also

$$H^1_{D;\mathrm{div}}(\Omega_0)^n := \left\{\mathbf{w} \in H^1_D(\Omega_0)^n : \mathrm{div}\,\mathbf{w} = 0\right\}. \tag{2.12}$$

Let Ξ be a relatively open $(n-1)$-dimensional subset of Γ_0, e.g., D or N. Let r_Ξ denote the operator of restriction of distributions from Γ_0 to Ξ. Then the boundary Sobolev spaces on Ξ are defined by

$$H^{\frac{1}{2}}(\Xi)^n := \left\{\boldsymbol{\varphi}|_\Xi : \boldsymbol{\varphi} \in H^{\frac{1}{2}}(\Gamma_0)^n\right\}, \tag{2.13}$$

$$\widetilde{H}^{\frac{1}{2}}(\Xi)^n := \left\{\boldsymbol{\varphi} \in H^{\frac{1}{2}}(\Gamma_0)^n : \boldsymbol{\varphi} = \mathbf{0} \text{ on } \Gamma_0 \setminus \overline{\Xi}\right\}, \tag{2.14}$$

$$H^{-\frac{1}{2}}(\Xi)^n := \left(\widetilde{H}^{\frac{1}{2}}(\Xi)^n\right)', \ \widetilde{H}^{-\frac{1}{2}}(\Xi)^n := \left(H^{\frac{1}{2}}(\Xi)^n\right)' \tag{2.15}$$

(cf., e.g., [37], [7, Definition 4.8, Theorem 5.1]).

Lemma 2.2. *The trace operator $\gamma_{\Omega_0} : H^1_D(\Omega_0)^n \to \widetilde{H}^{\frac{1}{2}}(N)^n$ is bounded, linear and surjective, having a (non-unique) bounded, linear right inverse $\gamma^{-1}_{\Omega_0} : \widetilde{H}^{\frac{1}{2}}(N)^n \to H^1_D(\Omega_0)^n$.*

Proof. Recall that the trace operator $\gamma_{\Omega_0} : H^1(\Omega_0)^n \to H^{\frac{1}{2}}(\partial\Omega_0)^n$ is linear, bounded and surjective (cf. [14], [38, Lemma 2.6], [42, Theorem 2.5.2]). Then the desired result is a direct consequence of this property. $\square$

The following lemma provides a variant of Bogovskii's result [6] in the case of vector fields with vanishing traces on a submanifold of a Lipschitz boundary (see also [30, Lemma 7.4] in the setting of compact Riemannian manifolds, [3, Theorem 3.1], and [35, Lemma 5.1], [44, Proposition 2.1], [8, (6.10)] for the mixed problem for the Stokes system in polyhedral domains, bounded Lipschitz domains in $\mathbb{R}^2$, or in creased Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$).

Lemma 2.3. (*i*) *The divergence operator*

$$\mathrm{div} : H^1_D(\Omega_0)^n \to L^2(\Omega_0) \tag{2.16}$$

is bounded, linear and surjective, having a bounded, linear right inverse $\mathcal{R}_{\Omega_0} : L^2(\Omega_0) \to H^1_D(\Omega_0)^n$. Thus, there exists a constant $C_D = C_D(\Omega_0, D, n) > 0$ such that

$$\mathrm{div}(\mathcal{R}_{\Omega_0} f) = f, \ \|\mathcal{R}_{\Omega_0} f\|_{H^1_D(\Omega_0)^n} \leq C_D \|f\|_{L^2(\Omega_0)}, \ \forall\, f \in L^2(\Omega_0). \tag{2.17}$$

(*ii*) *The operator* $\mathrm{div} : H^1_D(\Omega_0)^n / H^1_{D;\mathrm{div}}(\Omega_0)^n \to L^2(\Omega_0)$ *is an isomorphism.*

Proof. (i) The linearity and continuity of the operator in (2.16) are immediate. Let us now show that operator (2.16) is surjective, by using an argument similar to that for [30, Lemma 7.4] (see also [35, Lemma 5.1]). Let $h \in L^2(\Omega_0)$. Our purpose is to show that there exists $\mathbf{u} \in H^1_D(\Omega_0)^n$ such that

$$\mathrm{div}\,\mathbf{u} = h \text{ in } \Omega_0. \tag{2.18}$$

To this end, we analyze the following cases related to the constant

$$\langle h, 1\rangle_{\Omega_0} := \int_{\Omega_0} h\, dx. \tag{2.19}$$

If $\langle h, 1\rangle_{\Omega_0} = 0$, and, thus, $h \in L^2_0(\Omega_0)$, then the desired existence result follows from the surjectivity of the map $\operatorname{div} : \mathring{H}^1(\Omega_0)^n \to L^2_0(\Omega_0)$ (see Proposition 2.1) and the inclusion $\mathring{H}^1(\Omega_0)^n \subseteq H^1_D(\Omega_0)^n$.

Assume now that $\langle h, 1\rangle_{\Omega_0} \neq 0$. Let $\boldsymbol{\nu}_{\Gamma_0}$ be the outward unit normal to Ω_0, which exists a.e. on Γ_0. In view of the membership of $\boldsymbol{\nu}_{\Gamma_0}$ in $H^{-\frac{1}{2}}(\Gamma_0)^n$, we define $\boldsymbol{\nu}_{N} := r_{N}\boldsymbol{\nu}_{\Gamma_0} \in H^{-\frac{1}{2}}(N)^n = \big(\widetilde{H}^{\frac{1}{2}}(N)^n\big)'$ and then

$$\langle \boldsymbol{\nu}_{N}, \boldsymbol{\varphi}\rangle_N = \langle \boldsymbol{\nu}_{\Gamma_0}, \boldsymbol{\varphi}\rangle_{\Gamma_0}, \ \forall \boldsymbol{\varphi} \in \widetilde{H}^{\frac{1}{2}}(N)^n . \tag{2.20}$$

With respect to the inner product $(\cdot,\cdot)_{H^{-\frac{1}{2}}(N)^n}$ in the Hilbert space $H^{-\frac{1}{2}}(N)^n$, whose induced norm is $\|\cdot\|_{H^{-\frac{1}{2}}(N)^n}$, we have $\left(\boldsymbol{\nu}_{N}, \frac{1}{\|\boldsymbol{\nu}_N\|^2_{H^{-\frac{1}{2}}(N)^n}}\boldsymbol{\nu}_{N}\right)_{H^{-\frac{1}{2}}(N)^n} = 1$. The element $\frac{1}{\|\boldsymbol{\nu}_N\|^2_{H^{-\frac{1}{2}}(N)^n}}\boldsymbol{\nu}_{N}$ produces, through the inner product, a linear bounded functional in $H^{-\frac{1}{2}}(N)^n$ and, thus, is isomorphic with an element $\widetilde{\boldsymbol{\mu}}_{N}$ in the dual space $\widetilde{H}^{\frac{1}{2}}(N)^n$. Therefore, there exists $\widetilde{\boldsymbol{\mu}}_{N} \in \widetilde{H}^{\frac{1}{2}}(N)^n$ such that

$$\big\langle \boldsymbol{\nu}_{N}, \widetilde{\boldsymbol{\mu}}_{N}\big\rangle_N = 1. \tag{2.21}$$

According to the membership of $\widetilde{\boldsymbol{\mu}}_{N}$ in $\widetilde{H}^{\frac{1}{2}}(N)^n$ and Lemma 2.2, there exists $\mathbf{v} \in H^1_D(\Omega_0)^n$ such that $\gamma_{\Omega_0}\mathbf{v} = \widetilde{\boldsymbol{\mu}}_{N}$ a.e. on Γ_0, and

$$\mathbf{v} = \gamma^{-1}_{\Omega_0}(\widetilde{\boldsymbol{\mu}}_{N}) \in H^1_D(\Omega_0)^n , \tag{2.22}$$

where $\gamma^{-1}_{\Omega_0} : \widetilde{H}^{\frac{1}{2}}(N)^n \to H^1_D(\Omega_0)^n$ is a bounded right inverse of the trace operator $\gamma_{\Omega_0} : H^1_D(\Omega_0)^n \to \widetilde{H}^{\frac{1}{2}}(N)^n$ (see also [41, Proposition 5.4]). Now let $h_0 \in L^2(\Omega_0)$,

$$h_0 := h - \langle h, 1\rangle_{\Omega_0} \operatorname{div} \mathbf{v}. \tag{2.23}$$

Relations (2.19), (2.20), (2.21), (2.23) and the Divergence Theorem imply that $h_0 \in L^2_0(\Omega)$. Then there exists $\mathbf{u}_0 \in \mathring{H}^1(\Omega_0)^n \subset H^1_D(\Omega_0)^n$ such that

$$\operatorname{div} \mathbf{u}_0 = h_0 \text{ in } \Omega_0 \text{ and } \mathbf{u}_0 = R_{\Omega_0} h_0 , \tag{2.24}$$

where $R_{\Omega_0} : L^2_0(\Omega_0) \to \mathring{H}^1(\Omega_0)^n$ is a bounded right inverse of the operator $\operatorname{div} : \mathring{H}^1(\Omega_0)^n \to L^2_0(\Omega_0)$ (see also [6], [3, Theorem 3.1]).

We are now able to consider the field $\mathbf{u} \in H^1(\Omega_0)^n$,

$$\mathbf{u} := \mathbf{u}_0 + \langle h, 1\rangle_{\Omega_0}\mathbf{v}, \tag{2.25}$$

where $\mathbf{v}$ is given by (2.22), and show that it satisfies equation (2.18). Indeed, the membership relation $\mathbf{u}_0, \mathbf{v} \in H^1_D(\Omega_0)^n$ shows that $\mathbf{u} \in H^1_D(\Omega_0)^n$, and relations (2.23), (2.24) and (2.25) imply that $\operatorname{div} \mathbf{u} = h$. Hence, $\mathbf{u}$ given by (2.25) belongs to $H^1_D(\Omega_0)^n$ and satisfies equation (2.18). Moreover, relations (2.19), (2.22), (2.23) and (2.24) show that $\mathbf{u} = \mathcal{R}_{\Omega_0} h$, where the operator $\mathcal{R}_{\Omega_0} : L^2(\Omega_0) \to H^1_D(\Omega_0)^n$,

$$\mathcal{R}_{\Omega_0} := R_{\Omega_0} \circ \left\{\mathbb{I} - \left(\operatorname{div} \gamma^{-1}_{\Omega_0}\widetilde{\boldsymbol{\mu}}_{N}\right) \langle \cdot, 1\rangle_{\Omega_0}\right\} + \left(\gamma^{-1}_{\Omega_0}\widetilde{\boldsymbol{\mu}}_{N}\right) \langle \cdot, 1\rangle_{\Omega_0} , \tag{2.26}$$

is a right inverse of the operator $\operatorname{div} : H^1_D(\Omega_0)^n \to L^2(\Omega)$. Since the operators $\gamma^{-1}_{\Omega_0} : \widetilde{H}^{\frac{1}{2}}(N)^n \to H^1_D(\Omega_0)^n$, $R_{\Omega_0} : L^2_0(\Omega_0) \to \mathring{H}^1(\Omega_0)^n$ and $\operatorname{div} : H^1_D(\Omega_0)^n \to L^2(\Omega_0)$ are bounded, we conclude that the operator $\mathcal{R}_{\Omega_0} : L^2(\Omega_0) \to H^1_D(\Omega_0)^n$ is bounded as well, which completes the proof of item (i).

(ii) By item (i), operator (2.16) is bounded and surjective, and its kernel is the space $H^1_{D;\operatorname{div}}(\Omega_0)^n$. Thus, the operator $\operatorname{div} : H^1_D(\Omega_0)^n / H^1_{D;\operatorname{div}}(\Omega_0)^n \to L^2(\Omega_0)$ is an isomorphism. $\square$

3. Sobolev spaces, conormal derivatives and Green's identity for the Stokes system in a Lipschitz domain with a transversal Lipschitz interface

In the first part of this section we will mention useful results related to the generalized conormal derivative for the Stokes system with partially vanishing traces in a bounded Lipschitz domain, and in the second part we will describe useful Sobolev spaces and main results related to the conormal derivatives for the Stokes system in a bounded Lipschitz domain with a transversal Lipschitz interface.

3.1. Conormal derivative for the Stokes system with partially vanishing traces in a bounded Lipschitz domain

Assumption 3.1. Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary $\partial\Omega$. Let $\partial\Omega$ be divided into two non-empty relatively open subsets Γ^+ and Γ^-, such that Γ^+ has positive $(n-1)$-Hausdorff measure, $\Gamma^+ \cap \Gamma^- = \emptyset$, $\overline{\Gamma^+} \cup \overline{\Gamma^-} = \partial\Omega$, and $\overline{\Gamma^+} \cap \overline{\Gamma^-}$ is an $(n-2)$-dimensional closed Lipschitz submanifold of $\partial\Omega$ if $n > 2$, and two distinct points if $n = 2$.

Recall that the space $H^1_{\Gamma^+}(\Omega)^n$ is defined as in (2.11), while the boundary Sobolev spaces on $\Gamma^{\pm}$ are defined as in (2.13)-(2.15). Let us also define the space

$$\boldsymbol{H}^1_{\Gamma^+}(\Omega, \boldsymbol{\mathcal{L}}) := \left\{ (\mathbf{u}, \pi, \tilde{\mathbf{f}}) \in H^1(\Omega)^n \times L^2(\Omega) \times \left(H^1_{\Gamma^+}(\Omega)^n\right)' : \boldsymbol{\mathcal{L}}(\mathbf{u}, \pi) = \tilde{\mathbf{f}}|_\Omega \text{ in } \Omega \right\}, \tag{3.1}$$

where $\boldsymbol{\mathcal{L}} : H^1(\Omega)^n \times L^2(\Omega) \to H^{-1}(\Omega)^n$ is the operator defined in (1.6). Note that if $\tilde{\mathbf{f}} \in \left(H^1_{\Gamma^+}(\Omega)^n\right)'$, then $\tilde{\mathbf{f}}|_\Omega \in H^{-1}(\Omega)^n$.

The conormal derivative operator $\mathbf{t}_\Omega : \boldsymbol{H}^1(\Omega, \boldsymbol{\mathcal{L}}) \to H^{-\frac{1}{2}}(\partial\Omega)^n$ is given by Definition A.2, which assumes that the right hand side $\tilde{\mathbf{f}}$ is an element of $\widetilde{H}^{-1}(\Omega)^n$. For the case when $\tilde{\mathbf{f}}$ belongs to $\left(H^1_{\Gamma^+}(\Omega)^n\right)'$ but is not fixed as an element of $\widetilde{H}^{-1}(\Omega)^n$, the conormal derivative $\mathbf{t}_\Omega(\mathbf{u}, \pi, \tilde{\mathbf{f}})$ can not be uniquely defined on the entire boundary $\partial\Omega$, however we can uniquely define its restriction to Γ^- as follows (cf., e.g., [41, Proposition 8.1], [28, Definition 5.4]).

Definition 3.2. Let Assumption 3.1 and condition (1.2) hold. If $(\mathbf{u}, \pi, \tilde{\mathbf{f}}) \in \boldsymbol{H}^1_{\Gamma^+}(\Omega, \boldsymbol{\mathcal{L}})$, then the restriction on Γ^- of generalized conormal derivative, $\left(\mathbf{t}_\Omega(\mathbf{u}, \pi; \tilde{\mathbf{f}})\right)\big|_{\Gamma^-} \in H^{-\frac{1}{2}}(\Gamma^-)^n$, is defined in the weak form by the formula

$$\left\langle \left(\mathbf{t}_\Omega(\mathbf{u}, \pi, \tilde{\mathbf{f}})\right)\big|_{\Gamma^-}, \boldsymbol{\Phi} \right\rangle_{\Gamma^-} = \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\gamma_\Omega^{-1}\boldsymbol{\Phi}) \right\rangle_\Omega - \left\langle \pi, \operatorname{div}(\gamma_\Omega^{-1}\boldsymbol{\Phi}) \right\rangle_\Omega + \left\langle \tilde{\mathbf{f}}, \gamma_\Omega^{-1}\boldsymbol{\Phi} \right\rangle_\Omega, \quad \forall\, \boldsymbol{\Phi} \in \widetilde{H}^{\frac{1}{2}}(\Gamma^-)^n, \tag{3.2}$$

where $\gamma_\Omega^{-1} : \widetilde{H}^{\frac{1}{2}}(\Gamma^-)^n \to H^1_{\Gamma^+}(\Omega)^n$ is a bounded right inverse of the trace operator $\gamma_\Omega : H^1_{\Gamma^+}(\Omega)^n \to \widetilde{H}^{\frac{1}{2}}(\Gamma^-)^n$.

In addition, similar arguments to those for Lemma 2.5 in [25] imply the following Green formula whose proof will be omitted for the sake of brevity (see also [14], [38, Theorem 3.2], [39, Theorem 5.3], [41, Proposition 8.1], [42, Theorem 10.4.1]).

Lemma 3.3. *Let Assumption 3.1 and condition* (1.2) *hold. Then the generalized conormal derivative operator* $\mathbf{t}_\Omega : \boldsymbol{H}^1_{\Gamma^+}(\Omega, \boldsymbol{\mathcal{L}}) \to H^{-\frac{1}{2}}(\Gamma^-)^n$ *is linear and bounded, and definition* (3.2) *does not depend on the particular choice of a right inverse* $\gamma_\Omega^{-1} : \widetilde{H}^{\frac{1}{2}}(\Gamma^-)^n \to H^1_{\Gamma^+}(\Omega)^n$ *of the trace operator* $\gamma_\Omega : H^1_{\Gamma^+}(\Omega)^n \to \widetilde{H}^{\frac{1}{2}}(\Gamma^-)^n$. *Moreover, for* $\mathbf{w} \in H^1_{\Gamma^+}(\Omega)^n$ *and* $(\mathbf{u}, \pi, \tilde{\mathbf{f}}) \in \boldsymbol{H}^1_{\Gamma^+}(\Omega, \boldsymbol{\mathcal{L}})$, *the first Green identity holds,*

$$\left\langle \left(\mathbf{t}_\Omega(\mathbf{u}, \pi; \tilde{\mathbf{f}})\right)\big|_{\Gamma^-}, \gamma_\Omega \mathbf{w} \right\rangle_{\Gamma^-} = \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{w}) \right\rangle_\Omega - \langle \pi, \operatorname{div} \mathbf{w} \rangle_{\Omega_0} + \langle \tilde{\mathbf{f}}, \mathbf{w} \rangle_\Omega\,. \tag{3.3}$$

Note that the term $\langle\widetilde{\mathbf{f}},\mathbf{w}\rangle_\Omega$ is well defined in (3.3) for $\tilde{\mathbf{f}}\in\big(H^1_{\Gamma^+}(\Omega)^n\big)'=\widetilde{H}^{-1}(\Omega)^n/H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n$. Indeed, elements of the same class in the quotient space $\widetilde{H}^{-1}(\Omega)^n/H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n$ differ only on $\overline{\Gamma^+}$ and since $\mathbf{w}\in H^1_{\Gamma^+}(\Omega)^n$, different elements of the same class $\tilde{\mathbf{f}}\in\big(H^1_{\Gamma^+}(\Omega)^n\big)'$ will give the same value of the functional $\langle\widetilde{\mathbf{f}},\mathbf{w}\rangle_\Omega$.

Moreover, we have the following useful result for the mixed problem (cf. [7, Definition 7.1] for strongly elliptic higher-order systems in divergence form).

Lemma 3.4. *Let Assumption 3.1 and conditions (1.2) and (1.3) hold.*

(i) Let $(\mathbf{u},\pi)\in H^1(\Omega)^n\times L^2(\Omega)$. Let $\tilde{\mathbf{f}}_1,\tilde{\mathbf{f}}_2$ be such that $(\mathbf{u},\pi,\tilde{\mathbf{f}}_1),(\mathbf{u},\pi,\tilde{\mathbf{f}}_2)\in\boldsymbol{H}^1(\Omega,\boldsymbol{\mathcal{L}})$ and let $\mathbf{t}_\Omega(\mathbf{u},\pi,\tilde{\mathbf{f}}_1)\big|_{\Gamma^-}$, $\mathbf{t}_\Omega(\mathbf{u},\pi,\tilde{\mathbf{f}}_2)\big|_{\Gamma^-}\in H^{-\frac12}(\partial\Omega)^n$ be the corresponding conormal derivative restrictions introduced in Definition 3.2. If $\operatorname{supp}(\tilde{\mathbf{f}}_1-\tilde{\mathbf{f}}_2)\subseteq\overline{\Gamma^+}$, then $\big(\mathbf{t}_\Omega(\mathbf{u},\pi,\tilde{\mathbf{f}}_1)\big)\big|_{\Gamma^-}=\big(\mathbf{t}_\Omega(\mathbf{u},\pi,\tilde{\mathbf{f}}_2)\big)\big|_{\Gamma^-}$.

(ii) If $(\mathbf{u},\pi,\tilde{\mathbf{f}})\in\boldsymbol{H}^1_{\Gamma^+}(\Omega,\boldsymbol{\mathcal{L}})$, then the conormal derivative restriction $\big(\mathbf{t}_\Omega(\mathbf{u},\pi,\tilde{\mathbf{f}})\big)\big|_{\Gamma^-}$ is well defined, that is, it is the same when $\tilde{\mathbf{f}}$ is replaced by $\tilde{\mathbf{f}}+\tilde{\mathbf{f}}_0$ with any $\tilde{\mathbf{f}}_0\in H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n$.

Proof. (i) From definition (3.2) we obtain that

$$\begin{aligned}&\big\langle\big(\mathbf{t}_\Omega(\mathbf{u},\pi,\tilde{\mathbf{f}}_1)\big)\big|_{\Gamma^-}-\big(\mathbf{t}_\Omega(\mathbf{u},\pi,\tilde{\mathbf{f}}_2)\big)\big|_{\Gamma^-},\boldsymbol{\Phi}\big\rangle_{\Gamma^-}=\big\langle\big(\mathbf{t}_\Omega(\mathbf{0},0,\tilde{\mathbf{f}}_1-\tilde{\mathbf{f}}_2)\big|_{\Gamma^-},\boldsymbol{\Phi}\big\rangle_{\Gamma^-}\\&=\big\langle\mathbf{t}_\Omega(\mathbf{0},0,\tilde{\mathbf{f}}_1-\tilde{\mathbf{f}}_2),\boldsymbol{\Phi}\big\rangle_{\partial\Omega}=\big\langle\tilde{\mathbf{f}}_1-\tilde{\mathbf{f}}_2,\gamma_\Omega^{-1}\boldsymbol{\Phi}\big\rangle_\Omega=0\,,\quad\forall\,\boldsymbol{\Phi}\in\widetilde{H}^{\frac12}(\Gamma^-)^n\,,\end{aligned}\tag{3.4}$$

since $\operatorname{supp}(\tilde{\mathbf{f}}_1-\tilde{\mathbf{f}}_2)\subseteq\overline{\Gamma_+}$ and $\gamma_\Omega\gamma_\Omega^{-1}\boldsymbol{\Phi}=\boldsymbol{\Phi}=0$ on $\overline{\Gamma_+}$, where the last equality follows from the assumption that $\boldsymbol{\Phi}\in\widetilde{H}^{\frac12}(\Gamma^-)^n$. Here $\gamma_\Omega^{-1}:\widetilde{H}^{\frac12}(\Gamma^-)^n\to H^1_{\Gamma^+}(\Omega)^n$ is a right inverse of the trace operator $\gamma_\Omega:H^1_{\Gamma^+}(\Omega)^n\to\widetilde{H}^{\frac12}(\Gamma^-)^n$.

Relation (3.4) shows that $\big(\mathbf{t}_\Omega(\mathbf{u},\pi,\tilde{\mathbf{f}}_1)\big)\big|_{\Gamma^-}=\big(\mathbf{t}_\Omega(\mathbf{u},\pi,\tilde{\mathbf{f}}_2)\big)\big|_{\Gamma^-}$, as asserted.

(ii) If $(\mathbf{u},\pi,\tilde{\mathbf{f}})\in\boldsymbol{H}^1_{\Gamma^+}(\Omega,\boldsymbol{\mathcal{L}})$, then by (3.1) $\tilde{\mathbf{f}}$ belongs to the space $\big(H^1_{\Gamma^+}(\Omega)^n\big)'$, which can be identified with $\widetilde{H}^{-1}(\Omega)^n/H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n$ due to Lemma B.3. Hence $\tilde{\mathbf{f}}$ can be considered as a class in this quotient space, and elements of this class can differ only on $\overline{\Gamma^+}$. But by item (i), the conormal derivative restriction $\big(\mathbf{t}_\Omega(\mathbf{u},\pi,\tilde{\mathbf{f}})\big)\big|_{\Gamma^-}$ does not depend on this difference and hence is well defined. □

3.2. Sobolev spaces on a transversal interface in a Lipschitz domain

In the sequel, we adopt the following assumption on the geometric setting.

Assumption 3.5. Let $n\geq 2$ and $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary $\partial\Omega$. The domain Ω is divided into two disjoint Lipschitz sub-domains Ω^+ and Ω^- by an $(n-1)$-dimensional Lipschitz open interface Σ, such that $\partial\Sigma=\overline{\Sigma}\cap\partial\Omega$ is a non-empty $(n-2)$-dimensional Lipschitz manifold if $n>2$, and two distinct points if $n=2$. In this case $\overline{\Sigma}$ intersects $\partial\Omega$ transversally and $\Omega=\Omega^+\cup\Sigma\cup\Omega^-$. Let $\Gamma^+:=\partial\Omega^+\setminus\overline{\Sigma}$ and $\Gamma^-:=\partial\Omega^-\setminus\overline{\Sigma}$ denote the remaining parts of the boundaries $\partial\Omega^+$ and $\partial\Omega^-$, respectively (see Fig. 1).

Therefore, Γ^+ and Γ^- are non-empty relatively open subsets of $\partial\Omega$.

We need the following spaces defined on the domains Ω, Ω^+ and Ω^-,

$$H^1_{\Gamma^\pm}(\Omega)^n:=\big\{\mathbf{v}\in H^1(\Omega)^n:(\gamma_\Omega\mathbf{v})|_{\Gamma^\pm}=\mathbf{0}\text{ on }\Gamma^\pm\big\}\,,\tag{3.5}$$

$$H^1_{\Gamma^\pm}(\Omega^\pm)^n:=\big\{\mathbf{v}^\pm\in H^1(\Omega^\pm)^n:(\gamma_{\Omega^\pm}\mathbf{v}^\pm)|_{\Gamma^\pm}=\mathbf{0}\text{ on }\Gamma^\pm\big\}\,,\tag{3.6}$$

where $\gamma_{\Omega^\pm}:H^1(\Omega^\pm)\to H^{\frac12}(\partial\Omega^\pm)$ are the trace operators corresponding to the domains $\Omega^\pm$.

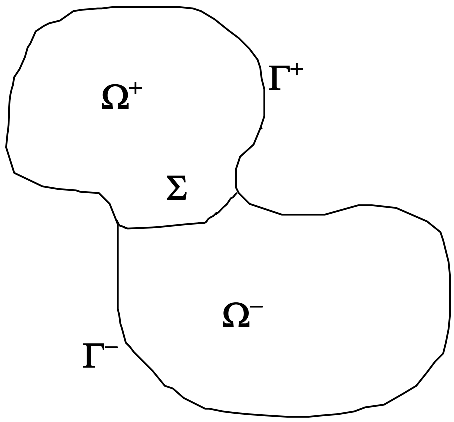

Fig. 1. Bounded domain $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$ with a transversal interface Σ.

We need also some Sobolev spaces defined on the interface Σ (cf., e.g., [7,37]). The space

$$H^{\frac{1}{2}}(\Sigma)^n := \left\{\phi \in L^2(\Sigma)^n : \exists\, \phi^+ \in H^{\frac{1}{2}}(\partial\Omega^+)^n \text{ such that } \phi = \phi^+\big|_{\Sigma}\right\} \tag{3.7}$$

can be identified with the space

$$\left\{\phi \in L^2(\Sigma)^n : \exists\, \phi^- \in H^{\frac{1}{2}}(\partial\Omega^-)^n \text{ such that } \phi = \phi^-\big|_{\Sigma}\right\}, \tag{3.8}$$

in view of the equivalence of each of them to the space defined as in (2.8), with Σ instead of $\partial\Omega$ (see also Lemma B.2). Let us also consider the space

$$\widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^+)^n := \left\{\widetilde{\phi}^+ \in H^{\frac{1}{2}}(\partial\Omega^+)^n : \operatorname{supp}\widetilde{\phi}^+ \subseteq \overline{\Sigma}\right\}, \tag{3.9}$$

which can be identified with the space

$$\widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^-)^n := \left\{\widetilde{\phi}^- \in H^{\frac{1}{2}}(\partial\Omega^-)^n : \operatorname{supp}\widetilde{\phi}^- \subseteq \overline{\Sigma}\right\}, \tag{3.10}$$

as Lemma B.2(ii) shows. Moreover, the norm of the space $\widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^\pm)^n$ is that of the space $H^{\frac{1}{2}}(\partial\Omega^\pm)^n$. Let $H_\bullet^{\frac{1}{2}}(\Sigma)^n$ be the space of all functions $\phi \in H^{\frac{1}{2}}(\Sigma)^n$ whose extensions by zero on $\partial\Omega^+$, $\mathring{E}_{\Sigma\to\partial\Omega^+}\phi$, belong to the space $\widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^+)^n$. Thus,

$$H_\bullet^{\frac{1}{2}}(\Sigma)^n := \{\phi \in H^{\frac{1}{2}}(\Sigma)^n : \mathring{E}_{\Sigma\to\partial\Omega^+}\phi \in \widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^+)^n\}.$$

According to Lemma B.2(ii) the space $H_\bullet^{\frac{1}{2}}(\Sigma)^n$ can be also described as

$$H_\bullet^{\frac{1}{2}}(\Sigma)^n := \{\phi \in H^{\frac{1}{2}}(\Sigma)^n : \mathring{E}_{\Sigma\to\partial\Omega^-}\phi \in \widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^-)^n\},$$

and can be endowed with the norm

$$\|\phi\|_{H_\bullet^{\frac{1}{2}}(\Sigma)^n} = \max\left\{\|\mathring{E}_{\Sigma\to\partial\Omega^+}\phi\|_{H^{\frac{1}{2}}(\partial\Omega^+)^n}, \|\mathring{E}_{\Sigma\to\partial\Omega^-}\phi\|_{H^{\frac{1}{2}}(\partial\Omega^-)^n}\right\}.$$

The operators of extension by zero $\mathring{E}_{\Sigma\to\partial\Omega^\pm} : H_\bullet^{\frac{1}{2}}(\Sigma)^n \to \widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^\pm)^n$ are continuous and surjective (cf. [38, Theorem 2.10(i)]). Therefore, the space $H_\bullet^{\frac{1}{2}}(\Sigma)^n$ can be identified with the spaces $\widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^\pm)^n$, and in view of [28, Theorem B.3], it can be also described as the weighted space $H^{\frac{1}{2}}_{00}(\Sigma)$ of all functions $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Sigma)^n$, such that $\delta^{-\frac{1}{2}}\boldsymbol{\phi} \in L^2(\Sigma)^n$, where $\delta(x)$ is the distance from $x \in \Sigma$ to $\partial\Sigma$. This is a Hilbert space endowed with the norm

$$\|\boldsymbol{\phi}\|^2_{H^{\frac{1}{2}}_{00}(\Sigma)^n} := \|\boldsymbol{\phi}\|^2_{H^{\frac{1}{2}}(\Sigma)^n} + \|\rho^{-\frac{1}{2}}\boldsymbol{\phi}\|^2_{L^2(\Sigma)^n}$$

(cf. [33, Chapter 1, Theorem 11.7], see also [13]).

In addition, we consider the spaces

$$H^{-\frac{1}{2}}(\Sigma)^n := \left(H^{\frac{1}{2}}_{\bullet}(\Sigma)^n\right)', \; \widetilde{H}^{-\frac{1}{2}}(\Sigma)^n := \left(H^{\frac{1}{2}}(\Sigma)^n\right)'. \tag{3.11}$$

Lemma 3.6. *The operator $\gamma_{_\Sigma} : H^1_{\Gamma^+}(\Omega)^n \to H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$ given by*

$$\gamma_{_\Sigma}\mathbf{v} := \left(\gamma_{_{\Omega^+}}\left(\mathbf{v}|_{\Omega^+}\right)\right)\big|_{\Sigma} = \left(\gamma_{_{\Omega^-}}\left(\mathbf{v}|_{\Omega^-}\right)\right)\big|_{\Sigma}, \quad \forall\, \mathbf{v} \in H^1_{\Gamma^+}(\Omega)^n, \tag{3.12}$$

is linear, bounded and surjective.

Proof. The linearity and boundedness of the operator $\gamma_{_\Sigma}$ follow from the linearity and boundedness of the trace operators

$$\gamma_{_{\Omega^+}} : H^1_{\Gamma^+}(\Omega^+)^n \to \widetilde{H}^{\frac{1}{2}}(\Sigma; \partial\Omega^+)^n, \; \gamma_{_{\Omega^-}} : H^1(\Omega^-)^n \to H^{\frac{1}{2}}(\partial\Omega^-)^n \tag{3.13}$$

(see Lemma 2.2). Moreover, the equality of restrictions to Σ of the traces from Ω^+ and Ω^- in (3.12) follows from Lemma B.1(ii) and the membership of $\mathbf{v}$ in $H^1(\Omega)^n$. In addition, the operators in (3.13) are surjective (for the first of them see Lemma 2.2), and then the operator $\gamma_{_\Sigma} : H^1_{\Gamma^+}(\Omega)^n \to H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$ is also surjective. To this end, assume that $\boldsymbol{\varphi} \in H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$. Therefore, $\mathring{E}_{\Sigma\to\partial\Omega^\pm}\boldsymbol{\varphi} \in \widetilde{H}^{\frac{1}{2}}(\Sigma; \partial\Omega^\pm)^n$ and Lemma 2.2 implies that there exist $\mathbf{v}^\pm \in H^1_{\Gamma^\pm}(\Omega^\pm)^n$ such that $\gamma_{_{\Omega^\pm}}\mathbf{v}^\pm = \mathring{E}_{\Sigma\to\partial\Omega^\pm}\boldsymbol{\varphi}$ on $\partial\Omega^\pm$. Consequently, $\left(\gamma_{_{\Omega^+}}\mathbf{v}^+\right)\big|_{\Sigma} = \left(\gamma_{_{\Omega^-}}\mathbf{v}^-\right)\big|_{\Sigma}$ on Σ, and by Lemma B.1(i), there exists $\mathbf{v} \in H^1(\Omega)^n$ such that $\mathbf{v}|_{\Omega^\pm} = \mathbf{v}^\pm$. Moreover, $\gamma_{_{\Omega^\pm}}\mathbf{v}^\pm = 0$ on $\Gamma^\pm$, and hence $\mathbf{v} \in \mathring{H}^1(\Omega)^n$, and $\gamma_{_\Sigma}\mathbf{v} = \boldsymbol{\varphi}$. Thus, the operator $\gamma_{_\Sigma} : \mathring{H}^1(\Omega)^n \to H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$ is surjective. Since $\mathring{H}^1(\Omega)^n \subset H^1_{\Gamma^+}(\Omega)^n$, the operator $\gamma_{_\Sigma} : H^1_{\Gamma^+}(\Omega)^n \to H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$ is surjective as well. □

Lemma 3.6 implies that the space $H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$ can be also characterized as

$$H^{\frac{1}{2}}_{\bullet}(\Sigma)^n = \left\{\boldsymbol{\phi} \in L^2(\Sigma) : \exists\, \mathbf{v} \in H^1_{\Gamma^+}(\Omega)^n \text{ such that } \boldsymbol{\phi} = \left(\gamma_{_{\Omega^+}}\left(\mathbf{v}|_{\Omega^+}\right)\right)\big|_{\Sigma} = \left(\gamma_{_{\Omega^-}}\left(\mathbf{v}|_{\Omega^-}\right)\right)\big|_{\Sigma}\right\}. \tag{3.14}$$

3.3. The generalized conormal derivative for the Stokes system on a transversal interface in a bounded Lipschitz domain

Recall that the space $\boldsymbol{H}^1(\Omega, \boldsymbol{\mathcal{L}})$ and the conormal derivative operator $\mathbf{t}_\Omega : \boldsymbol{H}^1(\Omega, \boldsymbol{\mathcal{L}}) \to H^{-\frac{1}{2}}(\partial\Omega)^n$ are given by Definition A.2, which assumes that the distribution $\tilde{\mathbf{f}}$ there is an element of $\widetilde{H}^{-1}(\Omega)^n$. For the case when $\tilde{\mathbf{f}}$ belongs to $\left(H^1_{\Gamma^+}(\Omega)^n\right)'$ but is not fixed as an element of $\widetilde{H}^{-1}(\Omega)^n$, the conormal derivative $\mathbf{t}_\Omega(\mathbf{u}, \pi; \tilde{\mathbf{f}})$ can not be uniquely defined on the entire boundary $\partial\Omega$ but its restriction to Γ^- can (see Definition 3.2).

Let us now consider the following counterpart of Definition 3.2 giving the restriction of conormal derivative to the interface Σ (cf. [28, Definition 5.4]).

Definition 3.7. Let Assumption 3.5 and condition (1.2) hold. Let

$$\boldsymbol{H}^1_{\Gamma^\pm}(\Omega^\pm, \boldsymbol{\mathcal{L}}) := \left\{(\mathbf{u}^\pm, \pi^\pm, \tilde{\mathbf{f}}^\pm) \in H^1_{\Gamma^\pm}(\Omega^\pm)^n \times L^2(\Omega^\pm) \times \left(H^1_{\Gamma^\pm}(\Omega^\pm)^n\right)' : \boldsymbol{\mathcal{L}}(\mathbf{u}^\pm, \pi^\pm) = \tilde{\mathbf{f}}^\pm|_{\Omega^\pm} \text{ in } \Omega^\pm\right\}. \tag{3.15}$$

If $(\mathbf{u}^{\pm}, \pi^{\pm}, \tilde{\mathbf{f}}^{\pm}) \in \boldsymbol{H}^1_{\Gamma^{\pm}}(\Omega^{\pm}, \boldsymbol{\mathcal{L}})$, then the formula

$$\left\langle \left(\mathbf{t}_{\Omega^{\pm}}(\mathbf{u}^{\pm}, \pi^{\pm}; \tilde{\mathbf{f}}^{\pm})\right)|_{\Sigma}, \mathbf{\Phi}^{\pm} \right\rangle_{\Sigma} := \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}^{\pm}), E_{i\alpha}(\gamma^{-1}_{\Omega^{\pm}} \mathbf{\Phi}^{\pm}) \right\rangle_{\Omega^{\pm}} - \left\langle \pi^{\pm}, \operatorname{div}(\gamma^{-1}_{\Omega^{\pm}} \mathbf{\Phi}^{\pm}) \right\rangle_{\Omega^{\pm}} + \left\langle \tilde{\mathbf{f}}^{\pm}, \gamma^{-1}_{\Omega^{\pm}} \mathbf{\Phi}^{\pm} \right\rangle_{\Omega^{\pm}}, \quad \forall\, \mathbf{\Phi}^{\pm} \in H^{\frac{1}{2}}_{\bullet}(\Sigma)^n , \tag{3.16}$$

defines the generalized conormal derivative $\left(\mathbf{t}_{\Omega^{\pm}}(\mathbf{u}^{\pm}, \pi^{\pm}; \tilde{\mathbf{f}}^{\pm})\right)|_{\Sigma} \in H^{-\frac{1}{2}}(\Sigma)^n$, where $\gamma^{-1}_{\Omega^{\pm}} : H^{\frac{1}{2}}_{\bullet}(\Sigma)^n \to H^1_{\Gamma^{\pm}}(\Omega^{\pm})^n$ represents a bounded right inverse of the trace operator $\gamma_{\Omega^{\pm}} : H^1_{\Gamma^{\pm}}(\Omega^{\pm})^n \to H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$.

According to Lemma 2.2, all duality pairings in formula (3.16) are well-defined. Moreover, we have the following result, whose proof is omitted for the sake of brevity (cf. [41, Proposition 8.1] for the Laplace operator, [30, Lemma 7.6] for extensions to compact Riemannian manifolds, and [7, Definition 7.1] in the case of higher order elliptic operators, see also [37, Lemma 4.3], [27, Lemma 2.3], [26, Lemma 1], [38, Definition 3.1, Theorem 3.2], [42, Theorem 10.4.1]).

Lemma 3.8. *Let Assumption 3.5 and conditions (1.2) and (1.3) hold.*

(*i*) *The generalized conormal derivative operator $\mathbf{t}_{\Omega^{\pm}} : \boldsymbol{H}^1_{\Gamma^{\pm}}(\Omega^{\pm}, \boldsymbol{\mathcal{L}}) \to H^{-\frac{1}{2}}(\Sigma)^n$ is linear and bounded, and definition (3.16) does not depend on the particular choice of the right inverse $\gamma^{-1}_{\Omega^{\pm}} : H^{\frac{1}{2}}_{\bullet}(\Sigma)^n \to H^1_{\Gamma^{\pm}}(\Omega^{\pm})^n$ of the trace operator $\gamma_{\Omega^{\pm}} : H^1_{\Gamma^{\pm}}(\Omega^{\pm})^n \to H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$.*

(*ii*) *Let $(\mathbf{u}^{+}, \pi^{+}, \tilde{\mathbf{f}}^{+}) \in \boldsymbol{H}^1_{\Gamma^{+}}(\Omega^{+}, \boldsymbol{\mathcal{L}})$ and $(\mathbf{u}^{-}, \pi^{-}, \tilde{\mathbf{f}}^{-}) \in \boldsymbol{H}^1(\Omega^{-}, \boldsymbol{\mathcal{L}})$. Let $\pi \in L^2(\Omega)$ and $\tilde{\mathbf{f}} \in \left(H^1_{\Gamma^{+}}(\Omega)^n\right)'$ be such that*

$$\pi|_{\Omega^{\pm}} = \pi^{\pm}, \quad \left\langle \tilde{\mathbf{f}}, \mathbf{w} \right\rangle_{\Omega} := \left\langle \tilde{\mathbf{f}}^{+}, \mathbf{w}|_{\Omega^{+}} \right\rangle_{\Omega^{+}} + \left\langle \tilde{\mathbf{f}}^{-}, \mathbf{w}|_{\Omega^{-}} \right\rangle_{\Omega^{-}}, \quad \forall\, \mathbf{w} \in H^1_{\Gamma^{+}}(\Omega)^n . \tag{3.17}$$

Then the following Green identity holds

$$\begin{aligned}&\left\langle \left(\mathbf{t}_{\Omega^{+}}(\mathbf{u}^{+}, \pi^{+}; \tilde{\mathbf{f}}^{+})\right)|_{\Sigma} + \left(\mathbf{t}_{\Omega^{-}}(\mathbf{u}^{-}, \pi^{-}; \tilde{\mathbf{f}}^{-})\right)|_{\Sigma}, \gamma_{\Sigma}\mathbf{w} \right\rangle_{\Sigma} + \left\langle \left(\mathbf{t}_{\Omega^{-}}(\mathbf{u}^{-}, \pi^{-}; \tilde{\mathbf{f}}^{-})\right)|_{\Gamma^{-}}, (\gamma_{\Omega}\mathbf{w})|_{\Gamma^{-}} \right\rangle_{\Gamma^{-}} \\ &= \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}^{+}), E_{i\alpha}(\mathbf{w}) \right\rangle_{\Omega^{+}} + \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}^{-}), E_{i\alpha}(\mathbf{w}) \right\rangle_{\Omega^{-}} - \langle \pi, \operatorname{div} \mathbf{w} \rangle_{\Omega} + \langle \tilde{\mathbf{f}}, \mathbf{w} \rangle_{\Omega}, \quad \forall\, \mathbf{w} \in H^1_{\Gamma^{+}}(\Omega)^n .\end{aligned} \tag{3.18}$$

Note that the existence of a function $\pi \in L^2(\Omega)$ as in (3.17) follows from Lemma B.1, while $\tilde{\mathbf{f}}$ defined in (3.17) belongs to the space $\left(H^1_{\Gamma^{+}}(\Omega)^n\right)'$. (Indeed, the relations $\tilde{\mathbf{f}}^{+} \in \left(H^1_{\Gamma^{+}}(\Omega^{+})^n\right)'$ and $\left(H^1_{\Gamma^{+}}(\Omega^{+})^n\right)' \hookrightarrow \left(H^1_{\Gamma^{+}}(\Omega)^n\right)'$ imply that $\tilde{\mathbf{f}}^{+} \in \left(H^1_{\Gamma^{+}}(\Omega)^n\right)'$. In addition, the embedding $\tilde{\mathbf{f}}^{-} \in \widetilde{H}^{-1}(\Omega^{-})^n$ implies that $\tilde{\mathbf{f}}^{-} \in \left(H^1_{\Gamma^{+}}(\Omega)^n\right)'$. Thus, $\tilde{\mathbf{f}} = \tilde{\mathbf{f}}^{+} + \tilde{\mathbf{f}}^{-}$ belongs indeed to $\left(H^1_{\Gamma^{+}}(\Omega)^n\right)'$.)

4. Mixed and mixed-transmission problems for the anisotropic Stokes system in bounded Lipschitz domains

Well-posedness results for the mixed problem for the Stokes and Brinkman systems with an L^{∞} scalar viscosity coefficient in Lipschitz domains on compact Riemannian manifolds have been obtained in [30, Theorem 7.9] and [31, Theorem 8.4]. Mixed problems for the Stokes system with constant coefficients in polyhedral domains, or in bounded Lipschitz domains of $\mathbb{R}^2$, have been analyzed in [35, Theorem 5.1] and [44, Theorem 3.1]. Well-posedness and regularity results for the elasticity equations with mixed boundary conditions on polyhedral domains have been obtained in [36]. Well-posedness results for the mixed problem for higher-order elliptic operators in (ϵ, δ)-domains have been established in [7, Theorem 7.3].

In this section we show the well-posedness of boundary value problems of mixed-transmission type for the anisotropic Stokes system in a compressible framework, in bounded Lipschitz domains with internal Lipschitz interfaces.

4.1. Mixed problem for the anisotropic compressible Stokes system

Let us now consider the following variational problem.

For given data $(\mathfrak{F}, g) \in \left(H^1_{\Gamma^+}(\Omega)^n\right)' \times L^2(\Omega)$, *find* $(\mathbf{u}, \pi) \in H^1_{\Gamma^+}(\Omega)^n \times L^2(\Omega)$ *such that*

$$\begin{cases} a_{\mathbb{A};\Omega}(\mathbf{u}, \mathbf{w}) + b_\Omega(\mathbf{w}, \pi) = \langle \mathfrak{F}, \mathbf{w}\rangle_\Omega \,, & \forall\, \mathbf{w} \in H^1_{\Gamma^+}(\Omega)^n \,, \\ b_{{}_\Omega}(\mathbf{u}, q) = -\langle g, q\rangle_\Omega, & \forall\, q \in L^2(\Omega) \,, \end{cases} \tag{4.1}$$

where $a_{\mathbb{A};\Omega} : H^1_{\Gamma^+}(\Omega)^n \times H^1_{\Gamma^+}(\Omega)^n \to \mathbb{R}$, $b_\Omega : H^1_{\Gamma^+}(\Omega)^n \times L^2(\Omega) \to \mathbb{R}$ *are the bilinear forms given by*

$$\begin{aligned} a_{\mathbb{A};\Omega}(\mathbf{v}, \mathbf{w}) &:= \left\langle A^{\alpha\beta}\partial_\beta(\mathbf{v}), \partial_\alpha(\mathbf{w})\right\rangle_\Omega \\ &= \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{v}), E_{i\alpha}(\mathbf{w})\right\rangle_\Omega \,, \quad \forall\, \mathbf{v},\, \mathbf{w} \in H^1_{\Gamma^+}(\Omega)^n \,, \end{aligned} \tag{4.2}$$

$$b_\Omega(\mathbf{v}, q) := -\, \langle \operatorname{div} \mathbf{v}, q\rangle_\Omega \,, \qquad \forall\, \mathbf{v} \in H^1_{\Gamma^+}(\Omega)^n \,,\ \forall\, q \in L^2(\Omega) \,. \tag{4.3}$$

Theorem 4.1. *Let conditions* (1.2)-(1.4) *and Assumption 3.1 hold. Then for all given data* $(\mathfrak{F}, g) \in \left(H^1_{\Gamma^+}(\Omega)^n\right)' \times L^2(\Omega)$, *the variational problem* (4.1) *is well-posed, that is, it has a unique solution* $(\mathbf{u}, \pi) \in H^1_{\Gamma^+}(\Omega)^n \times L^2(\Omega)$ *and there exists a constant* $C = C(\Omega, \Gamma^+, \Gamma^-, C_{\mathbb{A}}, \|\mathbb{A}\|, n) > 0$, *such that*

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\pi\|_{L^2(\Omega)} \le C \left(\|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + \|g\|_{L^2(\Omega)}\right) . \tag{4.4}$$

Proof. First, we note that condition (1.2) and the Hölder inequality imply that the bilinear form $a_{\mathbb{A};\Omega} : H^1_{\Gamma^+}(\Omega)^n \times H^1_{\Gamma^+}(\Omega)^n \to \mathbb{R}$ is bounded. In addition, assumption (1.4) combined with the first Korn inequality for functions in $H^1_{\Gamma^+}(\Omega)^n$ (cf., e.g., [36, Proposition 5, Eq. (53)]) implies that

$$\begin{aligned} C_0 C_{\mathbb{A}}^{-1} \|\mathbf{u}\|^2_{H^1(\Omega)^n} &\le C_{\mathbb{A}}^{-1} \|\mathbb{E}(\mathbf{u})\|^2_{L^2(\Omega)^{n\times n}} \\ &\le \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{u})\right\rangle_\Omega \\ &= a_{\mathbb{A};\Omega}(\mathbf{u}, \mathbf{u}) \,, \quad \forall\, \mathbf{u} \in H^1_{\Gamma^+;\mathrm{div}}(\Omega)^n \,, \end{aligned} \tag{4.5}$$

with some constant $C_0 = C_0(\Omega, \Gamma^+, \Gamma^-, C_{\mathbb{A}}, \|\mathbb{A}\|, n) > 0$.

Second, it is immediate that the bilinear form $b_{{}_\Omega} : H^1_{\Gamma^+}(\Omega)^n \times L^2(\Omega) \to \mathbb{R}$ given by (4.3) is also bounded. In addition, Lemma 2.3(ii) shows that the operator $\operatorname{div} : H^1_{\Gamma^+}(\Omega)^n / H^1_{\Gamma^+;\mathrm{div}}(\Omega)^n \to L^2(\Omega)$ is an isomorphism. Then by, e.g., [18, Theorem A.56, Remark 2.7] there exists a constant $C_{\Gamma^+} > 0$ such that the bilinear form $b_{{}_\Omega}(\cdot, \cdot)$ satisfies the inf-sup condition

$$\inf_{q \in L^2(\Omega)\setminus\{0\}} \ \sup_{\mathbf{v} \in H^1_{\Gamma^+}(\Omega)^n \setminus \{\mathbf{0}\}} \frac{b_{{}_\Omega}(\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1_{\Gamma^+}(\Omega)^n} \|q\|_{L^2(\Omega)}} \ge C_{\Gamma^+} \,. \tag{4.6}$$

Hence due to, e.g., Theorem 2.34 in [18] (with $X = H^1_{\Gamma^+}(\Omega)^n$, $V := \operatorname{Ker} B = H^1_{\Gamma^+;\mathrm{div}}(\Omega)^n$, which is the null space of the operator $B := -\operatorname{div} : X \to M$, and $M = L^2(\Omega)$) shows that there exists a unique solution $(\mathbf{u}, \pi) \in H^1_{\Gamma^+}(\Omega)^n \times L^2(\Omega)$ of the variational problem (4.1), which satisfies inequality (4.4). □

Let us prove the following well-posedness result for the mixed boundary value problem with homogeneous Dirichlet condition (see also [31, Theorem 8.4] for the isotropic Stokes system in the compact Riemannian setting).

Theorem 4.2. *Let conditions (1.2)-(1.4) and Assumption 3.1 hold. Then for all given data* $(\tilde{\mathbf{f}}, g, \boldsymbol{\psi}_{_{\Gamma^-}}) \in \big(H^1_{\Gamma^+}(\Omega)^n\big)' \times L^2(\Omega) \times H^{-\frac{1}{2}}(\Gamma^-)^n$, *the mixed Dirichlet-Neumann problem for the anisotropic Stokes system*

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u},\pi) = \widetilde{\mathbf{f}}|_{\Omega},\ \operatorname{div}\mathbf{u} = g & \text{in } \Omega\,,\\ (\gamma_{\Omega}\mathbf{u})|_{_{\Gamma^+}} = \mathbf{0} & \text{on } \Gamma^+\,,\\ \big(\mathbf{t}_{\Omega}(\mathbf{u},\pi;\widetilde{\mathbf{f}})\big)|_{\Gamma^-} = \boldsymbol{\psi}_{_{\Gamma^-}} & \text{on } \Gamma^- \end{cases} \tag{4.7}$$

has a unique solution $(\mathbf{u},\pi)$ *in* $H^1(\Omega)^n\times L^2(\Omega)$ *and there exists a constant* $C_0 = C_0(\Omega,\Gamma^+,\Gamma^-,C_{\mathbb{A}},\|\mathbb{A}\|,n) > 0$, *such that*

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\pi\|_{L^2(\Omega)} \le C_0\Big(\|\widetilde{\mathbf{f}}\|_{\widetilde{H}^{-1}(\Omega^+)^n} + \|g\|_{L^2(\Omega)} + \|\boldsymbol{\psi}_{_{\Gamma^-}}\|_{H^{-\frac{1}{2}}(\Gamma^-)^n}\Big)\,. \tag{4.8}$$

Proof. Let us prove that the mixed problem (4.7) is equivalent to variational problem (4.1) with $\boldsymbol{\mathfrak{F}} \in \big(H^1_{\Gamma^+}(\Omega)^n\big)'$ defined by

$$\langle \boldsymbol{\mathfrak{F}}, \mathbf{w}\rangle_{\Omega} = \Big\langle \boldsymbol{\psi}_{_{\Gamma^-}}, \gamma_{\Omega}\mathbf{w}\Big\rangle_{\Gamma^-} - \big\langle \widetilde{\mathbf{f}}, \mathbf{w}\big\rangle_{\Omega}\,, \quad \forall\, \mathbf{w} \in H^1_{\Gamma^+}(\Omega)^n\,. \tag{4.9}$$

First, let us prove that if $(\mathbf{u},\pi) \in H^1(\Omega)^n \times L^2(\Omega)$ is a solution of the boundary value problem (4.7) then it solves the variational problem (4.1) with $\boldsymbol{\mathfrak{F}}$ given by (4.9). Indeed, the homogeneous Dirichlet boundary condition in (4.7) implies that $\mathbf{u} \in H^1_{\Gamma^+}(\Omega)^n$. Then the first equation in (4.1) follows from the Green identity (3.3), the Neumann boundary condition in (4.7), and notation (4.9). The second equation in (4.1) follows from the equation $\operatorname{div}\mathbf{u} = g$ in (4.7) and a duality argument.

Conversely, assume that the pair $(\mathbf{u},\pi) \in H^1_{\Gamma^+}(\Omega)^n \times L^2(\Omega)$ satisfies the variational problem (4.1). Then $\mathbf{u}$ satisfies the Dirichlet boundary condition in (4.7), and the first equation in (4.1) can be written as

$$\big\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{w})\big\rangle_{\Omega} - \big\langle \pi, \operatorname{div}\mathbf{w}\big\rangle_{\Omega} - \big\langle \widetilde{\mathbf{f}}, \mathbf{w}\big\rangle_{\Omega} = 0, \quad \forall\, \mathbf{w} \in H^1_{\Gamma^+}(\Omega)^n. \tag{4.10}$$

Since $\mathcal{D}(\Omega)^n \subset H^1_{\Gamma^+}(\Omega)^n$, formula (4.10) holds also for any $\mathbf{w} \in \mathcal{D}(\Omega)^n$, which implies the anisotropic Stokes equation in (4.7) in the sense of distributions. Moreover, the second equation in (4.1) is the variational form of the equation $\operatorname{div}\mathbf{u} = g$ in Ω.

In addition, formula (3.2) and the first equation in (4.1) with $\mathbf{w} = \gamma_{\Omega}^{-1}\boldsymbol{\Phi} \in H^1_{\Gamma^+}(\Omega)^n$ yield

$$\big\langle \big(\mathbf{t}_{\Omega}(\mathbf{u},\pi;\widetilde{\mathbf{f}})\big)|_{\Gamma^-}, \boldsymbol{\Phi}\big\rangle_{\Gamma^-} = \Big\langle \boldsymbol{\psi}_{_{\Gamma^-}}, \boldsymbol{\Phi}\Big\rangle_{\Gamma^-}\,, \quad \forall\, \boldsymbol{\Phi} \in \widetilde{H}^{\frac{1}{2}}(\Gamma^-)^n\,,$$

and thus, $\big(\mathbf{t}_{\Omega}(\mathbf{u},\pi;\widetilde{\mathbf{f}})\big)|_{\Gamma^-} = \boldsymbol{\psi}_{_{\Gamma^-}}$ on Γ^-.

Consequently, the mixed problem (4.7) is equivalent to the variational problem (4.1) with $\boldsymbol{\mathfrak{F}}$ given by (4.9), as asserted. Theorem 4.1 shows then that the mixed problem (4.7) has a unique solution, given by the solution of the variational problem (4.1), and inequality (4.8) follows from inequality (4.4) and the continuity of the operators involved in relation (4.9). □

In order to analyze the fully non-homogeneous mixed problem, we need the following Bogovskii-type result.

Lemma 4.3. *Let Assumption 3.1 hold. Then for* $(g, \boldsymbol{\varphi}_{_{\Gamma^+}}) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma^+)^n$ *given, there exist a field* $\mathbf{v} \in H^1(\Omega)^n$ *and a constant* $C_{_{\Gamma^+}} = C_{_{\Gamma^+}}(\Omega,\Gamma^+,n) > 0$ *such that*

$$\operatorname{div}\mathbf{v} = g \ \text{ in } \Omega\,, \tag{4.11}$$

$$\gamma_{\Omega}\mathbf{v} = \boldsymbol{\varphi}_{_{\Gamma^+}} \ \text{ on } \Gamma^+\,, \tag{4.12}$$

and

$$\|\mathbf{v}\|_{H^1(\Omega)^n} \le C_{\Gamma^+}\Big(\|g\|_{L^2(\Omega)} + \|\boldsymbol{\varphi}_{\Gamma^+}\|_{H^{\frac{1}{2}}(\Gamma^+)^n}\Big)\,. \tag{4.13}$$

Proof. Let us introduce the function

$$\mathbf{v}_1 := \gamma_\Omega^{-1} E_{\Gamma^+\to\partial\Omega}\boldsymbol{\varphi}_{\Gamma^+}\,, \tag{4.14}$$

where $\gamma_\Omega^{-1} : H^{\frac{1}{2}}(\partial\Omega)^n \to H^1(\Omega)^n$ is a continuous right inverse of the trace operator $\gamma_\Omega : H^1(\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$, and $E_{\Gamma^+\to\partial\Omega} : H^{\frac{1}{2}}(\Gamma^+)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$ is a continuous extension operator (see, e.g., Proposition 4.1 in [41] for the existence of such an operator). Then $\mathbf{v}_1$ belongs to $H^1(\Omega)^n$. Let us now define

$$g_1 := \operatorname{div}\mathbf{v}_1 \in L^2(\Omega). \tag{4.15}$$

Hence $g - g_1 \in L^2(\Omega)$. Then due to Lemma 2.3 there exist $\mathbf{v}_0 \in H^1_{\Gamma^+}(\Omega)^n$ and a constant $C^0_{\Gamma^+} = C^0_{\Gamma^+}(\Omega,\Gamma^+,n) > 0$ such that

$$\operatorname{div}\mathbf{v}_0 = g - g_1 \text{ in } \Omega\,, \tag{4.16}$$

$$\|\mathbf{v}_0\|_{H^1(\Omega)^n} \le C^0_{\Gamma^+}\|g-g_1\|_{L^2(\Omega)} \le C^0_{\Gamma^+}\left(\|g\|_{L^2(\Omega)} + \|g_1\|_{L^2(\Omega)}\right). \tag{4.17}$$

Finally, choosing $\mathbf{v} := \mathbf{v}_1 + \mathbf{v}_0$ and using inequality (4.17) and the continuity of the operators involved in (4.14)-(4.15), we obtain the desired result. □

Let us consider the spaces

$$\mathcal{X}_\Omega := H^1(\Omega)^n \times L^2(\Omega)\,, \tag{4.18}$$

$$\mathcal{Y}_\Omega := \left(H^1_{\Gamma^+}(\Omega)^n\right)' \times L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma^+)^n \times H^{-\frac{1}{2}}(\Gamma^-)^n\,. \tag{4.19}$$

Then we obtain the following well-posedness result.

Theorem 4.4. *Let conditions (1.2)-(1.4) and Assumption 3.1 hold. Then for all given data* $(\widetilde{\mathbf{f}}, g, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}) \in \mathcal{Y}_\Omega$*, there exists a constant* $C = C(\Omega,\Gamma^+,\Gamma^-,C_{\mathbb{A}},\|\mathbb{A}\|,n) > 0$*, such that the mixed Dirichlet-Neumann problem for the anisotropic Stokes system*

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u},\pi) = \widetilde{\mathbf{f}}|_\Omega,\ \operatorname{div}\mathbf{u} = g & \textit{in } \Omega\,,\\ (\gamma_\Omega\mathbf{u})|_{\Gamma^+} = \boldsymbol{\varphi}_{\Gamma^+} & \textit{on } \Gamma^+\,,\\ \big(\mathbf{t}_\Omega(\mathbf{u},\pi;\widetilde{\mathbf{f}})\big)|_{\Gamma^-} = \boldsymbol{\psi}_{\Gamma^-} & \textit{on } \Gamma^- \end{cases} \tag{4.20}$$

has a unique solution $(\mathbf{u},\pi) \in \mathcal{X}_\Omega$*, which satisfies the inequality*

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\pi\|_{L^2(\Omega)} \le C\Big(\|\widetilde{\mathbf{f}}\|_{\widetilde{H}^{-1}(\Omega^+)^n} + \|g\|_{L^2(\Omega)} + \|\boldsymbol{\varphi}_{\Gamma^+}\|_{H^{\frac{1}{2}}(\Gamma^+)^n} + \|\boldsymbol{\psi}_{\Gamma^-}\|_{H^{-\frac{1}{2}}(\Gamma^-)^n}\Big)\,. \tag{4.21}$$

Moreover, the solution can be represented as $(\mathbf{u},\pi) = \boldsymbol{\mathcal{T}}_\Omega\left(\widetilde{\mathbf{f}}, g, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}\right)$*, where* $\boldsymbol{\mathcal{T}}_\Omega : \mathcal{Y}_\Omega \to \mathcal{X}_\Omega$ *is a linear and continuous operator.*

Proof. Let $\mathbf{v} \in H^1(\Omega)^n$ be the function given by Lemma 4.3. For the velocity-pressure couple $(\mathbf{v},0)$, let us also define the field $\check{\mathbf{f}}$ with the entries

$$\check{f}_i := \partial_\alpha \mathring{E}_\Omega\big(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{v})\big), \quad i = 1, \dots, n, \tag{4.22}$$

where $\mathring{E}_\Omega$ is the operator of zero extension from Ω to $\mathbb{R}^n$. Hence $\check{\mathbf{f}} \in \widetilde{H}^{-1}(\Omega)^n \hookrightarrow \big(H^1_{\Gamma^+}(\Omega)^n\big)'$, $\widetilde{\mathbf{f}} - \check{\mathbf{f}} \in \big(H^1_{\Gamma^+}(\Omega)^n\big)'$, $\check{\mathbf{f}}|_\Omega = \boldsymbol{\mathcal{L}}(\mathbf{v}, 0)$ in Ω, as follows from (1.7), and $\mathbf{t}_\Omega(\mathbf{v}, 0; \check{\mathbf{f}}) = \mathbf{0}$ due to Definition 3.2.

Using the notation $\mathbf{u}_0 := \mathbf{u} - \mathbf{v}$ and taking into account that

$$\mathbf{t}_\Omega(\mathbf{u}_0, \pi; \widetilde{\mathbf{f}} - \check{\mathbf{f}}) = \mathbf{t}_\Omega(\mathbf{u} - \mathbf{v}, \pi; \widetilde{\mathbf{f}} - \check{\mathbf{f}}) = \mathbf{t}_\Omega(\mathbf{u}, \pi; \widetilde{\mathbf{f}}) - \mathbf{t}_\Omega(\mathbf{v}, 0; \check{\mathbf{f}}) = \mathbf{t}_\Omega(\mathbf{u}, \pi; \widetilde{\mathbf{f}}),$$

we reduce problem (4.20) to the mixed problem

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u}_0, \pi) = (\widetilde{\mathbf{f}} - \check{\mathbf{f}})|_\Omega, \ \operatorname{div} \mathbf{u}_0 = 0 & \text{in } \Omega, \\ (\gamma_\Omega \mathbf{u}_0)|_{\Gamma^+} = \mathbf{0} & \text{on } \Gamma^+, \\ \big(\mathbf{t}_\Omega(\mathbf{u}_0, \pi; \widetilde{\mathbf{f}} - \check{\mathbf{f}})\big)|_{\Gamma^-} = \boldsymbol{\psi}_{\Gamma^-} & \text{on } \Gamma^- \end{cases} \tag{4.23}$$

for $(\mathbf{u}_0, \pi) \in H^1(\Omega)^n \times L^2(\Omega)$. Theorem 4.2 shows that problem (4.23) is uniquely solvable and its solution depends continuously on the right hand sides $\widetilde{\mathbf{f}} - \check{\mathbf{f}} \in \big(H^1_{\Gamma^+}(\Omega)^n\big)'$ and $\boldsymbol{\psi}_{\Gamma^-} \in H^{-\frac{1}{2}}(\Gamma^-)^n$.

Then the pair $(\mathbf{u}, \pi) = (\mathbf{v} + \mathbf{u}_0, \pi) \in H^1(\Omega)^n \times L^2(\Omega)$ is a solution of the mixed problem (4.20), which, due to Theorem 4.2 and Lemma 4.3 (for $\mathbf{v}$), depends continuously on the data $(\widetilde{\mathbf{f}}, g, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-})$, that is, estimate (4.21) holds. By using again Theorem 4.2, it follows that this solution is unique. Moreover, the linearity and boundedness of the solution operator $\boldsymbol{\mathcal{T}}_\Omega : \boldsymbol{\mathcal{Y}}_\Omega \to \boldsymbol{\mathcal{X}}_\Omega$ is an immediate consequence of the linearity of the mixed problem (4.20) and of estimate (4.21). □

4.2. Mixed-transmission problem with homogeneous Dirichlet and interface trace conditions

Let us consider the spaces

$$\boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-} := H^1(\Omega^+)^n \times L^2(\Omega^+) \times H^1(\Omega^-)^n \times L^2(\Omega^-), \tag{4.24}$$

$$\boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-} := \big(H^1_{\Gamma^+}(\Omega^+)^n\big)' \times \widetilde{H}^{-1}(\Omega^-)^n \times L^2(\Omega) \times H^{-\frac{1}{2}}(\Sigma)^n \times H^{-\frac{1}{2}}(\Gamma^-)^n, \tag{4.25}$$

and the mixed-transmission problem for the anisotropic Stokes system with homogeneous Dirichlet and interface trace conditions

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u}^+, \pi^+) = \widetilde{\mathbf{f}}^+|_{\Omega^+}, \ \operatorname{div} \mathbf{u}^+ = g|_{\Omega^+} & \text{in } \Omega^+, \\ \boldsymbol{\mathcal{L}}(\mathbf{u}^-, \pi^-) = \widetilde{\mathbf{f}}^-|_{\Omega^-}, \ \operatorname{div} \mathbf{u}^- = g|_{\Omega^-} & \text{in } \Omega^-, \\ (\gamma_{\Omega^+} \mathbf{u}^+)|_\Sigma - (\gamma_{\Omega^-} \mathbf{u}^-)|_\Sigma = \mathbf{0} & \text{on } \Sigma, \\ \big(\mathbf{t}_{\Omega^+}(\mathbf{u}^+, \pi^+; \widetilde{\mathbf{f}}^+)\big)|_\Sigma + \big(\mathbf{t}_{\Omega^-}(\mathbf{u}^-, \pi^-; \widetilde{\mathbf{f}}^-)\big)|_\Sigma = \boldsymbol{\psi}_\Sigma & \text{on } \Sigma, \\ (\gamma_{\Omega^+} \mathbf{u}^+)|_{\Gamma^+} = \mathbf{0} & \text{on } \Gamma^+, \\ \big(\mathbf{t}_{\Omega^-}(\mathbf{u}^-, \pi^-; \widetilde{\mathbf{f}}^-)\big)|_{\Gamma^-} = \boldsymbol{\psi}_{\Gamma^-} & \text{on } \Gamma^-, \end{cases} \tag{4.26}$$

with the given data $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\psi}_\Sigma, \boldsymbol{\psi}_{\Gamma^-}) \in \boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-}$ and the unknown $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-}$.

Note that the conormal derivative operators $\mathbf{t}_{\Omega^+}$ and $\mathbf{t}_{\Omega^-}$ involved in the second transmission condition in (4.26), are considered as in Definition 3.2 and correspond to the outward unit normals to Ω^+ and Ω^-, respectively, that have opposite directions on Σ. However, one can instead consider also the conormal derivatives with respect to unit normals of the same direction on Σ, but then the sum in the second transmission condition in (4.26) needs to be replaced by the difference and leads to the jump of the conormal derivatives. Such an approach has been considered in [25–27].

Now, let $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-}$ be such that $\mathbf{u}^+$ and $\mathbf{u}^-$ satisfy the interface condition $(\gamma_{\Omega^+} \mathbf{u}^+)|_\Sigma = (\gamma_{\Omega^-} \mathbf{u}^-)|_\Sigma$ on Σ. Then Lemma B.1 implies that there exists a unique pair $\big(\mathbf{u}, \pi\big) \in H^1(\Omega)^n \times L^2(\Omega)$ such that

$$\mathbf{u}|_{\Omega^+} = \mathbf{u}^+, \quad \mathbf{u}|_{\Omega^-} = \mathbf{u}^-, \quad \pi|_{\Omega^+} = \pi^+, \quad \pi|_{\Omega^-} = \pi^-. \tag{4.27}$$

Moreover, if $\mathbf{u}^+$ satisfies also the homogeneous Dirichlet condition in (4.26), then $(\mathbf{u}, \pi) \in H^1_{\Gamma^+}(\Omega)^n \times L^2(\Omega)$.

Theorem 4.5. *Let Assumption 3.5 and conditions (1.2)-(1.4) hold. Then for all given data* $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\psi}_{_\Sigma}, \boldsymbol{\psi}_{_{\Gamma^-}}) \in \boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-}$, *the mixed-transmission problem (4.26) for the anisotropic Stokes system has a unique solution* $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-)$ *in the space* $\mathcal{X}_{\Omega^+,\Omega^-}$ *and there exists a constant* $C = C(\Omega^+, \Omega^-, C_{\mathbb{A}}, \|\mathbb{A}\|, n) > 0$ *such that*

$$\left\| \left(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-\right) \right\|_{\mathcal{X}_{\Omega^+,\Omega^-}} \le C \left\| \left(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\psi}_{_\Sigma}, \boldsymbol{\psi}_{_{\Gamma^-}}\right) \right\|_{\boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-}}. \tag{4.28}$$

Proof. Let us prove that the mixed-transmission problem (4.26) with the unknown $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathcal{X}_{\Omega^+,\Omega^-}$ is equivalent, in the sense of relations (4.27), to the variational problem (4.1) with the unknown $(\mathbf{u}, \pi) \in H^1_{\Gamma^+}(\Omega)^n \times L^2(\Omega)$, and $\mathfrak{F} \in \left(H^1_{\Gamma^+}(\Omega)^n\right)'$ is given by

$$\begin{aligned}
\langle \mathfrak{F}, \mathbf{w}\rangle_\Omega &:= \langle \boldsymbol{\psi}_{_\Sigma}, \gamma_{_\Omega}\mathbf{w}\rangle_\Sigma + \left\langle \boldsymbol{\psi}_{_{\Gamma^-}}, \gamma_{_\Omega}\mathbf{w}\right\rangle_{\Gamma^-} - \left(\langle \tilde{\mathbf{f}}^+, \mathbf{w}|_{\Omega^+}\rangle_{\Omega^+} + \langle \tilde{\mathbf{f}}^-, \mathbf{w}|_{\Omega^-}\rangle_{\Omega^-}\right)\\
&= \langle \boldsymbol{\psi}_{_\Sigma}, \gamma_{_\Sigma}\mathbf{w}\rangle_\Sigma + \left\langle \boldsymbol{\psi}_{_{\Gamma^-}}, \gamma_{_\Omega}\mathbf{w}\right\rangle_{\Gamma^-} - \left(\langle \tilde{\mathbf{f}}^+, \mathbf{w}|_{\Omega^+}\rangle_{\Omega^+} + \langle \tilde{\mathbf{f}}^-, \mathbf{w}|_{\Omega^-}\rangle_{\Omega^-}\right)\\
&= \langle \gamma^*_{_\Sigma}\boldsymbol{\psi}_{_\Sigma}, \mathbf{w}\rangle_\Omega + \left\langle \gamma^*_{_\Omega}\boldsymbol{\psi}_{_{\Gamma^-}}, \mathbf{w}\right\rangle_{\Omega} - \left(\langle \tilde{\mathbf{f}}^+, \mathbf{w}|_{\Omega^+}\rangle_{\Omega^+} + \langle \tilde{\mathbf{f}}^-, \mathbf{w}|_{\Omega^-}\rangle_{\Omega^-}\right),\ \forall\, \mathbf{w} \in H^1_{\Gamma^+}(\Omega)^n,
\end{aligned} \tag{4.29}$$

where $\gamma^*_{_\Omega} : H^{-\frac{1}{2}}(\Gamma^-)^n \to \left(H^1_{\Gamma^+}(\Omega)^n\right)'$ is the adjoint of the operator $\gamma_{_\Omega} : H^1_{\Gamma^+}(\Omega)^n \to \widetilde{H}^{\frac{1}{2}}(\Gamma^-)^n$, and $\gamma^*_{_\Sigma} : H^{-\frac{1}{2}}(\Sigma)^n \to \left(H^1_{\Gamma^+}(\Omega)^n\right)'$ is the adjoint of the operator $\gamma_{_\Sigma} : H^1_{\Gamma^+}(\Omega)^n \to H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$ (see Lemma 2.2 and Lemma 3.6). Therefore,

$$\mathfrak{F} = \gamma^*_{_\Sigma}\boldsymbol{\psi}_{_\Sigma} + \gamma^*_{_\Omega}\boldsymbol{\psi}_{_{\Gamma^-}} - \left(\tilde{\mathbf{f}}^+ + \tilde{\mathbf{f}}^-\right).$$

Recall that the space $\left(H^1_{\Gamma^+}(\Omega^+)^n\right)'$ can be identified with a subspace of $H^{-1}(\Omega)^n$ given by (B.2) (see Lemma B.3).

First, assume that $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathcal{X}_{\Omega^+,\Omega^-}$ satisfies the mixed-transmission problem (4.26). Then the first equation of the variational problem (4.1) follows from the Green identity (3.18) applied to the pairs $(\mathbf{u}^+, \pi^+) \in H^1_{\Gamma^+}(\Omega^+)^n \times L^2(\Omega^+)$ and $(\mathbf{u}^-, \pi^-) \in H^1(\Omega^-)^n \times L^2(\Omega^-)$ in Ω^+ and Ω^-, respectively, with $\mathbf{w} \in H^1_{\Gamma^+}(\Omega)^n$ and $\widetilde{\mathbf{f}} = -\mathfrak{F}$, where $\mathfrak{F} \in \left(H^1_{\Gamma^+}(\Omega)^n\right)'$ is given by formula (4.29). The second equation in (4.1) follows from the equations $\operatorname{div}\mathbf{u}^\pm = g|_{\Omega^\pm}$ in $\Omega^\pm$.

Conversely, let $(\mathbf{u}, \pi) \in H^1_{\Gamma^+}(\Omega)^n \times L^2(\Omega)$ satisfy the variational problem (4.1) and let $(\mathbf{u}^\pm, \pi^\pm) := (\mathbf{u}|_{\Omega^\pm}, \pi|_{\Omega^\pm})$ in $\Omega^\pm$. Then the inclusion $\mathbf{u} \in H^1_{\Gamma^+}(\Omega)^n$ implies that the Dirichlet condition $(\gamma_{_{\Omega^+}}\mathbf{u}^+)|_{_{\Gamma^+}} = \mathbf{0}$ on Γ^+ and the interface condition $(\gamma_{_{\Omega^+}}\mathbf{u}^+)|_{_\Sigma} = (\gamma_{_{\Omega^-}}\mathbf{u}^-)|_{_\Sigma}$ on Σ are satisfied. Therefore, we obtain that $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathcal{X}_{\Omega^+,\Omega^-}$. In addition, the first equation in (4.1) can be written as

$$\left\langle a^{\alpha\beta}_{ij}E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{w})\right\rangle_\Omega - \left\langle \pi, \operatorname{div}\mathbf{w}\right\rangle_\Omega - \left\langle \mathfrak{F}, \mathbf{w}\right\rangle_\Omega = 0, \quad \forall\, \mathbf{w} \in H^1_{\Gamma^+}(\Omega)^n. \tag{4.30}$$

Since $\mathcal{D}(\Omega^\pm)^n \subset H^1_{\Gamma^+}(\Omega)^n$, formula (4.30) holds also for any $\mathbf{w} \in \mathcal{D}(\Omega^\pm)^n$. Then the distributional form of the anisotropic Stokes equation in (4.26), corresponding to each of the domains Ω^+ and Ω^-, follows from equation (4.30) written for all $\mathbf{w} \in \mathcal{D}(\Omega^+)^n$ and $\mathbf{w} \in \mathcal{D}(\Omega^-)^n$, respectively. The second variational equation in (4.1) yields the divergence equation $\operatorname{div}\mathbf{u} = g$ in Ω, and hence $\operatorname{div}\mathbf{u}^\pm = g|_{\Omega^\pm}$ in $\Omega^\pm$. Therefore, the pairs $(\mathbf{u}^\pm, \pi^\pm)$ satisfy the anisotropic Stokes system in $\Omega^\pm$. Then by using again the first equation in (4.1) and by applying the Green identity (3.18) to the pairs $(\mathbf{u}^\pm, \pi^\pm)$ in $\Omega^\pm$, we obtain for any $\mathbf{w} \in \mathcal{D}(\Omega)^n \subset \mathring{H}^1(\Omega)^n \subset H^1_{\Gamma^+}(\Omega)^n$ that

$$\left\langle \left(\mathbf{t}_{\Omega^+}(\mathbf{u}^+,\pi^+;\tilde{\mathbf{f}}^+)\right)\big|_{\Sigma} - \left(\mathbf{t}_{\Omega^-}(\mathbf{u}^-,\pi^-;\tilde{\mathbf{f}}^-)\right)\big|_{\Sigma}, (\gamma_{\Omega}\mathbf{w})|_{\Sigma} \right\rangle_{\Sigma} = \left\langle \boldsymbol{\psi}_{\Sigma}, \gamma_{\Sigma}\mathbf{w} \right\rangle_{\Sigma}. \tag{4.31}$$

The dense embedding of the space $\mathcal{D}(\Omega)^n$ in $\mathring{H}^1(\Omega)^n$ shows that formula (4.31) is satisfied also for any $\mathbf{w} \in \mathring{H}^1(\Omega)^n$. Moreover, since the trace operator $\gamma_{\Sigma} : \mathring{H}^1(\Omega)^n \to H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$ defined by (3.12) is surjective (see the proof of Lemma 3.6), formula (4.31) can be written as

$$\left\langle \left(\mathbf{t}_{\Omega^+}(\mathbf{u}^+,\pi^+;\tilde{\mathbf{f}}^+)\right)\big|_{\Sigma} - \left(\mathbf{t}_{\Omega^-}(\mathbf{u}^-,\pi^-;\tilde{\mathbf{f}}^-)\right)\big|_{\Sigma}, \boldsymbol{\varphi} \right\rangle_{\Sigma} = \left\langle \boldsymbol{\psi}_{\Sigma}, \boldsymbol{\varphi} \right\rangle_{\Sigma}, \ \forall\, \boldsymbol{\varphi} \in H^{\frac{1}{2}}_{\bullet}(\Sigma)^n.$$

Therefore, $\left(\mathbf{t}_{\Omega^+}(\mathbf{u}^+,\pi^+;\tilde{\mathbf{f}}^+)\right)_{\Sigma} - \left(\mathbf{t}_{\Omega^-}(\mathbf{u}^-,\pi^-;\tilde{\mathbf{f}}^-)\right)\big|_{\Sigma} = \boldsymbol{\psi}_{\Sigma}$ on Σ. Definition (3.2) and the first equation in (4.1) imply also that $\left(\mathbf{t}_{\Omega}(\mathbf{u},\pi;\widetilde{\mathbf{f}})\right)|_{\Gamma^-} = \boldsymbol{\psi}_{\Gamma^-}$ on Γ^-.

Consequently, the mixed-transmission problem (4.26) is indeed equivalent to the variational problem (4.1). According to Theorem 4.1, there exists a unique solution $(\mathbf{u},\pi) \in H^1_{\Gamma^+}(\Omega)^n \times L^2(\Omega)$ of the variational problem (4.1) with $\mathfrak{F} \in \left(H^1_{\Gamma^+}(\Omega)^n\right)'$ given by (4.29). Hence the equivalence just proved implies the well-posedness of the mixed-transmission problem (4.26) in the space $\mathcal{X}_{\Omega^+,\Omega^-}$, and estimate (4.28) follows from (4.4) and (4.29). □

4.3. Mixed-transmission problem with non-homogeneous Dirichlet and interface trace conditions

The remaining part of this section is devoted to the well-posedness of a fully non-homogeneous mixed-transmission problem for the anisotropic Stokes system. In order to analyze such a problem, we need the following Bogovskii-type result.

Lemma 4.6. *Let Assumption 3.5 hold. Then for all given data* $(g, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi}_{\Gamma^+}) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Sigma)^n \times H^{\frac{1}{2}}(\Gamma^+)^n$, *there exist two functions* $\mathbf{v}^+ \in H^1(\Omega^+)^n$ *and* $\mathbf{v}^- \in H^1(\Omega^-)^n$ *such that*

$$\begin{cases} \operatorname{div} \mathbf{v}^+ = g|_{\Omega^+} & \textit{in } \Omega^+ \\ \operatorname{div} \mathbf{v}^- = g|_{\Omega^-} & \textit{in } \Omega^- \\ (\gamma_{\Omega^+}\mathbf{v}^+)|_{\Sigma} - (\gamma_{\Omega^-}\mathbf{v}^-)|_{\Sigma} = \boldsymbol{\varphi}_{\Sigma} & \textit{on } \Sigma, \\ (\gamma_{\Omega^+}\mathbf{v}^+)|_{\Gamma^+} = \boldsymbol{\varphi}_{\Gamma^+} & \textit{on } \Gamma^+. \end{cases} \tag{4.32}$$

Moreover, there exists a constant $C_{\Sigma} = C_{\Sigma}(\Omega^+,\Omega^-,n) > 0$, *such that*

$$\|\mathbf{v}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{v}^-\|_{H^1(\Omega^-)^n} \le C_{\Sigma}\left(\|g\|_{L^2(\Omega)} + \|\boldsymbol{\varphi}_{\Sigma}\|_{H^{\frac{1}{2}}(\Sigma)^n} + \|\boldsymbol{\varphi}_{\Gamma^+}\|_{H^{\frac{1}{2}}(\Gamma^+)^n}\right).$$

Proof. Let us introduce the functions

$$\mathbf{v}_1^+ := \mathbf{0} \text{ in } \Omega^+, \quad \mathbf{v}_1^- := -\gamma_{\Omega^-}^{-1} E_{\Sigma\to\partial\Omega^-}\boldsymbol{\varphi}_{\Sigma} \text{ in } \Omega^-, \tag{4.33}$$

where $\gamma_{\Omega^-}^{-1} : H^{\frac{1}{2}}(\partial\Omega^-)^n \to H^1(\Omega^-)^n$ is a continuous right inverse of the trace operator $\gamma_{\Omega^-} : H^1(\Omega^-)^n \to H^{\frac{1}{2}}(\partial\Omega^-)^n$, and $E_{\Sigma\to\partial\Omega^-} : H^{\frac{1}{2}}(\Sigma)^n \to H^{\frac{1}{2}}(\partial\Omega^-)^n$ is a continuous extension operator. Then $\mathbf{v}_1^{\pm}$ belong to $H^1(\Omega^{\pm})^n$, respectively, and these functions satisfy the transmission condition in (4.32). Let us now define

$$g_1^+ := 0 \text{ in } \Omega^+, \quad g_1^- := \operatorname{div} \mathbf{v}_1^- \text{ in } \Omega^-, \tag{4.34}$$

and let $G \in L^2(\Omega)$ be such that

$$G|_{\Omega^{\pm}} = g|_{\Omega^{\pm}} - g_1^{\pm}. \tag{4.35}$$

Then by Lemma 4.3 there exists a solution $\mathbf{w} \in H^1(\Omega)^n$ of the boundary problem

$$\begin{cases} \operatorname{div} \mathbf{w} = G & \text{in } \Omega\,, \\ \gamma_\Omega \mathbf{w} = \boldsymbol{\varphi}_{\Gamma^+} & \text{on } \Gamma^+\,, \end{cases} \tag{4.36}$$

and, moreover, there exists a constant $C_{\Gamma^+} = C_{\Gamma^+}(\Omega, \Gamma^+, n) > 0$ such that

$$\|\mathbf{w}\|_{H^1(\Omega)^n} \le C_{\Gamma^+} \left(\|g\|_{L^2(\Omega)} + \|g_1^-\|_{L^2(\Omega^-)} + \|\boldsymbol{\varphi}_{\Gamma^+}\|_{H^{\frac{1}{2}}(\Gamma^+)^n} \right). \tag{4.37}$$

Finally, choosing $\mathbf{v}^\pm \in H^1(\Omega^\pm)^n$ such that $\mathbf{v}^\pm := \mathbf{v}_1^\pm + \mathbf{w}|_{\Omega^\pm}$, and using inequality (4.37) and the continuity of the operators involved in (4.33)-(4.34), we obtain the desired result. □

For a better presentation of the next result, let us recall that $\boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-}$ and $\boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-}$ are the spaces defined in (4.24) and respectively (4.25). Thus,

$$\boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-} := \left(H^1_{\Gamma^+}(\Omega^+)^n\right)' \times \widetilde{H}^{-1}(\Omega^-)^n \times L^2(\Omega) \times H^{\frac{1}{2}}(\Sigma)^n \times H^{-\frac{1}{2}}(\Sigma)^n \times H^{\frac{1}{2}}(\Gamma^+)^n \times H^{-\frac{1}{2}}(\Gamma^-)^n\,.$$

Then we have the following well-posedness result.

Theorem 4.7. *Let Assumption 3.5 and conditions (1.2)-(1.4) be satisfied. Then the following properties hold.*

(*i*) *For all given data $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}) \in \boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-}$, the mixed-transmission problem*

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u}^+, \pi^+) = \widetilde{\mathbf{f}}^+|_{\Omega^+},\ \operatorname{div} \mathbf{u}^+ = g|_{\Omega^+} & \textit{in } \Omega^+\,, \\ \boldsymbol{\mathcal{L}}(\mathbf{u}^-, \pi^-) = \widetilde{\mathbf{f}}^-|_{\Omega^-},\ \operatorname{div} \mathbf{u}^- = g|_{\Omega^-} & \textit{in } \Omega^-\,, \\ (\gamma_{\Omega^+} \mathbf{u}^+)|_\Sigma - (\gamma_{\Omega^-} \mathbf{u}^-)|_\Sigma = \boldsymbol{\varphi}_\Sigma & \textit{on } \Sigma\,, \\ \left(\mathbf{t}_{\Omega^+}(\mathbf{u}^+, \pi^+; \widetilde{\mathbf{f}}^+)\right)|_\Sigma + \left(\mathbf{t}_{\Omega^-}(\mathbf{u}^-, \pi^-; \widetilde{\mathbf{f}}^-)\right)|_\Sigma = \boldsymbol{\psi}_\Sigma & \textit{on } \Sigma\,, \\ (\gamma_{\Omega^+} \mathbf{u}^+)|_{\Gamma^+} = \boldsymbol{\varphi}_{\Gamma^+} & \textit{on } \Gamma^+\,, \\ \left(\mathbf{t}_{\Omega^-}(\mathbf{u}^-, \pi^-; \widetilde{\mathbf{f}}^-)\right)|_{\Gamma^-} = \boldsymbol{\psi}_{\Gamma^-} & \textit{on } \Gamma^- \end{cases} \tag{4.38}$$

has a unique solution $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-}$, and there exists a constant $C = C(\Sigma, \Gamma^+, \Gamma^-, C_{\mathbb{A}}, \|\mathbb{A}\|, n) > 0$, such that

$$\left\| \left(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-\right) \right\|_{\boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-}} \le C \left\| \left(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}\right) \right\|_{\boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-}}.$$

(*ii*) *The solution of the mixed-transmission problem (4.38) can be represented as*

$$(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) = \boldsymbol{\mathcal{T}} \left(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}\right),$$

where $\boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-} \to \boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-}$ is a linear and continuous operator.

Proof. (*i*) For $(g, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi}_{\Gamma^+}) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Sigma)^n \times H^{\frac{1}{2}}(\partial\Omega)^n$ given, there exist the functions $\mathbf{v}^\pm \in H^1(\Omega^\pm)^n$ satisfying the boundary value problem (4.32). For the velocity-pressure couples $(\mathbf{v}^\pm, 0)$, let

$$\check{f}_i^\pm := \partial_\alpha \mathring{E}_{\Omega^\pm}\left(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{v}^\pm)\right), \quad i = 1, \dots, n, \tag{4.39}$$

where $\mathring{E}_{\Omega^\pm}$ is the operator of zero extension from $\Omega^\pm$ to $\mathbb{R}^n$. Therefore, we have that $\check{\mathbf{f}}^+ \in \widetilde{H}^{-1}(\Omega^+)^n \hookrightarrow \left(H^1_{\Gamma^+}(\Omega^+)^n\right)'$ and $\check{\mathbf{f}}^- \in \widetilde{H}^{-1}(\Omega^-)^n$, where $\check{\mathbf{f}}^\pm = (\check{f}_1^\pm, \dots, \check{f}_n^\pm)$. Moreover, $\check{\mathbf{f}}^\pm|_{\Omega^\pm} = \boldsymbol{\mathcal{L}}(\mathbf{v}^\pm, 0)$ in $\Omega^\pm$ (see (1.7)), and $\mathbf{t}_{\Omega^\pm}(\mathbf{v}^\pm, 0; \check{\mathbf{f}}^\pm) = \mathbf{0}$ (due to Definition 3.2).

Then by considering the change of variables $\mathbf{w}^{\pm} := \mathbf{u}^{\pm} - \mathbf{v}^{\pm} \in H^1(\Omega^{\pm})^n$, the fully nonhomogeneous mixed-transmission problem (4.38) reduces to the following mixed-transmission problem with the homogeneous Dirichlet condition on Γ^+ and the homogeneous interface condition for the traces across Σ,

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{w}^+, \pi^+) = (\widetilde{\mathbf{f}}^+ - \check{\mathbf{f}}^+)|_{\Omega^+}, \ \operatorname{div} \mathbf{w}^+ = 0 & \text{in } \Omega^+, \\ \boldsymbol{\mathcal{L}}(\mathbf{w}^-, \pi^-) = (\widetilde{\mathbf{f}}^- - \check{\mathbf{f}}^-)|_{\Omega^-}, \ \operatorname{div} \mathbf{w}^- = 0 & \text{in } \Omega^-, \\ (\gamma_{\Omega^+} \mathbf{w}^+)|_{\Sigma} - (\gamma_{\Omega^-} \mathbf{w}^-)|_{\Sigma} = \mathbf{0} & \text{on } \Sigma, \\ \big(\mathbf{t}_{\Omega^+}(\mathbf{w}^+, \pi^+; \widetilde{\mathbf{f}}^+ - \check{\mathbf{f}}^+)\big)|_{\Sigma} + \big(\mathbf{t}_{\Omega^-}(\mathbf{w}^-, \pi^-; \widetilde{\mathbf{f}}^- - \check{\mathbf{f}}^-)\big)|_{\Sigma} = \boldsymbol{\psi}_{\Sigma} & \text{on } \Sigma, \\ (\gamma_{\Omega^+} \mathbf{w}^+)|_{\Gamma^+} = \mathbf{0} & \text{on } \Gamma^+, \\ \big(\mathbf{t}_{\Omega^-}(\mathbf{w}^-, \pi^-; \widetilde{\mathbf{f}}^- - \check{\mathbf{f}}^-)\big)|_{\Gamma^-} = \boldsymbol{\psi}_{\Gamma^-} & \text{on } \Gamma^-, \end{cases} \tag{4.40}$$

where $\widetilde{\mathbf{f}}^+ - \check{\mathbf{f}}^+ \in \big(H^1_{\Gamma^+}(\Omega^+)^n\big)'$ and $\widetilde{\mathbf{f}}^- - \check{\mathbf{f}}^- \in \widetilde{H}^{-1}(\Omega^-)^n$. In view of Theorem 4.5, the mixed-transmission problem (4.40) has a unique solution $(\mathbf{w}^+, \pi^+, \mathbf{w}^-, \pi^-)$ in the space $\boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-}$.

Then $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) = (\mathbf{v}^+ + \mathbf{w}^+, \pi^+, \mathbf{v}^- + \mathbf{w}^-, \pi^-) \in \boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-}$ is a solution of the mixed-transmission problem (4.38) and satisfies the asserted estimate. This solution is unique due to the uniqueness statement of Theorem 4.5. Moreover, the solution can be represented as in item (ii), and by estimate of item (i) and the linearity of the mixed-transmission problem (4.38), the solution operator $\boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-} \to \boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-}$ is continuous and linear, as asserted. □

5. Mixed and mixed-transmission problems for the anisotropic compressible Navier-Stokes system in bounded Lipschitz domains

In the first part of this section we describe the existence and uniqueness result of a weak solution of a fully non-homogeneous mixed Dirichlet-Neumann problem for the anisotropic Navier-Stokes system in a compressible, case with small data in L^2-based Sobolev spaces in a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$. The second part is concerned with a well-posedness result of a weak solution for a nonlinear mixed-transmission problem for the Navier-Stokes system in a bounded Lipschitz domain with a transversal interface satisfying Assumption 3.5.

5.1. Mixed problem for the anisotropic compressible Navier-Stokes system with small data in L^2-based Sobolev spaces on a bounded Lipschitz domain

Let us consider the nonlinear mixed Dirichlet-Neumann problem for the anisotropic compressible Navier-Stokes system

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u}, \pi) = \widetilde{\mathbf{f}}|_{\Omega} + (\mathbf{u} \cdot \nabla)\mathbf{u}, \ \operatorname{div} \mathbf{u} = g & \text{in } \Omega, \\ (\gamma_{\Omega} \mathbf{u})|_{\Gamma^+} = \boldsymbol{\varphi}_{\Gamma^+} & \text{on } \Gamma^+, \\ \big(\mathbf{t}_{\Omega}(\mathbf{u}, \pi; \widetilde{\mathbf{f}} + \mathring{E}_{\Omega \to \mathbb{R}^n}((\mathbf{u} \cdot \nabla)\mathbf{u}))\big)|_{\Gamma^-} = \boldsymbol{\psi}_{\Gamma^-} & \text{on } \Gamma^-, \end{cases} \tag{5.1}$$

with the couple of unknowns $(\mathbf{u}, \pi) \in H^1(\Omega)^n \times L^2(\Omega)$, $n = 2, 3$ (see also [31, Theorem 9.1] for the mixed problem for the incompressible isotropic Navier-Stokes system in Lipschitz domains on compact Riemannian manifolds, with L^∞ coefficients and homogeneous Dirichlet condition).

Theorem 5.1. *Let $n = 2, 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain satisfying Assumption 3.1. Then there exist two constants $\lambda, \gamma > 0$ depending on Ω, Γ^+, Γ^-, $\|A\|$, and the ellipticity constant $C_{\mathbb{A}}$, such that for all given data $(\widetilde{\mathbf{f}}, g, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}) \in \big(H^1_{\Gamma^+}(\Omega)^n\big)' \times L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma^+)^n \times H^{-\frac{1}{2}}(\Gamma^-)^n$ satisfying the condition*

$$\|\widetilde{\mathbf{f}}\|_{(H^1_{\Gamma^+}(\Omega)^n)'} + \|g\|_{L^2(\Omega)} + \|\boldsymbol{\varphi}_{\Gamma^+}\|_{H^{\frac{1}{2}}(\Gamma^+)^n} + \|\boldsymbol{\psi}_{\Gamma^-}\|_{H^{-\frac{1}{2}}(\Gamma^-)^n} \le \lambda,$$

the mixed problem (5.1) *for the anisotropic Navier-Stokes system has a unique solution* $(\mathbf{u},\pi)\in H^1(\Omega)^n\times L^2(\Omega)$, *such that* $\|\mathbf{u}\|_{H^1(\Omega)^n}\le\gamma$. *The solution depends continuously on the given data* $(\widetilde{\mathbf{f}},g,\boldsymbol{\varphi}_{\Gamma^+},\boldsymbol{\psi}_{\Gamma^-})$.

Proof. We use arguments similar to those for [31, Theorem 9.1] for a mixed problem for the incompressible isotropic Navier-Stokes system in a Lipschitz domain on a compact Riemannian manifold. Let $\mathring{E}_\Omega$ be the operator of extension by zero from Ω to $\mathbb{R}^n$. Let also

$$\boldsymbol{\mathcal{N}}_\Omega(\mathbf{v}):=\mathring{E}_\Omega\big((\mathbf{v}\cdot\nabla)\mathbf{v}\big),\quad \forall\,\mathbf{v}^{\pm}\in H^1(\Omega)^n\,. \tag{5.2}$$

According to estimate (C.5) we have that $\boldsymbol{\mathcal{N}}_\Omega(\mathbf{v})\in\widetilde{H}^{-1}(\Omega)^n\hookrightarrow\big(H^1_{\Gamma^+}(\Omega)^n\big)'$, and there exists a constant $C_\Omega>0$ depending only on Ω such that for all $\mathbf{v},\mathbf{w}\in H^1(\Omega)^n$, we have the estimates

$$\|\boldsymbol{\mathcal{N}}_\Omega(\mathbf{v})\|_{(H^1_{\Gamma^+}(\Omega)^n)'}\le C_\Omega\|\mathbf{v}\|_{H^1(\Omega)^n}\|\nabla\mathbf{v}\|_{L^2(\Omega)^{n\times n}}\le C_\Omega\|\mathbf{v}\|^2_{H^1(\Omega)^n}\,, \tag{5.3}$$

$$\|\boldsymbol{\mathcal{N}}_\Omega(\mathbf{v})-\boldsymbol{\mathcal{N}}_\Omega(\mathbf{w})\|_{(H^1_{\Gamma^+}(\Omega)^n)'}\le C_\Omega\left(\|\mathbf{v}\|_{H^1(\Omega)^n}+\|\mathbf{w}\|_{H^1(\Omega)^n}\right)\|\mathbf{v}-\mathbf{w}\|_{H^1(\Omega)^n}\,. \tag{5.4}$$

Therefore, the nonlinear operator $\boldsymbol{\mathcal{N}}_\Omega:H^1(\Omega)^n\to\big(H^1_{\Gamma^+}(\Omega)^n\big)'$ is bounded and continuous.

Next, for the given data $(\widetilde{\mathbf{f}},g,\boldsymbol{\varphi}_{\Gamma^+},\boldsymbol{\psi}_{\Gamma^-})\in\big(H^1_{\Gamma^+}(\Omega)^n\big)'\times L^2(\Omega)\times H^{\frac12}(\Gamma^+)^n\times H^{-\frac12}(\Gamma^-)^n$ and for a fixed $\mathbf{u}$ in the space $H^1(\Omega)^n$, such that $\operatorname{div}\mathbf{u}=g$ in Ω, we consider the following linear mixed problem for the Stokes system

$$\begin{cases}\boldsymbol{\mathcal{L}}(\mathbf{v},q)=\widetilde{\mathbf{f}}|_\Omega+(\boldsymbol{\mathcal{N}}_\Omega(\mathbf{u}))\,|_\Omega\,, \quad \operatorname{div}\mathbf{v}=g|_\Omega & \text{in } \Omega\,,\\ (\gamma_\Omega\mathbf{v})|_{\Gamma^+}=\boldsymbol{\varphi}_{\Gamma^+} & \text{on } \Gamma^+\,,\\ \big(\mathbf{t}_\Omega(\mathbf{v},\pi;\widetilde{\mathbf{f}}+\boldsymbol{\mathcal{N}}_\Omega(\mathbf{u}))\big)|_{\Gamma^-}=\boldsymbol{\psi}_{\Gamma^-} & \text{on } \Gamma^-\,,\end{cases} \tag{5.5}$$

with the couple of unknowns $(\mathbf{v},q)\in H^1(\Omega)^n\times L^2(\Omega)$.

Since $\big(\widetilde{\mathbf{f}}+\boldsymbol{\mathcal{N}}_\Omega(\mathbf{u})\big)\in\big(H^1_{\Gamma^+}(\Omega)^n\big)'$, Theorem 4.4 shows that the linear problem (5.5) has a unique solution $(\mathbf{v},q)\in\boldsymbol{\mathcal{X}}_\Omega$ that can be expressed in terms of the corresponding (bounded linear) solution operator $\boldsymbol{\mathcal{T}}_\Omega:\boldsymbol{\mathcal{Y}}_\Omega\to\boldsymbol{\mathcal{X}}_\Omega$, as follows

$$(\mathbf{v},q):=(\mathbf{U}(\mathbf{u}),P(\mathbf{u}))=\boldsymbol{\mathcal{T}}_\Omega\left(\tilde{\mathbf{f}}+\boldsymbol{\mathcal{N}}_\Omega(\mathbf{u})\,,g,\boldsymbol{\varphi}_{\Gamma^+},\boldsymbol{\psi}_{\Gamma^-}\right), \tag{5.6}$$

where $\boldsymbol{\mathcal{X}}_\Omega$ and $\boldsymbol{\mathcal{Y}}_\Omega$ are the spaces defined in (4.18) and (4.19), respectively.

Then the linearity and boundedness of the operator $\boldsymbol{\mathcal{T}}_\Omega$ and estimate (5.3) imply that there exists a constant $c>0$ depending on Ω, Γ^+, Γ^-, $\|A\|$, and the ellipticity constant $C_{\mathbb{A}}$, such that for every $\mathbf{w}\in H^1(\Omega)^n$ we have the estimate

$$\big\|\big(\mathbf{U}(\mathbf{w}),P(\mathbf{w})\big)\big\|_{\boldsymbol{\mathcal{X}}_\Omega}\le c\big\|\big(\widetilde{\mathbf{f}},g,\boldsymbol{\varphi}_{\Gamma^+},\boldsymbol{\psi}_{\Gamma^-}\big)\big\|_{\boldsymbol{\mathcal{Y}}_\Omega}+cC_\Omega\|\mathbf{w}\|^2_{H^1(\Omega)^n}\,. \tag{5.7}$$

Next we show that the nonlinear operator $\mathbf{U}:H^1(\Omega)^n\to H^1(\Omega)^n$ is invariant over a closed ball of the space $H^1(\Omega)^n$. To this end, let us consider the constants

$$\lambda:=\frac{3\gamma}{4c}\,,\quad \gamma:=\frac{1}{4C_\Omega c}\,, \tag{5.8}$$

where C_Ω and c are the constants from inequalities (5.3), (5.4), (5.7). Let also

$$\mathbf{B}_\gamma:=\left\{\mathbf{v}\in H^1(\Omega)^n:\operatorname{div}\mathbf{v}=g|_\Omega \text{ in } \Omega\,,\ \ \|\mathbf{v}\|_{H^1(\Omega)^n}\le\gamma\right\}. \tag{5.9}$$

Then the assumption $\|(\widetilde{\mathbf{f}}, g, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-})\|_{\boldsymbol{\mathcal{Y}}_\Omega} \le \lambda$, and inequality (5.7) imply that the operator $\mathbf{U}$ maps the closed ball $\mathbf{B}_\gamma$ into itself, as asserted. In addition, expression (5.6) of $\mathbf{U}$ and inequality (5.4) yield the estimate

$$\|\mathbf{U}(\mathbf{v}) - \mathbf{U}(\mathbf{w})\|_{H^1(\Omega)^n} \le \frac{1}{2}\|\mathbf{v} - \mathbf{w}\|_{H^1(\Omega)^n},\ \forall\, \mathbf{v},\, \mathbf{w} \in \mathbf{B}_\gamma\,,$$

which shows that the mapping $\mathbf{U} : \mathbf{B}_\gamma \to \mathbf{B}_\gamma$ is a contraction. The Banach fixed point theorem then yields the existence and uniqueness of a fixed point $\mathbf{u} \in \mathbf{B}_\gamma$ of $\mathbf{U}$, that is, $\mathbf{U}(\mathbf{u}) = \mathbf{u}$. Moreover, definition (5.6) of the operator $\mathbf{T}_\Omega$ implies that $(\mathbf{u}, P(\mathbf{u}))$ is a solution of the nonlinear mixed problem (5.1) in the space $\boldsymbol{\mathcal{X}}_\Omega$, such that $\|\mathbf{u}\|_{H^1(\Omega)^n} \le \gamma$. This solution is unique due to the uniqueness of the fixed point of the mapping $\mathbf{U}$ on $\mathbf{B}_\gamma$ (see the proof of [25, Theorem 4.2] for further details), and depends continuously on the given data $(\widetilde{\mathbf{f}}, g, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}) \in \boldsymbol{\mathcal{Y}}_\Omega$ by the continuity of the solution operator $\boldsymbol{\mathcal{T}}_\Omega$. □

5.2. Mixed-transmission problem for the anisotropic compressible Navier-Stokes system with small data in L^2-based Sobolev spaces on a bounded Lipschitz domain with a transversal Lipschitz interface

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded Lipschitz domain satisfying Assumption 3.5.

Let us recall the definition of our main spaces

$$\begin{aligned}
&\boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-} := H^1(\Omega^+)^n \times L^2(\Omega^+) \times H^1(\Omega^-)^n \times L^2(\Omega^-)\,,\\
&\boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-} := \big(H^1_{\Gamma^+}(\Omega^+)^n\big)' \times \widetilde{H}^{-1}(\Omega^-)^n \times L^2(\Omega) \times H^{\frac{1}{2}}(\Sigma)^n \times H^{-\frac{1}{2}}(\Sigma)^n \times H^{\frac{1}{2}}(\Gamma^+)^n \times H^{-\frac{1}{2}}(\Gamma^-)^n\,,
\end{aligned}$$

and consider the following non-homogeneous *Poisson problem of mixed-transmission type for the anisotropic Navier-Stokes system* in a compressible framework

$$\begin{cases}
\boldsymbol{\mathcal{L}}(\mathbf{u}^+, \pi^+) = \widetilde{\mathbf{f}}^+|_{\Omega^+} + (\mathbf{u}^+ \cdot \nabla)\mathbf{u}^+\,, \quad \operatorname{div} \mathbf{u}^+ = g|_{\Omega^+} & \text{in } \Omega^+,\\
\boldsymbol{\mathcal{L}}(\mathbf{u}^-, \pi^-) = \widetilde{\mathbf{f}}^-|_{\Omega^-} + (\mathbf{u}^- \cdot \nabla)\mathbf{u}^-\,, \quad \operatorname{div} \mathbf{u}^- = g|_{\Omega^-} & \text{in } \Omega^-,\\
(\gamma_{\Omega^+}\mathbf{u}^+)|_\Sigma - (\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma = \boldsymbol{\varphi}_\Sigma & \text{on } \Sigma\,,\\
\big(\mathbf{t}_{\Omega^+}\big(\mathbf{u}^+, \pi^+; \widetilde{\mathbf{f}}^+ + \mathring{E}_{\Omega^+\to\Omega^-}(\mathbf{u}^+ \cdot \nabla)\mathbf{u}^+\big)\big)\big|_\Sigma & \\
\qquad + \big(\mathbf{t}_{\Omega^-}\big(\mathbf{u}^-, \pi^-; \widetilde{\mathbf{f}}^- + \mathring{E}_{\Omega^-\to\Omega^+}(\mathbf{u}^- \cdot \nabla)\mathbf{u}^-\big)\big)\big|_\Sigma = \boldsymbol{\psi}_\Sigma & \text{on } \Sigma\,,\\
(\gamma_{\Omega^+}\mathbf{u}^+)|_{\Gamma^+} = \boldsymbol{\varphi}_{\Gamma^+} & \text{on } \Gamma^+\,,\\
\big(\mathbf{t}_{\Omega^-}(\mathbf{u}^-, \pi^-; \widetilde{\mathbf{f}}^-)\big)|_{\Gamma^-} = \boldsymbol{\psi}_{\Gamma^-} & \text{on } \Gamma^-\,,
\end{cases} \tag{5.10}$$

with the given data $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-})$ in the space $\boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-}$ and the unknown $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-)$ in $\boldsymbol{\mathcal{X}}_{\Omega^+,\Omega^-}$.

By combining Theorem 4.7 with the Banach fixed point theorem we prove the following well-posedness result for the nonlinear mixed-transmission problem (5.10) (see also [25, Theorem 4.2] for a transmission problem for the Navier-Stokes system in the Euclidean pseudostress setting). Recall that $C_{\mathbb{A}}$ is the constant in (1.4).

Theorem 5.2. *Let $n = 2, 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain satisfying Assumption 3.5. Let conditions (1.2)-(1.4) hold. Then there exist two constants $\alpha, \beta > 0$, depending on Ω^+, Ω^-, $\|A\|$, and $C_{\mathbb{A}}$, such that for all given data $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}) \in \boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-}$, with*

$$\Big\|(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-})\Big\|_{\boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-}} \le \alpha\,,$$

the mixed-transmission problem for the Navier-Stokes system (5.10) *has a unique solution* $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathcal{X}_{\Omega^+,\Omega^-}$, *such that* $\|\mathbf{u}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{u}^-\|_{H^1(\Omega^-)^n} \le \beta$. *Moreover, this solution depends continuously on* $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\psi}_{\Sigma}, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-})$.

Proof. We use arguments similar to those for Theorem 4.2 in [25] devoted to a transmission problem for the anisotropic Stokes and Navier-Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$, in a pseudostress approach. Recall that $\mathring{E}_{\Omega^\pm\to\Omega}$ are the operators of extensions by zero from $\Omega^\pm$ to Ω. Let

$$\mathcal{N}_{\Omega^\pm}(\mathbf{v}^\pm) := \mathring{E}_{\Omega^\pm\to\Omega}\big((\mathbf{v}^\pm \cdot \nabla)\mathbf{v}^\pm\big),\ \forall\, \mathbf{v}^\pm \in H^1(\Omega^\pm)^n\,. \tag{5.11}$$

Estimate (C.5) shows that $\mathcal{N}_{\Omega^+}(\mathbf{v}^+) \in \widetilde{H}^{-1}(\Omega^+)^n \hookrightarrow \big(H^1_{\Gamma^+}(\Omega^+)^n\big)'$ and $\mathcal{N}_{\Omega^-}(\mathbf{v}^-) \in \widetilde{H}^{-1}(\Omega^-)^n$. Moreover, there exists a constant $C_1 > 0$ depending only on Ω^+ and Ω^- such that for all $\mathbf{v}^\pm, \mathbf{w}^\pm \in H^1(\Omega^\pm)^n$, we have the estimates

$$\|\mathcal{N}_{\Omega^\pm}(\mathbf{v}^\pm)\|_{\widetilde{H}^{-1}(\Omega^\pm)^n} \le C_1 \|\mathbf{v}^\pm\|_{H^1(\Omega^\pm)^n} \|\nabla \mathbf{v}^\pm\|_{L^2(\Omega^\pm)^{n\times n}} \le C_1 \|\mathbf{v}^\pm\|^2_{H^1(\Omega^\pm)^n}\,, \tag{5.12}$$

$$\|\mathcal{N}_{\Omega^\pm}(\mathbf{v}^\pm) - \mathcal{N}_{\Omega^\pm}(\mathbf{v}^\pm)\|_{\widetilde{H}^{-1}(\Omega^\pm)^n} \le C_1 \left(\|\mathbf{v}^\pm\|_{H^1(\Omega^\pm)^n} + \|\mathbf{w}^\pm\|_{H^1(\Omega^\pm)^n}\right) \|\mathbf{v}^\pm - \mathbf{w}^\pm\|_{H^1(\Omega^\pm)^n}\,, \tag{5.13}$$

which show that the nonlinear operators $\mathcal{N}_{\Omega^+} : H^1_{\Gamma^+}(\Omega^+)^n \to \widetilde{H}^{-1}(\Omega^+)^n$ and $\mathcal{N}_{\Omega^-} : H^1(\Omega^-)^n \to \widetilde{H}^{-1}(\Omega^-)^n$ are bounded and continuous. Moreover, the continuity of the embeddings $H^1_{\Gamma^+}(\Omega^+)^n \hookrightarrow H^1(\Omega^+)^n$ and $\widetilde{H}^{-1}(\Omega^+)^n \hookrightarrow \big(H^1_{\Gamma^+}(\Omega^+)^n\big)'$ and the boundedness and continuity of the operator $\mathcal{N}_{\Omega^+} : H^1(\Omega^+)^n \to \widetilde{H}^{-1}(\Omega^+)^n$ imply also the boundedness and continuity of the operator $\mathcal{N}_{\Omega^+} : H^1_{\Gamma^+}(\Omega^+)^n \to \big(H^1_{\Gamma^+}(\Omega^+)^n\big)'$. We then have the estimates

$$\|\mathcal{N}_{\Omega^+}(\mathbf{v}^+)\|_{(H^1_{\Gamma^+}(\Omega^+)^n)'} \le C_1 \|\mathbf{v}^+\|^2_{H^1(\Omega^+)^n}\,, \tag{5.14}$$

$$\|\mathcal{N}_{\Omega^+}(\mathbf{v}^+) - \mathcal{N}_{\Omega^+}(\mathbf{w}^+)\|_{(H^1_{\Gamma^+}(\Omega^+)^n)'} \le C_1 \left(\|\mathbf{v}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{w}^+\|_{H^1(\Omega^+)^n}\right) \|\mathbf{v}^+ - \mathbf{w}^+\|_{H^1(\Omega^+)^n}\,, \tag{5.15}$$

where, for the sake of brevity, we have kept the same constant C_1 in all inequalities (5.12) up to (5.15).

Now, for $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\psi}_{\Sigma}, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}) \in \mathcal{Y}_{\Omega^+,\Omega^-}$ given and for a fixed pair $(\mathbf{u}^+, \mathbf{u}^-)$ in $H^1(\Omega^+)^n \times H^1(\Omega^-)^n$, such that $\operatorname{div} \mathbf{u}^\pm = g|_{\Omega^\pm}$ in $\Omega^\pm$, we consider the following linear mixed-transmission problem for the Stokes system

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{v}^+, q^+) = \widetilde{\mathbf{f}}^+|_{\Omega^+} + (\mathcal{N}_{\Omega^+}(\mathbf{u}^+))\,|_{\Omega^+}\,, \ \ \operatorname{div} \mathbf{v}^+ = g|_{\Omega^+} & \text{in } \Omega^+,\\ \boldsymbol{\mathcal{L}}(\mathbf{u}^-, \pi^-) = \widetilde{\mathbf{f}}^-|_{\Omega^-} + (\mathcal{N}_{\Omega^-}(\mathbf{u}^-))\,|_{\Omega^-}\,, \ \operatorname{div} \mathbf{v}^- = g|_{\Omega^-} & \text{in } \Omega^-,\\ (\gamma_{\Omega^+}\mathbf{v}^+)|_{\Sigma} - (\gamma_{\Omega^-}\mathbf{v}^-)|_{\Sigma} = \boldsymbol{\varphi}_{\Sigma} & \text{on } \Sigma\,,\\ \big(\mathbf{t}_{\Omega^+}\big(\mathbf{v}^+, q^+; \widetilde{\mathbf{f}}^+ + \mathcal{N}_{\Omega^+}(\mathbf{u}^+)\big)\big)\big|_{\Sigma} & \\ \qquad\qquad + \big(\mathbf{t}_{\Omega^-}\big(\mathbf{v}^-, q^-; \widetilde{\mathbf{f}}^- + \mathcal{N}_{\Omega^-}(\mathbf{u}^-)\big)\big|_{\Sigma} = \boldsymbol{\psi}_{\Sigma} & \text{on } \Sigma\,,\\ (\gamma_{\Omega^+}\mathbf{v}^+)|_{\Gamma^+} = \boldsymbol{\varphi}_{\Gamma^+} & \text{on } \Gamma^+\,,\\ \big(\mathbf{t}_{\Omega^-}(\mathbf{v}^-, \pi^-; \widetilde{\mathbf{f}}^- + \mathcal{N}_{\Omega^-}(\mathbf{u}_-))\big)|_{\Gamma^-} = \boldsymbol{\psi}_{\Gamma^-} & \text{on } \Gamma^-\,, \end{cases} \tag{5.16}$$

with the unknown $(\mathbf{v}^+, q^+, \mathbf{v}^-, q^-) \in \mathcal{X}_{\Omega^+,\Omega^-}$.

In view of the relations $\big(\tilde{\mathbf{f}}^+ + \mathcal{N}_{\Omega^+}(\mathbf{u}^+)\big) \in \widetilde{H}^{-1}(\Omega^+)^n$, $\big(\tilde{\mathbf{f}}^- + \mathcal{N}_{\Omega^-}(\mathbf{u}^-)\big) \in \widetilde{H}^{-1}(\Omega^-)^n$, and $H^{-1}(\Omega^+)^n \hookrightarrow \big(H^1_{\Gamma^+}(\Omega^+)^n\big)'$, Theorem 4.7 (ii) implies that problem (5.16) has a unique solution that can be expressed in terms of the corresponding (bounded linear) solution operator $\boldsymbol{\mathcal{T}} : \mathcal{Y}_{\Omega^+,\Omega^-} \to \mathcal{X}_{\Omega^+,\Omega^-}$, as follows

$$\begin{aligned}\big(\mathbf{v}^+, q^+, \mathbf{v}^-, \mathbf{v}^-, q_-\big) &:= \big(\mathbf{U}^+(\mathbf{u}^+, \mathbf{u}^-), P^+(\mathbf{u}^+, \mathbf{u}^-), \mathbf{U}^-(\mathbf{u}^+, \mathbf{u}^-), P^-(\mathbf{u}^+, \mathbf{u}^-)\big)\\ &= \boldsymbol{\mathcal{T}}\left(\tilde{\mathbf{f}}^+ + \mathcal{N}_{\Omega^+}(\mathbf{u}^+), \tilde{\mathbf{f}}^- + \mathcal{N}_{\Omega^-}(\mathbf{u}^-), \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\psi}_{\Sigma}, g, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}\right).\end{aligned} \tag{5.17}$$

Then the linearity and boundedness of the operator $\boldsymbol{\mathcal{T}}$ and estimates (5.12) and (5.14) imply that there exists a constant $C_2 = C_2(\Omega^+, \Omega^-, C_{\mathbb{A}}, \|A\|) > 0$ such that for all $(\mathbf{w}^+, \mathbf{w}^-) \in H^1(\Omega^+)^n \times H^1(\Omega^-)^n$ we have

$$\begin{aligned}
&\big\|\big(\mathbf{U}^+(\mathbf{w}^+,\mathbf{w}^-), P^+(\mathbf{w}^+,\mathbf{w}^-), \mathbf{U}^-(\mathbf{w}^+,\mathbf{w}^-), P^-(\mathbf{w}^+,\mathbf{w}^-)\big)\big\|_{\mathcal{X}_{\Omega^+,\Omega^-}} \\
&\qquad\le C_2\big\|\big(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\psi}_{\Sigma}, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}\big)\big\|_{\mathcal{Y}_{\Omega^+,\Omega^-}} + C_1C_2\left(\|\mathbf{w}^+\|^2_{H^1(\Omega^+)^n} + \|\mathbf{w}^-\|^2_{H^1(\Omega^-)^n}\right) \\
&\qquad\le C_2\big\|\big(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\psi}_{\Sigma}, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}\big)\big\|_{\mathcal{Y}_{\Omega^+,\Omega^-}} + C_1C_2\|(\mathbf{w}^+,\mathbf{w}^-)\|^2_{H^1(\Omega^+)^n\times H^1(\Omega^-)^n}\,.
\end{aligned} \tag{5.18}$$

The next step of our arguments is to show that the nonlinear operator $(\mathbf{U}^+, \mathbf{U}^-) : H^1(\Omega^+)^n \times H^1(\Omega^-)^n \to H^1(\Omega^+)^n \times H^1(\Omega^-)^n$ is invariant over a closed ball of the space $H^1(\Omega^+)^n \times H^1(\Omega^-)^n$. In order to prove this property, let

$$\alpha := \frac{3\beta}{4C_2}\,,\quad \beta := \frac{1}{4C_1C_2}\,, \tag{5.19}$$

where C_1 and C_2 are the constants from inequalities (5.12), (5.13), (5.14), (5.15), and (5.18). Let also

$$\mathbf{B}_\beta := \big\{(\mathbf{v}^+,\mathbf{v}^-) \in H^1(\Omega^+)^n \times H^1(\Omega^-)^n : \operatorname{div}\mathbf{v}^\pm = g|_{\Omega^\pm} \text{ in } \Omega^\pm,\ \|\mathbf{v}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{v}^-\|_{H^1(\Omega^-)^n} \le \beta\big\}\,. \tag{5.20}$$

Then by assuming that

$$\big\|\big(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\psi}_{\Sigma}, \boldsymbol{\varphi}_{\Gamma^+}, \boldsymbol{\psi}_{\Gamma^-}\big)\big\|_{\mathcal{Y}_{\Omega^+,\Omega^-}} \le \alpha\,, \tag{5.21}$$

and by using inequalities (5.18) and (5.21), we obtain that the operator $(\mathbf{U}^+, \mathbf{U}^-)$ maps the closed ball $\mathbf{B}_\beta$ into itself, as asserted.

In addition, by using expression (5.17) of the operator $(\mathbf{U}^+, P^+, \mathbf{U}^-, P^-)$, the linearity of the operator $\boldsymbol{\mathcal{T}}$, and inequalities (5.13) and (5.15), we obtain the following estimate

$$\begin{aligned}
&\|(\mathbf{U}^+, P^+, \mathbf{U}^-, P^-)(\mathbf{v}^+,\mathbf{v}^-) - (\mathbf{U}^+, P^+, \mathbf{U}^-, P^-)(\mathbf{w}^+,\mathbf{w}^-)\|_{\mathcal{X}_{\Omega^+,\Omega^-}} \\
&\quad\le C_2\Big(\|\mathcal{N}_{\Omega^+}(\mathbf{v}^+) - \mathcal{N}_{\Omega^+}(\mathbf{w}^+)\|_{H^1(\Omega^+)^n} + \|\mathcal{N}_{\Omega^-}(\mathbf{v}^-) - \mathcal{N}_{\Omega^-}(\mathbf{w}^-)\|_{H^1(\Omega^-)^n}\Big) \\
&\quad\le C_1C_2\Big(\big(\|\mathbf{v}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{w}^+\|_{H^1(\Omega^+)^n}\big)\|\mathbf{v}^+ - \mathbf{w}^+\|_{H^1(\Omega^+)^n} \\
&\qquad + \big(\|\mathbf{v}^-\|_{H^1(\Omega^+)^n} + \|\mathbf{w}^-\|_{H^1(\Omega^-)^n}\big)\|\mathbf{v}^- - \mathbf{w}^-\|_{H^1(\Omega^-)^n}\Big) \\
&\quad\le 2C_1C_2\beta\left(\|\mathbf{v}^+ - \mathbf{w}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{v}^- - \mathbf{w}^-\|_{H^1(\Omega^-)^n}\right) \\
&\quad= \frac{1}{2}\|(\mathbf{v}^+,\mathbf{v}^-) - (\mathbf{w}^+,\mathbf{w}^-)\|_{H^1(\Omega^+)^n\times H^1(\Omega^-)^n},\ \forall\,(\mathbf{v}^+,\mathbf{v}^-),\ (\mathbf{w}^+,\mathbf{w}^-) \in \mathbf{B}_\beta\,.
\end{aligned}$$

In particular, we deduce the estimate

$$\begin{aligned}
&\|(\mathbf{U}^+,\mathbf{U}^-)(\mathbf{v}^+,\mathbf{v}^-) - (\mathbf{U}^+,\mathbf{U}^-)(\mathbf{w}^+,\mathbf{w}^-)\|_{H^1(\Omega^+)^n\times H^1(\Omega^-)^n} \\
&\qquad\le \frac{1}{2}\|(\mathbf{v}^+,\mathbf{v}^-) - (\mathbf{w}^+,\mathbf{w}^-)\|_{H^1(\Omega^+)^n\times H^1(\Omega^-)^n},\ \forall\,(\mathbf{v}^+,\mathbf{v}^-),\ (\mathbf{w}^+,\mathbf{w}^-) \in \mathbf{B}_\beta\,,
\end{aligned}$$

which shows that the map $(\mathbf{U}^+,\mathbf{U}^-) : \mathbf{B}_\beta \to \mathbf{B}_\beta$ is a contraction. Then the Banach fixed point theorem yields that $(\mathbf{U}^+,\mathbf{U}^-)$ has a unique fixed point $(\mathbf{u}^+,\mathbf{u}^-) \in \mathbf{B}_\beta$, that is, $(\mathbf{U}^+(\mathbf{u}^+,\mathbf{u}^-), \mathbf{U}^-(\mathbf{u}^+,\mathbf{u}^-)) = (\mathbf{u}^+,\mathbf{u}^-)$.

Moreover, definition (5.17) of the operator $\boldsymbol{\mathcal{T}}$ implies that $(\mathbf{u}^+,\ P^+(\mathbf{u}^+,\mathbf{u}^-),\ \mathbf{u}^-,\ P^-(\mathbf{u}^+,\ \mathbf{u}^-))$ is a solution of the nonlinear mixed-transmission problem (5.10) in the space $\mathcal{X}_{\Omega^+,\Omega^-}$, such that $\|\mathbf{u}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{u}^-\|_{H^1(\Omega^-)^n} \leq \beta$. This solution is unique due to the uniqueness of the fixed point of the map $(\mathbf{U}^+, \mathbf{U}^-)$ on $\mathbf{B}_\beta$ (see the proof of [25, Theorem 4.2] for further details), and depends continuously on the given data $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\varphi}_{_\Sigma}, \boldsymbol{\psi}_{_\Sigma}, \boldsymbol{\varphi}_{_{\Gamma^+}}, \boldsymbol{\psi}_{_{\Gamma^-}}) \in \boldsymbol{\mathcal{Y}}_{\Omega^+,\Omega^-}$ by the continuity of the solution operator $\boldsymbol{\mathcal{T}}$. $\square$

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Appendix A. The generalized conormal derivative for the Stokes system in a bounded Lipschitz domain

Interpreting the Stokes equation in (1.6) in the sense of distributions and using the dense embedding of the space $\mathcal{D}(\Omega)^n$ into $\mathring{H}^1(\Omega)^n$, we obtain the following result.

Lemma A.1. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and that conditions (1.2), (1.3) are satisfied. Let $(\mathbf{u}, \pi) \in H^1(\Omega)^n \times L^2(\Omega)$ and $\mathbf{f} \in H^{-1}(\Omega)^n$ be such that $\boldsymbol{\mathcal{L}}(\mathbf{u}, \pi) = \mathbf{f}$ in Ω. Then the following Green identity holds*

$$\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{w})\right\rangle_\Omega - \langle \pi, \operatorname{div} \mathbf{w}\rangle_\Omega + \langle \mathbf{f}, \mathbf{w}\rangle_\Omega = 0\,,\ \forall\, \mathbf{w} \in \mathring{H}^1(\Omega)^n\,.$$

By following [27, Definition 2.2] and [26, Definition 1], we introduce the concept of the generalized conormal derivative for the anisotropic Stokes system as follows (see also [37, Lemma 4.3], [38, Definition 3.1, Theorem 3.2], [25, Definition 2.4], [42, Theorem 10.4.1]).

Definition A.2. Let conditions (1.2) and (1.3) be satisfied and let

$$\boldsymbol{H}^1(\Omega, \boldsymbol{\mathcal{L}}) := \Big\{(\mathbf{u}, \pi, \tilde{\mathbf{f}}) \in H^1(\Omega)^n \times L^2(\Omega) \times \widetilde{H}^{-1}(\Omega)^n : \boldsymbol{\mathcal{L}}(\mathbf{u}, \pi) = \tilde{\mathbf{f}}|_\Omega \text{ in } \Omega\Big\}.$$

If $(\mathbf{u}, \pi, \tilde{\mathbf{f}}) \in \boldsymbol{H}^1(\Omega, \boldsymbol{\mathcal{L}})$, then the generalized conormal derivative $\mathbf{t}_\Omega(\mathbf{u}, \pi; \tilde{\mathbf{f}}) \in H^{-\frac{1}{2}}(\partial\Omega)^n$ is defined by the formula

$$\left\langle \mathbf{t}_\Omega(\mathbf{u}, \pi; \tilde{\mathbf{f}}), \mathbf{\Phi}\right\rangle_{\partial\Omega} := \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\gamma_{_\Omega}^{-1}\mathbf{\Phi})\right\rangle_\Omega - \left\langle \pi, \operatorname{div}(\gamma_{_\Omega}^{-1}\mathbf{\Phi})\right\rangle_\Omega + \left\langle \tilde{\mathbf{f}}, \gamma_{_\Omega}^{-1}\mathbf{\Phi}\right\rangle_\Omega, \forall\, \mathbf{\Phi} \in H^{\frac{1}{2}}(\partial\Omega)^n\,, \tag{A.1}$$

where $\gamma_{_\Omega}^{-1} : H^{\frac{1}{2}}(\partial\Omega)^n \to H^1(\Omega)^n$ is a bounded right inverse of the trace operator $\gamma_{_\Omega} : H^1(\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$. We use the simplified notation $\mathbf{t}_\Omega(\mathbf{u}, \pi)$ for $\mathbf{t}_\Omega(\mathbf{u}, \pi; \mathbf{0})$.

Appendix B. Extension properties in Sobolev spaces on Lipschitz domains with internal Lipschitz interfaces

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain that satisfies Assumption 3.5. Hence, $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$, where Σ is an $(n-1)$-dimensional Lipschitz interface that intersects transversally $\partial\Omega$, and Ω^+ and Ω^- are disjoint Lipschitz sub-domains of Ω. Moreover, $\partial\Omega^\pm = \overline{\Sigma} \cup \Gamma^\pm$. Let $\gamma_{_{\Omega^\pm}}$ be the trace operator from $H^1(\Omega^\pm)$ to $H^{\frac{1}{2}}(\partial\Omega^\pm)$.

The result of the following lemma has been obtained in [28, Lemma B.1] (see also [26, Lemma C.1]).

Lemma B.1. *The following assertions hold.*

(i) *Let $u^+ \in H^1(\Omega^+)$ and $u^- \in H^1(\Omega^-)$ be such that $\gamma_{\Omega^+} u^+ = \gamma_{\Omega^-} u^-$ on Σ. Then there exists a unique function $u \in H^1(\Omega)$ such that $u|_{\Omega^\pm} = u^\pm$. Moreover, there exists $C = C(n, \Omega^\pm) > 0$ such that $\|u\|_{H^1(\Omega)} \le C\left(\|u^+\|_{H^1(\Omega^+)} + \|u^-\|_{H^1(\Omega^-)}\right)$.*

(ii) *If $u \in H^1(\Omega)$ then $[\gamma u]|_\Sigma = 0$, where $[\gamma_\Omega u]|_\Sigma := \gamma_{\Omega^+}(u|_{\Omega^+}) - \gamma_{\Omega^-}(u|_{\Omega^-})$ on Σ.*

The next result has been obtained in [28, Lemma B.2].

Lemma B.2. *Let $0 \le s \le 1$. Then the following assertions hold.*

(i) *Let Γ_1 and Γ_2 be the graphs of two Lipschitz functions $x_n = \zeta_1(x')$ and $x_n = \zeta_2(x')$, $x' \in \mathbb{R}^{n-1}$. Let the graphs coincide on a part Γ_0, which is the image of a set $S_0 \subset \mathbb{R}^{n-1}$, i.e., $x_n = \zeta_1(x') = \zeta_2(x')$ for $x' \in S_0$. Let $f_i \in L^2(\Gamma_i)$, $f_i = 0$ on $\Gamma_i \setminus \Gamma_0$, $i = 1, 2$, and $f_2 = f_1$ on Γ_0. Then $f_1 \in \widetilde{H}^s(\Gamma_0)$ if and only if $f_2 \in \widetilde{H}^s(\Gamma_0)$.*

(ii) *Let Γ_1 and Γ_2 be two compact $(n-1)$-dimensional Lipschitz surfaces in $\mathbb{R}^n$ that coincide on a relatively open subset Γ_0 (having a Lipschitz boundary if $n > 2$). Let $f_i \in L^2(\Gamma_i)$, $f_i = 0$ on $\Gamma_i \setminus \Gamma_0$, $i = 1, 2$, and $f_2 = f_1$ on Γ_0. Then $f_1 \in \widetilde{H}^s(\Gamma_0)$ if and only if $f_2 \in \widetilde{H}^s(\Gamma_0)$.*

Let us now consider the following space

$$H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n = \{\mathbf{\Phi} \in H^{-1}(\mathbb{R}^n)^n : \operatorname{supp} \mathbf{\Phi} \subseteq \overline{\Gamma^+}\} \tag{B.1}$$

(cf., e.g., [37, p. 76]). Then we have the following equivalence results (cf. [28, Lemmas B.5 and B.6]).

Lemma B.3. *Let Assumption 3.5 be satisfied. Then the following properties hold.*

(i) *The dual $\left(H^1_{\Gamma^+}(\Omega)^n\right)'$ of the space $H^1_{\Gamma^+}(\Omega)^n$ can be identified with $\widetilde{H}^{-1}(\Omega)^n / H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n$.*

(ii) *The dual $\left(H^1_{\Gamma^+}(\Omega^+)^n\right)'$ of the space $H^1_{\Gamma^+}(\Omega^+)^n$ can be identified with the space $\widetilde{H}^{-1}(\Omega^+)^n / H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n$ and with the space*

$$\left\{\boldsymbol{\varphi} \in H^{-1}(\Omega)^n : \boldsymbol{\varphi} = \mathbf{0} \ \text{on } \Omega^-\right\}. \tag{B.2}$$

Appendix C. Estimates of the nonlinear term in the Navier-Stokes equation

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \in \{2, 3\}$, and $\mathring{E}_\Omega$ be the zero extension operator from Ω to $\mathbb{R}^n$.

• By the Sobolev embedding theorem (see, e.g., [1, Theorem 6.3]), the space $H^1(\Omega)^n$ is compactly embedded in $L^4(\Omega)^n$ and there exists a constant $c_1 = c_1(\Omega, n) > 0$ such that

$$\|\mathbf{v}\|_{L^4(\Omega)^n} \le c_1 \|\mathbf{v}\|_{H^1(\Omega)^n}, \quad \forall\, \mathbf{v} \in H^1(\Omega)^n. \tag{C.1}$$

The equivalence in $\mathring{H}^1(\Omega)^n$ of the semi-norm $\|\nabla(\cdot)\|_{L^2(\Omega)^{n\times n}}$ with the norm $\|\cdot\|_{H^1(\Omega)^n}$ given by (2.3) and estimate (C.1) imply that there exists a constant $c_0 = c_0(\Omega, n) > 0$ such that

$$\|\mathbf{v}\|_{L^4(\Omega)^n} \le c_0 \|\nabla \mathbf{v}\|_{L^2(\Omega)^{n\times n}}, \quad \forall\, \mathbf{v} \in \mathring{H}^1(\Omega)^n. \tag{C.2}$$

- By the Hölder inequality, we obtain for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in H^1(\Omega)^n$,

$$|\langle(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_\Omega| \le \|\mathbf{v}_1\|_{L^4(\Omega)^n}\|\mathbf{v}_3\|_{L^4(\Omega)^n}\|\nabla\mathbf{v}_2\|_{L^2(\Omega)^{n\times n}} \le c_1\|\mathbf{v}_1\|_{L^4(\Omega)^n}\|\mathbf{v}_3\|_{H^1(\Omega)^n}\|\nabla\mathbf{v}_2\|_{L^2(\Omega)^{n\times n}}. \tag{C.3}$$

This also implies that

$$\begin{aligned}\left|\left\langle \mathring{E}_\Omega[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2], \mathbf{v}_3\right\rangle_\Omega\right| &\le \|\,|\mathring{E}_\Omega\mathbf{v}_1|\,|\mathbf{V}_3|\,\|_{L^2(\mathbb{R}^n)}\|\mathring{E}_\Omega\nabla\mathbf{v}_2\|_{L^2(\mathbb{R}^n)^{n\times n}}\\ &\le \|\mathbf{v}_1\|_{L^4(\Omega)^n}\|\mathbf{v}_3\|_{L^4(\Omega)^n}\|\nabla\mathbf{v}_2\|_{L^2(\Omega)^{n\times n}}\\ &\le c_1\|\mathbf{v}_1\|_{L^4(\Omega)^n}\|\mathbf{v}_3\|_{H^1(\Omega)^n}\|\nabla\mathbf{v}_2\|_{L^2(\Omega)^{n\times n}}, \quad \forall\, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in H^1(\Omega)^n,\end{aligned} \tag{C.4}$$

where $\mathbf{V}_3 \in H^1(\mathbb{R}^n)^n$ is such that $r_\Omega\mathbf{V}_3 = \mathbf{v}_3$. This shows that $\mathring{E}_\Omega[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2]$ belongs to the space $\widetilde{H}^{-1}(\Omega)^n = \big(H^1(\Omega)^n\big)'$. Moreover, inequality (C.1) implies for all $\mathbf{v}_1, \mathbf{v}_2 \in H^1(\Omega)^n$,

$$\left\|\mathring{E}_\Omega[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2]\right\|_{\widetilde{H}^{-1}(\Omega)^n} \le c_1^2\|\mathbf{v}_1\|_{H^1(\Omega)^n}\|\mathbf{v}_2\|_{H^1(\Omega)^n}. \tag{C.5}$$

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