Newtonian and Single Layer Potentials for the Stokes System with L^∞ Coefficients and the Exterior Dirichlet Problem

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Dedicated to Professor H. Begehr on the occasion of his 80*th birthday*

Abstract A mixed variational formulation of some problems in L^2-based Sobolev spaces is used to define the Newtonian and layer potentials for the Stokes system with L^∞ coefficients on Lipschitz domains in $\mathbb{R}^3$. Then the solution of the exterior Dirichlet problem for the Stokes system with L^∞ coefficients is presented in terms of these potentials and the inverse of the corresponding single layer operator.

Keywords Stokes system with L^∞ coefficients · Newtonian and layer potentials · Variational approach · Inf-sup condition · Sobolev spaces

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1 Introduction

Let $\mathbf{u}$ be an unknown vector field, π be an unknown scalar field, and $\mathbf{f}$ be a given vector field defined on an exterior Lipschitz domain $\Omega_- \subset \mathbb{R}^3$. Let also $\mathbb{E}(\mathbf{u})$ be the symmetric part of the gradient of $\mathbf{u}$, $\nabla\mathbf{u}$. Then the equations

$$\mathcal{L}_\mu(\mathbf{u}, \pi) := \operatorname{div}(2\mu\mathbb{E}(\mathbf{u})) - \nabla\pi = \mathbf{f},\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega_- \tag{1.1}$$

determine the *Stokes system* with a known viscosity coefficient $\mu \in L^\infty(\Omega_-)$. This linear PDE system describes the flows of viscous incompressible fluids, when the inertia of such a fluid can be neglected. The coefficient μ is related to the physical properties of the fluid (for further details we refer the reader to the books [45] and [23] and the references therein).

The methods of layer potential theory have a main role in the analysis of boundary value problems for elliptic partial differential equations (see, e.g., [13, 17, 30, 32, 39, 42, 48]). Fabes, Kenig and Verchota [21] obtained mapping properties of layer potentials for the constant coefficient Stokes system in L^p spaces. Mitrea and Wright [42] have used various methods of layer potentials in the analysis of the main boundary problems for the Stokes system with constant coefficients in arbitrary Lipschitz domains in $\mathbb{R}^n$. The authors in [34] have obtained mapping properties of the constant coefficient Stokes layer potential operators in standard and weighted Sobolev spaces by exploiting results of singular integral operators. Gatica and Wendland [24] used the coupling of mixed finite element and boundary integral methods for solving a class of linear and nonlinear elliptic boundary value problems. The authors in [33] used the Stokes and Brinkman integral layer potentials and a fixed point theorem to show an existence result for a nonlinear Neumann-transmission problem for the Stokes and Brinkman systems with data in L^p, Sobolev, and Besov spaces (see also [35, 36]). All above results are devoted to elliptic boundary value problems with constant coefficients.

Potential theory plays also a main role in the study of elliptic boundary value problems with variable coefficients. Dindoš and Mitrea [19] have obtained well-posedness results in Sobolev spaces for Poisson problems for the Stokes and Navier-Stokes systems with Dirichlet condition on C^1 and Lipschitz domains in compact Riemannian manifolds by using mapping properties of Stokes layer potentials in Sobolev and Besov spaces. Chkadua, Mikhailov and Natroshvili [14] obtained direct segregated systems of boundary-domain integral equations for a mixed boundary value problem of Dirichlet-Neumann type for a scalar second-order divergent elliptic partial differential equation with a variable coefficient in an exterior domain of $\mathbb{R}^3$ (see also [13]). Hofmann, Mitrea and Morris [29] considered layer potentials in L^p spaces for elliptic operators of the form $L = -\operatorname{div}(A\nabla u)$ acting in the upper half-space $\mathbb{R}^n_+$, $n \geq 3$, or in more general Lipschitz graph domains, with an L^∞ coefficient matrix A, which is t-independent, and solutions of the equation $Lu = 0$ satisfy interior De Giorgi-Nash-Moser estimates. They obtained a Calderón-Zygmund type theory associated to the layer potentials, and well-posedness results of boundary problems for the operator L in L^p and endpoint spaces.

Our variational approach is inspired by that developed by Sayas and Selgas in [46] for the constant coefficient Stokes layer potentials on Lipschitz boundaries, and is based on the technique of Nédélec [44]. Girault and Sequeira [26] obtained a well-posed result in weighted Sobolev spaces for the Dirichlet problem for the standard Stokes system in exterior Lipschitz domains of $\mathbb{R}^n$, $n = 2, 3$. Băcuţă, Hassell and Hsiao [7] developed a variational approach for the standard Brinkman single layer potential and used it in the analysis of the time dependent exterior Stokes problem with Dirichlet boundary condition in $\mathbb{R}^n$, $n = 2, 3$. Barton [8] constructed layer potentials for strongly elliptic differential operators in general settings by using the Lax-Milgram theorem, and generalized various properties of layer potentials for harmonic and second order elliptic equations. Brewster et al. in [9] have used a variational approach and a deep analysis to obtain well-posedness results for boundary problems of Dirichlet, Neumann and mixed type for higher order divergence-form elliptic equations with L^{∞} coefficients in locally (ϵ, δ)-domains and in Besov and Bessel potential spaces. Choi and Lee [15] have studied the Dirichlet problem for the Stokes system with nonsmooth coefficients, and proved the unique solvability of the problem in Sobolev spaces on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ with a small Lipschitz constant when the coefficients have vanishing mean oscillations with respect to all variables. Choi and Yang [16] obtained the existence and pointwise bound of the fundamental solution for the Stokes system with measurable coefficients in $\mathbb{R}^n$, $n \geq 3$, whenever the weak solutions of the system are locally Hölder continuous. Alliot and Amrouche [3] have used a variational approach to obtain weak solutions for the exterior Stokes problem in weighted Sobolev spaces. Also, Amrouche and Nguyen [5] proved existence and uniqueness results in weighted Sobolev spaces for the Poisson problem with Dirichlet boundary condition for the Navier-Stokes system in exterior Lipschitz domains in $\mathbb{R}^3$.

The purpose of this work is to show the well-posedness result of the Poisson problem of Dirichlet type for the Stokes system with L^{∞} coefficients in L^2-based Sobolev spaces on an exterior Lipschitz domain in $\mathbb{R}^3$. We use a variational approach that reduces this boundary value problem to a mixed variational formulation. A similar variational approach is used to define the Newtonian and layer potentials for the Stokes system with L^{∞} coefficients on Lipschitz surfaces in $\mathbb{R}^3$, by using the weak solutions of some transmission problems in L^2-based Sobolev spaces. Finally, the mapping properties of these layer potentials are used to construct explicitly the solution of the exterior Dirichlet problem for the Stokes system with L^{∞} coefficients. The analysis developed in this paper confines to the case $n = 3$, due to its practical interest, but the extension to the case $n \geq 3$ can be done with similar arguments.

2 Functional Setting and Useful Results

Let $\Omega_+ := \Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, i.e., an open connected set whose boundary $\partial\Omega$ is locally the graph of a Lipschitz function. Assume that $\partial\Omega$ is connected. Let $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}_+$ denote the exterior Lipschitz domain. Let $\mathring{E}_\pm$ denote the operators of extension by zero outside $\Omega_\pm$.

2.1 Standard L^2-Based Sobolev Spaces and Related Results

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse defined on the space of tempered distributions $\mathcal{S}^*(\mathbb{R}^3)$ (i.e., the topological dual of the space $\mathcal{S}(\mathbb{R}^3)$ of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^3$). The Lebesgue space of (equivalence classes of) measurable, square integrable functions on $\mathbb{R}^3$ is denoted by $L^2(\mathbb{R}^3)$, and by $L^\infty(\mathbb{R}^3)$ we denote the space of (equivalence classes of) essentially bounded measurable functions on $\mathbb{R}^3$. Let $H^1(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)^3$ denote the L^2-based Sobolev (Bessel potential) spaces

$$H^1(\mathbb{R}^3):=\left\{f\in\mathcal{S}^*(\mathbb{R}^3) : \|f\|_{H^1(\mathbb{R}^3)}=\left\|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{1}{2}}\mathcal{F}f]\right\|_{L^2(\mathbb{R}^3)}<\infty\right\}, \tag{2.1}$$

$$H^1(\mathbb{R}^3)^3 := \{f = (f_1, f_2, f_3) : f_j \in H^1(\mathbb{R}^3),\ j = 1, 2, 3\}. \tag{2.2}$$

The topological dual of a linear space X is denoted by X^*. Now let Ω' be Ω_+, Ω_- or $\mathbb{R}^3$. We denote by $\mathcal{D}(\Omega') := C_0^\infty(\Omega')$ the space of infinitely differentiable functions with compact support in Ω', equipped with the inductive limit topology. Let $\mathcal{D}^*(\Omega')$ denote the corresponding space of distributions on Ω', i.e., the dual space of $\mathcal{D}(\Omega')$. Let us consider the space

$$H^1(\Omega') := \{f \in \mathcal{D}^*(\Omega') : \exists\, F \in H^1(\mathbb{R}^3) \text{ such that } F|_{\Omega'} = f\}, \tag{2.3}$$

where $\cdot|_{\Omega'}$ is the restriction operator to Ω'. The space $\widetilde{H}^1(\Omega')$ is the closure of $\mathcal{D}(\Omega')$ in $H^1(\mathbb{R}^3)$. This space can be also characterized as

$$\widetilde{H}^1(\Omega') := \left\{\widetilde{f} \in H^1(\mathbb{R}^3) : \operatorname{supp}\widetilde{f} \subseteq \overline{\Omega'}\right\}. \tag{2.4}$$

Similar to definition (2.2), $H^1(\Omega')^3$ and $\widetilde{H}^1(\Omega')^3$ are the spaces of vector-valued functions whose components belong to the scalar spaces $H^1(\Omega')$ and $\widetilde{H}^1(\Omega')$, respectively (see, e.g., [38]). The Sobolev space $\widetilde{H}^1(\Omega')$ can be identified with the closure $\mathring{H}^1(\Omega')$ of $\mathcal{D}(\Omega')$ in the norm of $H^1(\Omega')$ (see, e.g., [43, (3.11), (3.13)], [38, Theorem 3.33]). The space $\mathcal{D}(\overline{\Omega'})$ is dense in $H^1(\Omega')$, and the following spaces can be isomorphically identified (cf., e.g., [38, Theorem 3.14])

$$(H^1(\Omega'))^* = \widetilde{H}^{-1}(\Omega'),\quad H^{-1}(\Omega') = (\widetilde{H}^1(\Omega'))^*. \tag{2.5}$$

For $s \in [0,1]$, the Sobolev space $H^s(\partial\Omega)$ on the boundary $\partial\Omega$ can be defined by using the space $H^s(\mathbb{R}^2)$, a partition of unity and the pull-backs of the local parametrization of $\partial\Omega$, and $H^{-s}(\partial\Omega) = (H^s(\partial\Omega))^*$. All the above spaces are Hilbert spaces. For further properties of Sobolev spaces on bounded Lipschitz domains and Lipschitz boundaries, we refer to [1, 31, 38, 42, 47].

A useful result for the next arguments is given below (see, e.g., [17], [31, Proposition 3.3]).

Lemma 2.1 *Assume that $\Omega := \Omega_+ \subset \mathbb{R}^3$ is a bounded Lipschitz domain with connected boundary $\partial\Omega$ and denote by $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$ the corresponding exterior domain. Then there exist linear and bounded trace operators $\gamma_\pm : H^1(\Omega_\pm) \to H^{\frac{1}{2}}(\partial\Omega)$ such that $\gamma_\pm f = f|_{\partial\Omega}$ for any $f \in C^\infty(\overline{\Omega}_\pm)$. These operators are surjective and have (non-unique) bounded linear right inverse operators $\gamma_\pm^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega_\pm)$.*

The jump of a function $u \in H^1(\mathbb{R}^3 \setminus \partial\Omega)$ across $\partial\Omega$ is denoted by $[\gamma(u)] := \gamma_+(u) - \gamma_-(u)$. For $u \in H^1_{\text{loc}}(\mathbb{R}^3)$, $[\gamma(u)] = 0$. The trace operator $\gamma : H^1(\mathbb{R}^3) \to H^{\frac{1}{2}}(\partial\Omega)$ can be also considered and is linear and bounded.[1]

If X is either an open subset or a surface in $\mathbb{R}^3$, then we use the notation $\langle\cdot,\cdot\rangle_X$ for the duality pairing of two dual Sobolev spaces defined on X.

2.2 Some Weighted Sobolev Spaces and Related Results

For a point $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, its distance to the origin is denoted by $|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$. Let ρ denote the weight function

$$\rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{\frac{1}{2}}. \tag{2.6}$$

For $\lambda \in \mathbb{R}$, we consider the weighted space $L^2(\rho^\lambda; \mathbb{R}^3)$ given by

$$f \in L^2(\rho^\lambda; \mathbb{R}^3) \Longleftrightarrow \rho^\lambda f \in L^2(\mathbb{R}^3), \tag{2.7}$$

which is a Hilbert space when it is endowed with the inner product and the associated norm,

$$(f, g)_{L^2(\rho^\lambda;\mathbb{R}^3)} := \int_{\mathbb{R}^3} fg\rho^{2\lambda} dx, \ \|f\|^2_{L^2(\rho^\lambda;\mathbb{R}^3)} := (f, f)_{L^2(\rho^\lambda;\mathbb{R}^3)}. \tag{2.8}$$

[1]The trace operators defined on Sobolev spaces of vector fields on $\Omega_\pm$ or $\mathbb{R}^3$ are also denoted by $\gamma_\pm$ and γ, respectively.

We also consider the weighted Sobolev space

$$\begin{aligned}\mathcal{H}^1(\mathbb{R}^3) &:= \left\{ f \in \mathcal{D}'(\mathbb{R}^3) : \rho^{-1} f \in L^2(\mathbb{R}^3),\ \nabla f \in L^2(\mathbb{R}^3)^3 \right\} \\ &= \left\{ f \in L^2(\rho^{-1}; \mathbb{R}^3) : \nabla f \in L^2(\mathbb{R}^3)^3 \right\}, \end{aligned} \tag{2.9}$$

which is a Hilbert space with respect to the inner product

$$(f, g)_{\mathcal{H}^1(\mathbb{R}^3)} := (f, g)_{L^2(\rho^{-1};\mathbb{R}^3)} + (\nabla f, \nabla g)_{L^2(\mathbb{R}^3)^3} \tag{2.10}$$

and the associated norm

$$\|f\|^2_{\mathcal{H}^1(\mathbb{R}^3)} := \left\| \rho^{-1} f \right\|^2_{L^2(\mathbb{R}^3)} + \|\nabla f\|^2_{L^2(\mathbb{R}^3)^3} \tag{2.11}$$

(cf. [28]; see also [5]). The spaces $L^2(\rho^\lambda; \Omega_-)$ and $\mathcal{H}^1(\Omega_-)$ can be similarly defined, and $\mathcal{D}(\overline{\Omega}_-)$ is dense in $\mathcal{H}^1(\Omega_-)$ (see, e.g., [28, Theorem I.1], [27, Ch.1, Theorem 2.1]). The seminorm

$$|f|_{\mathcal{H}^1(\Omega_-)} := \|\nabla f\|_{L^2(\Omega_-)^3} \tag{2.12}$$

is equivalent to the norm of $\mathcal{H}^1(\Omega_-)$ defined as in (2.11), with Ω_- in place of $\mathbb{R}^3$ (see, e.g., [18, Chapter XI, Part B, §1, Theorem 1]). The weighted spaces $L^2(\rho^{-1}; \Omega_+)$ and $\mathcal{H}^1(\Omega_+)$ coincide with the standard spaces $L^2(\Omega_+)$ and $H^1(\Omega_+)$, respectively (with equivalent norms).

Note that the result in Lemma 2.1 extends also to the weighted Sobolev space $\mathcal{H}^1(\Omega_-)$. Therefore, there exists a linear bounded *exterior trace operator*

$$\gamma_- : \mathcal{H}^1(\Omega_-) \to H^{\frac{1}{2}}(\partial\Omega), \tag{2.13}$$

which is also surjective (see [46, p. 69]). Moreover, the embedding of the space $H^1(\Omega_-)$ into $\mathcal{H}^1(\Omega_-)$ and Lemma 2.1 show the existence of a (non-unique) linear and bounded right inverse $\gamma_-^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to \mathcal{H}^1(\Omega_-)$ of operator (2.13) (see [34, Lemma 2.2], [40, Theorem 2.3, Lemma 2.6]).

Let $\mathring{\mathcal{H}}^1(\Omega_-) \subset \mathcal{H}^1(\Omega_-)$ denote the closure of $\mathcal{D}(\Omega_-)$ in $\mathcal{H}^1(\Omega_-)$. This space can be characterized as

$$\mathring{\mathcal{H}}^1(\Omega_-) = \left\{ v \in \mathcal{H}^1(\Omega_-) : \gamma_- v = 0 \text{ on } \partial\Omega \right\} \tag{2.14}$$

(cf., e.g., [38, Theorem 3.33]). Also let $\widetilde{\mathcal{H}}^1(\Omega_-) \subset \mathcal{H}^1(\mathbb{R}^3)$ denote the closure of $\mathcal{D}(\Omega_-)$ in $\mathcal{H}^1(\mathbb{R}^3)$. This space can be also characterized as

$$\widetilde{\mathcal{H}}^1(\Omega_-) = \{u \in \mathcal{H}^1(\mathbb{R}^3) : \operatorname{supp} u \subseteq \overline{\Omega}_-\}, \tag{2.15}$$

and can be isomorphically identified with the space $\mathring{\mathcal{H}}^1(\Omega_-)$ via the extension by zero operator $\mathring{E}_-$, i.e., $\widetilde{\mathcal{H}}^1(\Omega_-) = \mathring{E}_-\mathring{\mathcal{H}}^1(\Omega_-)$ (cf., e.g., [38, Theorem 3.29 (ii)]). In addition, consider the spaces (see, e.g., [5, p. 44], [37, Theorem 2.4])

$$\mathcal{H}^{-1}(\mathbb{R}^3) := \big(\mathcal{H}^1(\mathbb{R}^3)\big)^*,\ \mathcal{H}^{-1}(\Omega_-) := \big(\widetilde{\mathcal{H}}^1(\Omega_-)\big)^*,\ \widetilde{\mathcal{H}}^{-1}(\Omega_-) := \Big(\mathcal{H}^1(\Omega_-)\Big)^*.$$

3 The Conormal Derivative Operators for the Stokes System with L^∞ Coefficients

In the sequel we assume that the viscosity coefficient μ of the Stokes system (1.1) belongs to $L^\infty(\mathbb{R}^3)$ and there exists a constant $c_\mu > 0$, such that

$$c_\mu^{-1} \le \mu \le c_\mu \text{ a.e. in } \mathbb{R}^3. \tag{3.1}$$

Let $\mathbb{E}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)$ be the strain rate tensor. If $(\mathbf{u}, \pi) \in C^1(\overline{\Omega}_\pm)^3 \times C^0(\overline{\Omega}_\pm)$, we can define the *classical* interior and exterior conormal derivatives (i.e., *the boundary traction fields*) for the Stokes system (1.1) with continuously differentiable viscosity coefficient μ by the well-known formula

$$\mathbf{t}_\mu^{c\pm}(\mathbf{u}, \pi) := \gamma_\pm\left(-\pi\mathbb{I} + 2\mu\mathbb{E}(\mathbf{u})\right)\boldsymbol{\nu}, \tag{3.2}$$

where $\boldsymbol{\nu}$ is the outward unit normal to Ω_+, defined a.e. on $\partial\Omega$, and the symbol $\pm$ refers to the limit and conormal derivative from $\Omega_\pm$. Then for any function $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^3)^3$ we obtain the first Green identity

$$\pm\left\langle \mathbf{t}_\mu^{c\pm}(\mathbf{u}, \pi), \boldsymbol{\varphi}\right\rangle_{\partial\Omega} = 2\langle \mu\mathbb{E}(\mathbf{u}), \mathbb{E}(\boldsymbol{\varphi})\rangle_{\Omega_\pm} - \langle \pi, \operatorname{div}\boldsymbol{\varphi}\rangle_{\Omega_\pm} + \left\langle \mathcal{L}_\mu(\mathbf{u}, \pi), \boldsymbol{\varphi}\right\rangle_{\Omega_\pm}.$$

This formula suggests the following weak definition of the generalized conormal derivative for the Stokes system with L^∞ coefficients in the setting of L^2-weighted Sobolev spaces (cf., e.g., [17, Lemma 3.2], [34, Lemma 2.9], [35, Lemma 2.2], [40, Definition 3.1, Theorem 3.2], [42, Theorem 10.4.1]).

Definition 3.1 Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Let

$$\begin{aligned}\boldsymbol{\mathcal{H}}^1(\Omega_\pm, \mathcal{L}_\mu) := \Big\{(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \mathcal{H}^1(\Omega_\pm)^3 \times L^2(\Omega_\pm) \times \widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^3 :\\ \boldsymbol{\mathcal{L}}_\mu(\mathbf{u}_\pm, \pi_\pm) = \tilde{\mathbf{f}}_\pm|_{\Omega_\pm} \text{ and } \operatorname{div}\mathbf{u}_\pm = 0 \text{ in } \Omega_\pm\Big\}.\end{aligned} \tag{3.3}$$

Then define the conormal derivative operator $\mathbf{t}_\mu^\pm : \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}_\mu) \to H^{-\frac{1}{2}}(\partial\Omega)^3$,

$$\boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}_\mu) \ni (\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \longmapsto \mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm) \in H^{-\frac{1}{2}}(\partial\Omega)^3, \tag{3.4}$$

$$\begin{aligned}\pm\left\langle \mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm), \Phi \right\rangle_{\partial\Omega} &:= 2\langle \mu \mathbb{E}(\mathbf{u}_\pm), \mathbb{E}(\gamma_\pm^{-1}\Phi)\rangle_{\Omega_\pm} \\ &- \langle \pi_\pm, \operatorname{div}(\gamma_\pm^{-1}\Phi)\rangle_{\Omega_\pm} + \langle \tilde{\mathbf{f}}_\pm, \gamma_\pm^{-1}\Phi\rangle_{\Omega_\pm}, \ \forall\, \Phi \in H^{\frac{1}{2}}(\partial\Omega)^3,\end{aligned} \tag{3.5}$$

where $\gamma_\pm^{-1} : H^{\frac{1}{2}}(\partial\Omega)^3 \to \mathcal{H}^1(\Omega_\pm)^3$ is a (non-unique) bounded right inverse of the trace operator $\gamma_\pm : \mathcal{H}^1(\Omega_\pm)^3 \to H^{\frac{1}{2}}(\partial\Omega)^3$.

We use the simplified notation $\mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm)$ for $\mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm; \mathbf{0})$. The following assertion can be proved similar to [41, Theorem 5.3], [34, Lemma 2.9].

Lemma 3.2 *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Then for all $\mathbf{w}_\pm \in \mathcal{H}^1(\Omega_\pm)^3$ and $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}_\mu)$ the following identity holds*

$$\begin{aligned}\pm\left\langle \mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm), \gamma_\pm \mathbf{w}_\pm \right\rangle_{\partial\Omega} = 2\langle \mu \mathbb{E}(\mathbf{u}_\pm), \mathbb{E}(\mathbf{w}_\pm)\rangle_{\Omega_\pm} - \langle \pi_\pm, \operatorname{div} \mathbf{w}_\pm\rangle_{\Omega_\pm} \\ + \langle \tilde{\mathbf{f}}_\pm, \mathbf{w}_\pm\rangle_{\Omega_\pm}.\end{aligned} \tag{3.6}$$

Let γ denote the trace operator from $\mathcal{H}^1(\mathbb{R}^3)^3$ to $H^{\frac{1}{2}}(\partial\Omega)^3$ (cf., e.g., [40, Theorem 2.3, Lemma 2.6], [7, (2.2)]). For $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}_\mu)$, let

$$\mathbf{u} := \mathring{E}_+\mathbf{u}_+ + \mathring{E}_-\mathbf{u}_-, \ \pi := \mathring{E}_+\pi_+ + \mathring{E}_-\pi_-, \ \mathbf{f} := \tilde{\mathbf{f}}_+ + \tilde{\mathbf{f}}_- \tag{3.7}$$

$$[\mathbf{t}_\mu(\mathbf{u}, \pi; \mathbf{f})] := \mathbf{t}_\mu^+(\mathbf{u}_+, \pi_+; \tilde{\mathbf{f}}_+) - \mathbf{t}_\mu^-(\mathbf{u}_-, \pi_-; \tilde{\mathbf{f}}_-). \tag{3.8}$$

Moreover, if $\mathbf{f} = \mathbf{0}$, we define

$$[\mathbf{t}_\mu(\mathbf{u}, \pi)] := [\mathbf{t}_\mu(\mathbf{u}, \pi; \mathbf{0})] = \mathbf{t}_\mu^+(\mathbf{u}_+, \pi_+) - \mathbf{t}_\mu^-(\mathbf{u}_-, \pi_-). \tag{3.9}$$

Then Lemma 3.2 leads to the following result.

Lemma 3.3 *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Also let $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}_\mu)$ and let $(\mathbf{u}, \pi, \mathbf{f})$ be defined as in (3.7). Then for all $\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3$, the following formula holds*

$$\begin{aligned}\left\langle [\mathbf{t}_\mu(\mathbf{u}, \pi; \mathbf{f})], \gamma \mathbf{w} \right\rangle_{\partial\Omega} = 2\langle \mu \mathbb{E}(\mathbf{u}_+), \mathbb{E}(\mathbf{w})\rangle_{\Omega_+} + 2\langle \mu \mathbb{E}(\mathbf{u}_-), \mathbb{E}(\mathbf{w})\rangle_{\Omega_-} \\ - \langle \pi, \operatorname{div} \mathbf{w}\rangle_{\mathbb{R}^3} + \langle \mathbf{f}, \mathbf{w}\rangle_{\mathbb{R}^3}.\end{aligned} \tag{3.10}$$

We also need the following particular case of Lemma 3.3 when $\mathbf{f} = \mathbf{0}$.

Lemma 3.4 *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Also let $(\mathbf{u}_\pm, \pi_\pm, \mathbf{0}) \in \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}_\mu)$. Let $\mathbf{u}$ and π defined as in formula (3.7). Then for all $\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3$,*

$$\left\langle [\mathbf{t}_\mu(\mathbf{u}, \pi)], \gamma \mathbf{w} \right\rangle_{\partial\Omega} = 2\langle \mu \mathbb{E}(\mathbf{u}_+), \mathbb{E}(\mathbf{w})\rangle_{\Omega_+} + 2\langle \mu \mathbb{E}(\mathbf{u}_-), \mathbb{E}(\mathbf{w})\rangle_{\Omega_-} - \langle \pi, \operatorname{div} \mathbf{w}\rangle_{\mathbb{R}^3}\,. \tag{3.11}$$

4 Newtonian and Single Layer Potentials for the Stokes System with L^∞ Coefficients

Recall that the function $\mu \in L^\infty(\mathbb{R}^3)$ satisfies conditions (3.1). Next, we define the Newtonian and single layer potentials for the L^∞ coefficient Stokes system (1.1).

4.1 Variational Solution of the Variable-Coefficient Stokes System in $\mathbb{R}^3$

First we show the following useful well-posedness result.

Lemma 4.1 *Let $a_\mu(\cdot,\cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times \mathcal{H}^1(\mathbb{R}^3)^3 \to \mathbb{R}$ and $b(\cdot,\cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \to \mathbb{R}$ be the bilinear forms given by*

$$a_\mu(\mathbf{u}, \mathbf{v}) := 2\langle \mu \mathbb{E}(\mathbf{u}), \mathbb{E}(\mathbf{v})\rangle_{\mathbb{R}^3},\ \forall\, \mathbf{u}, \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3, \tag{4.1}$$

$$b(\mathbf{v}, q) := -\langle \operatorname{div} \mathbf{v}, q\rangle_{\mathbb{R}^3},\ \forall\, \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3,\ \forall\, q \in L^2(\mathbb{R}^3). \tag{4.2}$$

Also let $\boldsymbol{\ell} : \mathcal{H}^1(\mathbb{R}^3)^3 \to \mathbb{R}$ be a linear and bounded map. Then the mixed variational formulation

$$\begin{cases} a_\mu(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \boldsymbol{\ell}(\mathbf{v}),\ \forall\, \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3, \\ b(\mathbf{u}, q) = 0,\ \forall\, q \in L^2(\mathbb{R}^3) \end{cases} \tag{4.3}$$

is well-posed. Hence, (4.3) has a unique solution $(\mathbf{u}, p) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ and there exists a constant $C = C(c_\mu) > 0$ such that

$$\|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} + \|p\|_{L^2(\mathbb{R}^3)} \le C\|\boldsymbol{\ell}\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}. \tag{4.4}$$

Proof By using conditions (3.1) and definition (2.11) of the norm of the weighted Sobolev space $\mathcal{H}^1(\mathbb{R}^3)$ we obtain that

$$\begin{aligned} |a_\mu(\mathbf{u}, \mathbf{v})| &\le 2c_\mu \|\mathbb{E}(\mathbf{u})\|_{L^2(\mathbb{R}^3)^{3\times 3}} \|\mathbb{E}(\mathbf{v})\|_{L^2(\mathbb{R}^3)^{3\times 3}} \\ &\le 2c_\mu \|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \|\mathbf{v}\|_{\mathcal{H}^1(\mathbb{R}^3)^3},\ \forall\, \mathbf{u}, \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3. \end{aligned} \tag{4.5}$$

Moreover, by using the Korn type inequality for functions in $\mathcal{H}^1(\mathbb{R}^3)^3$,

$$\|\mathrm{grad}\,\mathbf{v}\|_{L^2(\mathbb{R}^3)^{3\times 3}} \le 2^{\frac{1}{2}} \|\mathbb{E}(\mathbf{v})\|_{L^2(\mathbb{R}^3)^{3\times 3}} \tag{4.6}$$

(cf., e.g., [46, (2.2)]) and since the seminorm

$$|g|_{\mathcal{H}^1(\mathbb{R}^3)} := \|\nabla g\|_{L^2(\mathbb{R}^3)^3} \tag{4.7}$$

is a norm in $\mathcal{H}^1(\mathbb{R}^3)^3$ equivalent to the norm defined by (2.11) (see, e.g., [18, Chapter XI, Part B, §1, Theorem 1]), there exists a constant $c_1 > 0$ such that

$$\begin{aligned} a_\mu(\mathbf{u},\mathbf{u}) \ge 2c_\mu^{-1}\|\mathbb{E}(\mathbf{u})\|^2_{L^2(\mathbb{R}^3)^{3\times 3}} &\ge c_\mu^{-1}\|\nabla \mathbf{u}\|^2_{L^2(\mathbb{R}^3)^{3\times 3}} \\ &\ge c_\mu^{-1} c_1 \|\mathbf{u}\|^2_{\mathcal{H}^1(\mathbb{R}^3)^3},\ \forall\, \mathbf{u} \in \mathcal{H}^1(\mathbb{R}^3)^3. \end{aligned} \tag{4.8}$$

Inequalities (4.5) and (4.8) show that $a_\mu(\cdot,\cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times \mathcal{H}^1(\mathbb{R}^3)^3 \to \mathbb{R}$ is a bounded and coercive bilinear form. Moreover, since the divergence operator

$$\mathrm{div} : \mathcal{H}^1(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3) \tag{4.9}$$

is bounded, then the bilinear form $b(\cdot,\cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \to \mathbb{R}$ is bounded as well. In addition, the operator in (4.9) is surjective (cf. [2, Proposition 2.1], [46, Proposition 2.4]) and also

$$\begin{aligned} \mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^3)^3 &:= \left\{\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3 : \mathrm{div}\,\mathbf{w} = 0\right\} \\ &= \left\{\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3 : b(\mathbf{w}, q) = 0,\ \forall\, q \in L^2(\mathbb{R}^3)\right\}. \end{aligned}$$

In addition, the operator in (4.9) is surjective (cf. [2, Proposition 2.1], [46, Proposition 2.4]), and hence the operator

$$-\mathrm{div} : \mathcal{H}^1(\mathbb{R}^3)^3 / \mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)$$

is an isomorphism. Then by Lemma 2(ii) the bounded bilinear form $b(\cdot,\cdot)$: $\mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \to \mathbb{R}$ satisfies the inf-sup condition (7). Hence, there exists $\beta_0 \in (0,\infty)$ such that

$$\inf_{q \in L^2(\mathbb{R}^3)\setminus\{0\}} \ \sup_{\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3 \setminus \{\mathbf{0}\}} \frac{b(\mathbf{w}, q)}{\|\mathbf{w}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \|q\|_{L^2(\mathbb{R}^3)}} \ge \beta_0. \tag{4.10}$$

By applying Theorem 4, with $X = \mathcal{H}^1(\mathbb{R}^3)^3$, $M = L^2(\mathbb{R}^3)$, $V = \mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^3)^3$, we conclude that the mixed variational formulation (4.3) has a unique solution $(\mathbf{u}, p) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ and there exists a constant $C = C(c_\mu) > 0$ such that $(\mathbf{u}, p)$ satisfies inequality (4.4). □

Next we use the result of Lemma 4.1 in order to show the well-posedness of the L^∞ coefficient Stokes system in the space $\mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ (see also [2, Theorem 3] for the constant-coefficient case).

Theorem 4.2 *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy conditions (3.1). Then the L^∞ coefficient Stokes system*

$$\begin{cases} \operatorname{div}(2\mu\mathbb{E}(\mathbf{u})) - \nabla\pi = \boldsymbol{\ell}, \ \boldsymbol{\ell} \in \mathcal{H}^{-1}(\mathbb{R}^3)^3, \\ \operatorname{div}\mathbf{u} = 0, \qquad\qquad\qquad\qquad\quad \mathit{in}\ \mathbb{R}^3, \end{cases} \tag{4.11}$$

has a unique solution $(\mathbf{u}, p) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$, and there exists a constant $C_0 = C_0(c_\mu) > 0$ such that

$$\|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} + \|p\|_{L^2(\mathbb{R}^3)} \le C_0\|\boldsymbol{\ell}\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}. \tag{4.12}$$

Proof Note that the Stokes system (4.11) is equivalent to the variational problem (4.3) as follows from the density of $\mathcal{D}(\mathbb{R}^3)^3$ in the space $\mathcal{H}^1(\mathbb{R}^3)^3$ (cf., e.g., [28], [46, Proposition 2.1]). Then the well-posedness result of the Stokes system with L^∞ coefficients (4.11) follows from Lemma 4.1. □

4.2 Newtonian Potential for the Stokes System with L^∞ Coefficients

The well-posedness of problem (4.11) allows us to define the *Newtonian potential for the Stokes system with L^∞ coefficients* as follows.

Definition 4.3 For any $\boldsymbol{\ell} \in \mathcal{H}^{-1}(\mathbb{R}^3)^3$, we define the *Newtonian velocity and pressure potentials* for the Stokes system with L^∞ coefficients as

$$\boldsymbol{\mathcal{N}}_{\mu;\mathbb{R}^3}\boldsymbol{\ell} := \mathbf{u}, \ \mathcal{Q}_{\mu;\mathbb{R}^3}\boldsymbol{\ell} := \pi, \tag{4.13}$$

where $(\mathbf{u}, \pi) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ is the unique solution of problem (4.11) with the given datum $\boldsymbol{\ell}$.

Moreover, the well-posedness of problem (4.11) yields the continuity of the above operators as stated in the following assertion (cf. also [34, Lemma A.3] for $\mu = 1$).

Lemma 4.4 *The Newtonian velocity and pressure potential operators*

$$\boldsymbol{\mathcal{N}}_{\mu;\mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \to \mathcal{H}^1(\mathbb{R}^3)^3, \ \mathcal{Q}_{\mu;\mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3) \tag{4.14}$$

are linear and continuous.

4.3 Single Layer Potential for the Stokes System with L^∞ Coefficients

For a given $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$, we now consider the following transmission problem for the Stokes system with L^∞ coefficients

$$\begin{cases} \operatorname{div}\left(2\mu\mathbb{E}(\mathbf{u}_{\boldsymbol{\varphi}})\right) - \nabla\pi_{\boldsymbol{\varphi}} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ \operatorname{div}\mathbf{u}_{\boldsymbol{\varphi}} = 0 & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ \left[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}})\right] = \boldsymbol{\varphi} & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

and show that this problem has a unique solution $\left(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}}\right) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ (cf. also [46, Proposition 5.1] for $\mu = 1$). Note that the membership of $\mathbf{u}_{\boldsymbol{\varphi}}$ in $\mathcal{H}^1(\mathbb{R}^3)^3$ implies the transmission condition

$$\left[\gamma(\mathbf{u}_{\boldsymbol{\varphi}})\right] = \mathbf{0} \text{ on } \partial\Omega\,, \tag{4.16}$$

and the first equation in (4.15) implies also that the jump $\left[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}})\right]$ is well defined.

Theorem 4.5 *Let $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$ be given. Then the transmission problem (4.15) has the following equivalent mixed variational formulation: Find $(\mathbf{u}_{\varphi}, \pi_{\boldsymbol{\varphi}}) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ such that*

$$\begin{cases} 2\langle\mu\mathbb{E}(\mathbf{u}_{\boldsymbol{\varphi}}), \mathbb{E}(\mathbf{v})\rangle_{\mathbb{R}^3} - \langle\pi_{\boldsymbol{\varphi}}, \operatorname{div}\mathbf{v}\rangle_{\mathbb{R}^3} = \langle\boldsymbol{\varphi}, \gamma\mathbf{v}\rangle_{\partial\Omega},\ \forall\, \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3, \\ \langle\operatorname{div}\mathbf{u}_{\boldsymbol{\varphi}}, q\rangle_{\mathbb{R}^3} = 0,\ \forall\, q \in L^2(\mathbb{R}^3). \end{cases} \tag{4.17}$$

Moreover, problem (4.17) is well-posed. Hence (4.17) has a unique solution $(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}}) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$, and there exists a constant $C = C(c_\mu)$ such that

$$\|\mathbf{u}_{\boldsymbol{\varphi}}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} + \|\pi_{\boldsymbol{\varphi}}\|_{L^2(\mathbb{R}^3)} \leq C\|\boldsymbol{\varphi}\|_{H^{-\frac{1}{2}}(\partial\Omega)^3}. \tag{4.18}$$

Proof The equivalence between the transmission problem (4.15) and the variational problem (4.17) follows from the density of the space $\mathcal{D}(\mathbb{R}^3)^3$ in $\mathcal{H}^1(\mathbb{R}^3)^3$ and formula (3.11), while the well-posedness of the variational problem (4.17) is an immediate consequence of Lemma 4.1 with the linear and continuous form $\boldsymbol{\ell} : \mathcal{H}^1(\mathbb{R}^3)^3 \to \mathbb{R}$ given by

$$\boldsymbol{\ell}(\mathbf{v}) := \langle\boldsymbol{\varphi}, \gamma\mathbf{v}\rangle_{\partial\Omega} = \langle\gamma^*\boldsymbol{\varphi}, \mathbf{v}\rangle_{\mathbb{R}^3},\ \forall\, \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3,$$

and hence $\boldsymbol{\ell} = \gamma^*\boldsymbol{\varphi}$, where $\gamma^* : H^{-\frac{1}{2}}(\partial\Omega)^3 \to \mathcal{H}^{-1}(\mathbb{R}^3)^3$ is the adjoint of the trace operator $\gamma : \mathcal{H}^1(\mathbb{R}^3)^3 \to H^{\frac{1}{2}}(\partial\Omega)^3$. ◻

Theorem 4.5 leads to the following definition (cf. [46, p. 75] for $\mu = 1$).

Definition 4.6 For any $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$, we define the *single layer velocity and pressure potentials* for the Stokes system with L^∞ coefficients (1.1) as

$$\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi} := \mathbf{u}_{\boldsymbol{\varphi}},\ \mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\varphi} := \pi_{\boldsymbol{\varphi}}, \tag{4.19}$$

and the *potential operators* $\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to H^{\frac{1}{2}}(\partial\Omega)^3$ and $\mathbf{K}^*_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to H^{-\frac{1}{2}}(\partial\Omega)^3$ as

$$\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}\boldsymbol{\varphi} := \gamma\mathbf{u}_{\boldsymbol{\varphi}},\ \mathbf{K}^*_{\mu;\partial\Omega}\boldsymbol{\varphi} := \frac{1}{2}\left(\mathbf{t}^+_\mu(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}}) + \mathbf{t}^-_\mu(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}})\right), \tag{4.20}$$

where $(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}})$ is the unique solution of problem (4.15) in $\mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$.

The next result shows the continuity of single layer velocity and pressure potential operators for the variable coefficient Stokes system (cf. [46, Proposition 5.2], [34, Lemma A.4, (A.10), (A.12)] and [42, Theorem 10.5.3] in the case $\mu = 1$).

Lemma 4.7 *The following operators are linear and continuous*

$$\mathbf{V}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to \mathcal{H}^1(\mathbb{R}^3)^3,\ \mathcal{Q}^s_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to L^2(\mathbb{R}^3), \tag{4.21}$$

$$\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to H^{\frac{1}{2}}(\partial\Omega)^3,\ \mathbf{K}^*_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to H^{-\frac{1}{2}}(\partial\Omega)^3. \tag{4.22}$$

Proof The continuity of operators (4.21) and (4.22) follows from the well-posedness of the transmission problem (4.15) and Definition 4.6. □

The next result yields the jump relations of the single layer potential and its conormal derivative across $\partial\Omega$ (see also [46, Proposition 5.3] for $\mu = 1$).

Lemma 4.8 *Let* $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$. *Then almost everywhere on* $\partial\Omega$,

$$\left[\gamma\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}\right] = \mathbf{0}, \tag{4.23}$$

$$\left[\mathbf{t}_\mu\left(\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}, \mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\varphi}\right)\right] = \boldsymbol{\varphi},\ \mathbf{t}^\pm_\mu\left(\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}, \mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\varphi}\right) = \pm\frac{1}{2}\boldsymbol{\varphi} + \mathbf{K}^*_{\mu;\partial\Omega}\boldsymbol{\varphi}. \tag{4.24}$$

Proof Formulas (4.23) and (4.24) follow from Definition 4.6 and the transmission condition in (4.16), as well as the transmission condition in the third line of (4.15). □

Let $\mathbb{R}\boldsymbol{\nu} = \{c\boldsymbol{\nu} : c \in \mathbb{R}\}$. Let $\mathrm{Ker}\{T : X \to Y\} := \{x \in X : T(x) = 0\}$ denote the null space of the map $T : X \to Y$.

We next obtain the main properties of the single layer potential operator (cf., e.g., [42, Theorem 10.5.3], and [7, Proposition 3.3(c)] and [46, Proposition 5.4] for $\mu = 1$ and $\alpha \in [0, \infty)$).

Lemma 4.9 *The following properties hold*

$$\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\nu}=\mathbf{0} \text{ in } \mathbb{R}^3,\ \mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\nu}=-\chi_{\Omega_+} \tag{4.25}$$

$$\operatorname{Ker}\left\{\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega)^3\to H^{\frac{1}{2}}(\partial\Omega)^3\right\}=\mathbb{R}\boldsymbol{\nu}, \tag{4.26}$$

$$\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}\boldsymbol{\varphi}\in H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3,\ \forall\,\boldsymbol{\varphi}\in H^{-\frac{1}{2}}(\partial\Omega)^3, \tag{4.27}$$

where $\chi_{\Omega_+}=1$ *in* Ω_+, $\chi_{\Omega_+}=0$ *in* Ω_-, *and*

$$H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3:=\left\{\boldsymbol{\phi}\in H^{\frac{1}{2}}(\partial\Omega)^3:\langle\boldsymbol{\nu},\boldsymbol{\phi}\rangle_{\partial\Omega}=0\right\}. \tag{4.28}$$

Proof First, we consider the transmission problem (4.15) with the datum $\boldsymbol{\varphi}=\boldsymbol{\nu}\in H^{-\frac{1}{2}}(\partial\Omega)^3$. Then the solution of this problem is given by

$$(\mathbf{u}_{\boldsymbol{\nu}},\pi_{\boldsymbol{\nu}})=\left(\mathbf{0},-\chi_{\Omega_+}\right)\in\mathcal{H}^1(\mathbb{R}^3)^3\times L^2(\mathbb{R}^3). \tag{4.29}$$

Indeed, the pair $(\mathbf{u}_{\boldsymbol{\nu}},\pi_{\boldsymbol{\nu}})$ satisfies the equations and the transmission condition in (4.15), as well as the transmission condition (4.16), and, in view of formula (3.11) and the divergence theorem,

$$\langle[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\nu}},\pi_{\boldsymbol{\nu}})],\gamma\mathbf{v}\rangle_{\partial\Omega}=-\langle\pi_{\boldsymbol{\nu}},\operatorname{div}\mathbf{v}\rangle_{\mathbb{R}^3}=\langle\boldsymbol{\nu},\gamma\mathbf{v}\rangle_{\partial\Omega},\ \forall\,\mathbf{v}\in\mathcal{D}(\mathbb{R}^3)^3. \tag{4.30}$$

Then by formula (2.3), Lemma 2.1, the dense embedding of the space $\mathcal{D}(\mathbb{R}^3)^3$ in $H^1(\mathbb{R}^3)^3$, and the above equality, we obtain that $\langle[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\nu}},\pi_{\boldsymbol{\nu}})],\Phi\rangle_{\partial\Omega}=\langle\boldsymbol{\nu},\Phi\rangle_{\partial\Omega}$ for any $\Phi\in H^{\frac{1}{2}}(\partial\Omega)^3$. Hence, $[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\nu}},\pi_{\boldsymbol{\nu}})]=\boldsymbol{\nu}$, as asserted. Then Definition 4.6 implies relations (4.25). Moreover, $\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}\boldsymbol{\nu}=\mathbf{0}$, i.e., $\mathbb{R}\boldsymbol{\nu}\subseteq\operatorname{Ker}\left\{\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega)^3\to H^{\frac{1}{2}}(\partial\Omega)^3\right\}$.

Now let $\boldsymbol{\varphi}_0\in\operatorname{Ker}\left\{\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega)^3\to H^{\frac{1}{2}}(\partial\Omega)^3\right\}$. Let $(\mathbf{u}_{\boldsymbol{\varphi}_0},\pi_{\boldsymbol{\varphi}_0})=\left(\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}_0,\mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\varphi}_0\right)\in\mathcal{H}^1(\mathbb{R}^3)^3\times L^2(\mathbb{R}^3)$ be the unique solution of problem (4.15) with datum $\boldsymbol{\varphi}_0$. Since $\gamma\mathbf{u}_{\boldsymbol{\varphi}_0}=\mathbf{0}$ on $\partial\Omega$, formula (3.11) yields that

$$0=\langle[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}_0},\pi_{\boldsymbol{\varphi}_0})],\gamma\mathbf{u}_{\boldsymbol{\varphi}_0}\rangle_{\partial\Omega}=a_\mu\left(\mathbf{u}_{\boldsymbol{\varphi}_0},\mathbf{u}_{\boldsymbol{\varphi}_0}\right), \tag{4.31}$$

and hence $\mathbf{u}_{\boldsymbol{\varphi}_0}=\mathbf{0}$, $\pi_{\boldsymbol{\varphi}_0}=c\chi_{\Omega_+}$ in $\mathbb{R}^3$, where $c\in\mathbb{R}$. In view of formula (3.11),

$$\langle[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}_0},\pi_{\boldsymbol{\varphi}_0})],\gamma\mathbf{w}\rangle_{\partial\Omega}=-\langle\pi_{\boldsymbol{\varphi}_0},\operatorname{div}\mathbf{w}\rangle_{\mathbb{R}^3}=-c\langle\boldsymbol{\nu},\gamma\mathbf{w}\rangle_{\partial\Omega},\ \forall\,\mathbf{w}\in\mathcal{D}(\mathbb{R}^3)^3,$$

and, thus, $\boldsymbol{\varphi}_0=[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}},\pi_{\boldsymbol{\varphi}_0})]=-c\boldsymbol{\nu}$. Hence, formula (4.26) follows.

Now let $\boldsymbol{\varphi}\in H^{-\frac{1}{2}}(\partial\Omega)^3$. By using the first formula in (4.20), we obtain for any $\boldsymbol{\varphi}\in H^{-\frac{1}{2}}(\partial\Omega)^3$ that $\left\langle\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}\boldsymbol{\varphi},\boldsymbol{\nu}\right\rangle_{\partial\Omega}=\left\langle\gamma\mathbf{u}_{\boldsymbol{\varphi}},\boldsymbol{\nu}\right\rangle_{\partial\Omega}=\left\langle\operatorname{div}\mathbf{u}_{\boldsymbol{\varphi}},1\right\rangle_{\Omega}=0$, where $\mathbf{u}_{\boldsymbol{\varphi}}=\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}$. Thus, we get relation (4.27). ◻

Next we use the notation $[\![\cdot]\!]$ for the equivalence classes of the space $H^{-\frac{1}{2}}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\boldsymbol{\nu}$. Thus, any $[\![\boldsymbol{\varphi}]\!] \in H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$ can be written as $[\![\boldsymbol{\varphi}]\!] = \boldsymbol{\varphi} + \mathbb{R}\boldsymbol{\nu}$, where $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$.

Exploiting properties (4.26) and (4.27), we now show the following invertibility result (cf. [42, Theorem 10.5.3], [7, Proposition 3.3(d)], [46, Proposition 5.5] for $\mu = 1$ and $\alpha \geq 0$ constant).

Lemma 4.10 *The following operator is an isomorphism*

$$\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\nu \to H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3. \tag{4.32}$$

Proof We use arguments similar to those for Proposition 5.5 in [46]. First, Lemma 4.7 and the membership relation (4.27) imply that the linear operator in (4.32) is continuous. We show that this operator is also $H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$-elliptic, i.e., that there exists a constant $c = c(\partial\Omega) > 0$ such that

$$\left\langle \boldsymbol{\mathcal{V}}_{\mu;\partial\Omega} [\![\boldsymbol{\varphi}]\!], [\![\boldsymbol{\varphi}]\!] \right\rangle_{\partial\Omega} \geq c \, \|[\![\boldsymbol{\varphi}]\!]\|^2_{H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}}, \ \forall \, [\![\boldsymbol{\varphi}]\!] \in H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}. \tag{4.33}$$

Let $[\![\boldsymbol{\varphi}]\!] \in H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$. Thus, $[\![\boldsymbol{\varphi}]\!] = \boldsymbol{\varphi} + \mathbb{R}\boldsymbol{\nu}$, where $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$. In view of formula (3.11), Definition 4.6, relations (4.26), (4.27), and inequality (4.8),

$$\begin{aligned}\left\langle \boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}([\![\boldsymbol{\varphi}]\!]), [\![\boldsymbol{\varphi}]\!] \right\rangle_{\partial\Omega} &= \left\langle \boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}(\boldsymbol{\varphi}), \boldsymbol{\varphi} \right\rangle_{\partial\Omega} = \langle \gamma \mathbf{u}_{\boldsymbol{\varphi}}, [\mathbf{t}_{\mu}(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}})] \rangle_{\partial\Omega} \\ &= a_{\mu}(\mathbf{u}_{\boldsymbol{\varphi}}, \mathbf{u}_{\boldsymbol{\varphi}}) \geq c_{\mu}^{-1} \|\mathbf{u}_{\boldsymbol{\varphi}}\|^2_{H^1(\mathbb{R}^3)^3},\end{aligned} \tag{4.34}$$

where $\mathbf{u}_{\boldsymbol{\varphi}} = \mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}$ and $\pi_{\boldsymbol{\varphi}} = \mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\varphi}$. Now we use the property that the trace operator

$$\gamma : \mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^3)^3 \to H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3 \tag{4.35}$$

is surjective having a bounded right inverse $\gamma^{-1} : H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3 \to \mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^3)^3$ (cf., e.g., [46, Proposition 4.4]). Hence, for any $\boldsymbol{\Phi} \in H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3$, we have that $\mathbf{w} = \gamma^{-1}\boldsymbol{\Phi} \in \mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^3)^3$. Then there exists $c' \equiv c'(\partial\Omega) \in (0, \infty)$ such that

$$\begin{aligned}|\langle [\![\boldsymbol{\varphi}]\!], \boldsymbol{\Phi} \rangle_{\partial\Omega}| &= |\langle \boldsymbol{\varphi}, \boldsymbol{\Phi} \rangle_{\partial\Omega}| = |\langle [\mathbf{t}_{\mu}(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}})], \gamma \mathbf{w} \rangle_{\partial\Omega}| = |a_{\mu}(\mathbf{u}_{\boldsymbol{\varphi}}, \mathbf{w})| \\ &\leq 2c_{\mu} \|\mathbf{u}_{\boldsymbol{\varphi}}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \|\gamma^{-1}\boldsymbol{\Phi}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \leq 2c_{\mu} c' \|\mathbf{u}_{\boldsymbol{\varphi}}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \|\boldsymbol{\Phi}\|_{H^{\frac{1}{2}}(\partial\Omega)^3},\end{aligned} \tag{4.36}$$

where the first equality in (4.36) follows from the relation $[\![\boldsymbol{\varphi}]\!] = \boldsymbol{\varphi} + \mathbb{R}\boldsymbol{\nu}$ and the membership of $\boldsymbol{\Phi}$ in $H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3$, the second equality follows from Definition 4.6, and the third equality is a consequence of formula (3.11). Since the space $H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3$ is the dual of the space $H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\nu$, formula (4.36) yields that

$$\| [\![\boldsymbol{\varphi}]\!] \|_{H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}} \leq 2c_\mu c' \|\mathbf{u}_{\boldsymbol{\varphi}}\|_{\mathcal{H}^1(\mathbb{R}^3)^3}. \tag{4.37}$$

Then by (4.34) and (4.37) we obtain inequality (4.33), and the Lax-Milgram lemma yields that operator (4.32) is an isomorphism. □

Remark 4.11 The fundamental solution of the constant-coefficient Stokes system in $\mathbb{R}^3$ is well known and leads to the construction of Newtonian and boundary layer potentials via the integral approach (see, e.g., [17, 32, 42, 48]). In view of Theorems 4.2 and 4.5, the Newtonian and single layer potentials provided by the variational approach (in the case $\mu = 1$) coincide with classical ones expressed in terms of the fundamental solution, since they satisfy the same boundary value problems (4.11) and (4.15), respectively (see also [46, Proposition 5.1] for $\mu = 1$). The assumption $\mu = 1$ is a particular case of a more general case of L^∞ coefficients analyzed in this paper. We also note that an alternative approach, reducing various boundary value problems for variable-coefficient elliptic partial differential equations to *boundary-domain integral equations*, by employing the explicit parametrix-based integral potentials, was explored in, e.g., [12–14].

5 Exterior Dirichlet Problem for the Stokes System with L^∞ Coefficients

In this section we analyze the exterior Dirichlet problem for the Stokes system with L^∞ coefficients

$$\begin{cases} \mathrm{div}\,(2\mu\mathbb{E}(\mathbf{u})) - \nabla\pi = \mathbf{f} & \text{in } \Omega_-, \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega_-, \\ \gamma_-\mathbf{u} = \boldsymbol{\phi} & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

with given data $(\mathbf{f}, \boldsymbol{\phi}) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}(\partial\Omega)^3$.

5.1 *Variational Approach*

First, we use a variational approach and show that problem (5.1) has a unique solution $(\mathbf{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$ (see also [26, Theorem 3.4] and [3, Theorem 3.16] for the constant-coefficient Stokes system).

Theorem 5.1 *Assume that $\mu \in L^\infty(\Omega_-)$ satisfies conditions (3.1). Then for all given data $(\mathbf{f}, \boldsymbol{\phi}) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}(\partial\Omega)^3$ the exterior Dirichlet problem for the L^∞ coefficient Stokes system (5.1) is well posed. Hence problem (5.1) has a unique solution $(\mathbf{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$ and there exists a constant $C \equiv C(\partial\Omega; c_\mu) > 0$ such that*

$$\|\mathbf{u}\|_{\mathcal{H}^1(\Omega_-)^3} + \|\pi\|_{L^2(\Omega_-)} \le C\left(\|\mathbf{f}\|_{\mathcal{H}^{-1}(\Omega_-)^3} + \|\boldsymbol{\phi}\|_{H^{\frac{1}{2}}(\partial\Omega)^3}\right). \tag{5.2}$$

Proof First, we note that the density of the space $\mathcal{D}(\Omega_-)^3$ in $\widetilde{\mathcal{H}}^1(\Omega_-)^3$ implies that the exterior Dirichlet problem (5.1) has the following equivalent variational formulation: Find $(\mathbf{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$ such that

$$\begin{cases} 2\langle \mu\mathbb{E}(\mathbf{u}), \mathbb{E}(\tilde{\mathbf{v}})\rangle_{\Omega_-} - \langle \pi, \operatorname{div}\tilde{\mathbf{v}}\rangle_{\Omega_-} = -\langle \mathbf{f}, \tilde{\mathbf{v}}\rangle_{\Omega_-}, \ \forall\, \tilde{\mathbf{v}} \in \widetilde{\mathcal{H}}^1(\Omega_-)^3, \\ \langle \operatorname{div}\mathbf{u}, q\rangle_{\Omega_-} = 0, \ \forall\, q \in L^2(\Omega_-), \\ \gamma_-(\mathbf{u}) = \boldsymbol{\phi} \text{ on } \partial\Omega. \end{cases} \tag{5.3}$$

Next, we consider $\mathbf{u}_0 \in \mathcal{H}^1(\Omega_-)^3$ such that

$$\begin{cases} \operatorname{div}\mathbf{u}_0 = 0 & \text{in } \Omega_-, \\ \gamma_-\mathbf{u}_0 = \boldsymbol{\phi} & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

Particularly, we can choose $\mathbf{u}_0$ as the solution of the Dirichlet problem for a constant-coefficient Brinkman system

$$\begin{cases} (\triangle - \alpha\mathbb{I})\mathbf{u}_0 - \nabla\pi_0 = 0, \ \operatorname{div}\mathbf{u}_0 = 0 & \text{in } \Omega_-, \\ \gamma_-\mathbf{u}_0 = \boldsymbol{\phi} & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

where $\alpha > 0$ is an arbitrary constant. The solution is given by the double layer potential

$$\mathbf{u}_0 = \mathbf{W}_{\alpha;\partial\Omega}\left(\frac{1}{2}\mathbb{I} + \mathbf{K}_{\alpha;\partial\Omega}\right)^{-1}\boldsymbol{\phi}, \tag{5.6}$$

where $\mathbf{K}_{\alpha;\partial\Omega} : H^{\frac{1}{2}}(\partial\Omega)^3 \to H^{\frac{1}{2}}(\partial\Omega)^3$ is the corresponding Brinkman double-layer boundary potential operator. Note that

$$(\mathbf{W}_\alpha\mathbf{h})_j(\mathbf{x}) := \int_{\partial\Omega} S^\alpha_{ij\ell}(\mathbf{x}, \mathbf{y})\nu_\ell(\mathbf{y})h_i(\mathbf{y})d\sigma_{\mathbf{y}}. \tag{5.7}$$

The explicit form of the kernel $S^\alpha_{ij\ell}(\mathbf{x}, \mathbf{y})$ can be found in [48, (2.14)–(2.18)] and [32, Section 3.2.1].

In addition, the operator $\frac{1}{2}\mathbb{I} + \mathbf{K}_{\alpha;\partial\Omega} : H^{\frac{1}{2}}(\partial\Omega)^3 \to H^{\frac{1}{2}}(\partial\Omega)^3$ is an isomorphism, and $\mathbf{u}_0$ belongs to the space $H^1(\Omega_-)^3$ and satisfies (5.5), and hence (5.4).

Moreover, the embedding $H^1(\Omega_-)^3 \subset \mathcal{H}^1(\Omega_-)^3$ shows that $\mathbf{u}_0$ belongs also to the space $\mathcal{H}^1(\Omega_-)^3$ (see also [26, Lemma 3.2, Remark 3.3]).

Then with the new variable $\mathring{\mathbf{u}} := \mathbf{u} - \mathbf{u}_0 \in \mathring{\mathcal{H}}^1(\Omega_-)^3$, the variational problem (5.3) reduces to the following mixed variational formulation (c.f. Problem (Q) in p. 324 of [26] for the constant-coefficient Stokes system): Find $(\mathring{\mathbf{u}}, \pi) \in \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-)$ such that

$$\begin{cases} a_{\mu;\Omega_-}(\mathring{\mathbf{u}}, \mathbf{v}) + b_{\Omega_-}(\mathbf{v}, \pi) = \mathfrak{F}_{\mu;\mathbf{u}_0}(\mathbf{v}), \ \forall\, \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3, \\ b_{\Omega_-}(\mathring{\mathbf{u}}, q) = 0, \ \forall\, q \in L^2(\Omega_-), \end{cases} \tag{5.8}$$

where $a_{\mu;\Omega_-} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times \mathring{\mathcal{H}}^1(\Omega_-)^3 \to \mathbb{R}$ and $b_{\Omega_-} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-) \to \mathbb{R}$ are the bilinear forms given by

$$a_{\mu;\Omega_-}(\mathbf{w}, \mathbf{v}) := 2\langle \mu \mathbb{E}(\mathbf{w}), \mathbb{E}(\mathbf{v})\rangle_{\Omega_-}, \ \forall\, \mathbf{v}, \mathbf{w} \in \mathring{\mathcal{H}}^1(\Omega_-)^3, \tag{5.9}$$

$$b_{\Omega_-}(\mathbf{v}, q) := -\langle \operatorname{div} \mathbf{v}, q\rangle_{\Omega_-}, \ \forall\, \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3, q \in L^2(\Omega_-), \tag{5.10}$$

and $\mathfrak{F}_{\mu;\mathbf{u}_0} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \to \mathbb{R}$ is the linear form given by

$$\mathfrak{F}_{\mu;\mathbf{u}_0}(\mathbf{v}) := -\left(\langle \mathbf{f}, \mathring{E}_- \mathbf{v}\rangle_{\Omega_-} + 2\langle \mu \mathbb{E}(\mathbf{u}_0), \mathbb{E}(\mathbf{v})\rangle_{\Omega_-}\right), \ \forall\, \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3. \tag{5.11}$$

Here we took into account that the spaces $\mathring{\mathcal{H}}^1(\Omega_-)^3$ and $\widetilde{\mathcal{H}}^1(\Omega_-)^3$ can be identified through the isomorphism $\mathring{E}_- : \mathring{\mathcal{H}}^1(\Omega_-)^3 \to \widetilde{\mathcal{H}}^1(\Omega_-)^3$. Note that

$$\begin{aligned} \mathring{\mathcal{H}}^1_{\operatorname{div}}(\Omega_-)^3 :&= \left\{\mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3 : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_-\right\} \\ &= \left\{\mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3 : b_{\Omega_-}(\mathbf{v}, q) = 0, \ \forall\, q \in L^2(\Omega_-)\right\}. \end{aligned} \tag{5.12}$$

Now, formula (2.11), inequality (3.1) and the Hölder inequality yield that

$$\begin{aligned} |a_{\mu;\Omega_-}(\mathbf{v}_1, \mathbf{v}_2)| &\le 2c_\mu \|\mathbb{E}(\mathbf{v}_1)\|_{L^2(\Omega_-)^{3\times 3}} \|\mathbb{E}(\mathbf{v}_2)\|_{L^2(\Omega_-)^{3\times 3}} \\ &\le 2c_\mu \|\mathbf{v}_1\|_{\mathcal{H}^1(\Omega_-)^3} \|\mathbf{v}\|_{\mathcal{H}^1(\Omega_-)^3}, \ \forall\, \mathbf{v}_1, \ \mathbf{v}_2 \in \mathring{\mathcal{H}}^1(\Omega_-)^3. \end{aligned} \tag{5.13}$$

Moreover, the formula

$$2\|\mathbb{E}(\mathbf{v})\|^2_{L^2(\Omega_-)^{3\times 3}} = \|\operatorname{grad} \mathbf{v}\|^2_{L^2(\Omega_-)^{3\times 3}} + \|\operatorname{div} \mathbf{v}\|^2_{L^2(\Omega_-)}, \ \forall\, \mathbf{v} \in \mathcal{D}(\Omega_-)^3 \tag{5.14}$$

(cf., e.g., the proof of Corollary 2.2 in [46]), and the density of the space $\mathcal{D}(\Omega_-)^3$ in $\mathring{\mathcal{H}}^1(\Omega_-)^3$ show that the same formula holds also for any function in $\mathring{\mathcal{H}}^1(\Omega_-)^3$. Therefore, we obtain the following Korn type inequality

$$\|\operatorname{grad} \mathbf{v}\|_{L^2(\Omega_-)^{3\times 3}} \le 2^{\frac{1}{2}} \|\mathbb{E}(\mathbf{v})\|_{L^2(\Omega_-)^{3\times 3}}, \ \forall\, \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3. \tag{5.15}$$

Then by using inequality (5.15), the equivalence of seminorm (2.12) to the norm (2.11) in the space $\mathcal{H}^1(\Omega_-)^3$, and assumption (3.1) we deduce that there exists a constant $C = C(\Omega_-) > 0$ such that

$$\|\mathbf{u}\|^2_{\mathcal{H}^1(\Omega_-)^3} \le C\|\mathrm{grad}\,\mathbf{u}\|^2_{L^2(\Omega_-)^{3\times 3}} \le 2C\|\mathbb{E}(\mathbf{u})\|^2_{L^2(\Omega_-)^{3\times 3}}$$
$$\le 2Cc_\mu\|\mu\mathbb{E}(\mathbf{u})\|^2_{L^2(\Omega_-)^{3\times 3}} = 2Cc_\mu a_{\mu;\Omega_-}(\mathbf{u},\mathbf{u}),\ \forall\,\mathbf{u}\in\mathring{\mathcal{H}}^1(\Omega_-)^3,$$

and accordingly that

$$a_{\mu;\Omega_-}(\mathbf{u},\mathbf{u}) \ge \frac{1}{2Cc_\mu}\|\mathbf{u}\|^2_{\mathcal{H}^1(\Omega_-)^3},\ \forall\,\mathbf{u}\in\mathring{\mathcal{H}}^1(\Omega_-)^3. \tag{5.16}$$

In view of inequalities (5.13) and (5.16) it follows that the bilinear form $a_{\mu;\Omega_-}(\cdot,\cdot) : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times \mathring{\mathcal{H}}^1(\Omega_-)^3 \to \mathbb{R}$ is bounded and coercive. Moreover, arguments similar to those for inequality (5.13) imply that the bilinear form $b_{\Omega_-}(\cdot,\cdot) : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-) \to \mathbb{R}$ and the linear form $\mathfrak{F}_{\mu;\mathbf{u}_0} : \mathring{\mathcal{H}}^1(\mathbb{R}^3)^3 \to \mathbb{R}$ given by (5.10) and (5.11), are also bounded. Since the operator

$$\mathrm{div} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \to L^2(\Omega_-) \tag{5.17}$$

is surjective (cf., e.g., [26, Theorem 3.2]), then by Lemma 2, the bounded bilinear form $b_{\Omega_-}(\cdot,\cdot) : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-) \to \mathbb{R}$ satisfies the inf-sup condition

$$\inf_{q\in L^2(\Omega_-)\setminus\{0\}}\ \sup_{\mathbf{v}\in\mathring{\mathcal{H}}^1(\Omega_-)^3\setminus\{\mathbf{0}\}} \frac{b_{\Omega_-}(\mathbf{v},q)}{\|\mathbf{v}\|_{\mathring{\mathcal{H}}^1(\Omega_-)^3}\|q\|_{L^2(\Omega_-)}} \ge \beta_D \tag{5.18}$$

with some constant $\beta_D > 0$ (cf. [26, Theorem 3.3]). Then Theorem 4 (with $X = \mathring{\mathcal{H}}^1(\Omega_-)^3$ and $M = L^2(\Omega_-)$) implies that the variational problem (5.8) has a unique solution $(\mathring{\mathbf{u}},\pi) \in \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-)$. Moreover, the pair $(\mathbf{u},\pi) = (\mathring{\mathbf{u}}+\mathbf{u}_0,\pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$, where $\mathbf{u}_0 \in \mathcal{H}^1(\Omega_-)^3$ satisfies relations (5.4), is the unique solution of the mixed variational formulation (5.3) and depends continuously on the given data $(\mathbf{f},\boldsymbol{\phi}) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}(\partial\Omega)^3$. The equivalence between the variational problem (5.3) and the exterior Dirichlet problem (5.1) shows that problem (5.1) is also well-posed, as asserted. □

5.2 *Potential Approach*

Theorem 5.1 asserts the well-posedness of the exterior Dirichlet problem for the Stokes system with L^∞ coefficients. However, if the given data $(\mathbf{f},\boldsymbol{\phi})$ belong to the space $\mathcal{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3$, then the solution can be expressed in terms of the

Newtonian and single layer potential and of the inverse of the single layer operator as follows (cf. [26, Theorem 3.4] for $\mu > 0$ constant, [22, Theorem 10.1] and [37, Theorem 5.1] for the Laplace operator).

Theorem 5.2 *If* $\mathbf{f} \in \mathcal{H}^{-1}(\Omega_-)^3$ *and* $\boldsymbol{\phi} \in H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3$ *then the exterior Dirichlet problem* (5.1) *has a unique solution* $(\mathbf{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$*, given by*

$$\mathbf{u} = \boldsymbol{\mathcal{N}}_{\mu;\mathbb{R}^3}(\tilde{\mathbf{f}})|_{\Omega_-} + \mathbf{V}_{\mu;\partial\Omega}\left(\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}^{-1}\big(\boldsymbol{\phi} - \gamma_-\big(\boldsymbol{\mathcal{N}}_{\mu;\mathbb{R}^3}(\tilde{\mathbf{f}})\big)\big)\right), \tag{5.19}$$

$$\pi = \mathcal{Q}_{\mu;\mathbb{R}^3}(\tilde{\mathbf{f}})|_{\Omega_-} + \mathcal{Q}^s_{\mu;\partial\Omega}\left(\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}^{-1}\big(\boldsymbol{\phi} - \gamma_-\big(\boldsymbol{\mathcal{N}}_{\mu;\mathbb{R}^3}(\tilde{\mathbf{f}})\big)\big)\right) \text{ in } \Omega_-, \tag{5.20}$$

where $\tilde{\mathbf{f}}$ *is an extension of* $\mathbf{f}$ *to an element of* $\mathcal{H}^1(\mathbb{R}^3)^3$.

Proof The result follows from Definition 4.3 and Lemmas 4.7, 4.8, and 4.10. □

Appendix: Mixed Variational Formulations and Their Well-Posedness Property

Here we make a brief review of well-posedness results due to Babuška [6] and Brezzi [10] for mixed variational formulations related to bounded bilinear forms in reflexive Banach spaces. We follow [20, Section 2.4], [11], and [25, §4].

Let X and $\mathcal{M}$ be reflexive Banach spaces, and let X^* and $\mathcal{M}^*$ be their dual spaces. Let $a(\cdot,\cdot) : X \times X \to \mathbb{R}$, $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be bounded bilinear forms. Then we consider the following abstract mixed variational formulation.

For $f \in X^*$, $g \in \mathcal{M}^*$ *given, find a pair* $(u, p) \in X \times \mathcal{M}$ *such that*

$$\begin{cases} a(u, v) + b(v, p) = f(v), & \forall\, v \in X, \\ b(u, q) \qquad\qquad\;\; = g(q), & \forall\, q \in \mathcal{M}. \end{cases} \tag{1}$$

Let $A : X \to X^*$ be the bounded linear operator defined by

$$\langle Av, w\rangle = a(v, w),\ \forall\, v, w \in X, \tag{2}$$

where $\langle\cdot,\cdot\rangle := {}_{X^*}\langle\cdot,\cdot\rangle_X$ is the duality pairing of the dual spaces X^* and X. We also use the notation $\langle\cdot,\cdot\rangle$ for the duality pairing ${}_{\mathcal{M}^*}\langle\cdot,\cdot\rangle_{\mathcal{M}}$. Let $B : X \to \mathcal{M}^*$ and $B^* : \mathcal{M} \to X^*$ be the bounded linear and transpose operators given by

$$\langle Bv, q\rangle = b(v, q),\ \langle v, B^*q\rangle = \langle Bv, q\rangle,\ \forall\, v \in X,\ \forall\, q \in \mathcal{M}. \tag{3}$$

In addition, we consider the spaces

$$V := \operatorname{Ker} B = \{v \in X : b(v, q) = 0,\ \forall\, q \in \mathcal{M}\}\,, \tag{4}$$

$$V^{\perp} := \left\{T \in X^* : \langle T, v\rangle = 0,\ \forall\, v \in V\right\}. \tag{5}$$

Then the following well-posedness result holds (cf., e.g., [20, Theorem 2.34]).

Theorem 1 *Let X and $\mathcal{M}$ be reflexive Banach spaces, $f \in X^*$ and $g \in \mathcal{M}^*$, and $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ and $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be bounded bilinear forms. Let V be the subspace of X defined by (4). Then the variational problem (1) is well-posed if and only if $a(\cdot,\cdot)$ satisfies the conditions*

$$\begin{cases} \exists\, \lambda > 0 \ \textit{ such that } \ \inf\limits_{u \in V\setminus\{0\}} \sup\limits_{v \in V\setminus\{0\}} \dfrac{a(u,v)}{\|u\|_X \|v\|_X} \geq \lambda, \\ \{v \in V : a(u,v) = 0,\ \forall\, u \in V\} = \{0\}, \end{cases} \tag{6}$$

and $b(\cdot,\cdot)$ satisfies the inf-sup (Ladyzhenskaya-Babuška-Brezzi) condition,

$$\exists\, \beta > 0 \ \textit{ such that } \ \inf_{q \in \mathcal{M}\setminus\{0\}} \sup_{v \in X\setminus\{0\}} \frac{b(v,q)}{\|v\|_X \|q\|_{\mathcal{M}}} \geq \beta. \tag{7}$$

Moreover, there exists a constant C depending on β, λ and the norm of $a(\cdot,\cdot)$, such that the unique solution $(u, p) \in X \times \mathcal{M}$ of (1) satisfies the inequality

$$\|u\|_X + \|p\|_{\mathcal{M}} \leq C\left(\|f\|_{X^*} + \|g\|_{\mathcal{M}^*}\right). \tag{8}$$

In addition, we have (see [20, Theorem A.56, Remark 2.7], [4, Theorem 2.7]).

Lemma 2 *Let $X, \mathcal{M}$ be reflexive Banach spaces. Let $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be a bounded bilinear form. Let $B : X \to \mathcal{M}^*$ and $B^* : \mathcal{M} \to X^*$ be the operators defined by (3), and let $V = \operatorname{Ker} B$. Then the following results are equivalent:*

- (i) *There exists a constant $\beta > 0$ such that $b(\cdot,\cdot)$ satisfies condition (7).*
- (ii) *$B : X/V \to \mathcal{M}^*$ is an isomorphism and $\|Bw\|_{\mathcal{M}^*} \geq \beta\|w\|_{X/V}$ for any $w \in X/V$.*
- (iii) *$B^* : \mathcal{M} \to V^{\perp}$ is an isomorphism and $\|B^*q\|_{X^*} \geq \beta\|q\|_{\mathcal{M}}$ for any $q \in \mathcal{M}$.*

Remark 3 Let X be a reflexive Banach space and V be a closed subspace of X. If a bounded bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is *coercive* on V, i.e., there exists a constant $c_a > 0$ such that

$$a(w,w) \geq c_a\|w\|_X^2,\ \forall\, w \in V, \tag{9}$$

then the conditions (6) are satisfied as well (see, e.g., [20, Lemma 2.8]).

The next result known as the *Babuška-Brezzi theorem* is the version of Theorem 1 for Hilbert spaces (see [6], [10, Theorems 0.1, 1.1, Corollary 1.2]).

Theorem 4 *Let X and $\mathcal{M}$ be two real Hilbert spaces. Let $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ and $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be bounded bilinear forms. Let $f \in X^*$ and $g \in \mathcal{M}^*$. Let V be the subspace of X defined by* (4)*. Assume that $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is coercive and that $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ satisfies the inf-sup condition* (7)*. Then the variational problem* (1) *is well-posed.*

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