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Layer potential theory for the anisotropic Stokes system with variable L_∞ symmetrically elliptic tensor coefficient

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The aim of this paper is to develop a layer potential theory in L_2-based weighted Sobolev spaces on Lipschitz bounded and exterior domains of $\mathbb{R}^n$, $n \geq 3$, for the anisotropic Stokes system with L_∞ viscosity tensor coefficient satisfying an ellipticity condition for symmetric matrices with zero matrix trace. To do this, we explore equivalent mixed variational formulations and prove the well-posedness of some transmission problems for the anisotropic Stokes system in Lipschitz domains of $\mathbb{R}^n$, with the given data in L_2-based weighted Sobolev spaces. These results are used to define the volume (Newtonian) and layer potentials and to obtain their properties. Then, we analyze the well-posedness of the exterior Dirichlet and Neumann problems for the anisotropic Stokes system with L_∞ symmetrically elliptic tensor coefficient by representing their solutions in terms of the obtained volume and layer potentials.

KEYWORDS

anisotropic Stokes system, discontinuous coefficients, exterior Dirichlet and Neumann problems, Newtonian and layer potentials, partial differential equations, potential theory, transmission problems, variational problem, weighted Sobolev spaces, well-posedness

MSC CLASSIFICATION

35J25; 35Q35; 42B20; 46E35; 76D; 76M

1 | INTRODUCTION

The layer potential methods play a fundamental role in the analysis of elliptic boundary value problems (see, e.g., previous studies[1-6]). Fabes et al.[7] obtained mapping properties of layer potential operators for the constant coefficient Stokes system in L_p spaces by using a technique of harmonic analysis. Further extensions of these results to L_p, Sobolev, Bessel potential, and Besov spaces have been obtained by Mitrea and Wright[5] using layer potential methods to obtain well-posedness results for the main boundary value problems for the standard Stokes system with constant coefficients in arbitrary Lipschitz domains in $\mathbb{R}^3$. Kohr et al.[8] obtained mapping properties of the constant-coefficient Stokes and Brinkman layer potential operators in standard and weighted Sobolev spaces in $\mathbb{R}^3$. Kohr et al.[9] combined a layer potential approach with a fixed point theorem to show an existence result for a nonlinear Neumann-transmission problem for the constant-coefficient Stokes and Brinkman systems in L_p, Sobolev, and Besov spaces (see also Kohr et al.[10]).

Choi and Lee[11] have studied the Dirichlet problem for the stationary Stokes system with irregular coefficients. They have proved the unique solvability of the problem in Sobolev spaces on a Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$, with a small Lipschitz constant, by assuming that the coefficients have vanishing mean oscillations (VMO) with respect to all variables. Existence and pointwise bounds of the fundamental solution for the stationary Stokes system with measurable coefficients in $\mathbb{R}^n$ ($n \geq 3$) have been obtained by Choi and Yang[12] under the assumption of local Hölder continuity of weak solutions of the Stokes system. They also discussed the existence and pointwise bounds of the Green function for the Stokes system with measurable coefficients on unbounded domains where the divergence equation is solvable, particularly on the half-space. The solvability in Sobolev spaces of the conormal derivative problem for the stationary Stokes system with nonsmooth coefficients on bounded Reifenberg flat domains have been proved by Choi et al.[13] (see also Choi et al.[14]).

The methods of layer potential theory play also a significant role in the study of elliptic boundary value problems with variable coefficients. Mitrea and Taylor[15] have obtained well-posedness results for the Dirichlet problem for the smooth coefficient Stokes system in L_p spaces on arbitrary Lipschitz domains in a compact Riemannian manifold and extended the well-posedness results by Fabes et al.[7] from the Euclidean setting to the compact Riemannian setting. Dindos and Mitrea[3] have used the mapping properties of Stokes layer potentials in Sobolev and Besov spaces to show well-posedness results for Poisson problems for the smooth coefficient Stokes and Navier–Stokes systems with Dirichlet boundary condition on C^1 and Lipschitz domains in compact Riemannian manifolds. Well-posedness results for transmission problems for the smooth coefficient Navier–Stokes and Darcy–Forchheimer–Brinkman systems in Lipschitz domains on compact Riemannian manifolds have been obtained by Kohr et al.[16]

An alternative approach was employed by Chkadua et al.,[17-22] where various boundary value problems for variable-coefficient elliptic partial differential equations were reduced to explicit parametrix-based *boundary-domain integral equations* (BDIEs). Equivalence of BDIEs to the boundary value problems and invertibility of BDIE operators in L_2 and L_p-based Sobolev spaces have been analyzed in these papers. Localized BDIEs based on a harmonic parametrix for divergence-form elliptic PDEs with variable matrix coefficients have been also developed; see Chkadua et al.[23] and the references therein.

Amrouche et al.[24] used a variational approach in the analysis of the exterior Dirichlet and Neumann problems for the n-dimensional Laplace operator in weighted Sobolev spaces. Mazzucato and Nistor[25] obtained well-posedness and regularity results for the elasticity equations with mixed conditions on polyhedral domains. Hofmann et al.[26] considered layer potentials in L_p spaces for elliptic operators of the form $L = -\mathrm{div}(A\nabla u)$ that act in the upper half-space $\mathbb{R}^{n+1}_+ := \{(x,t) : x \in \mathbb{R}^n, t \in \mathbb{R}_+\}$, $n \geq 2$, or in more general Lipschitz graph domains, where A is an $(n+1)\times(n+1)$ type matrix of L_∞ complex, t-independent coefficients satisfying a uniform ellipticity condition, and solutions of the equation $Lu = 0$ satisfying De Giorgi–Nash–Moser-type interior estimates. They developed a Calderón–Zygmund-type theory associated with the layer potentials and obtained well-posedness results for related boundary problems in L_p and endpoint spaces. Brewster et al.[27] have used a variational approach to obtain well-posedness results for Dirichlet, Neumann, and mixed boundary problems for higher order divergence-form elliptic equations with L_∞ coefficients in locally (ϵ, δ)-domains and in Besov and Bessel potential spaces (see also Haller-Dintelmann et al.[28]). Barton[29] has used the Lax–Milgram lemma to construct layer potentials for strongly elliptic differential operators in Banach spaces and generalized many properties of layer potentials for the harmonic equation. Barton and Mayboroda[30] developed layer potentials for second-order divergence elliptic operators with bounded measurable coefficients that are independent of the $(n+1)$st coordinate and well-posedness results for related boundary problems with data in Besov spaces.

Girault and Sequeira[31] obtained well-posedness of the exterior Dirichlet problem for the constant coefficient Stokes system in weighted Sobolev spaces on exterior Lipschitz domains in $\mathbb{R}^n$ for $n \in \{2, 3\}$, by applying a mixed variational formulation. Angot[32] analyzed some Stokes/Brinkman transmission problems with a scalar viscosity coefficient on bounded domains. Sayas and Selgas[33] developed a variational approach for the constant-coefficient Stokes layer potentials on Lipschitz boundaries, by using the technique of Nédélec.[34] The book by Sayas et al.[35] gives a comprehensive presentation of the basic variational theory for elliptic PDEs in Lipschitz domains. Băcuţă et al.[36] developed a variational approach for the constant-coefficient Brinkman single-layer potential and used it to analyze the corresponding time dependent exterior Dirichlet problem in $\mathbb{R}^n$, $n = 2, 3$. Alliot and Amrouche[37] have used a variational approach to obtain weak solutions for the exterior Stokes problem in weighted Sobolev spaces (see also Amrouche and Nguyen[38]).

Kohr et al.[39] obtained the well-posedness results for the *isotropic* Stokes system with a nonsmooth scalar viscosity coefficient $\mu \in L_\infty(\mathbb{R}^3)$ (see also previous studies[40-42] for the Stokes and Navier–Stokes systems with nonsmooth coefficients in compact Riemannian manifolds). Kohr et al.[43] also analyzed transmission problems in weighted Sobolev spaces for *anisotropic* Stokes and Navier–Stokes systems with an L_∞ strongly elliptic coefficient tensor, in the pseudostress setting.

In this paper, we proceed with the study of transmission and exterior boundary value problems for the anisotropic Stokes system. However, unlike paper,[43] we consider the L_∞ viscosity coefficient tensor satisfying an ellipticity condition only with respect to all *symmetric* matrices with zero matrix trace (see 1.4). Our purpose is to develop a generalized layer and volume potential theory in L_2-based weighted Sobolev spaces for such Stokes systems, which does not involve fundamental solutions and hence can be used when the fundamental solutions are not available. To do this, we explore equivalent mixed variational formulations and prove the well-posedness of some transmission problems for the anisotropic Stokes system in Lipschitz domains of $\mathbb{R}^n$, with the given data in L_2-based weighted Sobolev spaces. These results are used to define the volume and layer potentials in terms of solutions of the transmission problems and to obtain the potential properties, without introducing classical explicit integral potential operators. However, when the explicit integral representations of the potentials are available, they will coincide with the variational potentials developed here due to the uniqueness of solutions to the corresponding transmission problems.

Then, we analyze well-posedness of the exterior Dirichlet and Neumann problems for the anisotropic Stokes system with L_∞ tensor coefficient satisfying ellipticity condition (1.4) and represent their solutions in terms of the anisotropic Stokes Newtonian and layer potentials. Although these boundary value problems can be analyzed by variational methods directly, without employing the potential formalism, they are provided here as examples on how, using this formalism, one can easily generalize the classical potential approaches, available for constant-coefficient isotropic problems, to the discontinuous-coefficient anisotropic ones. The potential theory developed in this paper can be also useful when new fundamental solutions and potentials based on them become available. In this case, the potential properties can be obtained from the results developed here.

This paper deals with the potentials in $\mathbb{R}^n$, $n \geq 3$. Its results can be extended to $\mathbb{R}^2$ as well, but then, the analysis should be done in slightly different weighted Sobolev spaces.

Note that the boundary value problems for the anisotropic Stokes system with L_∞ coefficients considered in this paper can describe physical, engineering, or industrial processes related to the flow of immiscible fluids, or the flow of nonhomogeneous fluids with density dependent viscosity (cf., e.g., Choi et al.[13]). They appear also in modeling incompressible elastic anisotropic nonhomogeneous/composite materials.

1.1 | The anisotropic Stokes system with L_∞ symmetrically elliptic tensor coefficient

All along the paper, we use the Einstein summation convention for repeated indices from 1 to n, and the standard notation ∂_α for the first-order partial derivatives $\frac{\partial}{\partial x_\alpha}$, $\alpha = 1, \dots, n$.

Let $\mathbb{L}$ be a second-order differential operator in the divergence form in an open set $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$,

$$\mathbb{L}\mathbf{u} = \operatorname{div}(\mathbb{A}\mathbb{E}(\mathbf{u})) \iff (\mathbb{L}\mathbf{u})_i := \partial_\alpha \left(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}) \right),\ i = 1, \dots, n, \tag{1.1}$$

where $\mathbf{u} = (u_1, \dots, u_n)^\top$ and $\mathbb{E}(\mathbf{u}) = \left(E_{j\beta}(\mathbf{u})\right)_{1 \leq j,\beta \leq n}$ is the symmetric part of the gradient $\nabla \mathbf{u}$. Therefore, the components of the tensor field $\mathbb{E}(\mathbf{u})$ are defined by $E_{j\beta}(\mathbf{u}) := \frac{1}{2}(\partial_j u_\beta + \partial_\beta u_j)$.

The viscosity tensor coefficient $\mathbb{A}$ in the operator $\mathbb{L}$ consists of $n \times n$ matrix-valued functions $A^{\alpha\beta} = A^{\alpha\beta}(x)$ with essentially bounded, real-valued entries, that is,

$$\mathbb{A} = \left(A^{\alpha\beta}\right)_{1 \leq \alpha,\beta \leq n} = \left(a_{ij}^{\alpha\beta}\right)_{1 \leq \alpha,\beta,i,j \leq n};\ a_{ij}^{\alpha\beta} \in L_\infty(\Omega),\ 1 \leq \alpha, \beta, i, j \leq n, \tag{1.2}$$

satisfying the symmetry conditions

$$a_{ij}^{\alpha\beta}(x) = a_{\alpha j}^{i\beta}(x) = a_{i\beta}^{\alpha j}(x),\ x \in \Omega, \tag{1.3}$$

(cf. Oleinik et al.,[44], eq. (2.2) and Duffy,[45], eqs. (6) and (7)). Note that the symmetry conditions (1.3) *do not imply* the symmetry $a_{ij}^{\alpha\beta}(x) = a_{ji}^{\beta\alpha}(x)$, which will be generally not assumed in the paper.

We assume that the coefficients satisfy the following relaxed *ellipticity* condition, which asserts that there exists a constant $c_{\mathbb{A}} > 0$ such that for almost all $x \in \Omega$,

$$a_{ij}^{\alpha\beta}(x)\xi_{i\alpha}\xi_{j\beta} \geq c_{\mathbb{A}}^{-1}|\xi|^2 \quad \forall \xi = (\xi_{i\alpha})_{i,\alpha=1,\dots,n} \in \mathbb{R}^{n\times n} \text{ with } \xi = \xi^\top \text{ and } \sum_{i=1}^n \xi_{ii} = 0, \tag{1.4}$$

where $|\xi|^2 = \xi_{i\alpha}\xi_{i\alpha}$. Therefore, the ellipticity condition (1.4) is assumed only for all *symmetric* matrices $\xi = (\xi_{i\alpha})_{i,\alpha=1,\dots,n} \in \mathbb{R}^{n\times n}$ (cf. Oleinik et al.[44], eqs. (3.1) and (3.2)), having zero matrix trace, $\sum_{i=1}^{n}\xi_{ii} = 0$.

In view of (1.2), $\mathbb{A}$ is endowed with the norm

$$\|\mathbb{A}\|_{L_\infty(\Omega)} := \max_{i,j,\alpha,\beta\in\{1,\dots,n\}} \left\{ \|a_{ij}^{\alpha\beta}\|_{L_\infty(\Omega)} \right\}. \tag{1.5}$$

The symmetry conditions (1.3) allow us to express the operator $\mathbb{L}$ in the equivalent forms

$$(\mathbb{L}\mathbf{u})_i = \partial_\alpha \left(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}) \right) = \partial_\alpha \left(a_{ij}^{\alpha\beta} \partial_\beta u_j \right),\ i = 1, \dots, n, \tag{1.6}$$

$$\mathbb{L}\mathbf{u} = \partial_\alpha \left(A^{\alpha\beta} \partial_\beta \mathbf{u} \right). \tag{1.7}$$

Note that the first equality in (1.6) has not been encountered in our publication,[43] where the coefficients of the fourth-order tensor $\mathbb{A}$ have been assumed to satisfy the strong ellipticity condition similar to the second condition in (1.4) but for all (not only symmetric and zero-trace) matrices $\boldsymbol{\xi}$ (see Kohr et al.[43], eq. 2 and 3). The more restrictive ellipticity condition in paper[43] allowed to explore there the associated nonsymmetric pseudostress setting. In this paper, we require the symmetry conditions (1.3) and the ellipticity condition (1.4) only for symmetric zero-trace matrices $\boldsymbol{\xi}$ and develop our results in the symmetric stress setting. This approach allows us to obtain properties of layer potentials for the Stokes system with L_∞ variable coefficients generalizing well-known results for constant coefficients.

Let $\mathbf{u}$ be an unknown vector field, π be an unknown scalar field, and $\mathbf{f}$ and g be, respectively, vector and scalar fields defined in $\Omega \subseteq \mathbb{R}^n$. Then, the equations

$$\mathcal{L}(\mathbf{u}, \pi) := \mathbb{L}\mathbf{u} - \nabla\pi = \boldsymbol{f},\ \ \text{div }\ \mathbf{u} = g \text{ in } \ \Omega \tag{1.8}$$

determine the Stokes system which describes viscous compressible fluid flows with variable anisotropic viscosity tensor coefficient $\mathbb{A}$ depending on the physical properties of the fluid, such as, for example, the given fluid temperature.[45,46] If $g = 0$, then the fluid is incompressible.

According to (1.6) and (1.7), the Stokes operator $\mathcal{L}$ can be written in any of the equivalent forms

$$\mathcal{L}(\mathbf{u}, \pi) = \partial_\alpha \left(A^{\alpha\beta} \partial_\beta \mathbf{u} \right) - \nabla\pi,\ (\mathcal{L}(\mathbf{u}, \pi))_i = \partial_\alpha \left(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}) \right) - \partial_i \pi,\ i = 1, \dots, n. \tag{1.9}$$

Under condition (1.4), the anisotropic Stokes system (1.8) is Agmon–Douglis–Nirenberg elliptic (see Lemma 15).

1.2 | Isotropic case

For the *isotropic case*, the viscosity tensor $\mathbb{A}$ in (1.2) has the form (cf., e.g., appendix III, part I, section 1 in Temam[47])

$$a_{ij}^{\alpha\beta}(x) = \lambda(x)\delta_{i\alpha}\delta_{j\beta} + \mu(x)\left(\delta_{\alpha j}\delta_{\beta i} + \delta_{\alpha\beta}\delta_{ij} \right),\ 1 \le i, j, \alpha, \beta \le n, \tag{1.10}$$

where $\lambda, \mu \in L_\infty(\Omega)$ and

$$c_\mu^{-1} \le \mu(x) \le c_\mu \ \text{ for a.e.} x \in \Omega, \tag{1.11}$$

with a constant $c_\mu > 0$. Then,

$$a_{ij}^{\alpha\beta}(x)\xi_{i\alpha}\xi_{j\beta} = \lambda(x)(\xi_{ii})^2 + 2\mu(x)\xi_{i\alpha}\xi_{i\alpha} = 2\mu(x)\xi_{i\alpha}\xi_{i\alpha} = 2\mu(x)|\xi|^2 \ge 2c_\mu^{-1}|\xi|^2 \text{ for a.e. } x \in \Omega,$$

for any symmetric matrix $\xi = (\xi_{i\alpha})_{1\le i,\alpha\le n} \in \mathbb{R}^{n\times n}$ such that $\xi_{ii} = \sum_{i=1}^{n}\xi_{ii} = 0$. Therefore, the symmetric ellipticity condition (1.4) is satisfied as well, and hence, our results are also applicable to the *Stokes system in the isotropic case*. If $\mu > 0$ is a constant and $g = 0$, then (1.8) reduces to the well-known isotropic incompressible Stokes system with constant viscosity μ.

2 | FUNCTIONAL SETTING AND PRELIMINARY RESULTS

Let Ω_+ be a bounded Lipschitz domain in $\mathbb{R}^n$, that is, an open connected set whose boundary $\partial\Omega$ is locally the graph of a Lipschitz function and is connected. We further assume that $n \geq 3$ unless explicitly stated otherwise. Sometimes, we will write just Ω instead of Ω_+. Let $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}_+$ be the corresponding exterior Lipschitz domain. Let $\mathring{E}_\pm$ denote the operator of extension of functions by zero outside $\Omega_\pm$.

2.1 | L_2-based Sobolev spaces

Given a Banach space $\mathcal{X}$, its topological dual is denoted by $\mathcal{X}'$, and the notation $\langle \cdot , \cdot \rangle_X$ means the duality pairing of two dual spaces defined on a set $X \subseteq \mathbb{R}^n$.

Let Ω' be a nonempty open set in $\mathbb{R}^n$ or just $\mathbb{R}^n$. Let $L_2(\Omega')$ denote the Lebesgue space of (equivalence classes of) measurable, square-integrable functions on Ω', and $L_\infty(\Omega')$ denote the space of (equivalence classes of) essentially bounded measurable functions on Ω'. Let us define the L_2-based Sobolev space $H^1(\Omega') = W_2^1(\Omega') := \left\{ f \in L_2(\Omega') : \nabla f \in L_2(\Omega')^n \right\}$ endowed with the norm

$$\|f\|_{H^1(\Omega')} = \sqrt{\|f\|^2_{L_2(\Omega')} + \|\nabla f\|^2_{L_2(\Omega')^n}}. \tag{2.1}$$

Here, $L_2(\Omega')^n$ denotes the space of vector-valued functions whose components belong to the scalar space $L_2(\Omega')$. Similar notations are assumed also for other vector-valued and matrix-valued spaces.

Let $\mathcal{D}(\Omega') := C_0^\infty(\Omega')$ denote the space of infinitely differentiable functions with compact support in Ω', equipped with the inductive limit topology. Let $\mathcal{D}'(\Omega')$ denote the corresponding space of distributions on Ω', that is, the dual of the space $\mathcal{D}(\Omega')$.

Let Ω'' be either a bounded Lipschitz domain or the exterior of a bounded Lipschitz domain in $\mathbb{R}^n$. The space $\widetilde{H}^1(\Omega'')$ is the closure of $\mathcal{D}(\Omega'')$ in $H^1(\mathbb{R}^n)$. It can be also characterized as

$$\widetilde{H}^1\Omega'') := \left\{ \widetilde{f} \in H^1(\mathbb{R}^n) : \operatorname{supp} \widetilde{f} \subseteq \overline{\Omega'')} \right\}, \tag{2.2}$$

where $\operatorname{supp} f := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$ (see, e.g., Theorem 3.33 in McLean[48]).

The dual of $H^1(\mathbb{R}^n)$ is denoted as $H^{-1}(\mathbb{R}^n)$, while the dual of $H^1(\Omega'')$ as $\widetilde{H}^{-1}(\Omega'')$ and the dual of $\widetilde{H}^1(\Omega'')$ as $H^{-1}(\Omega'')$.

The boundary Sobolev space $H^s(\partial\Omega)$, $0 < s < 1$, can be defined by

$$H^s(\partial\Omega) = \left\{ f \in L_2(\partial\Omega) : \int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(\mathbf{x}) - f(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{n-1+2s}} d\sigma_{\mathbf{x}} d\sigma_{\mathbf{y}} < \infty \right\},$$

where $\sigma_{\mathbf{y}}$ is the surface measure on $\partial\Omega$ (see, e.g., Proposition 2.5.1 in Mitrea and Wright[5]). The dual of $H^s(\partial\Omega)$ is the space $H^{-s}(\partial\Omega)$, and we set $H^0(\partial\Omega) = L_2(\partial\Omega)$. Let $H^s(\partial\Omega)^n$ denote the space of vector-valued functions whose components belong to $H^s(\partial\Omega)$. The dual of $H^s(\partial\Omega)^n$ is the space $H^{-s}(\partial\Omega)^n$.

All L_2-based Sobolev spaces mentioned above are Hilbert spaces. The following well-known trace theorem holds true (cf. Costabel[1] and McLean[48]).

Theorem 1. *Let $\Omega := \Omega_+$ be a bounded Lipschitz domain of $\mathbb{R}^n$ with connected boundary $\partial\Omega$, and let $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ be the corresponding exterior domain. Then, there exist linear bounded trace operators $\gamma_\pm : H^1(\Omega_\pm) \to H^{\frac{1}{2}}(\partial\Omega)$ such that $\gamma_\pm f = f_{|\partial\Omega}$ for any $f \in C^\infty(\overline{\Omega}_\pm)$. The operators $\gamma_\pm$ are surjective and have (nonunique) linear and bounded right inverse operators $\gamma_\pm^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega_\pm)$. The trace operator $\gamma : H^1(\mathbb{R}^n) \to H^{\frac{1}{2}}(\partial\Omega)$ is linear and bounded as well.*[*]

Note that any function $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ has the jump

$$[\gamma(u)] := \gamma_+(u) - \gamma_-(u) \tag{2.3}$$

equal to zero across $\partial\Omega$.

Further properties of Sobolev spaces can be found in the literature.[5,48-50]

[*] The trace operators defined on Sobolev spaces of vector fields on $\Omega_\pm$ or $\mathbb{R}^n$ are also denoted by $\gamma_\pm$ and γ, respectively.

2.2 | Weighted Sobolev spaces

Let $|\mathbf{x}| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$ denote the Euclidean distance of a point $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ to the origin of $\mathbb{R}^n$. Let ρ be the weight function

$$\rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{\frac{1}{2}}. \tag{2.4}$$

2.2.1 | Weighted Sobolev spaces on $\mathbb{R}^n$

The weighted Lebesgue space $L_2(\rho^{-1}; \mathbb{R}^n)$ is defined by

$$L_2(\rho^{-1}; \mathbb{R}^n) := \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : \rho^{-1} f \in L_2(\mathbb{R}^n) \right\} \tag{2.5}$$

and has a Hilbert space structure with respect to the inner product and the associated norm

$$(f, g)_{L_2(\rho^{-1};\mathbb{R}^n)} := \int_{\mathbb{R}^n} f g \rho^{-2} dx, \quad \|f\|^2_{L_2(\rho^{-1};\mathbb{R}^n)} := (f, f)_{L_2(\rho^{-1};\mathbb{R}^n)}. \tag{2.6}$$

We also consider the weighted Sobolev space

$$\mathcal{H}^1(\mathbb{R}^n) := \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : \rho^{-1} f \in L_2(\mathbb{R}^n), \nabla f \in L_2(\mathbb{R}^n)^n \right\}, n \geq 3 \tag{2.7}$$

(cf. Definition 1.1 in Alliot and Amrouche[37] and Theorem I.1 in Hanouzet[51]), which is also a Hilbert space with the norm defined by

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} := \sqrt{\|\rho^{-1} f\|^2_{L_2(\mathbb{R}^n)} + \|\nabla f\|^2_{L_2(\mathbb{R}^n)^n}}. \tag{2.8}$$

The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{H}^1(\mathbb{R}^n)$ (cf., e.g., Alliot and Amrouche[52], p. 727; Theorem I.1 in Hanouzet[51] and Proposition 2.1 in Sayas and Selgas,[33] in the case $n = 3$), and thus, the dual $\mathcal{H}^{-1}(\mathbb{R}^n)$ of $\mathcal{H}^1(\mathbb{R}^n)$ is a space of distributions. Let us consider the seminorm

$$|f|_{\mathcal{H}^1(\mathbb{R}^n)} := \|\nabla f\|_{L_2(\mathbb{R}^n)^n}. \tag{2.9}$$

This seminorm is a norm on the space $\mathcal{H}^1(\mathbb{R}^n)$ and is equivalent to the norm $\|\cdot\|_{\mathcal{H}^1(\mathbb{R}^n)}$, given by (2.8) (cf., e.g., Theorem 1.1 in Alliot and Amrouche[52]).

In view of Lemma 2.5 of Kozono and Sohr,[53] the divergence operator div : $\mathcal{H}^1(\mathbb{R}^n)^n \to L_2(\mathbb{R}^n)$ is surjective and has a bounded right inverse. Moreover, Remark 3.8(i) of Alliot and Amrouche[52] and Proposition 2.4(i) of Kozono and Sohr[53] imply that for $n \geq 3$, the weighted Sobolev space $\mathcal{H}^1(\mathbb{R}^n)$ can be also characterized as

$$\mathcal{H}^1(\mathbb{R}^n) = \left\{ u \in L_{\frac{2n}{n-2}}(\mathbb{R}^n) : \nabla u \in L_2(\mathbb{R}^n)^n \right\}, \tag{2.10}$$

with equivalent norms.

2.2.2 | Weighted Sobolev spaces on exterior Lipschitz domains

The weighted Sobolev space $\mathcal{H}^1(\Omega_-)$ can be defined as in (2.7) with Ω_- in place of $\mathbb{R}^n$. Therefore,

$$\mathcal{H}^1(\Omega_-) := \left\{ v \in \mathcal{D}'(\Omega_-) : \rho^{-1} v \in L_2(\Omega_-), \nabla v \in L_2(\Omega_-)^n \right\}, n \geq 3 \tag{2.11}$$

is a Hilbert space with a norm given by (2.8) with Ω_- in place of $\mathbb{R}^n$ (see, e.g., Definition 1.1 in Alliot and Amrouche[37]). The space $\widetilde{\mathcal{H}}^{-1}(\Omega_-)$ is the dual of the space $\mathcal{H}^1(\Omega_-)$.

Next, we mention some useful properties of these spaces. First, note that the space $\mathcal{D}(\overline{\Omega}_-)$ is dense in $\mathcal{H}^1(\Omega_-)$. Moreover, the functions of $\mathcal{H}^1(\Omega_-)$ belong to $H^1(D)$ for any bounded domain D contained in Ω_- (see also Alliot and Amrouche[37]). Since $H^1(\Omega_-) \subset \mathcal{H}^1(\Omega_-)$, the statement of Theorem 1 extends also to the weighted Sobolev space $\mathcal{H}^1(\Omega_-)$. Therefore, there exists a bounded linear and surjective exterior trace operator

$$\gamma_- : \mathcal{H}^1(\Omega_-) \to H^{\frac{1}{2}}(\partial\Omega), \tag{2.12}$$

which has a (nonunique) bounded linear right inverse $\gamma_-^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to \mathcal{H}^1(\Omega_-)$ (see Lemma 2.2 in Kohr et al.,[8] Theorem 2.3 and Lemma 2.6 in Mikhailov,[54] and p.69 in Sayas and Selgas[33]). The trace operator $\gamma : \mathcal{H}^1(\mathbb{R}^n) \to H^{\frac{1}{2}}(\partial\Omega)$ defined by $\gamma(u) = \gamma_+(u_+) = \gamma_-(u_-)$ for any $u \in \mathcal{H}^1(\mathbb{R}^n)$, where $u_\pm := u|_{\Omega_\pm}$, is bounded linear and surjective as well (cf., e.g., Theorem 2.3 and Lemma 2.6 in Mikhailov[54] and formula (2.2) in Băcuţă et al.[36]).

Let us now consider the space $\mathring{\mathcal{H}}^1(\Omega_-)$ as the closure of the space $\mathcal{D}(\Omega_-)$ with respect to the norm $\|\cdot\|_{\mathcal{H}^1(\Omega_-)}$ defined as in (2.8), with Ω_- in place of $\mathbb{R}^n$ (cf., e.g., Amrouche et al.,[24] Definition 1.1 in Alliot and Amrouche,[37] and Theorem 2.1 in Ch. 1 in Giroire[55]). This is a Hilbert space that can be also characterized as

$$\mathring{\mathcal{H}}^1(\Omega_-) = \left\{ v \in \mathcal{H}^1(\Omega_-) : \gamma_- v = 0 \text{ on } \partial\Omega \right\} \tag{2.13}$$

(see Amrouche et al.[24], eq. (1.2) and Theorem 4.2 in Brewster et al.[27]). The space $\mathcal{D}(\Omega_-)$ is dense in $\mathring{\mathcal{H}}^1(\Omega_-)$. Hence, the dual of $\mathring{\mathcal{H}}^1(\Omega_-)$ denoted by $\mathcal{H}^{-1}(\Omega_-)$ is a subspace of $\mathcal{D}'(\Omega_-)$. In addition, the seminorm

$$|f|_{\mathcal{H}^1(\Omega_-)} := \|\nabla f\|_{L_2(\Omega_-)^n} \tag{2.14}$$

is a norm on $\mathring{\mathcal{H}}^1(\Omega_-)$ that is equivalent to the full norm $\|\cdot\|_{\mathcal{H}^1(\Omega_-)}$ given by (2.8) with Ω_- in place of $\mathbb{R}^n$ (cf., e.g., Theorem 1.2 in Amrouche et al.[24] and Theorem 1.2 (ii) in Alliot and Amrouche[37]).

We need also the space $\widetilde{\mathcal{H}}^1(\Omega_-) \subset \mathcal{H}^1(\mathbb{R}^n)$, defined as the closure of $\mathcal{D}(\Omega_-)$ in $\mathcal{H}^1(\mathbb{R}^n)$. This space can be also characterized as (see, e.g., formula (2.9) in Brewster et al.[27])

$$\widetilde{\mathcal{H}}^1(\Omega_-) = \left\{ u \in \mathcal{H}^1(\mathbb{R}^n) : \operatorname{supp} u \subseteq \overline{\Omega}_- \right\} \tag{2.15}$$

and can be identified isomorphically with $\mathring{\mathcal{H}}^1(\Omega_-)$ via the operator $\mathring{E}_-$ of extension by zero outside Ω_-.

By $\mathcal{H}^{\pm 1}(\mathbb{R}^n)^n$ and $\mathcal{H}^{\pm 1}(\Omega_-)^n$, we denote the spaces of vector-valued functions or distributions whose components belong to the spaces $\mathcal{H}^1(\mathbb{R}^n)$ and $\mathcal{H}^1(\Omega_-)$, respectively.

Remark 1. The weighted Sobolev space $\mathcal{H}^1(\Omega_+)$ can be defined as in formula (2.7) with Ω_+ in place of $\mathbb{R}^n$. The dual of the space $\mathcal{H}^1(\Omega_+)$ is denoted by $\widetilde{\mathcal{H}}^{-1}(\Omega_+)$. Let also $\mathring{\mathcal{H}}^1(\Omega_+)$ be the weighted space defined as the closure of the space $\mathcal{D}(\Omega_+)$ in $\mathcal{H}^1(\Omega_+)$, and let $\mathcal{H}^{-1}(\Omega_+)$ be its dual. Since Ω_+ is a bounded Lipschitz domain, we have that $\mathcal{H}^1(\Omega_+) = H^1(\Omega_+)$ and $\mathcal{H}^{-1}(\Omega_+) = H^{-1}(\Omega_+)$ (with equivalent norms).

2.2.3 | Weighted Sobolev spaces on $\mathbb{R}^n \backslash \partial\Omega$

We also consider the weighted space

$$\mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega) := \left\{ f \in L_2(\rho^{-1}; \mathbb{R}^n) : \nabla f \in L_2(\Omega_\pm)^n \right\}, \; n \geq 3. \tag{2.16}$$

This is a Hilbert space with the norm defined by

$$\|f\|^2_{\mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)} = \|\rho^{-1} f\|^2_{L_2(\mathbb{R}^n)} + \|\nabla f\|^2_{L_2(\Omega_-)^n} + \|\nabla f\|^2_{L_2(\Omega_+)^n}, \tag{2.17}$$

which is equivalent to the norm $(\|f\|^2_{H^1(\Omega_+)} + \|f\|^2_{\mathcal{H}^1(\Omega_-)})^{1/2}$ on $\mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)$.

Note that $f|_{\Omega_+} \in H^1(\Omega_+)$ and $f|_{\Omega_-} \in \mathcal{H}^1(\Omega_-)$, whenever $f \in \mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)$, and f could have a *jump across* $\partial\Omega$ denoted by $[\gamma(f)] := \gamma_+(f) - \gamma_-(f) = \gamma_+(f_+) - \gamma_-(f_-)$, where $f_\pm := f|_{\Omega_\pm}$. However, if $\mathbf{f} \in \mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)$ and $[\gamma(f)] = 0$, then $\mathbf{f} \in \mathcal{H}^1(\mathbb{R}^n)$, and conversely, if $\mathbf{f} \in \mathcal{H}^1(\mathbb{R}^n)$, then $[\gamma f)] = 0$ (see Lemma B1 and Theorem 5.13 in Brewster et al.[27]).

2.2.4 | Rigid motion fields

Let $\mathcal{R}$ be the linear space of rigid body motion fields in $\mathbb{R}^n$,

$$\mathcal{R} := \left\{ \mathbf{b} + \mathbf{B}\mathbf{x} : \mathbf{b} \in \mathbb{R}^n \text{ and } \mathbf{B} \in \mathbb{R}^{n \times n} \text{ such that } \mathbf{B} = -\mathbf{B}^\top \right\}. \tag{2.18}$$

It is easy to see that $\dim \mathcal{R} = n(n+1)/2$; compare book Oleinik et al.[44]

It is well known that if $\mathbf{v} \in H^1(\Omega')^n$ and $\mathbb{E}(\mathbf{v}) = 0$ in a bounded domain Ω', then $\mathbf{v} \in \mathcal{R}|_{\Omega'}$ (see, e.g., the proof of Theorem 2.5, chapter I in Oleinik et al.[44]). This immediately implies that if Ω'' is $\mathbb{R}^n$ or an exterior domain in $\mathbb{R}^n$ and $\mathbb{E}(\mathbf{v}) = 0$ for $\mathbf{v} \in \mathcal{H}^1(\Omega'')^n \subset H^1_{\rm loc}(\Omega'')^n$, then $\mathbf{v} \in \mathcal{R}|_{\Omega''}$ as well. Moreover, since $\mathbf{v}$ belongs to the space $\mathcal{H}^1(\Omega'')^n$, which is embedded in $L_{\frac{2n}{n-2}}(\Omega'')^n$ (see (2.10)), it follows that $\mathbf{v} = \mathbf{0}$ in Ω''.

2.3 | The conormal derivative for the Stokes system with L_∞ viscosity tensor coefficient

As above, $\mathbb{L}$ is the divergence form of a second-order elliptic differential operator given by (1.7), and the coefficients $A^{\alpha\beta}$ of the anisotropic tensor $\mathbb{A} = \left(A^{\alpha\beta}\right)_{1\le\alpha,\beta\le n}$ are $n \times n$ matrix-valued functions in $L_\infty(\mathbb{R}^n)^{n\times n}$, with bounded measurable, real-valued entries $a_{ij}^{\alpha\beta}$, satisfying the symmetry and ellipticity conditions (1.3) and (1.4). Moreover, $\mathcal{L}$ is the Stokes operator given by (1.9). Let $\nu = (\nu_1, \dots, \nu_n)^\top$ denote the outward unit normal to Ω_+, which is defined a.e. on $\partial\Omega$.

In the special case when $(\mathbf{u}, \pi) \in C^1(\overline{\Omega}_\pm)^n \times C^0(\overline{\Omega}_\pm)$ and the coefficients $a_{ij}^{\alpha\beta}$ are also continuous up to the boundary, the *classical* interior and exterior conormal derivatives (i.e., the *boundary traction fields*) for the Stokes system

$$\mathcal{L}(\mathbf{u}, \pi) = \mathbb{L}\mathbf{u} - \nabla\pi = \mathbf{f},\ \mathrm{div}\,\mathbf{u} = g \text{ in } \Omega_\pm, \tag{2.19}$$

where $\mathbf{f} \in L_2(\Omega_\pm)^n$, $g \in L_2(\Omega_\pm)$ are defined by the formula

$$\mathbf{t}^{c\pm}(\mathbf{u}, \pi) := -\gamma_\pm \pi\, \nu + \mathrm{T}^{c\pm}\mathbf{u}, \tag{2.20}$$

where $\mathrm{T}^{c\pm}\mathbf{u}$ are the conormal derivatives of $\mathbf{u}$ on $\partial\Omega$ associated with the operator $\mathbb{L}$ and defined by

$$\mathrm{T}^{c\pm}\mathbf{u} := \gamma_\pm(A^{\alpha\beta}\partial_\beta\mathbf{u})\nu_\alpha \tag{2.21}$$

(cf., e.g., Choi et al.[14]). In view of (1.3), we obtain that[†]

$$\left(\mathrm{T}^{c\pm}\mathbf{u}\right)_i = \gamma_\pm\left(a_{ij}^{\alpha\beta}\partial_\beta u_j\right)\nu_\alpha = \gamma_\pm\left(a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u})\right)\nu_\alpha, \tag{2.22}$$

where $E_{j\beta}(\mathbf{u}) := \frac{1}{2}\left(\partial_j u_\beta + \partial_\beta u_j\right)$. [‡]

Note that for the isotropic case (1.10), the classical conormal derivatives $\mathbf{t}^{c\pm}(\mathbf{u}, \pi)$ reduce to the well-known formulas in the isotropic compressible case (cf., e.g., Appendix III, Part I, Section 1 in Temam[47]),

$$\left(\mathbf{t}^{c\pm}(\mathbf{u}, \pi)\right)_i = -\gamma_\pm\pi\,\nu_i + \gamma_\pm\left(\lambda(\mathrm{div}\,\mathbf{u})\right)\nu_i + 2\gamma_\pm\left(\mu E_{i\alpha}(\mathbf{u})\right)\nu_\alpha,\ i = 1, \dots, n. \tag{2.23}$$

For the classical conormal derivatives defined by (2.20)–(2.22), the *first Green formula*

$$\pm\left\langle \mathbf{t}^{c\pm}(\mathbf{u}, \pi), \varphi\right\rangle_{\partial\Omega} = \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}), E_{i\alpha}(\varphi)\right\rangle_{\Omega_\pm} - \langle \pi, \mathrm{div}\,\varphi\rangle_{\Omega_\pm} + \langle \mathcal{L}(\mathbf{u}, \pi), \varphi\rangle_{\Omega_\pm}\ \forall \varphi \in \mathcal{D}(\mathbb{R}^n)^n \tag{2.24}$$

holds and suggests the following definition of the *generalized conormal derivative* for the Stokes system with L_∞ viscosity tensor coefficient in the setting of weighted Sobolev spaces (cf., e.g., Lemma 4.3 in McLean,[48] Lemma 2.9 in Kohr et al.,[8] Definition 3.1 and Theorem 3.2 in Mikhailov,[54] and Theorem 10.4.1 in Mitrea and Wright[5]; see also Definition 2.4 in Kohr et al.[43]).

Definition 1. Let conditions (1.2) and (1.3) hold. Then, for any $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \mathcal{H}^1(\Omega_\pm)^n \times L_2(\Omega_\pm) \times \widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^n$, the formal conormal derivatives $\mathbf{t}^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm) \in H^{-\frac{1}{2}}(\partial\Omega)^n$ are defined in the weak form by

$$\pm\left\langle \mathbf{t}^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm), \mathbf{\Phi}\right\rangle_{\partial\Omega} := \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}_\pm), E_{i\alpha}(\gamma_\pm^{-1}\mathbf{\Phi})\right\rangle_{\Omega_\pm} - \left\langle \pi_\pm, \mathrm{div}(\gamma_\pm^{-1}\mathbf{\Phi})\right\rangle_{\Omega_\pm} + \left\langle \tilde{\mathbf{f}}_\pm, \gamma_\pm^{-1}\mathbf{\Phi}\right\rangle_{\Omega_\pm}, \tag{2.25}$$

[†]Here and in the sequel, the notation $\pm$ applies to the conormal derivatives from $\Omega_\pm$, respectively.

[‡]Note that another type of conormal derivative, where $E_{j\beta}(\mathbf{u})$ is replaced by its deviator, $D_{j\beta}(\mathbf{u}) = E_{j\beta}(\mathbf{u}) - \frac{1}{n}\delta_{j\beta}E_{mm}(\mathbf{u})$ in the formulas like (2.22) and further on, has been considered in Fresdeda-Portillo and Mikhailov[56] for the isotropic case. Both types of conormal derivatives coincide for incompressible fluids.

for any $\mathbf{\Phi} \in H^{\frac{1}{2}}(\partial\Omega)^n$, where $\gamma_{\pm}^{-1} : H^{\frac{1}{2}}(\partial\Omega)^n \to \mathcal{H}^1(\Omega_{\pm})^n$ are bounded right inverses to the trace operators $\gamma_{\pm} : \mathcal{H}^1(\Omega_{\pm})^n \to H^{\frac{1}{2}}(\partial\Omega)^n$.

Moreover, if $(\mathbf{u}_{\pm}, \pi_{\pm}, \widetilde{\mathbf{f}}_{\pm}) \in \mathcal{H}^1(\Omega_{\pm}, \mathcal{L})$, where

$$\mathcal{H}^1(\Omega_{\pm}, \mathcal{L}) := \left\{ (\mathbf{v}_{\pm}, q_{\pm}, \widetilde{\phi}_{\pm}) \in \mathcal{H}^1(\Omega_{\pm})^n \times L_2(\Omega_{\pm}) \times \widetilde{\mathcal{H}}^{-1}(\Omega_{\pm})^n : \mathcal{L}(\mathbf{v}_{\pm}, q_{\pm}) = \widetilde{\phi}_{\pm}|_{\Omega_{\pm}} \text{in } \ \Omega_{\pm} \right\}, \tag{2.26}$$

then relations (2.25) define the generalized conormal derivatives $\mathbf{t}^{\pm}(\mathbf{u}_{\pm}, \pi_{\pm}; \widetilde{\mathbf{f}}_{\pm}) \in H^{-\frac{1}{2}}(\partial\Omega)^n$.

Some properties of the conormal derivatives are presented in the following assertion (cf. Lemma 4.3 in McLean,[48] Theorem 3.9 in Mikhailov,[54] Theorem 5.3 in Mikhailov,[57] Lemma 2.9 in Kohr et al.,[8] and Theorem 10.4.1 in Mitrea and Wright[5]).

Lemma 1. *Let conditions (1.2) and (1.3) hold.*

(i) The formal conormal derivative operators $\boldsymbol{t}^{\pm} : \mathcal{H}^1(\Omega_{\pm})^n \times L_2(\Omega_{\pm}) \times \widetilde{\mathcal{H}}^{-1}(\Omega_{\pm})^n \to H^{-\frac{1}{2}}(\partial\Omega)^n$ *are linear and bounded.*

(ii) The generalized conormal derivative operators $\boldsymbol{t}^{\pm} : \mathcal{H}^1(\Omega_{\pm}, \mathcal{L}) \to H^{-\frac{1}{2}}(\partial\Omega)^n$ *with* $\mathcal{L}$ *given by (1.8) are linear and bounded and do not depend on the choice of the right inverse operators* $\gamma_{\pm}^{-1}$ *in (2.25). In addition, for all* $\boldsymbol{w}_{\pm} \in \mathcal{H}^1(\Omega_{\pm})^n$ *and* $(\boldsymbol{u}_{\pm}, \pi_{\pm}, \widetilde{\boldsymbol{f}}_{\pm}) \in \mathcal{H}^1(\Omega_{\pm}, \mathcal{L})$*, the following Green formula holds:*

$$\pm \left\langle \boldsymbol{t}^{\pm}(\boldsymbol{u}_{\pm}, \pi_{\pm}; \widetilde{\boldsymbol{f}}_{\pm}), \gamma_{\pm}\boldsymbol{w}_{\pm} \right\rangle_{\partial\Omega} = \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\boldsymbol{u}_{\pm}), E_{i\alpha}(\boldsymbol{w}_{\pm}) \right\rangle_{\Omega_{\pm}} - \langle \pi_{\pm}, \mathrm{div}\boldsymbol{w}_{\pm} \rangle_{\Omega_{\pm}} + \langle \widetilde{\boldsymbol{f}}_{\pm}, \boldsymbol{w}_{\pm} \rangle_{\Omega_{\pm}}. \tag{2.27}$$

Proof. We use similar arguments to those for Lemma 2.2 in Kohr et al.[10] (see also Definition 3.1 and Theorem 3.2 in Mikhailov[54,57] and Theorem 10.4.1 in Mitrea and Wright[5]). First, we note that for $(\mathbf{u}_{\pm}, \pi_{\pm}; \widetilde{\mathbf{f}}_{\pm}) \in \mathcal{H}^1(\Omega_{\pm})^n \times L_2(\Omega_{\pm}) \times \widetilde{\mathcal{H}}^{-1}(\Omega_{\pm})^n$, the right-hand side in (2.25) defines a bounded linear functional acting on $\mathbf{\Phi} \in H^{\frac{1}{2}}(\partial\Omega)^n$, and hence, the left-hand side determines the formal conormal derivatives $\mathbf{t}^{\pm}(\mathbf{u}_{\pm}, \pi_{\pm}; \widetilde{\mathbf{f}}_{\pm})$ in $H^{-\frac{1}{2}}(\partial\Omega)^n$ and the formal conormal derivative operators $\mathbf{t}^{\pm} : \mathcal{H}^1(\Omega_{\pm})^n \times L_2(\Omega_{\pm}) \times \widetilde{\mathcal{H}}^{-1}(\Omega_{\pm})^n \to H^{-\frac{1}{2}}(\partial\Omega)^n$ given by (2.25) are bounded. Therefore, the generalized conormal derivative operators $\mathbf{t}^{\pm} : \mathcal{H}^1(\Omega_{\pm}, \mathcal{L}) \to H^{-\frac{1}{2}}(\partial\Omega)^n$ are bounded as well.

Further, the property that the generalized conormal derivative operators $\mathbf{t}^{\pm} : \mathcal{H}^1(\Omega_{\pm}, \mathcal{L}) \to H^{-\frac{1}{2}}(\partial\Omega)^n$ defined by (2.25) are invariant with respect to the choice of a right inverse of the trace operator $\gamma_{\pm} : \mathcal{H}^1(\Omega_{\pm})^n \to H^{\frac{1}{2}}(\partial\Omega)^n$ can be obtained with an argument similar to that for Theorem 3.2 in Mikhailov.[54]

Now, let $(\mathbf{u}_{\pm}, \pi_{\pm}, \widetilde{\mathbf{f}}_{\pm}) \in \mathcal{H}^1(\Omega_{\pm}, \mathcal{L})$. According to formula (2.25), we deduce the following equality:

$$\begin{aligned} \pm \left\langle \mathbf{t}^{\pm}(\mathbf{u}_{\pm}, \pi_{\pm}; \widetilde{\mathbf{f}}_{\pm}), \gamma_{\pm}\mathbf{w}_{\pm} \right\rangle_{\partial\Omega} &= \left\langle A^{\alpha\beta}\partial_\beta(\mathbf{u}_{\pm}), \partial_\alpha \left(\gamma_{\pm}^{-1}(\gamma_{\pm}\mathbf{w}_{\pm}) \right) \right\rangle_{\Omega_{\pm}} \\ &\quad - \left\langle \pi_{\pm}, \mathrm{div} \left(\gamma_{\pm}^{-1}(\gamma_{\pm}\mathbf{w}_{\pm}) \right) \right\rangle_{\Omega_{\pm}} + \left\langle \widetilde{\mathbf{f}}_{\pm}, \gamma_{\pm}^{-1}(\gamma_{\pm}\mathbf{w}_{\pm}) \right\rangle_{\Omega_{\pm}} \\ &= \left\langle A^{\alpha\beta}\partial_\beta(\mathbf{u}_{\pm}), \partial_\alpha(\mathbf{w}_{\pm}) \right\rangle_{\Omega_{\pm}} - \langle \pi_{\pm}, \mathrm{div}\mathbf{w}_{\pm} \rangle_{\Omega_{\pm}} + \left\langle \widetilde{\mathbf{f}}_{\pm}, \mathbf{w}_{\pm} \right\rangle_{\Omega_{\pm}} \\ &\quad + \left\langle A^{\alpha\beta}\partial_\beta(\mathbf{u}_{\pm}), \partial_\alpha \left(\gamma_{\pm}^{-1}(\gamma_{\pm}\mathbf{w}_{\pm}) - \mathbf{w}_{\pm} \right) \right\rangle_{\Omega_{\pm}} \\ &\quad - \left\langle \pi_{\pm}, \mathrm{div} \left(\gamma_{\pm}^{-1}(\gamma_{\pm}\mathbf{w}_{\pm}) - \mathbf{w}_{\pm} \right) \right\rangle_{\Omega_{\pm}} + \left\langle \widetilde{\mathbf{f}}_{\pm}, \gamma_{\pm}^{-1} \left(\gamma_{\pm}\mathbf{w}_{\pm} \right) - \mathbf{w}_{\pm} \right\rangle_{\Omega_{\pm}}, \end{aligned} \tag{2.28}$$

for all $\mathbf{w} \in \mathcal{H}^1(\Omega_{\pm})^n$. According to the property (2.13) and the equality $\gamma_{\pm} \left(\gamma_{\pm}^{-1}(\gamma_{\pm}\mathbf{w}_{\pm}) - \mathbf{w}_{\pm} \right) = \mathbf{0}$ on $\partial\Omega$, as well as the following equivalent description of the space $\mathring{\mathcal{H}}^1(\Omega_{\pm})^n$:

$$\mathring{\mathcal{H}}^1(\Omega_{\pm})^n = \left\{ \mathbf{v}_{\pm} \in \mathcal{H}^1(\Omega_{\pm})^n : \gamma_{\pm}\mathbf{v}_{\pm} = \mathbf{0} \text{ on } \partial\Omega \right\} \tag{2.29}$$

(cf., e.g., Alliot and Amrouche[58], 1.2), we obtain the inclusion

$$\gamma_{\pm}^{-1} \left(\gamma_{\pm}\mathbf{w}_{\pm} \right) - \mathbf{w}_{\pm} \in \mathring{\mathcal{H}}^1(\Omega_{\pm})^n. \tag{2.30}$$

Therefore, the Green formula (2.27) will follow from formula (2.28) if we show that

$$\left\langle A^{\alpha\beta}\partial_\beta(\mathbf{u}_\pm),\partial_\alpha(\mathbf{v}_\pm)\right\rangle_{\partial\Omega}-\langle\pi_\pm,\operatorname{div}\mathbf{v}_\pm\rangle_{\Omega_\pm}+\left\langle\widetilde{\mathbf{f}}_\pm,\mathbf{v}_\pm\right\rangle_{\Omega_\pm}=0\ \ \forall\mathbf{v}_\pm\in\mathring{\mathcal{H}}^1(\Omega_\pm)^n. \tag{2.31}$$

Since the space $\mathcal{D}(\Omega_\pm)^n$ is dense in $\mathring{\mathcal{H}}^1(\Omega_\pm)^n$, we need to show identity (2.31) only for the test functions $\mathbf{v}_\pm$ in $\mathcal{D}(\Omega_\pm)^n$. Indeed, the membership of $(\mathbf{u}_\pm,\pi_\pm,\widetilde{\mathbf{f}}_\pm)$ in $\mathcal{H}^1(\Omega_\pm,\mathcal{L})$ implies the equality $\mathcal{L}(\mathbf{u}_\pm,\pi_\pm)=\widetilde{\mathbf{f}}_\pm|_{\Omega_\pm}$ in the sense of distributions, and accordingly, identity (2.31) holds for any $\mathbf{v}_\pm\in\mathcal{D}(\Omega_\pm)^n$. □

In the sequel, we use the simplified notation $\mathbf{t}^\pm(\mathbf{u}_\pm,\pi_\pm)$ for $\mathbf{t}^\pm(\mathbf{u}_\pm,\pi_\pm;\mathbf{0})$.

Let $\mathring{E}_\pm$ denote the operator of extension by zero outside $\Omega_\pm$. Thus, for a function $v_\pm$ from $\Omega_\pm$ to $\mathbb{R}^n$,

$$\mathring{E}_\pm(v_\pm)(x):=\begin{cases}v_\pm(x) & \text{if } x\in\Omega_\pm,\\ 0 & \text{if } x\in\mathbb{R}^n\setminus\Omega_\pm.\end{cases} \tag{2.32}$$

Let γ be the trace operator from $\mathcal{H}^1(\mathbb{R}^n)^n$ to $H^{\frac{1}{2}}(\partial\Omega)^n$. For any $(\mathbf{u}_\pm,\pi_\pm,\widetilde{\mathbf{f}}_\pm)\in\mathcal{H}^1(\Omega_\pm)^n\times L_2(\Omega_\pm)\times\widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^n$, let

$$\mathbf{u}:=\mathring{E}_+\mathbf{u}_++\mathring{E}_-\mathbf{u}_-,\ \ \pi:=\mathring{E}_+\pi_++\mathring{E}_-\pi_-,\ \ \widetilde{\mathbf{f}}:=\widetilde{\mathbf{f}}_++\widetilde{\mathbf{f}}_-, \tag{2.33}$$

and the jump of the corresponding formal or generalized conormal derivatives is denoted by

$$[\mathbf{t}(\mathbf{u},\pi;\widetilde{\mathbf{f}})]:=\mathbf{t}^+(\mathbf{u}_+,\pi_+;\widetilde{\mathbf{f}}_+)-\mathbf{t}^-(\mathbf{u}_-,\pi_-;\widetilde{\mathbf{f}}_-). \tag{2.34}$$

Note that the inclusions $\widetilde{\mathbf{f}}_\pm\in\widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^n\subset\mathcal{H}^{-1}(\mathbb{R}^n)^n$ imply that $\widetilde{\mathbf{f}}=\widetilde{\mathbf{f}}_++\widetilde{\mathbf{f}}_-$ belongs to the space $\mathcal{H}^{-1}(\mathbb{R}^n)^n$. In the special case $\widetilde{\mathbf{f}}=\mathbf{0}$, we use the notation

$$[\mathbf{t}(\mathbf{u},\pi)]:=[\mathbf{t}(\mathbf{u},\pi;\mathbf{0})]=\mathbf{t}^+(\mathbf{u}_+,\pi_+)-\mathbf{t}^-(\mathbf{u}_-,\pi_-). \tag{2.35}$$

Then, Lemma 1 implies the following assertion.

Lemma 2. *Let conditions (1.2) and (1.3) hold. For $(\boldsymbol{u}_\pm,\pi_\pm,\widetilde{\boldsymbol{f}}_\pm)\in\mathcal{H}^1(\Omega_\pm,\mathcal{L})$ given, let $(\boldsymbol{u},\pi,\widetilde{\boldsymbol{f}})$ be defined as in (2.33). Then, the following identity holds for any $\boldsymbol{w}\in\mathcal{H}^1(\mathbb{R}^n)^n$:*

$$\left\langle[\boldsymbol{t}(\boldsymbol{u},\pi;\widetilde{\boldsymbol{f}})],\gamma\boldsymbol{w}\right\rangle_{\partial\Omega}=\left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\boldsymbol{u}_+),E_{i\alpha}(\boldsymbol{w})\right\rangle_{\Omega_+}+\left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\boldsymbol{u}_-),E_{i\alpha}(\boldsymbol{w})\right\rangle_{\Omega_-}-\langle\pi,\operatorname{div}\boldsymbol{w}\rangle_{\mathbb{R}^n}+\langle\widetilde{\boldsymbol{f}},\boldsymbol{w}\rangle_{\mathbb{R}^n}. \tag{2.36}$$

Proof. Note that $\gamma_+\mathbf{w}=\gamma_-\mathbf{w}=\gamma\mathbf{w}$ for any function $\mathbf{w}\in\mathcal{H}^1(\mathbb{R}^n)^n$. Then, formula (2.27) implies the desired result. □

The following assertion is immediately implied by Lemma 2 and the symmetry conditions (1.3).

Lemma 3. *Let conditions (1.2) and (1.3) hold. Let the pair $(\boldsymbol{u},\pi)$ in $\mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n\times L_2(\mathbb{R}^n)$ be such that $\mathcal{L}(\boldsymbol{u},\pi)\in L_2(\mathbb{R}^n\setminus\partial\Omega)^n$ and $\operatorname{div}\boldsymbol{u}=0$ in $\mathbb{R}^n\setminus\partial\Omega$. Let $\boldsymbol{u}_\pm:=r_{\Omega_\pm}\boldsymbol{u}$, $\pi_\pm:=r_{\Omega_\pm}\pi$, $\widetilde{\boldsymbol{f}}_\pm:=\mathring{E}_\pm r_{\Omega_\pm}\mathcal{L}(\boldsymbol{u},\pi)$, and $[\boldsymbol{t}(\boldsymbol{u},\pi;\boldsymbol{f})]:=\boldsymbol{t}^+(\boldsymbol{u}_+,\pi_+;\widetilde{\boldsymbol{f}}_+)-\boldsymbol{t}^-(\boldsymbol{u}_-,\pi_-;\widetilde{\boldsymbol{f}}_-)$. Then, for all $\boldsymbol{w}\in\mathcal{H}^1(\mathbb{R}^n)^n$, the following formula holds:*

$$\langle[\boldsymbol{t}(\boldsymbol{u},\pi;\boldsymbol{f})],\gamma\boldsymbol{w}\rangle_{\partial\Omega}=\left\langle A^{\alpha\beta}\partial_\beta\boldsymbol{u}_+,\partial_\alpha\boldsymbol{w}\right\rangle_{\Omega_+}+\left\langle A^{\alpha\beta}\partial_\beta\boldsymbol{u}_-,\partial_\alpha\boldsymbol{w}\right\rangle_{\Omega_-}-\langle\pi,\operatorname{div}\boldsymbol{w}\rangle_{\mathbb{R}^n}+\langle\mathcal{L}(\boldsymbol{u},\pi),\boldsymbol{w}\rangle_{\mathbb{R}^n\setminus\partial\Omega} \tag{2.37}$$

$$=\left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\boldsymbol{u}_+),E_{i\alpha}(\boldsymbol{w})\right\rangle_{\Omega_+}+\left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\boldsymbol{u}_-),E_{i\alpha}(\boldsymbol{w})\right\rangle_{\Omega_-}-\langle\pi,\operatorname{div}\boldsymbol{w}\rangle_{\mathbb{R}^n}+\langle\mathcal{L}(\boldsymbol{u},\pi),\boldsymbol{w}\rangle_{\mathbb{R}^n\setminus\partial\Omega}. \tag{2.38}$$

2.4 | Conormal derivative related to the adjoint Stokes operator

Let $\mathbb{L}$ be the divergence-type elliptic operator given by (1.7). Then, the formally adjoint $\mathbb{L}^*$ of the operator $\mathbb{L}$ is defined by

$$\mathbb{L}^*\mathbf{u}=\partial_\alpha\left(A^{*\alpha\beta}\partial_\beta\mathbf{u}\right):=\partial_\alpha\left(\left(A^{\beta\alpha}\right)^\top\partial_\beta\mathbf{u}\right), \tag{2.39}$$

where $A^{*\alpha\beta} = \left(A^{\beta\alpha}\right)^\top$ is the transpose of the matrix $A^{\beta\alpha}$ for all $\alpha, \beta = 1, \dots, n$, that is,

$$A^{*\alpha\beta} = \left(A^{\beta\alpha}\right)^\top = \left(a_{ij}^{*\alpha\beta}\right)_{1\le i,j\le n} = \left(a_{ji}^{\beta\alpha}\right)_{1\le i,j\le n}. \tag{2.40}$$

Note that the coefficients of the operator $\mathbb{L}^*$ belong to $L_\infty(\Omega)^{n\times n}$ (cf. (1.2)) and satisfy the ellipticity condition (1.4) with the same constant $c_{\mathbb{A}}$. Moreover, the operator

$$\begin{pmatrix} \mathbb{L}^* & -\nabla \\ \mathrm{div} & 0 \end{pmatrix} \tag{2.41}$$

is the adjoint of the Stokes operator

$$\begin{pmatrix} \mathbb{L} & -\nabla \\ \mathrm{div} & 0 \end{pmatrix}. \tag{2.42}$$

If a pair $(\mathbf{v}, q) \in C^1(\overline{\Omega}_\pm)^n \times C^0(\overline{\Omega}_\pm)$ satisfies the following equation, related to the adjoint Stokes operator (2.41),

$$\mathbb{L}^*\mathbf{v} - \nabla q = \mathbf{f}_* \text{ in } \Omega_\pm, \tag{2.43}$$

where $\mathbf{f}_* \in L_2(\Omega_\pm)^n$, then the corresponding classical conormal derivative is defined by

$$\mathbf{t}^{c*\pm}(\mathbf{v}, q) := -\gamma_\pm q\nu + \mathrm{T}^{c*\pm}\mathbf{v},\ \mathrm{T}^{c*\pm}\mathbf{v} := \gamma_\pm\left(\left(A^{\beta\alpha}\right)^\top\partial_\beta\mathbf{v}\right)\nu_\alpha. \tag{2.44}$$

If $(\mathbf{v}_\pm, q_\pm, \widetilde{\mathbf{f}}_{*\pm}) \in \mathcal{H}^1(\Omega_\pm)^n \times L_2(\Omega_\pm) \times \widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^n$ satisfies the following system (in distributional sense):

$$\mathcal{L}_*(\mathbf{v}_\pm, q_\pm) := \mathbb{L}^*\mathbf{v}_\pm - \nabla q_\pm = \widetilde{\mathbf{f}}_{*\pm}|_{\Omega_\pm} \text{ in } \Omega_\pm, \tag{2.45}$$

then we define the corresponding *generalized conormal derivative* $\mathbf{t}^{*\pm}(\mathbf{v}_\pm, q_\pm; \widetilde{\mathbf{f}}_{*\pm}) \in H^{-\frac{1}{2}}(\partial\Omega)^n$ by setting

$$\begin{aligned}\pm\left\langle \mathbf{t}^{*\pm}(\mathbf{v}_\pm, q_\pm; \widetilde{\mathbf{f}}_{*\pm}), \mathbf{\Phi}\right\rangle_{\partial\Omega} &:= \left\langle (A^{\beta\alpha})^\top\partial_\alpha\mathbf{v}_\pm, \partial_\beta(\gamma_\pm^{-1}\mathbf{\Phi})\right\rangle_{\Omega_\pm} - \left\langle q_\pm, \mathrm{div}(\gamma_\pm^{-1}\mathbf{\Phi})\right\rangle_{\Omega_\pm} + \left\langle \widetilde{\mathbf{f}}_{*\pm}, \gamma_\pm^{-1}\mathbf{\Phi}\right\rangle_{\Omega_\pm} \\ &= \left\langle A^{\alpha\beta}\partial_\beta(\gamma_\pm^{-1}\mathbf{\Phi}), \partial_\alpha\mathbf{v}_\pm\right\rangle_{\Omega_\pm} - \left\langle q_\pm, \mathrm{div}(\gamma_\pm^{-1}\mathbf{\Phi})\right\rangle_{\Omega_\pm} + \left\langle \widetilde{\mathbf{f}}_{*\pm}, \gamma_\pm^{-1}\mathbf{\Phi}\right\rangle_{\Omega_\pm},\end{aligned} \tag{2.46}$$

for any $\mathbf{\Phi} \in H^{\frac{1}{2}}(\partial\Omega)^n$. In addition, an argument similar to that for (2.27) along with relations (2.40) imply the Green formula

$$\pm\left\langle \mathbf{t}^{*\pm}(\mathbf{v}_\pm, q_\pm; \widetilde{\mathbf{f}}_{*\pm}), \gamma_\pm\mathbf{w}_\pm\right\rangle_{\partial\Omega} = \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{w}_\pm), E_{i\alpha}(\mathbf{v}_\pm)\right\rangle_{\Omega_\pm} - \langle q_\pm, \mathrm{div}\mathbf{w}_\pm\rangle_{\Omega_\pm} + \langle\widetilde{\mathbf{f}}_{*\pm}, \mathbf{w}_\pm\rangle_{\Omega_\pm}, \tag{2.47}$$

and the following variant of Lemma 3.

Lemma 4. *Let conditions (1.2) and (1.3) hold. Let the pair $(\boldsymbol{v}, q)$ in $\mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n \times L_2(\mathbb{R}^n)$ be such that $\mathcal{L}_*(\boldsymbol{v}, q) \in L_2(\mathbb{R}^n\backslash\partial\Omega)^n$ in $\mathbb{R}^n\backslash\partial\Omega$. Let $\boldsymbol{v}_\pm := r_{\Omega_\pm}\boldsymbol{v}$, $q_\pm := r_{\Omega_\pm}q$, $\widetilde{\boldsymbol{f}}_{*\pm} := \mathring{E}_\pm r_{\Omega_\pm}\mathcal{L}_*(\boldsymbol{v}, q)$, and $[\boldsymbol{t}^*(\boldsymbol{v}, q; \boldsymbol{f}_*)] := \boldsymbol{t}^{*+}(\boldsymbol{v}_+, q_+; \widetilde{\boldsymbol{f}}_{*+}) - \boldsymbol{t}^{*-}(\boldsymbol{v}_-, q_-; \widetilde{\boldsymbol{f}}_{*-})$. Then, for any $\boldsymbol{w} \in \mathcal{H}^1(\mathbb{R}^n)^n$,*

$$\langle[\boldsymbol{t}^*(\boldsymbol{v}, q; \boldsymbol{f}_*)], \gamma\boldsymbol{w}\rangle_{\partial\Omega} = \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\boldsymbol{w}), E_{i\alpha}(\boldsymbol{v}_+)\right\rangle_{\Omega_+} + \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\boldsymbol{w}), E_{i\alpha}(\boldsymbol{v}_-)\right\rangle_{\Omega_-} - \langle q, \mathrm{div}\,\boldsymbol{w}\rangle_{\mathbb{R}^n} + \langle\mathcal{L}_*(\boldsymbol{v}, q), \boldsymbol{w}\rangle_{\mathbb{R}^n\backslash\partial\Omega}. \tag{2.48}$$

3 | VARIATIONAL VOLUME AND LAYER POTENTIALS FOR THE ANISOTROPIC STOKES SYSTEM WITH L_∞ TENSOR COEFFICIENT

As in the previous sections, $\Omega_+ \subset \mathbb{R}^n$, $n \ge 3$, is a bounded Lipschitz domain with connected boundary $\partial\Omega$, and $\Omega_- := \mathbb{R}^n\backslash\overline{\Omega_+}$. Recall that $\mathcal{L}$ is the Stokes operator defined in (1.9). In this section, we define the Newtonian and layer potentials for the Stokes system (1.8) by means of a variational approach.

3.1 | Bilinear forms and weak solutions for the anisotropic Stokes system with L_∞ tensor coefficient in $\mathbb{R}^n$

Let $\mathbb{A}$ satisfy conditions (1.2)–(1.4) and $a_{\mathbb{A};\mathbb{R}^n} : \mathcal{H}^1(\mathbb{R}^n)^n \times \mathcal{H}^1(\mathbb{R}^n)^n \to \mathbb{R}$, $b_{\mathbb{R}^n} : \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n) \to \mathbb{R}$ be the bilinear forms given by

$$a_{\mathbb{A};\mathbb{R}^n}(\mathbf{u},\mathbf{v}) := \left\langle A^{\alpha\beta}\partial_\beta \mathbf{u}, \partial_\alpha \mathbf{v}\right\rangle_{\mathbb{R}^n} = \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v})\right\rangle_{\mathbb{R}^n},\ \forall \mathbf{u} \in \mathcal{H}^1(\mathbb{R}^n)^n, \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^n)^n, \tag{3.1}$$

$$b_{\mathbb{R}^n}(\mathbf{v},q) := -\langle \operatorname{div}\mathbf{v}, q\rangle_{\mathbb{R}^n},\ \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^n)^n, \forall q \in L_2(\mathbb{R}^n). \tag{3.2}$$

Let us denote

$$\mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n := \left\{\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^n)^n : \operatorname{div}\mathbf{w} = 0 \text{ in } \mathbb{R}^n\right\}.$$

The subspace $\mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n$ of $\mathcal{H}^1(\mathbb{R}^n)^n$ has also the characterization

$$\mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n = \left\{\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^n)^n : b_{\mathbb{R}^n}(\mathbf{w},q) = 0\ \forall q \in L_2(\mathbb{R}^n)\right\}. \tag{3.3}$$

An important role in the forthcoming analysis is played by the following well-posedness result (see also Lemma 4.1 in Kohr et al.[39] and Lemma 3.1 in Kohr et al.[43])

Lemma 5. *Let conditions (1.2)–(1.4) hold on $\mathbb{R}^n$. Let $a_{\mathbb{A};\mathbb{R}^n}$ and $b_{\mathbb{R}^n}$ be the bilinear forms defined in (3.1) and (3.2), respectively. Then, for all given data $\boldsymbol{F} \in \mathcal{H}^{-1}(\mathbb{R}^n)^n$ and $\eta \in L_2(\mathbb{R}^n)$, the mixed variational formulation*

$$\begin{cases} a_{\mathbb{A};\mathbb{R}^n}(\boldsymbol{u},\boldsymbol{v}) + b_{\mathbb{R}^n}(\boldsymbol{v},\pi) = \langle \boldsymbol{F},\boldsymbol{v}\rangle_{\mathbb{R}^n}\ \forall \boldsymbol{v} \in \mathcal{H}^1(\mathbb{R}^n)^n, \\ b_{\mathbb{R}^n}(\boldsymbol{u},q) = \langle \eta, q\rangle_{\mathbb{R}^n}\ \forall q \in L_2(\mathbb{R}^n) \end{cases} \tag{3.4}$$

is well-posed. Therefore, (3.4) has a unique solution $(\boldsymbol{u},\pi) \in \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$, and there exists a constant $C = C(c_{\mathbb{A}}, n) > 0$ such that

$$\|\boldsymbol{u}\|_{\mathcal{H}^1(\mathbb{R}^n)^n} + \|\pi\|_{L_2(\mathbb{R}^n)} \le C\left\{\|\boldsymbol{F}\|_{\mathcal{H}^{-1}(\mathbb{R}^n)^n} + \|\eta\|_{L_2(\mathbb{R}^n)}\right\}. \tag{3.5}$$

Proof. We intend to use Theorem 10, which requires the coercivity of the bilinear form $a_{\mathbb{A};\mathbb{R}^n}(\cdot,\cdot)$ from $\mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n \times \mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n$ to $\mathbb{R}$. Indeed, the following Korn-type inequality for functions in $\mathcal{H}^1(\mathbb{R}^n)^n$ holds:

$$\|\nabla \mathbf{w}\|^2_{L_2(\mathbb{R}^n)^{n\times n}} \le 2\|\mathbb{E}(\mathbf{w})\|^2_{L_2(\mathbb{R}^n)^{n\times n}}. \tag{3.6}$$

This inequality is available, e.g., in Sayas and Selgas[33], eq. (2.2) for $n = 3$. For arbitrary $n \ge 1$, the Korn inequality is proved in Theorem 10.1 of McLean[48] for any function $\mathbf{w} \in \mathcal{D}(\mathbb{R}^n)^n$. Hence, by the density of $\mathcal{D}(\mathbb{R}^n)^n$ in $\mathcal{H}^1(\mathbb{R}^n)^n$, this implies that inequality (3.6) is valid also in $\mathcal{H}^1(\mathbb{R}^n)^n$.

Note that if $\mathbf{w} \in \mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n$, then $\sum_{i=1}^n E_{ii}(\mathbf{w}) = 0$. Then, the ellipticity condition (1.4), inequality (3.6), and equivalence of the seminorm $\|\nabla(\cdot)\|_{L_2(\mathbb{R}^n)^{n\times n}}$ to the norm $\|\cdot\|_{\mathcal{H}^1(\mathbb{R}^n)^n}$ in $\mathcal{H}^1(\mathbb{R}^n)^n$ (see Section 2.2.1) imply that there exists a constant $c_1 = c_1(n) > 0$ such that

$$a_{\mathbb{A};\mathbb{R}^n}(\mathbf{w},\mathbf{w}) \ge c_{\mathbb{A}}^{-1}\|\mathbb{E}(\mathbf{w})\|^2_{L_2(\mathbb{R}^n)^{n\times n}} \ge \frac{1}{2}c_{\mathbb{A}}^{-1}\|\nabla \mathbf{w}\|^2_{L_2(\mathbb{R}^n)^{n\times n}} \ge \frac{1}{2}c_{\mathbb{A}}^{-1}c_1\|\mathbf{w}\|^2_{\mathcal{H}^1(\mathbb{R}^n)^n}\ \forall \mathbf{w} \in \mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n. \tag{3.7}$$

Inequality (3.7) shows that the bilinear form $a_{\mathbb{A};\mathbb{R}^n}(\cdot,\cdot) : \mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n \times \mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n \to \mathbb{R}$ is coercive.

The continuity of the operator $\nabla : \mathcal{H}^1(\mathbb{R}^n)^n \to L_2(\mathbb{R}^n)^{n\times n}$ and the Hölder inequality imply that

$$|a_{\mathbb{A};\mathbb{R}^n}(\mathbf{u},\mathbf{v})| \le C\|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^n)^n}\|\mathbf{v}\|_{\mathcal{H}^1(\mathbb{R}^n)^n}\ \forall \mathbf{u},\mathbf{v} \in \mathcal{H}^1(\mathbb{R}^n)^n, \tag{3.8}$$

where $C = n^4\|\mathbb{A}\|_{L_\infty(\mathbb{R}^n)}$. Thus, the bilinear form $a_{\mathbb{A};\mathbb{R}^n}(\cdot,\cdot) : \mathcal{H}^1(\mathbb{R}^n)^n \times \mathcal{H}^1(\mathbb{R}^n)^n \to \mathbb{R}$ is bounded. Moreover, the boundedness of the divergence operator div $: \mathcal{H}^1(\mathbb{R}^n)^n \to L_2(\mathbb{R}^n)$ implies that the bilinear form $b : \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n) \to \mathbb{R}$ is bounded as well.

The isomorphism property of the divergence operator

$$-\mathrm{div}\,:\,\mathcal{H}^1(\mathbb{R}^n)^n/\mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^n)^n\to L_2(\mathbb{R}^n) \tag{3.9}$$

(cf. Proposition 2.1 in Alliot and Amrouche[52] and Lemma 2.5 in Kozono and Shor[53]) implies that there exists a constant $c_0>0$ such that for any $q\in L_2(\mathbb{R}^n)$ there exists $\mathbf{v}\in\mathcal{H}^1(\mathbb{R}^n)^n$ satisfying the equation $-\mathrm{div}\,\mathbf{v}=q$ and the inequality $\|\mathbf{v}\|_{\mathcal{H}^1(\mathbb{R}^n)^n}\le c_0\|q\|_{L_2(\mathbb{R}^n)}$. Therefore, the following inequality holds for such $\mathbf{v}$:

$$b_{\mathbb{R}^n}(\mathbf{v},q)=-\langle\mathrm{div}\mathbf{v},q\rangle_{\mathbb{R}^n}=\langle q,q\rangle_{\mathbb{R}^n}=\|q\|^2_{L_2(\mathbb{R}^n)}\ge c_0^{-1}\|\mathbf{v}\|_{\mathcal{H}^1(\mathbb{R}^n)^n}\|q\|_{L_2(\mathbb{R}^n)}.$$

This implies that the bounded bilinear form $b_{\mathbb{R}^n}\,:\,\mathcal{H}^1(\mathbb{R}^n)^n\times L_2(\mathbb{R}^n)\to\mathbb{R}$ satisfies the inf-sup condition

$$\inf_{q\in L_2(\mathbb{R}^n)\setminus\{0\}}\ \sup_{\mathbf{w}\in\mathcal{H}^1(\mathbb{R}^n)^n\setminus\mathbf{0}}\frac{b_{\mathbb{R}^n}(\mathbf{w},q)}{\|\mathbf{w}\|_{\mathcal{H}^1(\mathbb{R}^n)^n}\|q\|_{L_2(\mathbb{R}^n)}}\ge c_0^{-1} \tag{3.10}$$

(see also Lemma 14(ii) and Proposition 2.4 in Sayas and Selgas[33] for $n=2,3$). Then, Theorem 10 with $X=\mathcal{H}^1(\mathbb{R}^n)^n$, $\mathcal{M}=L_2(\mathbb{R}^n)$, and $V=\mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^n)^n$ implies that problem (3.4) is well-posed, as asserted. □

3.2 | Volume potential operators for the anisotropic Stokes system with L_∞ tensor coefficient

Recall that $\mathcal{L}$ is the anisotropic Stokes operator defined in (1.9).

Theorem 2. *Let conditions (1.2)–(1.4) hold in $\mathbb{R}^n$. Then, for each $\boldsymbol{f}\in\mathcal{H}^{-1}(\mathbb{R}^n)^n$ and $g\in L_2(\mathbb{R}^n)$, the anisotropic Stokes system*

$$\mathcal{L}(\boldsymbol{u},\pi)=\boldsymbol{f},\ \mathrm{div}\,\boldsymbol{u}=g\ \ in\,\mathbb{R}^n \tag{3.11}$$

is well-posed, which means that (3.11) has a unique solution $(\boldsymbol{u},\pi)\in\mathcal{H}^1(\mathbb{R}^n)^n\times L_2(\mathbb{R}^n)$, and there exists a constant $C=C(c_{\mathbb{A}},n)>0$ such that

$$\|\boldsymbol{u}\|_{\mathcal{H}^1(\mathbb{R}^n)^n}+\|\pi\|_{L_2(\mathbb{R}^n)}\le C\left(\|\boldsymbol{f}\|_{\mathcal{H}^{-1}(\mathbb{R}^n)^n}+\|g\|_{L_2(\mathbb{R}^n)}\right). \tag{3.12}$$

Proof. The dense embedding of the space $\mathcal{D}(\mathbb{R}^n)^n$ in $\mathcal{H}^1(\mathbb{R}^n)^n$ shows that system (3.11) has the equivalent mixed variational formulation (3.4), with $\mathbf{F}=-\mathbf{f}$ and $\eta=-g$. Then, the well-posedness of the Stokes system (3.11) follows from Lemma 5. □

Theorem 2 allows us to define the volume potential operators for the Stokes system with L_∞ coefficients and obtain their continuity as follows.

Definition 2. Let conditions (1.2)–(1.4) hold.

(i) The Newtonian velocity and pressure potential operators,

$$\mathcal{N}_{\mathbb{R}^n}\,:\,\mathcal{H}^{-1}(\mathbb{R}^n)^n\to\mathcal{H}^1(\mathbb{R}^n)^n,\ \mathcal{Q}_{\mathbb{R}^n}\,:\,\mathcal{H}^{-1}(\mathbb{R}^n)^n\to L_2(\mathbb{R}^n), \tag{3.13}$$

are defined as

$$\mathcal{N}_{\mathbb{R}^n}\mathbf{f}:=\mathbf{u_f},\ \mathcal{Q}_{\mathbb{R}^n}\mathbf{f}:=\pi_{\mathbf{f}}\ \ \forall\mathbf{f}\in\mathcal{H}^{-1}(\mathbb{R}^n)^n, \tag{3.14}$$

where $(\mathbf{u_f},\pi_{\mathbf{f}})\in\mathcal{H}^1(\mathbb{R}^n)^n\times L_2(\mathbb{R}^n)$ is the unique solution of problem (3.11) with $\mathbf{f}\in\mathcal{H}^{-1}(\mathbb{R}^n)^n$ and $g=0$.

(ii) The velocity and pressure compressibility potential operators,

$$\mathcal{G}_{\mathbb{R}^n}\,:\,L_2(\mathbb{R}^n)\to\mathcal{H}^1(\mathbb{R}^n)^n,\ \mathcal{G}^0_{\mathbb{R}^n}\,:\,L_2(\mathbb{R}^n)\to L_2(\mathbb{R}^n), \tag{3.15}$$

are defined as

$$\mathcal{G}_{\mathbb{R}^n}g:=\mathbf{u}_g,\ \mathcal{G}^0_{\mathbb{R}^n}g:=\pi_g\ \ \forall g\in L_2(\mathbb{R}^n), \tag{3.16}$$

where $(\mathbf{u}_g,\pi_g)\in\mathcal{H}^1(\mathbb{R}^n)^n\times L_2(\mathbb{R}^n)$ is the unique solution of problem (3.11) with $g\in L_2(\mathbb{R}^n)$ and $\mathbf{f}=\mathbf{0}$.

Lemma 6. *Operators (3.13) and (3.15) are linear and continuous and for any $\boldsymbol{f} \in \mathcal{H}^{-1}(\mathbb{R}^n)^n$ and $g \in L_2(\mathbb{R}^n)^n$,*

$$\mathcal{L}(\mathcal{N}_{\mathbb{R}^n}\boldsymbol{f}, \mathcal{Q}_{\mathbb{R}^n}\boldsymbol{f}) = \boldsymbol{f},\ \operatorname{div}\mathcal{N}_{\mathbb{R}^n}\boldsymbol{f} = 0 \text{ in } \mathbb{R}^n,$$
$$\mathcal{L}(\mathcal{G}_{\mathbb{R}^n}g, \mathcal{G}^0_{\mathbb{R}^n}g) = \boldsymbol{0},\ \operatorname{div}\mathcal{G}_{\mathbb{R}^n}g = g \text{ in } \mathbb{R}^n.$$

3.3 | The single-layer potential operator for the anisotropic Stokes system with L_∞ tensor coefficient

Recall that $\Omega_+ \subset \mathbb{R}^n\,(n \geq 3)$ is a bounded Lipschitz domain with connected boundary $\partial\Omega$, $\Omega_- := \mathbb{R}^n\backslash\overline{\Omega_+}$, the notation $[\cdot]$ is used for jumps (see formulas (2.3) and (2.34–2.35)), and $\mathcal{L}$ is the anisotropic Stokes operator defined in (1.9).

The next well-posedness result for the transmission problem (3.17) plays a major role in the definition of the single-layer potentials for the L_∞ coefficient Stokes system in the Sobolev space $H^{-\frac{1}{2}}(\partial\Omega)^n$ (see also Theorem 3.5, Definition 3.7, and Lemma 3.8 in Kohr et al.[43] for the Stokes system with strongly elliptic tensor coefficient; Theorem 4.5 in Kohr et al.,[10] section 5 in Sayas and Selgas,[33] Section 2 in Băcuţă et al.,[36] and Theorem 10.5.3 in Mitrea and Wright[5] for the isotropic case (1.10) with $\mu = 1$ and $\lambda = 0$).

Theorem 3. *Let conditions (1.2)–(1.4) hold in $\mathbb{R}^n$. Then, for any $\psi \in H^{-\frac{1}{2}}(\partial\Omega)^n$, the transmission problem,*

$$\begin{cases} \mathcal{L}(\boldsymbol{u}_\psi, \pi_\psi) = \boldsymbol{0},\ \operatorname{div}\boldsymbol{u}_\psi = 0 & \text{in } \mathbb{R}^n\backslash\partial\Omega, \\ \left[\gamma\boldsymbol{u}_\psi\right] = \boldsymbol{0} & \text{on } \partial\Omega, \\ \left[\boldsymbol{t}(\boldsymbol{u}_\psi, \pi_\psi)\right] = \psi & \text{on } \partial\Omega, \end{cases} \tag{3.17}$$

has a unique solution $(\boldsymbol{u}_\psi, \pi_\psi) \in \mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n \times L_2(\mathbb{R}^n)$, and there exists a constant $C = C(\partial\Omega, c_{\mathbb{A}}, n) > 0$ such that

$$\|\boldsymbol{u}_\psi\|_{\mathcal{H}^1(\mathbb{R}^n)^n} + \|\pi_\psi\|_{L_2(\mathbb{R}^n)} \leq C\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)^n}. \tag{3.18}$$

Proof. Transmission problem (3.17) has the following equivalent mixed variational formulation.

Find $(\mathbf{u}_\psi, \pi_\psi) \in \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$ such that

$$\begin{cases} \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v})\right\rangle_{\mathbb{R}^n} - \langle \pi_\psi, \operatorname{div}\mathbf{v}\rangle_{\mathbb{R}^n} = \langle \psi, \gamma\mathbf{v}\rangle_{\partial\Omega}\ \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^n)^n, \\ \langle \operatorname{div}\mathbf{u}_\psi, q\rangle_{\mathbb{R}^n} = 0\ \forall q \in L_2(\mathbb{R}^n). \end{cases} \tag{3.19}$$

To show this equivalence, let us first assume that $(\mathbf{u}_\psi, \pi_\psi) \in \mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n \times L_2(\mathbb{R}^n)$ satisfy transmission problem (3.17). Then, the first transmission condition in (3.17) implies the membership of $\mathbf{u}_\psi$ in $\mathcal{H}^1(\mathbb{R}^n)^n$; compare Lemma 16(ii). Moreover, formula (2.37) shows that the the pair $(\mathbf{u}_\psi, \pi_\psi)$ satisfies also the first equation in (3.19). The second equation in (3.19) follows from the second equation in the first line of (3.17).

Let us show the converse property. To this end, we assume that the pair $(\mathbf{u}_\psi, \pi_\psi) \in \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$ is a solution of the variational problem (3.19). By using the density of the space $\mathcal{D}(\mathbb{R}^n)^n$ in $\mathcal{H}^1(\mathbb{R}^n)^n$, and by considering in the first equation of (3.19) any $\mathbf{v} \in C^\infty(\mathbb{R}^n)^n$ with compact support in $\Omega_\pm$, we obtain the following variational equation:

$$\left\langle \partial_\alpha \left(a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}_\psi)\right) - \partial_i \pi_\psi, w_i \right\rangle_{\Omega_\pm} = 0\ \forall \mathbf{w} \in C^\infty_0(\Omega_\pm)^n,$$

which leads to the first equation of the transmission problem (3.17). The second equation in (3.17) is an immediate consequence of the second equation in (3.19). On the other hand, the membership of $\mathbf{u}_\psi$ in $\mathcal{H}^1(\mathbb{R}^n)^n$ yields the first transmission condition in (3.17). In addition, formula (2.37) and the first equation in (3.19) show that

$$\left\langle [\mathbf{t}(\mathbf{u}_\psi, \pi_\psi)], \gamma\mathbf{v}\right\rangle_{\partial\Omega} = \langle \psi, \gamma\mathbf{v}\rangle_{\partial\Omega} \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^n)^n. \tag{3.20}$$

Since the trace operator $\gamma : \mathcal{H}^1(\mathbb{R}^n)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$ is surjective (see Theorem 1), Equation (3.20) can be written in the form $\left\langle [\mathbf{t}(\mathbf{u}_\psi, \pi_\psi)] - \psi, \boldsymbol{\Phi}\right\rangle_{\partial\Omega} = 0$ for any $\boldsymbol{\Phi} \in H^{\frac{1}{2}}(\partial\Omega)^n$, which implies the second transmission condition in (3.17).

Thus, the transmission problem (3.17) has the equivalent mixed variational formulation (3.19), which can be written as

$$\begin{cases} a_{\mathbb{A};\mathbb{R}^n}(\mathbf{u}_\psi,\mathbf{v})+b_{\mathbb{R}^n}(\mathbf{v},\pi_\psi)=\langle\mathbf{F},\mathbf{v}\rangle_{\mathbb{R}^n} & \forall\mathbf{v}\in\mathcal{H}^1(\mathbb{R}^n)^n,\\ b_{\mathbb{R}^n}(\mathbf{u}_\psi,q)=0 & \forall q\in L_2(\mathbb{R}^n),\end{cases} \tag{3.21}$$

where $a_{\mathbb{A};\mathbb{R}^n}$ and $b_{\mathbb{R}^n}$ are the bounded bilinear forms given by (3.1) and (3.2), and $\mathbf{F}\in\mathcal{H}^{-1}(\mathbb{R}^n)^n$ is defined as

$$\langle\mathbf{F},\mathbf{v}\rangle_{\mathbb{R}^n} := \langle\psi,\gamma\mathbf{v}\rangle_{\partial\Omega}=\langle\gamma^*\psi,\mathbf{v}\rangle_{\mathbb{R}^n}\ \forall\mathbf{v}\in\mathcal{H}^1(\mathbb{R}^n)^n, \tag{3.22}$$

where $\gamma^*: H^{-\frac{1}{2}}(\partial\Omega)^n\to\mathcal{H}^{-1}(\mathbb{R}^n)^n$ is the adjoint of the trace operator $\gamma:\mathcal{H}^1(\mathbb{R}^n)^n\to H^{\frac{1}{2}}(\partial\Omega)^n$. Then, by Lemma 5, the variational problem (3.19) is well-posed. Therefore, problem (3.17) has a unique solution $(\mathbf{u}_\psi,\pi_\psi)\in\mathcal{H}^1(\mathbb{R}^n)^n\times L_2(\mathbb{R}^n)$, which depends continuously on $\boldsymbol{\psi}$. □

Theorem 3 allows to define the single-layer potentials for L_∞ coefficient Stokes system and to obtain their continuity.

Definition 3. Let conditions (1.2)–(1.4) hold. The single-layer velocity and pressure potentials,

$$\mathbf{V}_{\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega)^n\to\mathcal{H}^1(\mathbb{R}^n)^n,\ \mathcal{Q}^s_{\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega)^n\to L_2(\mathbb{R}^n), \tag{3.23}$$

are defined as

$$\mathbf{V}_{\partial\Omega}\psi := \mathbf{u}_\psi,\ \mathcal{Q}^s_{\partial\Omega}\psi := \pi_\psi\ \forall\psi\in H^{-\frac{1}{2}}(\partial\Omega)^n, \tag{3.24}$$

and the boundary operators,

$$\mathcal{V}_{\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega)^n\to H^{\frac{1}{2}}(\partial\Omega)^n,\ \mathcal{K}_{\partial\Omega}: H^{-\frac{1}{2}}(\partial\Omega)^n\to H^{-\frac{1}{2}}(\partial\Omega)^n, \tag{3.25}$$

are defined as

$$\mathcal{V}_{\partial\Omega}\psi := \gamma\mathbf{u}_\psi,\ \mathcal{K}_{\partial\Omega}\psi := \frac{1}{2}\left(\mathbf{t}^+(\mathbf{u}_\psi,\pi_\psi)+\mathbf{t}^-(\mathbf{u}_\psi,\pi_\psi)\right)\quad\forall\psi\in H^{-\frac{1}{2}}(\partial\Omega)^n, \tag{3.26}$$

where $(\mathbf{u}_{\boldsymbol{\psi}},\pi_{\boldsymbol{\psi}})$ is the unique solution of the transmission problem (3.17) in $\mathcal{H}^1(\mathbb{R}^n)^n\times L_2(\mathbb{R}^n)$.

Lemma 7. *Operators (3.23) and (3.25) are linear and continuous and for any $\psi\in H^{-\frac{1}{2}}(\partial\Omega)^n$,*

$$\mathcal{L}(\boldsymbol{V}_{\partial\Omega}\psi,\mathcal{Q}^s_{\partial\Omega}\psi)=\boldsymbol{0},\ \operatorname{div}\boldsymbol{V}_{\partial\Omega}\psi=0\ \text{in}\ \Omega_\pm.$$

In addition, the following jump relations, that are similar to the case of the Stokes system with constant coefficients (see also Lemma 3.8 in Kohr et al.,[43] Mitrea and Wright,[5] Propositions 5.2 and 5.3 in Sayas and Selgas[33]), are implied by relations (3.26) and the transmission conditions in (3.17).

Lemma 8. *Let conditions (1.2)–(1.4) hold. If $\psi\in H^{-\frac{1}{2}}(\partial\Omega)^n$, then the following formulas hold on $\partial\Omega$:*

$$[\gamma\boldsymbol{V}_{\partial\Omega}\psi]=\boldsymbol{0}, \tag{3.27}$$

$$\left[\boldsymbol{t}\left(\boldsymbol{V}_{\partial\Omega}\psi,\mathcal{Q}^s_{\partial\Omega}\psi\right)\right]=\psi,\ \boldsymbol{t}^\pm_{\mathbb{A}}\left(\boldsymbol{V}_{\partial\Omega}\psi,\mathcal{Q}^s_{\partial\Omega}\psi\right)=\pm\frac{1}{2}\psi+\mathcal{K}_{\partial\Omega}\psi. \tag{3.28}$$

3.3.1 | The single-layer potential for the adjoint Stokes system

Recall that $\mathbb{L}^*$ is the operator defined in (2.39), and $\mathbf{t}^*$ is the conormal derivative operator for the adjoint Stokes system (see formula (2.47)). The next well-posedness result follows with an argument similar to that for Theorem 3 and is based on the Green formula (2.47).

Theorem 4. *Let conditions (1.2)–(1.4) hold in $\mathbb{R}^n$. Then, for any $\psi_*\in H^{-\frac{1}{2}}(\partial\Omega)^n$, the transmission problem for the adjoint Stokes system,*

$$\begin{cases}\mathbb{L}^*\boldsymbol{v}_{\psi_*}-\nabla q_{\psi_*}=\boldsymbol{0},\ \operatorname{div}\boldsymbol{v}_{\psi_*}=0 & \text{in}\ \mathbb{R}^n\setminus\partial\Omega,\\ \left[\gamma(\boldsymbol{v}_{\psi_*})\right]=\boldsymbol{0} & \text{on}\ \partial\Omega,\\ \left[\boldsymbol{t}^*(\boldsymbol{v}_{\psi_*},q_{\psi_*})\right]=\psi_* & \text{on}\ \partial\Omega,\end{cases} \tag{3.29}$$

has a unique solution $(\boldsymbol{v}_{\psi_*}, q_{\psi_*}) \in \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$*, and there exists* $C_* = C_*(\partial\Omega, c_{\mathbb{A}}, n) > 0$ *such that*

$$\|\boldsymbol{v}_{\psi_*}\|_{\mathcal{H}^1(\mathbb{R}^n)^n} + \|q_{\psi_*}\|_{L_2(\mathbb{R}^n)} \le C_* \|\psi_*\|_{H^{-\frac{1}{2}}(\partial\Omega)^n}. \tag{3.30}$$

Definition 4. Let conditions (1.2)–(1.4) hold. The *single-layer velocity and pressure potential operators* for the adjoint Stokes system (2.43),

$$\mathbf{V}_{*\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to \mathcal{H}^1(\mathbb{R}^n)^n,\ \mathcal{Q}^s_{*\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to L_2(\mathbb{R}^n), \tag{3.31}$$

are defined as

$$\mathbf{V}_{*\partial\Omega}\psi_* := \mathbf{v}_{\psi_*},\ \mathcal{Q}^s_{*\partial\Omega}\psi_* := q_{\psi_*}\ \forall \psi_* \in H^{-\frac{1}{2}}(\partial\Omega)^n, \tag{3.32}$$

and the boundary operators,

$$\mathcal{V}_{*\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n,\ \mathcal{K}_{*\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{-\frac{1}{2}}(\partial\Omega)^n, \tag{3.33}$$

are defined as

$$\mathcal{V}_{*\partial\Omega}\psi_* := \gamma\mathbf{v}_{\psi_*},\ \mathcal{K}_{*\partial\Omega}\psi_* := \frac{1}{2}\left(\mathbf{t}^{*+}(\mathbf{v}_{\psi_*}, q_{\psi_*}) + \mathbf{t}^{*-}(\mathbf{v}_{\psi_*}, q_{\psi_*})\right)\ \forall \psi_* \in H^{-\frac{1}{2}}(\partial\Omega)^n, \tag{3.34}$$

where $(\mathbf{v}_{\psi_*}, q_{\psi_*})$ is the unique solution of the transmission problem (3.29) in $\mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)^n \times L_2(\mathbb{R}^n)$.

Lemma 9. *Let conditions (1.2)–(1.4) hold. Then, the following formulas hold on* $\partial\Omega$*:*

$$[\gamma \boldsymbol{V}_{*\partial\Omega}\psi_*] = \boldsymbol{0},\ \boldsymbol{t}^{*\pm}\left(\boldsymbol{V}_{*\partial\Omega}\psi_*, \mathcal{Q}^s_{*\partial\Omega}\psi_*\right) = \pm\frac{1}{2}\psi_* + \mathcal{K}_{*\partial\Omega}\psi_*\ \forall \psi_* \in H^{-\frac{1}{2}}(\partial\Omega)^n, \tag{3.35}$$

$$\langle \psi, \mathcal{V}_{*\partial\Omega}\psi_* \rangle_{\partial\Omega} = \langle \psi_*, \mathcal{V}_{\partial\Omega}\psi \rangle_{\partial\Omega}\ \forall \psi, \psi_* \in H^{-\frac{1}{2}}(\partial\Omega)^n. \tag{3.36}$$

Proof. First, formulas (3.35) are implied by relations (3.34) and the transmission conditions in (3.29).

Now, let $\left(\mathbf{V}_{\partial\Omega}\psi, \mathcal{Q}^s_{\partial\Omega}\psi\right)$ be the unique solution in $\mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$ of transmission problem (3.17) with the given datum $\psi \in H^{-\frac{1}{2}}(\partial\Omega)^n$. Also let $\left(\mathbf{V}_{*\partial\Omega}\psi_*, \mathcal{Q}^s_{*\partial\Omega}\psi_*\right)$ denote the unique solution in $\mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$ of transmission problem (3.29) with the given datum $\psi_* \in H^{-\frac{1}{2}}(\partial\Omega)^n$. Then, the Green formulas (2.37) and (2.48) imply that

$$\left\langle \left[\mathbf{t}\left(\mathbf{V}_{\partial\Omega}\psi, \mathcal{Q}^s_{\partial\Omega}\psi\right)\right], \mathcal{V}_{*\partial\Omega}\psi_* \right\rangle_{\partial\Omega} = \left\langle a^{\alpha\beta}_{ij} E_{j\beta}\left(\mathbf{V}_{\partial\Omega}\psi\right), E_{i\alpha}\left(\mathbf{V}_{*\partial\Omega}\psi_*\right) \right\rangle_{\mathbb{R}^n}, \tag{3.37}$$

$$\left\langle \left[\mathbf{t}^*\left(\mathbf{V}_{*\partial\Omega}\psi_*, \mathcal{Q}^s_{*\partial\Omega}\psi_*\right)\right], \mathcal{V}_{\partial\Omega}\psi \right\rangle_{\partial\Omega} = \left\langle a^{\alpha\beta}_{ij} E_{j\beta}\left(\mathbf{V}_{\partial\Omega}\psi\right), E_{i\alpha}\left(\mathbf{V}_{*\partial\Omega}\psi_*\right) \right\rangle_{\mathbb{R}^n}. \tag{3.38}$$

Moreover, by the second formulas in (3.28) and (3.35),

$$\left[\mathbf{t}\left(\mathbf{V}_{\partial\Omega}\psi, \mathcal{Q}^s_{\partial\Omega}\psi\right)\right] = \psi,\ \left[\mathbf{t}^*\left(\mathbf{V}_{*\partial\Omega}\psi_*, \mathcal{Q}^s_{*\partial\Omega}\psi_*\right)\right] = \psi_*. \tag{3.39}$$

Then, equality (3.36) follows from (3.37)–(3.39) (cf. Proposition 5.4 in Sayas and Selgas[33] in the case (1.10) with $\mu = 1$ and $\lambda = 0$). □

Remark 2.

(*i*) Formula (3.36) shows that the adjoint of the single-layer operator $\mathcal{V}_{\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$ corresponding to the Stokes system from (3.17) is the operator $\mathcal{V}_{*\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$ given by formula (3.34) (see Definition 4) and corresponding to the adjoint Stokes system from (3.29).

(*ii*) In the isotropic case (1.10), Definition 4 reduces to Definition 3, and the single-layer operator $\mathcal{V}_{\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$ is self adjoint. Thus, formula (3.36) becomes

$$\langle \psi, \mathcal{V}_{\partial\Omega}\psi_* \rangle_{\partial\Omega} = \langle \psi_*, \mathcal{V}_{\partial\Omega}\psi \rangle_{\partial\Omega}\ \forall \psi, \psi_* \in H^{-\frac{1}{2}}(\partial\Omega)^n. \tag{3.40}$$

For a given operator $T:X\to Y$, the set $\mathrm{Ker}\{T : X \to Y\} := \{x \in X : T(x) = 0\}$ is the null space of T. Let $\boldsymbol{\nu}$ be the outward unit normal to Ω, which exists a.e. on $\partial\Omega$, and let $\mathrm{span}\{\nu\} := \{c\nu : c \in \mathbb{R}\}$. Let also

$$\chi_{\Omega_+} := \begin{cases} 1 \text{ in } \Omega_+ \\ 0 \text{ in } \Omega_-, \end{cases} \qquad H_\nu^{\frac{1}{2}}(\partial\Omega)^n := \left\{ \boldsymbol{\Phi} \in H^{\frac{1}{2}}(\partial\Omega)^n : \langle \boldsymbol{\Phi}, \nu \rangle_{\partial\Omega} = 0 \right\}. \tag{3.41}$$

Next, we mention the main properties of the single-layer operator, similar to the ones provided in Lemma 3.12 in Kohr et al.[43] in the case of a strongly elliptic viscosity tensor coefficient (see also Lemma 4.9 in Kohr et al.,[39] Theorem 10.5.3 in Mitrea and Wright,[5] Proposition 3.3(c) in Băcuţă et al.,[36] and Proposition 5.4 in Sayas and Selgas[33] in the case (1.10) with $\mu = 1$ and $\lambda = 0$).

Lemma 10. *Let conditions (1.2)–(1.4) hold. Then,*

$$\boldsymbol{V}_{\partial\Omega}\nu = \boldsymbol{0} \text{ in } \mathbb{R}^n, \ \ \mathcal{Q}^s_{\partial\Omega}\nu = -\chi_{\Omega_+}, \tag{3.42}$$

$$\mathcal{V}_{\partial\Omega}\nu = \boldsymbol{0} \text{ on } \partial\Omega, \tag{3.43}$$

$$\mathcal{V}_{\partial\Omega}\psi \in H_\nu^{\frac{1}{2}}(\partial\Omega)^n \ \ \forall \psi \in H^{-\frac{1}{2}}(\partial\Omega)^n. \tag{3.44}$$

Proof. First, we note that the transmission problem (3.17) with the given datum $\psi = \nu \in H^{-\frac{1}{2}}(\partial\Omega)^n$ is well-posed. Let us show that the pair

$$(\mathbf{u}_\nu, \pi_\nu) = \left(\mathbf{0}, -\chi_{\Omega_+}\right) \in \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n) \tag{3.45}$$

is the unique solution of this transmission problem. Indeed, $(\mathbf{u}_\nu, \pi_\nu)$ satisfies the equations and the first transmission condition in (3.17), and by formulas (2.25), (2.35), and (3.45), and by the divergence theorem, we obtain that

$$\langle [\mathbf{t}(\mathbf{u}_\nu, \pi_\nu)], \boldsymbol{\Phi} \rangle_{\partial\Omega} = -\left\langle \pi_\nu, \mathrm{div}(\gamma_+^{-1}\boldsymbol{\Phi}) \right\rangle_{\Omega_+} = \langle \nu, \boldsymbol{\Phi} \rangle_{\partial\Omega} \ \ \forall \boldsymbol{\Phi} \in H^{\frac{1}{2}}(\partial\Omega)^n, \tag{3.46}$$

and hence, $[\mathbf{t}(\mathbf{u}_\nu, \pi_\nu)] = \nu$. Consequently, the pair $(\mathbf{u}_\nu, \pi_\nu)$ given by (3.45) is the unique solution of the transmission problem (3.17) with the given datum $\psi = \nu \in H^{-\frac{1}{2}}(\partial\Omega)^n$. Then, relations (3.42) and (3.43) follow from Definition 3. Thus, $\mathrm{span}\{\nu\} \subseteq \mathrm{Ker}\left\{ \mathcal{V}_{\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n \right\}$. Similarly, we obtain

$$\mathbf{V}_{*\partial\Omega}\nu = \mathbf{0} \text{ in } \mathbb{R}^n, \ \mathcal{V}_{*\partial\Omega}\nu = \mathbf{0} \text{ on } \partial\Omega. \tag{3.47}$$

Next, we apply formula (3.36) for the densities $\psi \in H^{-\frac{1}{2}}(\partial\Omega)^n$ and $\nu \in H^{-\frac{1}{2}}(\partial\Omega)^n$ and use the second relation in (3.47). Then, we obtain that $\langle \mathcal{V}_{\partial\Omega}\psi, \nu \rangle_{\partial\Omega} = \langle \psi, \mathcal{V}_{*\partial\Omega}\nu \rangle_{\partial\Omega} = 0$, and hence, (3.44) follows. □

3.3.2 | Isomorphism property of the single-layer operator

Next, we show the following invertibility property of the single-layer potential operator (cf. Lemma 3.13 in Kohr et al.,[43] Theorem 10.5.3 in Mitrea and Wright,[5] Proposition 3.3 (d) in Băcuţă et al.,[36] and Proposition 5.5 in Sayas and Selgas[33] in the case 1.10 with $\mu = 1$ and $\lambda = 0$).

Lemma 11. *Let conditions (1.2)–(1.4) hold in $\mathbb{R}^n$. Then,*

$$\mathrm{Ker}\left\{ \mathcal{V}_{\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n \right\} = span\{\nu\}, \tag{3.48}$$

and the following operator is an isomorphism:

$$\mathcal{V}_{\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n / span\{\nu\} \to H_\nu^{\frac{1}{2}}(\partial\Omega)^n. \tag{3.49}$$

Proof.

(i) Let $\psi_0 \in \operatorname{Ker}\left\{\mathcal{V}_{\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n\right\}$ and let $(\mathbf{u}_{\psi_0}, \pi_{\psi_0}) = \left(\mathbf{V}_{\partial\Omega}\psi_0, \mathcal{Q}^s_{\partial\Omega}\psi_0\right)$ be the unique solution in $\mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$ of the transmission problem (3.17) with the given datum $\boldsymbol{\psi}_0$. In view of formula (2.37) and since $\gamma\mathbf{u}_{\psi_0} = \mathbf{0}$ on $\partial\Omega$, we obtain that

$$a_{\mathbb{A};\mathbb{R}^n}\left(\mathbf{u}_{\psi_0}, \mathbf{u}_{\psi_0}\right) = \left\langle [\mathbf{t}(\mathbf{u}_{\psi_0}, \pi_{\psi_0})], \gamma\mathbf{u}_{\psi_0}\right\rangle_{\partial\Omega} = 0. \tag{3.50}$$

In addition, since $\operatorname{div}\mathbf{u}_{\psi_0} = 0$, we have $E_{ii}(\mathbf{u}_{\psi_0}) = 0$, and due to assumption (1.4),

$$a_{\mathbb{A};\mathbb{R}^n}\left(\mathbf{u}_{\psi_0}, \mathbf{u}_{\psi_0}\right) \geq c_{\mathbb{A}}^{-1}\|\mathbb{E}(\mathbf{u}_{\psi_0})\|^2_{L_2(\mathbb{R}^n)^n}, \tag{3.51}$$

which implies that $\mathbb{E}(\mathbf{u}_{\psi_0}) = 0$ and hence $\mathbf{u}_{\psi_0} = \mathbf{0}$ in $\mathbb{R}^n$; compare Section 2.2.4.
Moreover, $\mathbf{u}_{\psi_0}$ and π_{ψ_0} satisfy the Stokes equation in $\mathbb{R}^n\setminus\partial\Omega$ and π_{ψ_0} belongs to $L_2(\mathbb{R}^n)$. Thus, there exists $c_0 \in \mathbb{R}$ such that $\pi_{\psi_0} = c_0\chi_{\Omega_+}$ in $\mathbb{R}^n$. Then, formulas (2.25) and (2.35) and the divergence theorem yield that

$$\langle [\mathbf{t}(\mathbf{u}_{\psi_0}, \pi_{\psi_0})], \boldsymbol{\Phi}\rangle_{\partial\Omega} = -\left\langle \pi_{\psi_0}, \operatorname{div}(\gamma_+^{-1}\boldsymbol{\Phi})\right\rangle_{\Omega_+} = -c_0\langle \nu, \boldsymbol{\Phi}\rangle_{\partial\Omega} \quad \forall \boldsymbol{\Phi} \in H^{\frac{1}{2}}(\partial\Omega)^n,$$

and accordingly that $\psi_0 = [\mathbf{t}(\mathbf{u}_{\psi_0}, \pi_{\psi_0})] = -c_0\nu$. Taking into account (3.43), formula (3.48) follows.

(ii) Next, we use the notation $[\![\cdot]\!]$ for the classes of the space $H^{-\frac{1}{2}}(\partial\Omega)^n/\operatorname{span}\{\nu\}$. Thus, $[\![\psi]\!] = \psi + \operatorname{span}\{\nu\}$, with $\psi \in H^{-\frac{1}{2}}(\partial\Omega)^n$. We show that there exists a constant $c = c(\partial\Omega, c_{\mathbb{A}}, n) > 0$ such that the single-layer potential operator satisfies the coercivity inequality

$$\langle [\![\psi]\!], \mathcal{V}_{\partial\Omega}[\![\psi]\!]\rangle_{\partial\Omega} \geq c\,\|[\![\psi]\!]\|^2_{H^{-\frac{1}{2}}(\partial\Omega)^n/\operatorname{span}\{\nu\}} \quad \forall [\![\psi]\!] \in H^{-\frac{1}{2}}(\partial\Omega)^n/\operatorname{span}\{\nu\} \tag{3.52}$$

(cf. Lemma 4.10 in Kohr et al.[39] and Proposition 5.5 in Sayas and Selgas[33]).

Let $[[\psi]] \in H^{-\frac{1}{2}}(\partial\Omega)^n/\operatorname{span}\{\nu\}$. In view of formula (2.37), Definition 3, relations (3.44) and (3.48), and the Korn inequality, we obtain (cf. 3.7),

$$\begin{aligned}\langle [\![\psi]\!], \mathcal{V}_{\partial\Omega}[\![\psi]\!]\rangle_{\partial\Omega} &= \langle \psi, \mathcal{V}_{\partial\Omega}\psi\rangle_{\partial\Omega} = \langle [\mathbf{t}(\mathbf{u}_\psi, \pi_\psi)], \gamma\mathbf{u}_\psi\rangle_{\partial\Omega}\\ &= a_{\mathbb{A};\mathbb{R}^n}(\mathbf{u}_\psi, \mathbf{u}_\psi) \geq c_{\mathbb{A}}^{-1}\|\mathbb{E}(\mathbf{u}_\psi)\|^2_{L_2(\mathbb{R}^n)^{n\times n}} \geq 2^{-1}c_{\mathbb{A}}^{-1}c_1\|\mathbf{u}_\psi\|^2_{\mathcal{H}^1(\mathbb{R}^n)^n},\end{aligned} \tag{3.53}$$

where $\mathbf{u}_\psi = \mathbf{V}_{\partial\Omega}\psi$ and $\pi_\psi = \mathcal{Q}^s_{\partial\Omega}\psi$. Moreover, the trace operator $\gamma : \mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n \to H^{\frac{1}{2}}_\nu(\partial\Omega)^n$ is surjective having a bounded right inverse $\gamma^{-1} : H^{\frac{1}{2}}_\nu(\partial\Omega)^n \to \mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^n)^n$ (cf., e.g., Proposition 4.4 in Sayas and Selgas[33] in the case $n = 3$. Arguments similar to those for Proposition 4.4 of Sayas and Selgas[33] imply that the result remains valid also in the case $n \geq 3$). Moreover, there exists $c' = c'(\partial\Omega, n) > 0$ such that

$$\begin{aligned}|\langle [\![\psi]\!], \boldsymbol{\Phi}\rangle_{\partial\Omega}| &= |\langle \psi, \boldsymbol{\Phi}\rangle_{\partial\Omega}| = |\langle [\mathbf{t}(\mathbf{u}_\psi, \pi_\psi)], \boldsymbol{\Phi}\rangle_{\partial\Omega}| = |a_{\mathbb{A};\mathbb{R}^n}(\mathbf{u}_\psi, \gamma^{-1}\boldsymbol{\Phi})|\\ &\leq \|\mathbb{A}\|_{L_\infty(\mathbb{R}^n)}c'\|\mathbf{u}_\psi\|_{\mathcal{H}^1(\mathbb{R}^n)^n}\|\boldsymbol{\Phi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n} \quad \forall \boldsymbol{\Phi} \in H^{\frac{1}{2}}_\nu(\partial\Omega)^n.\end{aligned} \tag{3.54}$$

Inequality (3.54) and the duality of the spaces $H^{\frac{1}{2}}_\nu(\partial\Omega)^n$ and $H^{-\frac{1}{2}}(\partial\Omega)^n/\operatorname{span}\{\nu\}$ show that

$$\|[\![\psi]\!]\|_{H^{-\frac{1}{2}}(\partial\Omega)^n/\operatorname{span}\{\nu\}} \leq \|\mathbb{A}\|_{L_\infty(\mathbb{R}^n)}c'\|\mathbf{u}_\psi\|_{\mathcal{H}^1(\mathbb{R}^n)^n}. \tag{3.55}$$

Then, the coercivity inequality (3.52) follows from inequalities (3.53) and (3.55), and the Lax–Milgram lemma yields that the single-layer potential operator (3.49) is an isomorphism, as asserted. □

3.4 | The double-layer potential operator for the anisotropic Stokes system with L_∞ viscosity tensor coefficient

Note that if $\mathbf{u} \in L_{2,\mathrm{loc}}(\mathbb{R}^n)^n$ is such that $\mathbf{u}|_{\Omega_+} \in H^1(\Omega_+)^n$, $\mathbf{u}|_{\Omega_-} \in \mathcal{H}^1(\Omega_-)^n$, then, due to Definition (2.16), $\mathbf{u} \in \mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n$ and can be endowed with the norm $\|\mathbf{u}\|^2_{\mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n} := \|\mathbf{u}\|^2_{H^1(\Omega_+)^n} + \|\mathbf{u}\|^2_{\mathcal{H}^1(\Omega_-)^n}$ that is equivalent to the norm (2.17).

By following a similar approach to that used to define the Stokes single-layer potentials, we now show the well-posedness of a transmission problem that allows us to define the L_∞-coefficient Stokes double-layer potentials with the densities in the space $H^{\frac{1}{2}}(\partial\Omega)^n$, $n \geq 3$ (cf. Theorem 3.14 in Kohr et al.[43] for the Stokes system with strongly elliptic tensor coefficient and Propositions 6.1 and 7.1 in Sayas and Selgas[33] for the isotropic case 1.10 with $\mu = 1$, $\lambda = 0$, and $n = 2, 3$).

Theorem 5. *Let conditions (1.2)–(1.4) hold on $\mathbb{R}^n$. Then, for any $\varphi \in H^{\frac{1}{2}}(\partial\Omega)^n$, the transmission problem,*

$$\begin{cases} \mathcal{L}(\boldsymbol{u}_\varphi, \pi_\varphi) = \boldsymbol{0},\ \operatorname{div}\boldsymbol{u}_\varphi = 0 & in\ \mathbb{R}^n\setminus\partial\Omega, \\ \left[\gamma\boldsymbol{u}_\varphi\right] = -\varphi & on\ \partial\Omega, \\ \left[\boldsymbol{t}(\boldsymbol{u}_\varphi, \pi_\varphi)\right] = \boldsymbol{0} & on\ \partial\Omega, \end{cases} \tag{3.56}$$

has a unique solution $(\boldsymbol{u}_\varphi, \pi_\varphi) \in \mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n \times L_2(\mathbb{R}^n)$, and there exists $C = C(\partial\Omega, c_{\mathbb{A}}, n) > 0$ such that

$$\|\boldsymbol{u}_\varphi\|_{\mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n} + \|\pi_\varphi\|_{L_2(\mathbb{R}^n)} \leq C\|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)^n}. \tag{3.57}$$

Proof. First, we show the uniqueness. Let $(\mathbf{u}_0, \pi_0) \in \mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n \times L_2(\mathbb{R}^n)$ be a solution of the homogeneous version of problem (3.56). Therefore, the couple $(\mathbf{u}_0, \pi_0)$ is a solution of the homogeneous version of the transmission problem (3.17), which, in view of Theorem 3, has only the trivial solution.

Next, we show that the transmission problem (3.56) has the following equivalent variational formulation.

Find $(\mathbf{u}_\varphi, \pi_\varphi) \in \mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n \times L_2(\mathbb{R}^n)$ such that

$$\begin{cases} \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}_\varphi), E_{i\alpha}(\mathbf{v})\right\rangle_{\Omega_+} + \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}_\varphi), E_{i\alpha}(\mathbf{v})\right\rangle_{\Omega_-} - \langle \pi_\varphi, \operatorname{div}\mathbf{v}\rangle_{\mathbb{R}^n} = 0\ \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^n)^n, \\ \left\langle \operatorname{div}\mathbf{u}_\varphi, q\right\rangle_{\mathbb{R}^n\setminus\partial\Omega} = 0\ \forall q \in L_2(\mathbb{R}^n), \\ \left[\gamma\mathbf{u}_\varphi\right] = -\varphi \operatorname{on} \partial\Omega. \end{cases} \tag{3.58}$$

Indeed, if $(\mathbf{u}_\varphi, \pi_\varphi) \in \mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n \times L_2(\mathbb{R}^n)$ satisfies transmission problem (3.56), then the Green formula (2.37) yields the first equation of problem (3.58). The second equation of (3.58) is the distributional form of the second equation of (3.56). Conversely, assume that $(\mathbf{u}_\varphi, \pi_\varphi) \in \mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n \times L_2(\mathbb{R}^n)$ satisfies the variational problem (3.58). Then, from the first equation of (3.58), we deduce that

$$\left\langle \left(\partial_\alpha\left(a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}_\varphi)\right) - \partial_i\pi_\varphi\right)\Big|_{\Omega_\pm}, v_i\right\rangle_{\Omega_\pm} = 0\ \forall \mathbf{v} = (v_1, \ldots, v_n) \in \mathcal{D}(\Omega_\pm)^n, \tag{3.59}$$

which is the distributional form of the first equation in (3.56). The second equation of (3.56) follows from the second equation of (3.58). In addition, the first equation of (3.58) and the Green formula (2.37) applied to the pair $(\mathbf{u}_\varphi, \pi_\varphi)$ yield that

$$\left\langle [\mathbf{t}(\mathbf{u}_\varphi, \pi_\varphi)], \gamma\mathbf{v}\right\rangle_{\partial\Omega} = 0\ \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^n)^n. \tag{3.60}$$

Moreover, the surjectivity property of the trace operator $\gamma : \mathcal{H}^1(\mathbb{R}^n)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$ shows that Equation (3.60) can be written in the equivalent form

$$\left\langle [\mathbf{t}(\mathbf{u}_\varphi, \pi_\varphi)], \boldsymbol{\Psi}\right\rangle_{\partial\Omega} = 0\ \forall \boldsymbol{\Psi} \in H^{\frac{1}{2}}(\partial\Omega)^n, \tag{3.61}$$

which yields the second transmission condition of (3.56). The first transmission condition in (3.56) follows from the transmission condition in (3.58). Therefore, problems (3.56) and (3.58) are equivalent.

By using again the existence of a right inverse $\gamma_\pm^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to \mathcal{H}^1(\Omega_\pm)$ of the trace operator $\gamma_\pm : \mathcal{H}^1(\Omega_\pm) \to H^{\frac{1}{2}}(\partial\Omega)$ (see Theorem 1), we deduce that for $\varphi \in H^{\frac{1}{2}}(\partial\Omega)^n$ given, there exists $\mathbf{w}_\varphi \in \mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n$ continuously depending on $\boldsymbol{\varphi}$ such that $\left[\gamma\mathbf{w}_\varphi\right] = -\varphi \operatorname{on} \partial\Omega$. For example, we can take $\mathbf{w}_\varphi = 0$ in Ω_- and $\mathbf{w}_\varphi = -\gamma_+^{-1}\varphi$ in Ω_+.

Therefore, $\mathbf{v}_\varphi := \mathbf{u}_\varphi - \mathbf{w}_\varphi$ satisfies the condition $[\gamma\mathbf{v}_\varphi] = \mathbf{0}$, and hence, by Lemma 16 can be extended to $\mathbf{v}_\varphi \in \mathcal{H}^1(\mathbb{R}^n)^n$. In addition, (3.58) reduces to the following variational problem:

$$\begin{cases} a_{\mathbb{A};\mathbb{R}^n}(\mathbf{v}_\varphi, \mathbf{v}) + b_{\mathbb{R}^n}(\mathbf{v}, \pi_\varphi) = \xi_\varphi(\mathbf{v}) \ \forall \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^n)^n, \\ b_{\mathbb{R}^n}(\mathbf{v}_\varphi, q) = \zeta_\varphi(q) \ \forall q \in L_2(\mathbb{R}^n), \end{cases} \tag{3.62}$$

with the unknown $(\mathbf{v}_\varphi, \pi_\varphi) \in \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$, where $a_{\mathbb{A};\mathbb{R}^n}(\cdot,\cdot) : \mathcal{H}^1(\mathbb{R}^n)^n \times \mathcal{H}^1(\mathbb{R}^n)^n \to \mathbb{R}$ and $b_{\mathbb{R}^n} : \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n) \to \mathbb{R}$ are the bounded bilinear forms given by (3.1) and (3.2), respectively. Conditions (1.2) and the Hölder inequality show the boundedness of the linear forms

$$\xi_\varphi : \mathcal{H}^1(\mathbb{R}^n)^n \to \mathbb{R},\ \xi_\varphi(\mathbf{v}) := -\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{w}_\varphi), E_{i\alpha}(\mathbf{v}) \right\rangle_{\Omega_+} - \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{w}_\varphi), E_{i\alpha}(\mathbf{v}) \right\rangle_{\Omega_-}, \tag{3.63}$$

$$\zeta_\varphi : L_2(\mathbb{R}^n)^n \to \mathbb{R},\ \zeta_\varphi(q) := -\left(\langle \operatorname{div}\mathbf{w}_\varphi, q\rangle_{\Omega_+} + \langle \operatorname{div}\mathbf{w}_\varphi, q\rangle_{\Omega_-} \right) \ \forall q \in L_2(\mathbb{R}^n). \tag{3.64}$$

Then, Lemma 5 implies that the variational problem (3.62) has a unique solution $(\mathbf{v}_\varphi, \pi_\varphi) \in \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$. Hence, the pair $(\mathbf{u}_\varphi, \pi_\varphi) = (\mathbf{w}_\varphi + \mathbf{v}_\varphi, \pi_\varphi)$ is a solution of the variational problem (3.58) in the space $\mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)^n \times L_2(\mathbb{R}^n)$ and depends continuously on $\boldsymbol{\varphi}$. The equivalence between problems (3.56) and (3.58) show that $(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}})$ is the unique solution of the transmission problem (3.56). □

Theorem 5 suggests the following definition of the double-layer potential operator for the anisotropic Stokes system (1.8) in the case $n \geq 3$ (cf. Sayas and Selgas[33], p.77 for the constant-coefficient Stokes system in $\mathbb{R}^3$, formula (4.5) and Lemma 4.6 in Barton[29] for general strongly elliptic differential operators, and Definition 3.15 in Kohr et al.[43] for the Stokes system with L_∞ strongly elliptic viscosity coefficient).

Definition 5. Let conditions (1.2)–(1.4) hold. Then, the double-layer velocity and pressure potential operators,

$$\mathbf{W}_{\partial\Omega} : H^{\frac{1}{2}}(\partial\Omega)^n \to \mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)^n,\ \mathcal{Q}^d_{\partial\Omega} : H^{\frac{1}{2}}(\partial\Omega)^n \to L_2(\mathbb{R}^n), \tag{3.65}$$

are defined as

$$\mathbf{W}_{\partial\Omega}\varphi := \mathbf{u}_\varphi,\ \mathcal{Q}^d_{\partial\Omega}\varphi := \pi_\varphi \ \forall \varphi \in H^{\frac{1}{2}}(\partial\Omega)^n, \tag{3.66}$$

and the boundary operators,

$$\mathbf{K}_{\partial\Omega} : H^{\frac{1}{2}}(\partial\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n,\ \mathbf{D}_{\partial\Omega} : H^{\frac{1}{2}}(\partial\Omega)^n \to H^{-\frac{1}{2}}(\partial\Omega)^n, \tag{3.67}$$

are defined as

$$\mathbf{K}_{\partial\Omega}\varphi := \frac{1}{2}\left(\gamma_+\mathbf{u}_\varphi + \gamma_-\mathbf{u}_\varphi\right) \ \forall \varphi \in H^{\frac{1}{2}}(\partial\Omega)^n, \tag{3.68}$$

$$\mathbf{D}_{\partial\Omega}\varphi := \mathbf{t}^+\left(\mathbf{W}_{\partial\Omega}\varphi, \mathcal{Q}^d_{\partial\Omega}\varphi\right) = \mathbf{t}^-\left(\mathbf{W}_{\partial\Omega}\varphi, \mathcal{Q}^d_{\partial\Omega}\varphi\right) \ \forall \varphi \in H^{\frac{1}{2}}(\partial\Omega)^n, \tag{3.69}$$

where $(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}})$ is the unique solution of the transmission problem (3.56) in $\mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)^n \times L_2(\mathbb{R}^n)$.

Moreover, the well-posedness of the transmission problem (3.56) and Definition 5 lead to the next result (see also formulas (10.81) and (10.82) in Mitrea and Wright[5] and Propositions 6.2 and 6.3 in Sayas and Selgas[33] for the constant-coefficient Stokes system in $\mathbb{R}^3$, and Lemma 5.8 in Barton[29] for strongly elliptic operators).

Lemma 12. *Let conditions (1.2)–(1.4) are satisfied. Then, the following assertions hold.*

(i) Operators (3.65) and (3.67) are linear and continuous and for any $\varphi \in H^{-\frac{1}{2}}(\partial\Omega)^n$,

$$\mathcal{L}(\boldsymbol{W}_{\partial\Omega}\varphi, \mathcal{Q}^d_{\partial\Omega}\varphi) = \boldsymbol{0},\ \operatorname{div}\boldsymbol{W}_{\partial\Omega}\varphi = 0 \ in\ \Omega_\pm.$$

(ii) For any $\varphi \in H^{\frac{1}{2}}(\partial\Omega)^n$, the following jump formulas hold on $\partial\Omega$

$$\gamma_\pm\left(\boldsymbol{W}_{\partial\Omega}\varphi\right) = \mp\frac{1}{2}\varphi + \boldsymbol{K}_{\partial\Omega}\varphi,\ \boldsymbol{t}^\pm\left(\boldsymbol{W}_{\partial\Omega}\varphi, \mathcal{Q}^d_{\partial\Omega}\varphi\right) = \boldsymbol{D}_{\partial\Omega}\varphi. \tag{3.70}$$

(iii) The operator $\mathcal{K}_{*\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{-\frac{1}{2}}(\partial\Omega)^n$ *defined in (3.34) is the transpose of the double-layer operator* $\boldsymbol{K}_{\partial\Omega}$: $H^{\frac{1}{2}}(\partial\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$ *defined in (3.68), that is,*

$$\langle \psi_*, \boldsymbol{K}_{\partial\Omega}\varphi\rangle_{\partial\Omega} = \langle \mathcal{K}_{*\partial\Omega}\psi_*, \varphi\rangle_{\partial\Omega} \ \forall \varphi \in H^{\frac{1}{2}}(\partial\Omega)^n, \ \psi_* \in H^{-\frac{1}{2}}(\partial\Omega)^n. \tag{3.71}$$

Proof. The continuity of operators (3.65) and (3.67) follows from the well-posedness of transmission problem (3.56) and Definition 5. By invoking again Definition 5 and the transmission conditions in (3.56), we obtain jump formulas (3.70).

Next, we show equality (3.71), by using an argument similar to that in the proof of Proposition 6.7 in Sayas and Selgas[33] for the constant-coefficient Stokes system. Let $\varphi \in H^{\frac{1}{2}}(\partial\Omega)^n$ be given, and let $(\mathbf{u}_\varphi, \pi_\varphi) = \left(\mathbf{W}_{\partial\Omega}\varphi, \mathcal{Q}^d_{\partial\Omega}\varphi\right) \in \mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n \times L_2(\mathbb{R}^n)$ be the unique solution of the problem (3.56) with datum $\boldsymbol{\varphi}$. Let also $\psi_* \in H^{-\frac{1}{2}}(\partial\Omega)^n$ and $(\mathbf{v}_{\psi_*}, q_{\psi_*}) = (\mathbf{V}_{*\partial\Omega}\psi_*, \mathcal{Q}^s_{*\partial\Omega}\psi_*) \in \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$ be the solution of the problem (3.29) with datum $\boldsymbol{\psi}_*$, that is, the single-layer velocity and pressure potentials with density $\boldsymbol{\psi}_*$ for the adjoint Stokes system (see Definition 4). Then, by formulas (2.37) and (3.69),

$$0 = \langle [\mathbf{t}(\mathbf{W}_{\partial\Omega}\varphi, \mathcal{Q}^d_{\partial\Omega}\varphi)], \gamma\mathbf{V}_{*\partial\Omega}\psi_*\rangle_{\partial\Omega} = \left\langle a^{\alpha\beta}_{ij} E_{j\beta}\left(\mathbf{W}_{\partial\Omega}\varphi\right), E_{i\alpha}\left(\mathbf{V}_{*\partial\Omega}\psi_*\right)\right\rangle_{\mathbb{R}^n\backslash\partial\Omega}. \tag{3.72}$$

Moreover, the Green formula (2.47) for the adjoint Stokes system and equality (3.72) yield that

$$\begin{aligned}
&\left\langle \mathbf{t}^{*+}\left(\mathbf{V}_{*\partial\Omega}\psi_*, \mathcal{Q}^s_{*\partial\Omega}\psi_*\right), \gamma_+(\mathbf{W}_{\partial\Omega}\varphi)\right\rangle_{\partial\Omega} = \left\langle a^{\alpha\beta}_{ij} E_{i\alpha}\left(\mathbf{V}_{*\partial\Omega}\psi_*\right), E_{j\beta}\left(\mathbf{W}_{\partial\Omega}\varphi\right)\right\rangle_{\Omega_+} \\
&= -\left\langle a^{\alpha\beta}_{ij} E_{i\alpha}\left(\mathbf{V}^*_{\partial\Omega}\psi_*\right), E_{j\beta}\left(\mathbf{W}_{\partial\Omega}\varphi\right)\right\rangle_{\Omega_-} = \left\langle \mathbf{t}^{*-}\left(\mathbf{V}_{*\partial\Omega}\psi, \mathcal{Q}^s_{*\partial\Omega}\psi_*\right), \gamma_-\left(\mathbf{W}_{\partial\Omega}\varphi\right)\right\rangle_{\partial\Omega}.
\end{aligned} \tag{3.73}$$

Therefore, we obtain the equality

$$\left\langle \mathbf{t}^{*+}\left(\mathbf{V}_{*\partial\Omega}\psi_*, \mathcal{Q}^s_{*\partial\Omega}\psi_*\right), \gamma_+(\mathbf{W}_{\partial\Omega}\varphi)\right\rangle_{\partial\Omega} = \left\langle \mathbf{t}^{*-}\left(\mathbf{V}_{*\partial\Omega}\psi, \mathcal{Q}^s_{*\partial\Omega}\psi_*\right), \gamma_-\left(\mathbf{W}_{\partial\Omega}\varphi\right)\right\rangle_{\partial\Omega}. \tag{3.74}$$

Then, the second formula (3.35), the first formula (3.70), and formula (3.74) lead to the equality

$$\left\langle \frac{1}{2}\psi_* + \mathcal{K}_{*\partial\Omega}\psi_*, -\frac{1}{2}\varphi + \mathbf{K}_{\partial\Omega}\varphi\right\rangle_{\partial\Omega} = \left\langle -\frac{1}{2}\psi_* + \mathcal{K}_{*\partial\Omega}\psi_*, \frac{1}{2}\varphi + \mathbf{K}_{\partial\Omega}\varphi\right\rangle_{\partial\Omega}, \tag{3.75}$$

and hence to equality (3.71), as asserted. □

Remark 3. If the operator $\mathbb{L}$ is self-adjoint, that is, $A^{*\alpha\beta} = A^{\beta\alpha}$, $a^{\beta\alpha}_{ji} = a^{\alpha\beta}_{ij}$, $\alpha, \beta, i, j = 1, \ldots, n$, see (2.40), and particularly in the isotropic case (1.10), then Definition 4 reduces to Definition 3 and the operator $\mathcal{K}_{*\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{-\frac{1}{2}}(\partial\Omega)^n$ given by (3.34) coincides with $\mathcal{K}_{\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^n \to H^{-\frac{1}{2}}(\partial\Omega)^n$ given by (3.26).

3.4.1 | Invertibility of the operator $\mathbf{D}_{\partial\Omega}$

Let $\mathcal{R}$ be the set of rigid body motion fields in $\mathbb{R}^n$, see (2.18), and let

$$\mathcal{R}_{\partial\Omega} := \gamma\mathcal{R}, \qquad \mathcal{R}^{\perp}_{\partial\Omega} := \left\{\boldsymbol{\Psi} \in H^{-\frac{1}{2}}(\partial\Omega)^n : \langle \boldsymbol{\Psi}, \mathbf{r}\rangle_{\partial\Omega} = 0 \ \forall \mathbf{r} \in \mathcal{R}_{\partial\Omega}\right\}. \tag{3.76}$$

Also let $H^{\frac{1}{2}}_{\mathcal{R}}(\partial\Omega)^n$ be the closed subspace of $H^{\frac{1}{2}}(\partial\Omega)^n$ defined by

$$H^{\frac{1}{2}}_{\mathcal{R}}(\partial\Omega)^n := \{\varphi \in H^{\frac{1}{2}}(\partial\Omega)^n : \int_{\partial\Omega} \varphi \cdot \mathbf{r} d\sigma = 0 \ \forall \mathbf{r} \in \mathcal{R}_{\partial\Omega}\}. \tag{3.77}$$

It is easy to see that

$$\mathbb{E}(\mathbf{r}) = 0, \ \operatorname{div}\mathbf{r} = 0 \ \forall \mathbf{r} \in \mathcal{R}. \tag{3.78}$$

Next, we show the isomorphism property of the operator $\mathbf{D}_{\partial\Omega}$ defined in (3.69) (cf. Lemma 3.17 in Kohr et al.[43] for a different structure of the kernel and range of the similar operator when $\mathbb{A}$ is an L_∞ strongly elliptic viscosity tensor coefficient and Propositions 6.4 and 6.5 in Sayas and Selgas[33] for the Stokes system with constant coefficients).

Lemma 13. *Let conditions (1.2)–(1.4) hold. Then,*

$$\mathrm{Ker}\ \left\{\boldsymbol{D}_{\partial\Omega} : H^{\frac{1}{2}}(\partial\Omega)^n \to H^{-\frac{1}{2}}(\partial\Omega)^n\right\} = \mathcal{R}_{\partial\Omega}, \tag{3.79}$$

$$\boldsymbol{D}_{\partial\Omega}\varphi \in \mathcal{R}_{\partial\Omega}^{\perp}\ \forall\varphi \in H^{\frac{1}{2}}(\partial\Omega)^n, \tag{3.80}$$

and the following operator is an isomorphism,

$$\boldsymbol{D}_{\partial\Omega} : H_{\mathcal{R}}^{\frac{1}{2}}(\partial\Omega)^n \to \mathcal{R}_{\partial\Omega}^{\perp}. \tag{3.81}$$

Proof.

(*i*) First, we show formula (3.79). Let us assume that $\varphi \in H^{\frac{1}{2}}(\partial\Omega)^n$ satisfies the equation $\mathbf{D}_{\partial\Omega}\varphi = \mathbf{0}$ on $\partial\Omega$. Let $\mathbf{u}_\varphi := \mathbf{W}_{\partial\Omega}\varphi$ and $\pi_\varphi := \mathcal{Q}^d_{\partial\Omega}\varphi$. Since $\mathrm{div}\mathbf{u}_\varphi = 0$ in $\Omega_\pm$, we have $E_{ii}(\mathbf{u}_\varphi) = 0$ implying that assumption (1.4) is applicable for $E_{i\alpha}(\mathbf{u}_{\boldsymbol{\varphi}})$. According to Lemma 1, the jump relations (3.70) and (3.69), and assumption (1.4), we obtain that

$$\begin{aligned} 0 = \langle -\mathbf{D}_{\partial\Omega}\varphi, \varphi\rangle_{\partial\Omega} &= \left\langle a_{ij}^{\alpha\beta}E_{j\beta}\left(\mathbf{u}_\varphi\right), E_{i\alpha}\left(\mathbf{u}_\varphi\right)\right\rangle_{\Omega_+} + \left\langle a_{ij}^{\alpha\beta}E_{j\beta}\left(\mathbf{u}_\varphi\right), E_{i\alpha}\left(\mathbf{u}_\varphi\right)\right\rangle_{\Omega_-} \\ &\geq c_{\mathbb{A}}^{-1}\left(\|\mathbb{E}(\mathbf{u}_\varphi)\|^2_{L_2(\Omega_+)^{n\times n}} + \|\mathbb{E}(\mathbf{u}_\varphi)\|^2_{L_2(\Omega_-)^{n\times n}}\right) \end{aligned} \tag{3.82}$$

and accordingly $\mathbb{E}(\mathbf{u}_\varphi) = 0$ in $\Omega_\pm$. Hence, by the statement in Section 2.2.4, there exist a constant $\mathbf{b} \in \mathbb{R}^n$ and an antisymmetric matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ such that $\mathbf{u}_\varphi = \mathbf{b} + \mathbf{Bx}$ in Ω_+, while $\mathbf{u}_\varphi = \mathbf{0}$ in Ω_-. Then, by using again the jump relations (3.70), we obtain that $\varphi = -(\mathbf{b} + \mathbf{Bx})|_{\partial\Omega}$. This relation shows that

$$\mathrm{Ker}\,\mathbf{D}_{\partial\Omega} \subseteq \mathcal{R}_{\partial\Omega}. \tag{3.83}$$

Now, let $\mathbf{r} \in \mathcal{R}$ and let $\mathbf{u_r}$ and $\pi_\mathbf{r}$ be the fields given by

$$\mathbf{u_r} := \begin{cases} -\mathbf{r} & \text{in } \Omega_+ \\ \mathbf{0} & \text{in } \Omega_-, \end{cases} \quad \text{and}\ \pi_\mathbf{r} = 0 \text{ in } \mathbb{R}^n. \tag{3.84}$$

By (3.78), $\mathbb{E}(\mathbf{u_r}) = 0$ and $\mathrm{div}\mathbf{u_r} = 0$ in $\mathbb{R}^n\backslash\partial\Omega$, and hence, in view of Lemma 1,

$$\pm\langle \mathbf{t}^{\pm}(\mathbf{u_r}, \pi_\mathbf{r}), \gamma_{\pm}\mathbf{v}_{\pm}\rangle_{\partial\Omega} = 0\ \forall\mathbf{v}_{\pm} \in \mathcal{H}^1(\Omega_{\pm})^n, \tag{3.85}$$

which show that $\mathbf{t}^{\pm}(\mathbf{u_r}, \pi_\mathbf{r}) = \mathbf{0}$, and accordingly that $[\mathbf{t}(\mathbf{u_r}, \pi_\mathbf{r})] = \mathbf{0}$ on $\partial\Omega$. Moreover, we have that $[\gamma\mathbf{u_r}] = -\mathbf{r}|_{\partial\Omega}$ on $\partial\Omega$. Consequently, the pair $(\mathbf{u_r}, \pi_\mathbf{r})$ belongs to $\mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n\times L_2(\mathbb{R}^n)$ and satisfies the transmission problem (3.56) with given boundary datum $\mathbf{r}|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)^n$. Then, Definition 5 yields that $\mathbf{W}_{\partial\Omega}(\mathbf{r}|_{\partial\Omega}) = \mathbf{u_r}$ and $\mathcal{Q}^d_{\partial\Omega}(\mathbf{r}|_{\partial\Omega}) = 0$ in $\mathbb{R}^n\backslash\partial\Omega$, and by formula (3.69), we obtain that $\mathbf{D}_{\partial\Omega}(\mathbf{r}|_{\partial\Omega}) = \mathbf{0}$ on $\partial\Omega$. Therefore,

$$\mathcal{R}_{\partial\Omega} \subseteq \mathrm{Ker}\,\mathbf{D}_{\mathbb{A};\partial\Omega}. \tag{3.86}$$

Relations (3.83) and (3.86) imply (3.79).

Now, let $\varphi \in H^{\frac{1}{2}}(\partial\Omega)^n$. By applying the Green formula (2.27) to the pair $(\mathbf{W}_{\partial\Omega}\varphi, \mathcal{Q}^d_{\partial\Omega}\varphi)$ and by using relation (3.69) along with (3.78), we obtain the formula

$$\langle \mathbf{D}_{\partial\Omega}\varphi, \gamma_+\mathbf{r}\rangle_{\partial\Omega} = \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{W}_{\partial\Omega}\varphi), E_{i\alpha}(\mathbf{r})\right\rangle_{\Omega_+} - \left\langle \mathcal{Q}^d_{\partial\Omega}\varphi, \mathrm{div}\mathbf{r}\right\rangle_{\Omega_+} = 0\ \forall\mathbf{r} \in \mathcal{R}, \tag{3.87}$$

implying formula (3.80).

(*ii*)

To prove that operator (3.81) is an isomorphism, we show that there exists a constant $C = C(\partial\Omega, c_{\mathbb{A}}, n) > 0$ such that

$$\langle -\mathbf{D}_{\partial\Omega}\varphi, \varphi\rangle_{\partial\Omega} \geq C\|\varphi\|^2_{H^{\frac{1}{2}}(\partial\Omega)^n} \ \forall \varphi \in H^{\frac{1}{2}}_{\mathcal{R}}(\partial\Omega)^n \tag{3.88}$$

(cf. Sayas and Selgas[33] Proposition 6.5 in the constant-coefficient Stokes system). Indeed, by applying Lemma 1 to the pair $(\mathbf{u}_\varphi, \pi_\varphi) := (\mathbf{W}_{\partial\Omega}\varphi, Q^d_{\partial\Omega}\varphi)$ with $\varphi \in H^{\frac{1}{2}}_{\mathcal{R}}(\partial\Omega)^n$, and using the jump relations (3.70) and condition (1.4), we obtain the inequality

$$\langle -\mathbf{D}_{\partial\Omega}\varphi, \varphi\rangle_{\partial\Omega} \geq c_{\mathbb{A}}^{-1}\left(\|\mathbb{E}(\mathbf{u}_\varphi)\|^2_{L_2(\Omega_+)^{n\times n}} + \|\mathbb{E}(\mathbf{u}_\varphi)\|^2_{L_2(\Omega_-)^{n\times n}}\right). \tag{3.89}$$

In addition, the continuity of the trace operators $\gamma_\pm : \mathcal{H}^1(\Omega_\pm)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$ and the jump formulas (3.70) imply that there exists a constant $C_1 = C_1(\partial\Omega, c_{\mathbb{A}}, n) > 0$ such that

$$\|\varphi\|^2_{H^{\frac{1}{2}}(\partial\Omega)^n} = \left\|[\gamma\mathbf{u}_\varphi]\right\|^2_{H^{\frac{1}{2}}(\partial\Omega)^n} \leq C_1\left(\|\mathbf{u}_\varphi\|^2_{\mathcal{H}^1(\Omega_+)^n} + \|\mathbf{u}_\varphi\|^2_{\mathcal{H}^1(\Omega_-)^n}\right) = C_1\|\mathbf{u}_\varphi\|^2_{\mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n}. \tag{3.90}$$

Now, let $\left\{\mathbf{r}_j : j = 1, \dots, n(n+1)/2\right\}$ be a basis of the $n(n+1)/2$-dimensional space $\mathcal{R}$. Then, the formula

$$\|\mathbf{w}\|^2_{1;\rho;\mathbb{R}^n\setminus\partial\Omega} := \|\mathbb{E}(\mathbf{w})\|^2_{L_2(\mathbb{R}^n\setminus\partial\Omega)^{n\times n}} + \sum_{j=1}^{n(n+1)/2} \left|\int_{\partial\Omega}[\gamma\mathbf{w}]\cdot\mathbf{r}_j d\sigma\right|^2 \ \forall \mathbf{w} \in \mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n \tag{3.91}$$

defines a norm on the space $\mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n$, which is equivalent to the norm $\|\cdot\|_{\mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n}$ (see Lemma 18, cf. also Sayas and Selgas[33], p.78 for $n = 3$). Therefore, there exists a constant $C_2 > 0$ such that

$$\|\mathbf{w}\|_{\mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n} \leq C_2\|\mathbf{w}\|_{1;\rho;\mathbb{R}^n\setminus\partial\Omega} \ \forall \mathbf{w} \in \mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n. \tag{3.92}$$

Now, by considering $\mathbf{w} = \mathbf{u}_\varphi$ in (3.91) and by using the jump formulas (3.70), and the assumption that $\varphi \in H^{\frac{1}{2}}_{\mathcal{R}}(\partial\Omega)^n$, as well as inequality (3.92), we obtain that

$$\|\mathbb{E}(\mathbf{u}_\varphi)\|^2_{L_2(\Omega_+)^{n\times n}} + \|\mathbb{E}(\mathbf{u}_\varphi)\|^2_{L_2(\Omega_-)^{n\times n}} = \|\mathbf{u}_\varphi\|^2_{1;\rho;\mathbb{R}^n\setminus\partial\Omega} \geq C_2^{-2}\|\mathbf{u}_\varphi\|^2_{\mathcal{H}^1(\mathbb{R}^n\setminus\partial\Omega)^n}. \tag{3.93}$$

Finally, by exploiting inequalities (3.89), (3.90), and (3.93), we obtain the coercivity inequality (3.88) with the constant $C = c_{\mathbb{A}}^{-1}C_1^{-1}C_2^{-2}$. Then, the Lax–Milgram lemma and the isomorphic identification of the dual of the space $H^{\frac{1}{2}}_{\mathcal{R}}(\partial\Omega)^n$ with $\mathcal{R}^\perp_{\partial\Omega}$, imply that operator (3.81) is an isomorphism, as asserted. □

3.5 | Poisson problems of transmission type for the anisotropic Stokes system in $\mathbb{R}^n$

For given data $\widetilde{\mathbf{f}}_\pm, \widetilde{\mathbf{f}}_-, g_+, g_-, \varphi, \psi$, we consider the following Poisson problem of transmission type:

$$\begin{cases} \mathcal{L}(\mathbf{u}_\pm, \pi_\pm) = \widetilde{\mathbf{f}}_\pm|_{\Omega_\pm}, \ \mathrm{div}\,\mathbf{u}_\pm = g_\pm & \text{in } \Omega_\pm, \\ \gamma_+\mathbf{u}_+ - \gamma_-\mathbf{u}_- = \varphi & \text{on } \partial\Omega, \\ \mathbf{t}^+(\mathbf{u}_+, \pi_+; \widetilde{\mathbf{f}}_+) - \mathbf{t}^-(\mathbf{u}_-, \pi_-; \widetilde{\mathbf{f}}_-) = \psi & \text{on } \partial\Omega, \end{cases} \tag{3.94}$$

where $\mathcal{L}$ denotes the Stokes operator defined in (1.9). The left-hand side in the last transmission condition in (3.94) is understood in the sense of Definition 1.

Theorem 6. *Let conditions (1.2)–(1.4) hold. Then, for all given data $(\widetilde{\boldsymbol{f}}_+, \widetilde{\boldsymbol{f}}_-, g_+, g_-, \varphi, \psi)$ in the space $\widetilde{H}^{-1}(\Omega_+)^n \times \widetilde{\mathcal{H}}^{-1}(\Omega_-)^n \times L_2(\Omega_+) \times L_2(\Omega_-) \times H^{\frac{1}{2}}(\partial\Omega)^n \times H^{-\frac{1}{2}}(\partial\Omega)^n$, the transmission problem (3.94) has a unique solution*

$(\boldsymbol{u}_+, \pi_+, \boldsymbol{u}_-, \pi_-) \in H^1(\Omega_+)^n \times L_2(\Omega_+) \times \mathcal{H}^1(\Omega_-)^n \times L_2(\Omega_-)$. *Moreover, there exists a constant* $C = C(\partial\Omega, c, n) > 0$ *such that*

$$\|\boldsymbol{u}_+\|_{H^1(\Omega_+)^n} + \|\pi_+\|_{L_2(\Omega_+)} + \|\boldsymbol{u}_-\|_{\mathcal{H}^1(\Omega_-)^n} + \|\pi_-\|_{L_2(\Omega_-)} \leq C\Big(\|\widetilde{\boldsymbol{f}}_+\|_{\widetilde{H}^{-1}(\Omega_+)^n} + \|\widetilde{\boldsymbol{f}}_-\|_{\widetilde{\mathcal{H}}^{-1}(\Omega_-)^n} + \|g_+\|_{L_2(\Omega_+)} + \|g_-\|_{L_2(\Omega_-)} + \|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)^n} + \|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)^n}\Big). \tag{3.95}$$

Proof. Theorem 3 yields uniqueness. Now, we show existence, by considering the potentials

$$\mathbf{u}_\pm = \left(\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}}_\pm\right)|_{\Omega_\pm} + \left(\mathcal{G}_{\mathbb{R}^n}\mathring{E}_\pm g_\pm\right)|_{\Omega_\pm} + \mathbf{V}_{\partial\Omega}\psi^0 - \mathbf{W}_{\partial\Omega}\varphi^0 \text{ in } \Omega_\pm, \tag{3.96}$$

$$\pi_\pm = \left(\mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}}_\pm\right)|_{\Omega_\pm} + \left(\mathcal{G}^0_{\mathbb{R}^n}\mathring{E}_\pm g_\pm\right)|_{\Omega_\pm} + \mathcal{Q}^s_{\partial\Omega}\psi^0 - \mathcal{Q}^d_{\partial\Omega}\varphi^0 \text{ in } \Omega_\pm, \tag{3.97}$$

where

$$\begin{aligned}
\varphi^0 &:= \varphi - \gamma_+\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}}_+ + \gamma_-\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}}_- - \gamma_+\mathcal{G}_{\mathbb{R}^n}\mathring{E}_+g_+ + \gamma_-\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g_-,\\
\psi^0 &:= \psi - \mathbf{t}^+\left(\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}}_+, \mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}}_+; \widetilde{\mathbf{f}}_+\right) + \mathbf{t}^-\left(\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}}_-, \mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}}_-; \widetilde{\mathbf{f}}_-\right) - \mathbf{t}^+\left(\mathcal{G}_{\mathbb{R}^n}\mathring{E}_+g_+, \mathcal{G}^0_{\mathbb{R}^n}\mathring{E}_+g_+\right) + \mathbf{t}^-\left(\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g_-, \mathcal{G}^0_{\mathbb{R}^n}\mathring{E}_-g_-\right).
\end{aligned}$$

Note that $\varphi^0 \in H^{\frac{1}{2}}(\partial\Omega)^n$ and $\psi^0 \in H^{-\frac{1}{2}}(\partial\Omega)^n$. From Lemmas 6,7, and 12, we deduce that $(\mathbf{u}_\pm, \pi_\pm)$ given in (3.96) and (3.97) provide a solution of the transmission problem (3.94) in the space $(H^1(\Omega_+)^n \times L_2(\Omega_+)) \times (\mathcal{H}^1(\Omega_-)^n \times L_2(\Omega_-))$ satisfying inequality (3.95). □

3.6 | The third Green identities for the anisotropic Stokes system

Next, we prove the representation formulas (the third Green identities) for solutions of the anisotropic Stokes system with L_∞ tensor coefficient (cf. Proposition 6.8 in Sayas and Selgas[33] for the homogeneous Stokes system in case (1.10) with $\mu = 1$, $\lambda = 0$, and $n = 3$ and Theorem 6.10 in McLean[48] for the strongly elliptic systems with smooth coefficients). They can be employed, for example, for reduction of the boundary and transmission problems to *direct* boundary equations, similar to the classical direct boundary integral equation approach, see, for example, Costabel,[1] McLean,[48] and Hsiao and Wendland.[4]

Theorem 7. *Let conditions (1.2)–(1.4) hold and let* $\mathcal{L}$ *denote the Stokes operator defined in (1.9). Let* $\boldsymbol{u}_+ \in H^1(\Omega_+)^n$, $\boldsymbol{u}_- \in \mathcal{H}^1(\Omega_-)^n$ *and* $\pi_\pm \in L_2(\Omega_\pm)$ *satisfy the Stokes system*

$$\mathcal{L}(\boldsymbol{u}_\pm, \pi_\pm) = \widetilde{\mathbf{f}}_\pm|_{\Omega_\pm}, \ \mathrm{div}\,\boldsymbol{u}_\pm = g_\pm \ \text{ in } \ \Omega_\pm \tag{3.98}$$

for some $\widetilde{\mathbf{f}}_+ \in \widetilde{H}^{-1}(\Omega_+)^n$, $\widetilde{\mathbf{f}}_- \in \widetilde{\mathcal{H}}^{-1}(\Omega_-)^n$, $g_+ \in L_2(\Omega_+)$, $g_- \in L_2(\Omega_-)$. *Let* $\widetilde{\boldsymbol{f}} := \widetilde{\mathbf{f}}_+ + \widetilde{\mathbf{f}}_-$, $g := \mathring{E}_+g_+ + \mathring{E}_-g_-$. *Then, the following representations in terms of jumps hold:*

$$\boldsymbol{u}_\pm = -\boldsymbol{W}_{\partial\Omega}[\gamma\boldsymbol{u}] + \boldsymbol{V}_{\partial\Omega}\left[\boldsymbol{t}(\boldsymbol{u}, \pi; \widetilde{\boldsymbol{f}})\right] + \mathcal{N}_{\mathbb{R}^n}\widetilde{\boldsymbol{f}} + \mathcal{G}_{\mathbb{R}^n}g \text{ in } \Omega_\pm, \tag{3.99}$$

$$\pi_\pm = -\mathcal{Q}^d_{\partial\Omega}[\gamma\boldsymbol{u}] + \mathcal{Q}^s_{\partial\Omega}\left[\boldsymbol{t}(\boldsymbol{u}, \pi; \widetilde{\boldsymbol{f}})\right] + \mathcal{Q}_{\mathbb{R}^n}\widetilde{\boldsymbol{f}} + \mathcal{G}^0_{\mathbb{R}^n}g \text{ in } \Omega_\pm. \tag{3.100}$$

Moreover, the following single-side representations also hold:

$$\boldsymbol{u}_\pm = \mp\boldsymbol{W}_{\partial\Omega}\gamma_\pm\boldsymbol{u}_\pm \pm \boldsymbol{V}_{\partial\Omega}\boldsymbol{t}^\pm(\boldsymbol{u}_\pm, \pi_\pm; \widetilde{\boldsymbol{f}}_\pm) + \mathcal{N}_{\mathbb{R}^n}\widetilde{\boldsymbol{f}}_\pm + \mathcal{G}_{\mathbb{R}^n}\mathring{E}_\pm g_\pm \text{ in } \Omega_\pm, \tag{3.101}$$

$$\pi_\pm = \mp\mathcal{Q}^d_{\partial\Omega}\gamma_\pm\boldsymbol{u}_\pm \pm \mathcal{Q}^s_{\partial\Omega}\boldsymbol{t}^\pm(\boldsymbol{u}_\pm, \pi_\pm; \widetilde{\boldsymbol{f}}_\pm) + \mathcal{Q}_{\mathbb{R}^n}\widetilde{\boldsymbol{f}}_\pm + \mathcal{G}^0_{\mathbb{R}^n}\mathring{E}_\pm g_\pm \text{ in } \Omega_\pm. \tag{3.102}$$

Proof. In view of the assumptions on $\mathbf{u}_{\pm}$, $\pi_{\pm}$, and $\widetilde{\mathbf{f}}_{\pm}$, we have the inclusions $\varphi := [\gamma\mathbf{u}] \in H^{\frac{1}{2}}(\partial\Omega)^n$ and $\psi := \left[\mathbf{t}(\mathbf{u},\pi;\widetilde{\mathbf{f}})\right] \in H^{-\frac{1}{2}}(\partial\Omega)^n$. Let

$$\mathbf{v} := -\mathbf{W}_{\partial\Omega}\varphi + \mathbf{V}_{\partial\Omega}\psi + \mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}_{\mathbb{R}^n}g,\ q := -\mathcal{Q}^d_{\partial\Omega}\varphi + \mathcal{Q}^s_{\partial\Omega}\psi + \mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}^0_{\mathbb{R}^n}g \text{ in } \mathbb{R}^n\backslash\partial\Omega. \tag{3.103}$$

By definitions of the potentials and according to Lemmas 6, 7, and 12(i), the pair $(\mathbf{v}, q)$ belongs to the space $\mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n \times L_2(\mathbb{R}^n)$ and satisfies the Stokes system

$$\mathcal{L}(\mathbf{v}_{\pm}, q_{\pm}) = \widetilde{\mathbf{f}},\ \operatorname{div}\mathbf{v}_{\pm} = g \text{ in } \Omega_{\pm}. \tag{3.104}$$

Due to Lemma 6, $\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}}, \mathcal{G}_{\mathbb{R}^n}g \in \mathcal{H}^1(\mathbb{R}^n)^n$ implying that $[\gamma\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}}] = \mathbf{0}$, $[\gamma\mathcal{G}_{\mathbb{R}^n}g] = \mathbf{0}$. Then, by formulas (3.27) and (3.70),

$$[\gamma\mathbf{v}] = \varphi \text{ on } \partial\Omega. \tag{3.105}$$

Let $r_{\Omega_{\pm}}$ be restriction operators to $\Omega_{\pm}$, that is, $r_{\Omega_{\pm}}g := g|_{\Omega_{\pm}}$. By Definition 1, the generalized conormal derivative is linear with respect to the triple of its arguments, implying that

$$\begin{aligned}\mathbf{t}^{\pm}(\mathbf{v}|_{\Omega_{\pm}}, q|_{\Omega_{\pm}}; \widetilde{\mathbf{f}}_{\pm}) = -\mathbf{t}^{\pm}(r_{\Omega_{\pm}}\mathbf{W}_{\partial\Omega}\varphi, r_{\Omega_{\pm}}\mathcal{Q}^d_{\partial\Omega}\varphi; \mathbf{0}) + \mathbf{t}^{\pm}(r_{\Omega_{\pm}}\mathbf{V}_{\partial\Omega}\psi, r_{\Omega_{\pm}}\mathcal{Q}^s_{\partial\Omega}\psi; \mathbf{0})\\ +\mathbf{t}^{\pm}(r_{\Omega_{\pm}}(\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}_{\mathbb{R}^n}g), r_{\Omega_{\pm}}(\mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}^0_{\mathbb{R}^n}g); \widetilde{\mathbf{f}}_{\pm}).\end{aligned} \tag{3.106}$$

By formulas (3.28) and (3.69), we obtain

$$[\mathbf{t}(\mathbf{V}_{\partial\Omega}\psi, \mathcal{Q}^s_{\partial\Omega}\psi; \mathbf{0})] = \psi,\ [\mathbf{t}(\mathbf{W}_{\partial\Omega}\varphi, \mathcal{Q}^d_{\partial\Omega}\varphi; \mathbf{0})] = \mathbf{0}. \tag{3.107}$$

On the other hand, from (2.25), we have for any $\mathbf{w} \in H^{\frac{1}{2}}(\partial\Omega)^n$ that

$$\begin{aligned}&\left\langle [\mathbf{t}(\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}_{\mathbb{R}^n}g, \mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}^0_{\mathbb{R}^n}g; \widetilde{\mathbf{f}})], \mathbf{w}\right\rangle_{\partial\Omega}\\ &= \left\langle A^{\alpha\beta}\partial_\beta r_{\Omega_+}(\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}_{\mathbb{R}^n}g), \partial_\alpha(\gamma^{-1}\mathbf{w})\right\rangle_{\Omega_+} + \left\langle A^{\alpha\beta}\partial_\beta r_{\Omega_-}(\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}_{\mathbb{R}^n}g), \partial_\alpha(\gamma^{-1}\mathbf{w})\right\rangle_{\Omega_-}\\ &- \left\langle r_{\Omega_+}(\mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}^0_{\mathbb{R}^n}g), \operatorname{div}(\gamma^{-1}\mathbf{w})\right\rangle_{\Omega_+} - \left\langle r_{\Omega_-}(\mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}^0_{\mathbb{R}^n}g), \operatorname{div}(\gamma^{-1}\mathbf{w})\right\rangle_{\Omega_-} + \left\langle \widetilde{\mathbf{f}}_+, \gamma^{-1}\mathbf{w}\right\rangle_{\Omega_+} + \left\langle \widetilde{\mathbf{f}}_-, \gamma^{-1}\mathbf{w}\right\rangle_{\Omega_-}\\ &= \left\langle A^{\alpha\beta}\partial_\beta(\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}_{\mathbb{R}^n}g), \partial_\alpha(\gamma^{-1}\mathbf{w})\right\rangle_{\mathbb{R}^n} - \left\langle \mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}^0_{\mathbb{R}^n}g, \operatorname{div}(\gamma^{-1}\mathbf{w})\right\rangle_{\mathbb{R}^n} + \left\langle \widetilde{\mathbf{f}}, \gamma^{-1}\mathbf{w}\right\rangle_{\mathbb{R}^n}\\ &= \left\langle -\mathcal{L}(\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}_{\mathbb{R}^n}g, \mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}^0_{\mathbb{R}^n}g) + \widetilde{\mathbf{f}}, \gamma^{-1}\mathbf{w}\right\rangle_{\mathbb{R}^n} = \mathbf{0},\end{aligned} \tag{3.108}$$

where $\gamma^{-1} : H^{\frac{1}{2}}(\partial\Omega)^n \to \mathcal{H}^1(\mathbb{R}^n)^n$ is a (nonunique) bounded right inverse of the trace operator $\gamma : \mathcal{H}^1(\mathbb{R}^n)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$. The last equality in (3.108) follows since $\mathcal{L}(\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}}+\mathcal{G}_{\mathbb{R}^n}g, \mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}}+\mathcal{G}^0_{\mathbb{R}^n}g) = \widetilde{\mathbf{f}}$ in $\mathbb{R}^n$. Combining (3.106)–(3.108), we obtain that the couple $(\mathbf{v}, q)$ satisfies the transmission condition

$$\left[\mathbf{t}(\mathbf{v}, q; \widetilde{\mathbf{f}})\right] = \psi \text{ on } \partial\Omega, \tag{3.109}$$

and thus, the transmission problem (3.104), (3.105), and (3.109). The pair $(\mathbf{u}, \pi)$ satisfies the same transmission problem, which, in view of Theorem 6, has at most one solution in $\mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n \times L_2(\mathbb{R}^n)$. Consequently, $\mathbf{u} = \mathbf{v}$ and $\pi = q$, and then, formulas (3.103) yield the representation formulas (3.99)–(3.100).

To obtain formulas (3.101) and (3.102) for $(\mathbf{u}_+, \pi_+)$, we can employ representations (3.99) and (3.100) with $\mathbf{u}_- = \mathbf{0}$, $\pi_- = 0$, $\widetilde{\mathbf{f}}_- = \mathbf{0}$, and $g_- = 0$. Formulas (3.101) and (3.102) for $(\mathbf{u}_-, \pi_-)$ can be obtained in a similar way. □

4 | BOUNDARY VALUE PROBLEMS FOR THE ANISOTROPIC STOKES SYSTEM IN WEIGHTED SOBOLEV SPACES

Girault and Sequeira[31] used in Theorem 3.4 a variational approach to show the well-posedness in $\mathcal{H}^1(\Omega')^n \times L_2(\Omega')$ for the exterior Dirichlet problem for the constant coefficient Stokes system in an exterior Lipschitz domain Ω' of $\mathbb{R}^n$, $n = 2, 3$. Dindos and Mitrea[3] (see Theorems 5.1, 5.6, 7.1, and 7.3) used a boundary integral approach and properties of Calderón–Zygmund-type singular integral operators to show well-posedness results in Sobolev and Besov spaces for Poisson problems of Dirichlet type for the Stokes and Navier–Stokes systems with smooth coefficients in Lipschitz domains on compact Riemannian manifolds (see also Theorem 7.1 in Mitrea and Taylor[15] and Proposition 4.5 in Băcuţă et al.[36] for an evolutionary exterior Stokes problem).

Recall that $\mathcal{L}$ is the Stokes operator defined in (1.9), such that the corresponding viscosity tensor coefficient $\mathbb{A} = \left(A^{\alpha\beta}\right)_{1\le\alpha,\beta\le n}$ satisfies conditions (1.2)–(1.4).

4.1 | Exterior Dirichlet problem for the anisotropic Stokes system in the compressible case

Let us consider the following Dirichlet problem for the anisotropic Stokes system with L_∞ coefficients:

$$\begin{cases} \mathcal{L}(\mathbf{u},\pi) = \mathbf{f},\ \operatorname{div}\mathbf{u} = g & \text{in } \Omega_-, \\ \gamma_-\mathbf{u} = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\mathcal{L}$ is the Stokes operator defined in (1.9) and the given data $(\mathbf{f}, g, \boldsymbol{\varphi})$ belong to $\mathcal{H}^{-1}(\Omega_-)^n \times L_2(\Omega_-) \times H^{\frac{1}{2}}(\partial\Omega)^n$. We show the well-posedness of this problem in the space $\mathcal{H}^1(\Omega_-)^n \times L_2(\Omega_-)$ and express its solution in terms of the Newtonian and single-layer potentials defined in Section 3 (cf. Theorem 5.2 in Kohr et al.[39] in the isotropic case 1.10, Theorem 3.4 in Girault and Sequeira[31] for the constant coefficient Stokes system, Theorem 10.1 in Fabes et al.,[59] and Theorem 5.1 in Lang and Méndez[60] for the Laplace operator).

As in the previous sections, $\Omega_+ \subset \mathbb{R}^n$ $(n \ge 3)$ is a bounded Lipschitz domain with connected boundary $\partial\Omega$, and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega_+}$. Recall that $\mathcal{L}$ is the Stokes operator defined in (1.9), and that $\mathring{E}_-$ is the operator of extension by zero outside Ω_-.

Theorem 8. *Let conditions (1.2)–(1.4) hold in Ω_-. Let $\mathbf{f} \in \mathcal{H}^{-1}(\Omega_-)^n$, $g \in L_2(\Omega_-)$ and $\varphi \in H^{\frac{1}{2}}(\partial\Omega)^n$. If $\varphi \in H_\nu^{\frac{1}{2}}(\partial\Omega)^n$, then the exterior Dirichlet problem (4.1) has a unique solution $(\boldsymbol{u}, \pi) \in \mathcal{H}^1(\Omega_-)^n \times L_2(\Omega_-)$, given by*

$$\mathbf{u} = \mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g + \boldsymbol{V}_{\partial\Omega}\mathcal{V}_{\partial\Omega}^{-1}\left(\varphi - \gamma_-\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} - \gamma_-\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g\right) \text{ in } \Omega_-, \tag{4.2}$$

$$\pi = \mathcal{Q}_{\mathbb{R}^n}\widetilde{\mathbf{f}} + \mathcal{G}^0_{\mathbb{R}^n}\mathring{E}_-g + \mathcal{Q}^s_{\partial\Omega}\mathcal{V}_{\partial\Omega}^{-1}\left(\varphi - \gamma_-\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} - \gamma_-\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g\right) \text{ in } \Omega_-, \tag{4.3}$$

where $\widetilde{\mathbf{f}}$ is an extension of $\mathbf{f}$ to an element of $\widetilde{\mathcal{H}}^{-1}(\Omega^n_-) \subset \mathcal{H}^{-1}(\mathbb{R}^n)^n$. In addition, there exists a constant $C = C(\partial\Omega, c_{\mathbb{A}}, n) > 0$ such that

$$\|\mathbf{u}\|_{\mathcal{H}^1(\Omega_-)^n} + \|\pi\|_{L_2(\Omega_-)} \le C\left(\|\mathbf{f}\|_{\mathcal{H}^{-1}(\Omega_-)^n} + \|g\|_{L_2(\Omega_-)} + \|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega)^n}\right). \tag{4.4}$$

Proof. Let $\mathbf{f} \in \mathcal{H}^{-1}(\Omega_-)^n$ and $g \in L_2(\Omega_-)$. Then, Theorem 3.2 and Definition 3.3 imply that

$$\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}},\ \mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g \in \mathcal{H}^1(\mathbb{R}^n)^n \tag{4.5}$$

and $\operatorname{div}\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} = 0$, $\operatorname{div}\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g = \mathring{E}_-g$ in $\mathbb{R}^n$. Hence, both potentials are divergence free vector fields in Ω_+ and the divergence theorem in Ω_+ implies that

$$\gamma_+\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}},\quad \gamma_+\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g \in H_\nu^{\frac{1}{2}}(\partial\Omega)^n. \tag{4.6}$$

From inclusions (4.5), we have

$$\gamma_-\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} = \gamma_+\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}},\quad \gamma_-\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g = \gamma_+\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g,$$

which, together with (4.6), implies that $\gamma_-\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}}, \gamma_-\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g \in H_\nu^{\frac{1}{2}}(\partial\Omega)^n$. Then, by the assumption $\varphi \in H_\nu^{\frac{1}{2}}(\partial\Omega)^n$, we conclude that $\varphi - \gamma_-\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} - \gamma_-\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g \in H_\nu^{\frac{1}{2}}(\partial\Omega)^n$, and, in view of Lemma 11, $\mathcal{V}_{\partial\Omega}^{-1}\left(\varphi - \gamma_-\mathcal{N}_{\mathbb{R}^n}\widetilde{\mathbf{f}} - \gamma_-\mathcal{G}_{\mathbb{R}^n}\mathring{E}_-g\right)$ is a well-defined element of the space $H^{-\frac{1}{2}}(\partial\Omega)^n/\mathrm{span}\{\nu\}$.

Moreover, Lemmas 6, 7, 8, and 11 imply that $(\mathbf{u}, \pi)$ represented by formulas (4.2) and (4.3) solve the exterior Dirichlet problem (4.1) in $\mathcal{H}^1(\Omega_-)^n \times L_2(\Omega_-)$, and the continuity of the operators involved in these formulas yields inequality (4.4).

Let us now show uniqueness. Assume that problem (4.1) has two weak solutions in $\mathcal{H}^1(\Omega_-)^n \times L_2(\Omega_-)$ and let $(\mathbf{u}^0, \pi^0)$ be their difference. Therefore, $(\mathbf{u}^0, \pi^0) \in \mathring{\mathcal{H}}^1(\Omega_-)^n \times L_2(\Omega_-)$ and by the Green formula (2.27), we obtain that

$$\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}^0), E_{i\alpha}(\mathbf{u}^0)\right\rangle_{\Omega_-} = 0. \tag{4.7}$$

Moreover, the ellipticity condition (1.4) implies that

$$\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}^0), E_{i\alpha}(\mathbf{u}^0)\right\rangle_{\Omega_-} \geq c_{\mathbb{A}}^{-1} \|\mathbb{E}(\mathbf{u}^0)\|^2_{L_2(\Omega_-)^{n\times n}}. \tag{4.8}$$

Therefore, $\mathbb{E}(\mathbf{u}^0) = 0$ in Ω_- and hence $\mathbf{u}^0 = \mathbf{0}$ in Ω_-; compare Section 2.2.4. In addition, since $\mathbf{u}^0$ and π^0 satisfy the Stokes equation in Ω_- and π^0 belongs to $L_2(\Omega_-)$, we conclude that $\pi^0 = 0$ in Ω_-, as asserted. □

4.2 | Exterior Neumann problem for the anisotropic Stokes system

The Neumann problem for the constant coefficient Stokes system in an exterior Lipschitz domain in $\mathbb{R}^n$, with boundary datum in L_p spaces, has been studied in Theorem 9.2.6 of Mitrea and Wright[5] by a potential approach (see also Theorem 10.6.4 in Mitrea and Wright[5] for the Neumann problem for the same system in a bounded Lipschitz domain). Next, we consider the following exterior Neumann problem for the L_∞ coefficient Stokes system:

$$\begin{cases} \mathcal{L}(\mathbf{u}, \pi) = \mathbf{0},\ \mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega_-, \\ \mathbf{t}^-(\mathbf{u}, \pi) = \psi \in \mathcal{R}_{\partial\Omega}^{\perp} \quad \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

Recall that $\mathcal{R}_{\partial\Omega}^{\perp}$ is defined in (3.76), $\mathbf{D}_{\partial\Omega} : H_{\mathcal{R}}^{\frac{1}{2}}(\partial\Omega)^n \to \mathcal{R}_{\partial\Omega}^{\perp}$ is given by (3.70) and $\mathbf{D}_{\partial\Omega}^{-1} : \mathcal{R}_{\partial\Omega}^{\perp} \to H_{\mathcal{R}}^{\frac{1}{2}}(\partial\Omega)^n$ is a continuous operator due to Lemma 13.

Theorem 9. *Let conditions (1.2)–(1.4) hold in Ω_-. If $\psi \in \mathcal{R}_{\partial\Omega}^{\perp}$, then problem (4.9) has a unique solution $(\boldsymbol{u}, \pi) \in \mathcal{H}^1(\Omega_-)^n \times L_2(\Omega_-)$, given by*

$$\boldsymbol{u} = \boldsymbol{W}_{\partial\Omega}\left(\boldsymbol{D}_{\partial\Omega}^{-1}\psi\right),\ \pi = \mathcal{Q}_{\partial\Omega}^{d}\left(\boldsymbol{D}_{\partial\Omega}^{-1}\psi\right) \text{ in } \Omega_-. \tag{4.10}$$

Moreover, there exists a constant $C = C(\Omega_-, c_{\mathbb{A}}, n) > 0$ such that

$$\|\boldsymbol{u}\|_{\mathcal{H}^1(\Omega_-)^n} + \|\pi\|_{L_2(\Omega_-)} \leq C\|\psi\|_{\mathcal{R}_{\partial\Omega}^{\perp}}. \tag{4.11}$$

Proof. Lemmas 12 and 13 imply that $(\mathbf{u}, \pi)$ represented by (4.10) solve problem (4.9) and the operators involved in (4.10) are continuous, which implies inequality (4.11).

To show uniqueness, let us assume that a pair $(\mathbf{u}_0, \pi_0) \in \mathcal{H}_{\mathrm{div}}^1(\Omega_-)^n \times L_2(\Omega_-)$ satisfies the homogeneous version of the exterior Neumann problem (4.9). Then, Lemma 1 and assumption (1.4) imply that

$$0 = -\langle \mathbf{t}^-(\mathbf{u}_0, \pi_0), \gamma_-(\mathbf{u}_0)\rangle_{\partial\Omega} = \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}_0), E_{i\alpha}(\mathbf{u}_0)\right\rangle_{\Omega_-} \geq c_{\mathbb{A}}^{-1}\|\mathbb{E}(\mathbf{u}_0)\|^2_{L_2(\Omega_-)^{n\times n}},$$

and hence, $\mathbb{E}(\mathbf{u}_0) = 0$ in Ω_-. Then, compare Section 2.2.4, $\mathbf{u}_0 = \mathbf{0}$ in Ω_-. Moreover, the Stokes equation in (4.9) shows that π_0 reduces to a constant $c_0 \in \mathbb{R}$, but the membership of π_0 in $L_2(\Omega_-)$ yields that $c_0 = 0$, and accordingly that $\pi_0 = 0$ in Ω_-. □

5 | AUXILIARY RESULTS

5.1 | Abstract mixed variational formulations

A major role in our analysis of mixed variational formulations is played by the following well-posedness result by Babuska[61] and Theorem 1.1 in Brezzi[62] (see also Theorem 2.34 and Remark 2.35(i) in Ern and Guermond[63] and Brezzi and Fortin[64]).

Theorem 10. *Let X and $\mathcal{M}$ be two real Hilbert spaces. Let $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ and $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be bounded bilinear forms. Let $f \in X'$ and $g \in \mathcal{M}'$. Let V be the subspace of X defined by*

$$V := \{v \in X : b(v,q) = 0 \ \forall q \in \mathcal{M}\}. \tag{5.1}$$

Assume that $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is coercive, which means that there exists a constant $C_a > 0$ such that

$$a(w,w) \geq C_a^{-1}\|w\|_X^2 \ \forall w \in V, \tag{5.2}$$

and that $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ satisfies the Babuska–Brezzi condition

$$\inf_{q\in\mathcal{M}\setminus\{0\}} \sup_{v\in X\setminus\{0\}} \frac{b(v,q)}{\|v\|_X\|q\|_{\mathcal{M}}} \geq C_b^{-1}, \tag{5.3}$$

with some constant $C_b > 0$. Then, the mixed variational formulation,

$$\begin{cases} a(u,v) + b(v,p) = f(v) \ \forall v \in X, \\ b(u,q) \qquad\qquad\ \ = g(q) \ \forall q \in \mathcal{M}, \end{cases} \tag{5.4}$$

has a unique solution $(u,p) \in X \times \mathcal{M}$ and

$$\|u\|_X \leq C_a\|f\|_{X'} + C_b(1 + \|a\|C_a)\|g\|_{\mathcal{M}'}, \tag{5.5}$$

$$\|p\|_{\mathcal{M}} \leq C_b(1 + \|a\|C_a)\|f\|_{X'} + \|a\|C_b^2(1 + \|a\|C_a)\|g\|_{\mathcal{M}'}, \tag{5.6}$$

where $\|a\|$ is the norm of the bilinear form $a(\cdot,\cdot)$.

We need also the following extension of the Babuška–Brezzi result (see Theorem 4.2 in Amrouche and Seloula;[65] see also Lemma A.40 in Ern and Guermond[63]).

Lemma 14. *Let X and $\mathcal{M}$ be reflexive Banach spaces. Let $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be a bounded bilinear form. Let $B : X \to \mathcal{M}'$ and $B^* : \mathcal{M} \to X'$ be the linear bounded operator and its transpose operator defined by*

$$\langle Bv, q\rangle = b(v,q), \ \langle v, B^*q\rangle = \langle Bv, q\rangle \ \forall v \in X, \forall q \in \mathcal{M}, \tag{5.7}$$

where $\langle\cdot,\cdot\rangle := {}_{X'}\langle\cdot,\cdot\rangle_X$ denotes the duality pairing between the dual spaces X' and X. The duality pairing between the spaces $\mathcal{M}'$ and $\mathcal{M}$ is also denoted by $\langle\cdot,\cdot\rangle$. Let $V := \operatorname{Ker} B$ and $V^\perp = X' \perp V := \{g \in X' : \langle g, v\rangle = 0 \forall v \in V\}$. Then, the following assertions are equivalent:

(i) There exists a constant $C_b > 0$ such that $b(\cdot,\cdot)$ satisfies the inf-sup condition (5.3).

(ii) The operator $B : X/V \to \mathcal{M}'$ is an isomorphism and

$$\|Bw\|_{\mathcal{M}'} \geq C_b^{-1}\|w\|_{X/V} \ \forall w \in X/V. \tag{5.8}$$

(iii) The operator $B^* : \mathcal{M} \to V^\perp$ is an isomorphism and

$$\|B^*q\|_{X'} \geq C_b^{-1}\|q\|_{\mathcal{M}} \ \forall q \in \mathcal{M}. \tag{5.9}$$

5.2 | The Agmon–Douglis–Nirenberg ellipticity of the anisotropic Stokes system

The principal symbol of the anisotropic Stokes system (1.1) and (1.8) is the $(n+1)\times(n+1)$ matrix

$$\sigma_{\ell j}(\mathbf{x},\xi)=\begin{cases}\xi_\alpha a_{\ell j}^{\alpha\beta}(\mathbf{x})\xi_\beta, & \ell,j=1,\ \dots\ ,n;\\ -i\xi_\ell, & \ell=1,\ \dots\ ,n,\ j=n+1;\\ -i\xi_j, & \ell=n+1,\ j=1,\ \dots\ ,n;\\ 0, & \ell=j=n+1.\end{cases} \tag{5.10}$$

Here, $i^2=-1$, and $\xi=(\xi_1,\ \dots\ ,\xi_n)$.

The Stokes system is *elliptic in the sense of Agmon–Douglis–Nirenberg* at $\mathbf{x}\in\mathbb{R}^n$ if $\sigma(\mathbf{x},\xi)$ is defined and nonsingular for any $\xi\in\mathbb{R}^n\setminus\{\mathbf{0}\}$ (see, e.g., Definition 6.2.3 in Hsiao and Wendland[4]). This property is well known for the Stokes system in the isotropic case (1.10) with $\mu=1$ and $\lambda=0$ (cf., e.g., Hsiao and Wendland[4], p.329). Next, we show that this ellipticity property remains valid also in the more general anisotropic case.

Lemma 15. *Let conditions (1.2)–(1.4) hold on $\mathbb{R}^n$. Then, the anisotropic Stokes system defined by (1.1) and (1.8) is elliptic in the sense of Agmon–Douglis–Nirenberg at almost any $\mathbf{x}\in\mathbb{R}^n$.*

Proof. First, we observe that the symbol matrix given by (5.10) is nonsingular if and only if the modified symbol matrix

$$\widetilde{\sigma}_{\ell j}(\mathbf{x},\xi)=\begin{cases}\xi_\alpha a_{\ell j}^{\alpha\beta}(\mathbf{x})\xi_\beta, & \ell,j=1,\ \dots\ ,n;\\ \xi_\ell, & \ell=1,\ \dots\ ,n,\ j=n+1;\\ \xi_j, & \ell=n+1,\ j=1,\ \dots\ ,n;\\ 0, & \ell=j=n+1\end{cases} \tag{5.11}$$

is nonsingular as well. Let $\mathbf{x}\in\mathbb{R}^n$ be such that the coefficients $a_{\ell j}^{\alpha\beta}(\mathbf{x})$ are well defined and finite and the ellipticity condition (1.4) holds. In order to show that $\widetilde{\sigma}_{\ell j}(\mathbf{x},\xi)$ is nonsingular for any $\xi\in\mathbb{R}^n\setminus\{0\}$, we use Theorem 10. To this end, for a fixed $\xi\in\mathbb{R}^n\setminus\{0\}$, we consider the bilinear forms $a_0:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and $b_0:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$,

$$a_0(\hat{\mathbf{u}},\hat{\mathbf{v}}):=\hat{u}_\ell\xi_\alpha a_{\ell j}^{\alpha\beta}(\mathbf{x})\xi_\beta\hat{v}_j\ \ \forall\hat{\mathbf{u}},\hat{\mathbf{v}}\in\mathbb{R}^n, \tag{5.12}$$

$$b_0(\hat{\mathbf{v}},\hat{q}):=-\xi_j\hat{v}_j\hat{q}\ \forall\hat{\mathbf{v}}\in\mathbb{R}^n,\ \hat{q}\in\mathbb{R}, \tag{5.13}$$

as well as the closed subspace V_ξ of $\mathbb{R}^n$ given by

$$V_\xi:=\left\{\hat{\mathbf{v}}\in\mathbb{R}^n:\ b_0(\hat{\mathbf{v}},\hat{q})=0,\ \forall\hat{q}\in\mathbb{R}\right\}=\left\{\hat{\mathbf{v}}\in\mathbb{R}^n:\ \xi_j\hat{v}_j=0\right\}. \tag{5.14}$$

It is immediate that these bilinear forms are bounded, as they satisfy the estimates:

$$|a_0(\hat{\mathbf{u}},\hat{\mathbf{v}})|\le\|\mathbb{A}\|_{L_\infty(\mathbb{R}^n)}|\xi|^2|\hat{\mathbf{u}}|\,|\hat{\mathbf{v}}|,\ |b_0(\hat{\mathbf{v}},\hat{q})|\le|\xi|\,|\hat{\mathbf{v}}|\,|\hat{q}|\ \forall\hat{\mathbf{u}},\hat{\mathbf{v}}\in\mathbb{R}^n,\ \forall\hat{q}\in\mathbb{R}.$$

The symmetry conditions (1.3) allow us to write the bilinear form a_0 as

$$a_0(\hat{\mathbf{u}},\hat{\mathbf{v}})=a_{\ell j}^{\alpha\beta}(\mathbf{x})(\hat{\mathbf{u}}\otimes\xi)^s_{\ell\alpha}(\hat{\mathbf{v}}\otimes\xi)^s_{\beta j}, \tag{5.15}$$

where $(\hat{\mathbf{u}}\otimes\xi)^s$ is the symmetric part of the matrix $\hat{\mathbf{u}}\otimes\xi$, that is,

$$(\hat{\mathbf{u}}\otimes\xi)^s_{\ell\alpha}:=\frac{1}{2}\left(\hat{u}_\ell\xi_\alpha+\hat{u}_\alpha\xi_\ell\right),\ \ell,\alpha=1,\ \dots\ ,n. \tag{5.16}$$

According to (5.15) and the ellipticity condition (1.4), we obtain that a_0 satisfies the estimate

$$a_0(\hat{\mathbf{v}},\hat{\mathbf{v}})\ge c_{\mathbb{A}}^{-1}|(\hat{\mathbf{v}}\otimes\xi)^s|^2=\frac{1}{2}c_{\mathbb{A}}^{-1}|\hat{\mathbf{v}}|^2|\xi|^2\ \forall\hat{\mathbf{v}}\in\mathbb{R}^n\text{such that }\ \hat{\mathbf{v}}\cdot\xi=0, \tag{5.17}$$

where $\hat{\mathbf{v}} \cdot \xi = \sum_{\ell=1}^{n} (\hat{\mathbf{v}} \otimes \xi)^s_{\ell\ell}$ is the trace of the symmetric matrix $(\hat{\mathbf{v}} \otimes \xi)^s$. Therefore, the bounded bilinear form $a_0 : V_\xi \times V_\xi \to \mathbb{R}$ is coercive when $\xi \neq \mathbf{0}$.

In addition, an elementary computation shows that

$$\inf_{\hat{q}\in\mathbb{R}\setminus\{0\}} \sup_{\hat{\mathbf{v}}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{b_0(\hat{\mathbf{v}}, \hat{q})}{|\hat{\mathbf{v}}|\,|\hat{q}|} = |\xi|, \tag{5.18}$$

and accordingly that the bilinear form b_0 satisfies the inf-sup condition with the inf-sup constant $|\xi|$.

By applying Theorem 10, we conclude that the modified symbol matrix $\widetilde{\sigma}(\mathbf{x}, \xi)$ given by (5.11) is invertible for any $\xi \neq \mathbf{0}$ and hence that the symbol matrix $\sigma(\mathbf{x},\xi)$ given by (5.10) has the same property. Thus, the anisotropic Stokes system is elliptic in the sense of Agmon–Douglis–Nirenberg, as asserted. □

5.3 | Extension result in the weighted Sobolev space $\mathcal{H}^1(\mathbb{R}^n)$

Lemma 16. *Let $\Omega_+ \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega_+}$.*

(i) Let $q_+ \in L_2(\Omega_+)$ and $q_- \in L_2(\Omega_-)$. Then, there exists a unique function $q \in L_2(\mathbb{R}^n)$ such that $q|_{\Omega_\pm} = q_\pm$. Moreover, $\|q\|^2_{L_2(\mathbb{R}^n)} = \|q_+\|^2_{L_2(\Omega_+)} + \|q_-\|^2_{L_2(\Omega_-)}$.

(ii) Let $u_+ \in H^1(\Omega_+)$ and $u_- \in \mathcal{H}^1(\Omega_-)$ be such that $\gamma_+ u_+ = \gamma_- u_-$ on $\partial\Omega$. Then, there exists a unique function $u \in \mathcal{H}^1(\mathbb{R}^n)$ such that $u|_{\Omega_\pm} = u_\pm$. Moreover, there exists a constant $C > 0$ depending on n and $\Omega_\pm$, such that

$$\|u\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \left(\|u_+\|_{H^1(\Omega_+)} + \|u_-\|_{\mathcal{H}^1(\Omega_-)} \right). \tag{5.19}$$

(iii) If $u \in \mathcal{H}^1(\mathbb{R}^n)$, then $[\gamma u] = 0$, where $[\gamma u] = \gamma_+(u|_{\Omega_+}) - \gamma_-(u|_{\Omega_-})$.

Proof.

(i) We can take $q = \mathring{E}_{\Omega_+} q_+ + \mathring{E}_{\Omega_-} q_- \in L_2(\mathbb{R})$, where $\mathring{E}_{\Omega\pm}$ are the operators of extension by zero defined in (2.32). Then, evidently $q|_{\Omega_\pm} = q_\pm$. To prove the uniqueness, let us assume that there are two such functions, q_1 and q_2. Then, $q_0 := q_1 - q_2$ belongs to $L_2(\mathbb{R}^n)$ and $q_0|_{\Omega_\pm} = 0$. Hence, $q_0 = 0$ in $\mathbb{R}^n$ in the sense of Lebesgue classes.

(ii) We follow similar arguments to those for Theorem 5.13 in Brewster et al.[27] Let $\mathcal{E}_{\Omega_+}$ be a bounded linear extension operator from $H^1(\Omega_+)$ to $H^1(\mathbb{R}^n)$ (see, e.g., Theorem 2.4.1 in Mitrea and Wright[5]). Let us take

$$u_-^* := (\mathcal{E}_{\Omega_+} u_+)|_{\Omega_-} \text{ in } \Omega_-. \tag{5.20}$$

Then, $u_-^* \in H^1(\Omega_-) \subset \mathcal{H}^1(\Omega_-)$. Moreover, there exists a constant $c > 0$ depending on n and $\Omega_\pm$, such that

$$\|u_-^*\|_{\mathcal{H}^1(\Omega_-)} \leq c\|u_+\|_{H^1(\Omega_+)}.$$

In addition, in view of (5.20), we have $\gamma_- u_-^* = \gamma_-(\mathcal{E}_{\Omega_+} u_+) = \gamma_+ u_+ = \gamma_- u_-$, and hence, $u_- - u_-^*$ belongs to $\mathring{\mathcal{H}}^1(\Omega_-)$. Thus, $\mathring{E}_{\Omega_-}(u_- - u_-^*)$ belongs to $\mathcal{H}^1(\mathbb{R}^n)$, and there exists a constant $c_1 = c_1(n, \Omega_\pm)$, such that

$$\|\mathring{E}_{\Omega_-}(u_- - u_-^*)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq c_1 \left(\|u_+\|_{H^1(\Omega_+)} + \|u_-\|_{\mathcal{H}^1(\Omega_-)} \right). \tag{5.21}$$

Let us now define the function

$$u := \mathring{E}_{\Omega_-}(u_- - u_-^*) + \mathcal{E}_{\Omega_+} u_+. \tag{5.22}$$

It belongs to $\mathcal{H}^1(\mathbb{R}^n)$, and there exists a constant $C_1 > 0$ depending on n and $\Omega_\pm$, such that the inequality (5.19) holds. According to (5.20) and (5.22), we have also the following relations:

$$\begin{aligned} u|_{\Omega_+} &= 0 + (\mathcal{E}_{\Omega_+} u_+)|_{\Omega_+} = u_+ \text{ a.e. in } \Omega_+, \\ u|_{\Omega_-} &= u_- - u_-^* + (\mathcal{E}_{\Omega_+} u_+)|_{\Omega_-} = u_- - u_-^* + u_-^* = u_- \text{ a.e. in } \Omega_-, \end{aligned}$$

and thus, the existence of a function u is proved.

To prove that the function u is unique, let us assume that there are two such functions, u_1 and u_2. Then, $u_0 := u_1 - u_2$ belongs to $\mathcal{H}^1(\mathbb{R}^n)$ and $u_0|_{\Omega_\pm} = 0$. Thus, $u_0 \in H^1(\mathbb{R}^n) \subset L_2(\mathbb{R}^n)$ and its support is a subset of $\partial\Omega$. Hence, $u_0 = 0$ in $\mathbb{R}^n$ in the sense of Lebesgue classes (cf. also Theorem 2.10(i) in Mikhailov[54]).

(iii) Let $u \in \mathcal{H}^1(\mathbb{R}^n)$. Consequently, $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$, and then, $\gamma_+ u = \gamma_- u$, that is, $[\gamma u] = 0$. □

5.4 | Equivalent norms in the weighted Sobolev space $\mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)^n$

We will further employ the following assertion concerning the equivalence of norms in Banach spaces (cf. Lemma 11.1 in Tartar[66]).

Lemma 17. *Let $(X, \|\cdot\|_X)$ be a Banach space, and let $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed spaces. Let $\mathcal{P} : X \to Y$, $\mathfrak{C} : X \to Z$, and $\mathcal{T} : X \to \mathcal{Y}$ be linear and continuous operators, such that*

(i) The operator $\mathfrak{C} : X \to Z$ is compact.
(ii) $\|P(\cdot)\|_Y + \|\mathfrak{C}(\cdot)\|_Z$ is a norm on X equivalent to the norm $\|\cdot\|_X$.
(iii) The operator $\mathcal{T} : X \to \mathcal{Y}$ satisfies the condition $\mathcal{T}(u) \neq 0$ whenever $P(u) = 0$ and $u \neq 0$.

Then, $\|u\| := \|P(u)\|_Y + \|\mathcal{T}(u)\|_{\mathcal{Y}}$, $u \in X$, is a norm on X equivalent to the given norm $\|\cdot\|_X$.

The following result for $n = 3$ is implied by Proposition 2.7(a) in Sayas and Selgas,[33] and its proof is based on the Korn inequalities (see, e.g., Theorems 10.1 and 10.2 in McLean[48]) and Lemma 17. The result for $n > 3$ follows with the same arguments.

Theorem 11. *Let $n \geq 3$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary $\partial\Omega$, and $\Omega_- := \mathbb{R}^n \backslash \overline{\Omega}$. Then, $\|\mathbb{E}(\cdot)\|_{L_2(\Omega_-)^{n\times n}}$ is a norm in the weighted Sobolev space $\mathcal{H}^1(\Omega_-)^n$, which is equivalent to the norm $\|\cdot\|_{\mathcal{H}^1(\Omega_-)^n}$ given by (2.8) with Ω_- in place of $\mathbb{R}^n$. Therefore, there exists a constant $C = C(\Omega_-, n) > 0$ such that*

$$C\|\boldsymbol{u}\|_{\mathcal{H}^1(\Omega_-)^n} \leq \|\mathbb{E}(\boldsymbol{u})\|_{L_2(\Omega_-)^{n\times n}} \leq \|\boldsymbol{u}\|_{\mathcal{H}^1(\Omega_-)^n}\ \forall \boldsymbol{u} \in \mathcal{H}^1(\Omega_-)^n. \tag{5.23}$$

Recall that ρ is the weight function given by (2.4), $\mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)$ is the space defined in (2.16)–(2.17), $\mathcal{R}$ is the space of rigid body motion fields in $\mathbb{R}^n$ defined in (2.18), and $\mathcal{R}_{\partial\Omega}$ is its trace. Note that $\dim \mathcal{R} = n(n+1)/2$, compare Section 2.2.4, and let $\left\{\mathbf{r}_j : j = 1, \dots, n(n+1)/2\right\}$ be a basis of $\mathcal{R}$.

Lemma 18. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded Lipschitz domain with connected boundary $\partial\Omega$. Then, the formula*

$$\|\boldsymbol{w}\|^2_{1;\rho;\mathbb{R}^n\backslash\partial\Omega} := \|\mathbb{E}(\boldsymbol{w})\|^2_{L_2(\mathbb{R}^n\backslash\partial\Omega)^{n\times n}} + \sum_{j=1}^{n(n+1)/2} \Big| \int_{\partial\Omega} [\gamma \boldsymbol{w}] \cdot \gamma \boldsymbol{r}_j d\sigma \Big|^2\ \forall \boldsymbol{w} \in \mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)^n \tag{5.24}$$

defines a norm in the weighted Sobolev space $\mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)^n$, which is equivalent to the norm

$$\|\boldsymbol{w}\|^2_{\mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n} = \|\rho^{-1}\boldsymbol{w}\|^2_{L_2(\mathbb{R}^n\backslash\partial\Omega)^n} + \|\nabla \boldsymbol{w}\|^2_{L_2(\mathbb{R}^n\backslash\partial\Omega)^n}. \tag{5.25}$$

Proof. First, we note that by Theorem 11, $\|\mathbb{E}(\cdot)\|_{L_2(\Omega_-)^{n\times n}}$ is a norm in $\mathcal{H}^1(\Omega_-)^n$, which is equivalent to the norm $\|\cdot\|_{\mathcal{H}^1(\Omega_-)^n}$, defined as in (5.25) with Ω_- in place of $\mathbb{R}^n \backslash \partial\Omega$. Moreover, in view of the second Korn inequality (see, e.g., Theorem 10.2 in McLean[48] and Proposition 11.4.2 in Mitrea and Wright[5]), $\|\mathbb{E}(\cdot)\|_{L_2(\Omega_+)^{n\times n}} + \|\cdot\|_{L_2(\Omega_+)^n}$ is an equivalent norm in the space $H^1(\Omega_+)^n$. Therefore,

$$\|\mathbb{E}(\mathbf{w})\|_{L_2(\Omega_-)^{n\times n}} + \|\mathbb{E}(\mathbf{w})\|_{L_2(\Omega_+)^{n\times n}} + \|\mathbf{w}\|_{L_2(\Omega_+)^n} = \|\mathbb{E}(\mathbf{w})\|_{L_2(\mathbb{R}^n\backslash\partial\Omega)^{n\times n}} + \|\mathbf{w}\|_{L_2(\Omega_+)^n} \tag{5.26}$$

is a norm in the space $\mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)^n$, equivalent to the norm (5.25) of this space.

Now, we consider the operators

$$P : \mathcal{H}^1(\mathbb{R}^n \backslash \partial\Omega)^n \to L_2(\mathbb{R}^n \backslash \partial\Omega)^{n\times n},\ P(\mathbf{w}) := \mathbb{E}(\mathbf{w}), \tag{5.27}$$

$$\mathfrak{C} : \mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n \to L_2(\Omega_+)^n,\ \mathfrak{C}(\mathbf{w}) = \mathbf{w}|_{\Omega_+}, \tag{5.28}$$

$$\mathcal{T} : \mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n \to \mathbb{R}^{n(n+1)/2},\ \mathcal{T}(\mathbf{w}) := \left(\int_{\partial\Omega}[\gamma\mathbf{w}]\cdot\gamma\mathbf{r}_1 d\sigma, \dots, \int_{\partial\Omega}[\gamma\mathbf{w}]\cdot\gamma\mathbf{r}_{n(n+1)/2} d\sigma\right), \tag{5.29}$$

which are linear and continuous. Moreover, the operator $\mathfrak{C}$ is compact due to the compact embedding of the space $H^1(\Omega_+)^n$ in $L_2(\Omega_+)^n$. In terms of these operators, the norm in (5.26) becomes

$$\|\mathbb{E}(\mathbf{w})\|_{L_2(\mathbb{R}^n\backslash\partial\Omega)^{n\times n}} + \|\mathbf{w}\|_{L_2(\Omega_+)^n} = \|P(\mathbf{w})\|_{L_2(\mathbb{R}^n\backslash\partial\Omega)^{n\times n}} + \|\mathfrak{C}(\mathbf{w})\|_{L_2(\Omega_+)^n}. \tag{5.30}$$

In addition, the operator $\mathcal{T}$ satisfies the condition $\mathcal{T}(\mathbf{w}) \neq 0$ whenever $P(\mathbf{w}) = 0$ and $\mathbf{w} \neq \mathbf{0}$. Indeed, the condition $P(\mathbf{w}) = 0$ is equivalent to $\mathbf{w}|_{\Omega_+} \in \mathcal{R}|_{\Omega_+}$ and $\mathbf{w}|_{\Omega_-} = \mathbf{0}$; compare Section 2.2.4. Assume that $\mathcal{T}(\mathbf{w}) = 0$ and $P(\mathbf{w}) = 0$. Then, $\gamma_\pm\mathbf{w} \in \mathcal{R}_{\partial\Omega}$ and

$$\int_{\partial\Omega}[\gamma\mathbf{w}]\cdot\gamma\mathbf{r}_j d\sigma = 0,\ j = 1, \dots, n(n+1)/2. \tag{5.31}$$

Since $\mathcal{R}_{\partial\Omega} = \operatorname{span}\{\gamma\mathbf{r}_j : j = 1, \dots, n(n+1)/2\}$, (5.31) yields that $[\gamma\mathbf{w}] = 0$ on $\partial\Omega$, and accordingly that $\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^n)^n$ (cf. Lemma 16) implying that $\mathbf{w} \in \widetilde{\mathcal{H}}^1(\Omega_+)^n$. Then, by the first Korn inequality (see, e.g., Theorem 10.1 in McLean[48]),

$$2\|\operatorname{grad}(\mathbf{w})\|_{L_2(\Omega_+)^{n\times n}} \le \|\mathbb{E}(\mathbf{w})\|_{L_2(\Omega_+)^{n\times n}} = \|P(\mathbf{w})\|_{L_2(\Omega_+)^{n\times n}} = 0,$$

and thus, $\mathbf{w}|_{\Omega_+} = \mathbf{a}_+$, which, together with the condition $\gamma_+\mathbf{w} = \gamma_-\mathbf{w} = \mathbf{0}$, implies that $\mathbf{w}|_{\Omega_+} = \mathbf{0}$. Hence, $\mathbf{w} = \mathbf{0}$ in $\mathbb{R}^n$, which contradicts the assumption $\mathbf{w} \neq \mathbf{0}$. Thus, $\mathcal{T}(\mathbf{w}) \neq 0$ whenever $P(\mathbf{w}) = 0$ and $\mathbf{w} \neq \mathbf{0}$, as asserted.

Consequently, the conditions of Lemma 17 with $X := \mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n$, $Y = L_2(\mathbb{R}^n\backslash\partial\Omega)^{n\times n}$, $Z = L_2(\Omega_+)^n$, and $\mathcal{Y} := \mathbb{R}^{n(n+1)/2}$ are satisfied, and hence,

$$\|P(\mathbf{w})\|_Y + \|\mathcal{T}(\mathbf{w})\|_{\mathcal{Y}} = \|\mathbb{E}(\mathbf{w})\|_{L_2(\mathbb{R}^n\backslash\partial\Omega)^{n\times n}} + \sum_{j=1}^{n(n+1)/2}\left|\int_{\partial\Omega}[\gamma\mathbf{w}]\cdot\gamma\mathbf{r}_j d\sigma\right| \tag{5.32}$$

is a norm on $\mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n$ equivalent to norm (5.25). This result and the equivalence of the norms (5.24) and (5.32) show that (5.24) is also a norm in $\mathcal{H}^1(\mathbb{R}^n\backslash\partial\Omega)^n$ equivalent to norm (5.25). □

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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