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DIRICHLET AND TRANSMISSION PROBLEMS FOR ANISOTROPIC STOKES AND NAVIER-STOKES SYSTEMS WITH L_∞ TENSOR COEFFICIENT UNDER RELAXED ELLIPTICITY CONDITION

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Abstract. The first aim of this paper is to show well-posedness of Dirichlet and transmission problems in bounded and exterior Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, for the anisotropic *Stokes* system with L_∞ viscosity tensor coefficient satisfying an ellipticity condition in terms of symmetric matrices with zero matrix trace, with data from standard and weighted Sobolev spaces. To this end we reduce the linear problems to equivalent mixed variational formulations and show that the variational problems are well-posed. Then we use the Leray-Schauder fixed point theorem and establish the existence of a weak solution for nonlinear Dirichlet and transmission problems for the anisotropic *Navier-Stokes* system in bounded Lipschitz domains in $\mathbb{R}^3$, with general (including *large*) data in Sobolev spaces. For exterior domains in $\mathbb{R}^3$, the analysis of the nonlinear Dirichlet and transmission problems in weighted Sobolev spaces relies on the existence result for the Dirichlet problem for the anisotropic Navier-Stokes system in a family of bounded Lipschitz domains. The obtained estimates for pressure in $\mathbb{R}^3$ look new also for the classical isotropic case.

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1. **Introduction.** Variational methods play a main role in the analysis of boundary value problems for the Stokes and Navier-Stokes systems. Girault and Sequeira [25] have used a mixed variational formulation for the exterior Dirichlet problem for the constant coefficient Stokes system in weighted Sobolev spaces on exterior Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$, and obtained well-posedness and regularity results for such a problem. Alliot and Amrouche [4] have used a variational approach to obtain weak solutions for the exterior Stokes problem in weighted Sobolev spaces. Existence and pointwise bounds of the fundamental solution for the Stokes system with measurable coefficients in $\mathbb{R}^n$ $(n \geq 3)$ have been obtained by Choi and Yang in [17] under the assumption of local Hölder continuity of weak solutions of the Stokes system. They also discussed the existence and pointwise bounds of the Green function for the Stokes system with measurable coefficients on unbounded domains where the divergence equation is solvable, particularly on the half-space (see also [15, 16]).

Mitrea and Wright [41] have used the layer potential theory and obtained well-posedness results for boundary problems for the constant coefficient Stokes system in arbitrary Lipschitz domains in $\mathbb{R}^n$ and in L_p, Sobolev, and Besov spaces. The authors in [29] used a layer potential method and a fixed point technique to show an existence result for a nonlinear Neumann-transmission problem for the constant-coefficient Stokes and Brinkman systems in L_p, Sobolev, and Besov spaces (see also [30]).

Dindoš and Mitrea [19] developed layer potential methods for the smooth coefficient Stokes system on compact Riemannian manifolds, and used them to show well-posedness of the Poisson problems for the Navier-Stokes system with Dirichlet condition and large data in Sobolev spaces on Lipschitz domains in compact Riemannian manifolds with dimension at most 4.

An alternative approach, which reduces various boundary problems for variable coefficient elliptic partial differential equations to *boundary-domain integral equations* (BDIEs), by means of explicit parametrix-based integral potentials, was explored, e.g., in [13, 14, 40] and the references therein.

In [31] we have used a variational approach in the analysis of transmission problems in weighted Sobolev spaces and in the pseudostress setting for *anisotropic* Stokes and Navier-Stokes systems with an L_∞ strongly elliptic tensor coefficient. For the nonlinear problems we have considered *small data* (see also the well-posedness results in [33, 34] for the Stokes and Navier-Stokes systems with non-smooth coefficients in compact Riemannian manifolds).

In this paper we continue the analysis of transmission problems for anisotropic Stokes and Navier-Stokes systems, by imposing a less restrictive ellipticity condition than that in [31]. Indeed, we consider the L_∞ viscosity tensor coefficient satisfying a strong ellipticity condition only with respect to all *symmetric* matrices in $\mathbb{R}^{n\times n}$ with *zero matrix trace* (see (1.4)). First, we explore equivalent mixed variational formulations and prove the well-posedness of some Dirichlet and transmission problems for the anisotropic Stokes system in bounded Lipschitz domains of $\mathbb{R}^n$ with given data in standard Sobolev spaces. Then we analyze the well-posedness of the exterior Dirichlet and transmission problems for the Stokes system in weighted Sobolev spaces. Next, we use the Leray-Schauder fixed point theorem and prove the existence of weak solutions of the Dirichlet and transmission problems in bounded Lipschitz domains in $\mathbb{R}^3$ for the anisotropic Navier-Stokes system with general (including *large*) data in weighted Sobolev spaces. Finally, we prove the existence of weak solutions $\mathbf{u}$ of the exterior Dirichlet and transmission problems in $\mathbb{R}^3$ and in exterior Lipschitz domains in $\mathbb{R}^3$ for the anisotropic Navier-Stokes system with general data in L_2-based weighted Sobolev spaces. The analysis relies on the existence result for a Dirichlet problem for the anisotropic Navier-Stokes system in a family of bounded Lipschitz domains (or, equivalently, for a Dirichlet-transmission problem for the Navier-Stokes system in a family of bounded composite Lipschitz domains) in $\mathbb{R}^3$. For $\mathbb{R}^3$ and exterior domains in $\mathbb{R}^3$, we particularly show the existence of a pressure field π, which belongs locally to L_2, such that $(\mathbf{u}, \pi)$ solves the transmission problem, or the exterior Dirichlet problem, for the Navier-Stokes system.

The main results presented in this paper include:

- Analysis of the Stokes and Navier-Stokes equations for anisotropic fluids with a *new non-standard relaxed ellipticity condition* for the viscosity tensor coefficient.
- Analysis of rather general anisotropic transmission problems with *jumps of generalized conormal derivatives.*
- *Explicit estimates* for solutions of the linear problems for the anisotropic Stokes system that are used to obtain *new estimates for the pressure in the nonlinear problems for the anisotropic Navier-Stokes system* in exterior Lipschitz domains. These estimates look new also for the classical isotropic case.

The well-posedness results in the Hilbert L_2-based Sobolev spaces developed in this paper for linear boundary value problems can be extended to a more general setting of L_p-based Sobolev or Besov spaces with p in some open interval containing 2, by exploiting the continuity in $H_p^1 \times L_p$-Sobolev spaces, $p > 1$, of an operator related to such a boundary value problem, its invertibility for $p = 2$, and the property that such spaces and their duals determine complex interpolation scales (see [41], [31] and [34] for further details). One can also extend the variational approach to mixed conditions on polyhedral domains and domains with cuts following [36].

The analysis of boundary problems for the anisotropic Stokes and Navier-Stokes systems may be employed in modelling physical, engineering, or industrial phenomena related to immiscible fluid flows, as well as inhomogeneous fluid flows with density dependent viscosity (see, e.g., [15]).

1.1. **The anisotropic Stokes system with L_∞ symmetrically elliptic tensor coefficient.** All along the paper we use the Einstein summation convention for repeated indices from 1 to n, and the standard notation ∂_α for the first order partial derivative $\frac{\partial}{\partial x_\alpha}$, $\alpha = 1, \ldots, n$. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an open set and let $\mathbb{L}$ be a second order differential operator in the divergence form

$$\mathbb{L}\mathbf{u} = \operatorname{div}(\mathbb{A}\mathbb{E}(\mathbf{u})) \quad \Longleftrightarrow \quad (\mathbb{L}\mathbf{u})_i := \partial_\alpha \left(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}) \right), \quad i = 1, \ldots, n, \tag{1.1}$$

where $\mathbf{u} = (u_1, \ldots, u_n)^\top$, and $\mathbb{E}(\mathbf{u}) = (E_{j\beta}(\mathbf{u}))_{1 \leq j, \beta \leq n}$ is the symmetric part of the gradient $\nabla\mathbf{u}$. Therefore, the components of the tensor field $\mathbb{E}(\mathbf{u})$ are defined by $E_{j\beta}(\mathbf{u}) := \frac{1}{2}(\partial_j u_\beta + \partial_\beta u_j)$.

The viscosity tensor coefficient $\mathbb{A}$ in the operator $\mathbb{L}$ consists of $n \times n$ matrix-valued functions $A^{\alpha\beta} = A^{\alpha\beta}(x)$ with essentially bounded, real-valued entries, i.e.,

$$\mathbb{A} = \left(A^{\alpha\beta}\right)_{1 \leq \alpha, \beta \leq n} = \left(a_{ij}^{\alpha\beta}\right)_{1 \leq \alpha, \beta, i, j \leq n}, \ a_{ij}^{\alpha\beta} \in L_\infty(\Omega), \ 1 \leq \alpha, \beta, i, j \leq n, \tag{1.2}$$

satisfying the symmetry conditions

$$a_{ij}^{\alpha\beta}(x) = a_{\alpha j}^{i\beta}(x) = a_{i\beta}^{\alpha j}(x), \quad x \in \Omega \tag{1.3}$$

(cf. [42, Eq. (3.2)], [20, Eqs. (6), (7)]). Note that the symmetry conditions (1.3) *do not imply* the symmetry $a_{ij}^{\alpha\beta}(x)=a_{ji}^{\beta\alpha}(x)$, which will be generally not assumed in the paper. In addition, we assume that the coefficients satisfy the following *ellipticity* condition, which asserts that there exists a constant $c_{\mathbb{A}}>0$ such that for almost all $x\in\Omega$,

$$a_{ij}^{\alpha\beta}(x)\xi_{i\alpha}\xi_{j\beta}\geq c_{\mathbb{A}}^{-1}|\boldsymbol{\xi}|^2,\quad \forall\, \boldsymbol{\xi}=(\xi_{i\alpha})_{i,\alpha=1,\dots,n}\in\mathbb{R}^{n\times n} \text{ with } \boldsymbol{\xi}=\boldsymbol{\xi}^\top \text{ and } \sum_{i=1}^n\xi_{ii}=0, \tag{1.4}$$

where $|\boldsymbol{\xi}|^2=\xi_{i\alpha}\xi_{i\alpha}$. Note that the ellipticity condition (1.4) is assumed only for all *symmetric* matrices $\boldsymbol{\xi}=(\xi_{i\alpha})_{i,\alpha=1,\dots,n}\in\mathbb{R}^{n\times n}$, cf. [42, Eqs. (3.1), (3.3)], having zero matrix trace, $\sum_{i=1}^n\xi_{ii}=0$.

In view of (1.2), $\mathbb{A}$ is endowed with the norm

$$\|\mathbb{A}\|_{L_\infty(\Omega)}:=\max_{i,j,\alpha,\beta\in\{1,\dots,n\}}\left\{\|a_{ij}^{\alpha\beta}\|_{L_\infty(\Omega)}\right\}. \tag{1.5}$$

The conditions (1.3) allow us to express the operator $\mathbb{L}$ in the equivalent forms

$$(\mathbb{L}\mathbf{u})_i=\partial_\alpha\left(a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u})\right)=\partial_\alpha\left(a_{ij}^{\alpha\beta}\partial_\beta u_j\right),\quad i=1,\dots,n, \tag{1.6}$$

$$\mathbb{L}\mathbf{u}=\partial_\alpha\left(A^{\alpha\beta}\partial_\beta\mathbf{u}\right). \tag{1.7}$$

Note that the first equality in (1.6) has not been encountered in [31], where the coefficients of the forth order tensor $\mathbb{A}$ have been assumed to satisfy the strong ellipticity condition similar to the second condition in (1.4) but for all (not only symmetric and zero-trace) matrices $\boldsymbol{\xi}$ (see [31, Eqs. (2)-(3)]). The more restrictive ellipticity condition in [31] allowed to explore there the associated non-symmetric pseudostress setting. In this paper we require the symmetry conditions (1.3) and the ellipticity condition (1.4) only for symmetric zero-trace matrices $\boldsymbol{\xi}$, and develop our results in the symmetric stress setting. This approach allows us to obtain properties of layer potentials for the Stokes system with L_∞ variable coefficients generalizing well known results for constant coefficients.

Let $\mathbf{u}$ be an unknown vector field, π be an unknown scalar field, and $\mathbf{f}$ be a given vector field defined in $\Omega\subseteq\mathbb{R}^n$. Then the equations

$$\boldsymbol{\mathcal{L}}(\mathbf{u},\pi):=\mathbb{L}\mathbf{u}-\nabla\pi=\mathbf{f},\ \operatorname{div}\,\mathbf{u}=g \text{ in } \Omega \tag{1.8}$$

determine the Stokes system, which, under conditions (1.4), is elliptic in the sense of Agmon-Douglis-Nirenberg, see [32, Theorem 5.3]. In the case $n=3$, this system describes viscous compressible fluid flows with variable anisotropic viscosity tensor coefficient $\mathbb{A}$ depending on the physical properties of the fluid, such as, e.g., the given fluid temperature (cf. [20]). If $g=0$ then the fluid is incompressible.

In view of (1.6) and (1.7), the Stokes operator $\boldsymbol{\mathcal{L}}$ can be written in any of forms

$$\begin{aligned}&\boldsymbol{\mathcal{L}}(\mathbf{u},\pi)=\partial_\alpha\left(A^{\alpha\beta}\partial_\beta\mathbf{u}\right)-\nabla\pi,\\ &(\boldsymbol{\mathcal{L}}(\mathbf{u},\pi))_i=\partial_\alpha\left(a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u})\right)-\partial_i\pi,\ i=1,\dots,n.\end{aligned} \tag{1.9}$$

1.2. **Isotropic case.** For the *isotropic case*, the viscosity tensor $\mathbb{A}$ in (1.2) has the form (cf., e.g., Appendix III, Part I, Section 1 in [48]),

$$a_{ij}^{\alpha\beta}(x)=\lambda(x)\delta_{i\alpha}\delta_{j\beta}+\mu(x)\left(\delta_{\alpha j}\delta_{\beta i}+\delta_{\alpha\beta}\delta_{ij}\right),\ 1\leq i,j,\alpha,\beta\leq n \tag{1.10}$$

where $\lambda, \mu \in L_\infty(\Omega)$ and $c_\mu^{-1} \le \mu(x) \le c_\mu$ for a.e. $x \in \Omega$ with some constant $c_\mu > 0$. Then

$$a_{ij}^{\alpha\beta}(x)\xi_{i\alpha}\xi_{j\beta} = \lambda(x)(\xi_{ii})^2 + 2\mu(x)\xi_{i\alpha}\xi_{i\alpha} = 2\mu(x)\xi_{i\alpha}\xi_{i\alpha} = 2\mu(x)|\boldsymbol{\xi}|^2 \ge c_\mu^{-1}|\boldsymbol{\xi}|^2$$

for a.e. $x \in \Omega$ and for any symmetric matrix $\boldsymbol{\xi} = (\xi_{i\alpha})_{1\le i,\alpha\le n} \in \mathbb{R}^{n\times n}$ such that $\sum_{i=1}^n \xi_{ii} = 0$. Therefore, the symmetric ellipticity condition (1.4) is satisfied as well, and hence our results are also applicable to the *Stokes system in the isotropic case*. If $\mu > 0$ is a constant and $g = 0$, then (1.8) reduces to the well known isotropic incompressible Stokes system with constant viscosity μ.

2. **Functional setting and preliminary results.** Given a Banach space $\mathcal{X}$, its topological dual is denoted by $\mathcal{X}'$, and the notation $\langle\cdot,\cdot\rangle_X$ means the duality pairing of two dual spaces defined on a set $X \subseteq \mathbb{R}^n$. We further assume that $n \ge 2$ unless explicitly stated otherwise.

In this paper we are concerned with Lipschitz domains in $\mathbb{R}^n$ as defined in, e.g., [22, Definition II.1.1]. Therefore, a *Lipschitz domain* in $\mathbb{R}^n$ is a non-empty open connected set, either bounded or unbounded, with compact (not necessarily connected) boundary, which can be locally represented as the graph of a Lipschitz function.

We say that Ω is a *Lipschitz set in* $\mathbb{R}^n$ if it is a finite union of Lipschitz domains in $\mathbb{R}^n$ with disjoint closures. Particularly, any Lipschitz domain can be considered as a Lipschitz set. The notion of Lipschitz set will be useful in the description of transmission problems.

2.1. L_2**-based Sobolev spaces.** Let $\mathcal{D}(\mathbb{R}^n) := C_0^\infty(\mathbb{R}^n)$ denote the space of infinitely differentiable functions with compact support in $\mathbb{R}^n$, equipped with the inductive limit topology. Let $\mathcal{D}'(\mathbb{R}^n)$ denote the space of distributions, i.e., the dual of the space $\mathcal{D}(\mathbb{R}^n)$. Let $L_2(\mathbb{R}^n)$ denote the Lebesgue space of equivalence classes of measurable, square-integrable functions in $\mathbb{R}^n$, and $L_\infty(\mathbb{R}^n)$ is the space of equivalence classes of essentially bounded measurable functions in $\mathbb{R}^n$.

Let $H^1(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)^n$ denote the L_2-based Sobolev spaces

$$\begin{aligned} H^1(\mathbb{R}^n) &:= \{f \in L_2(\mathbb{R}^n) : \nabla f \in L_2(\mathbb{R}^n)^n\} \\ &= \Big\{f \in L_2(\mathbb{R}^n) : \|f\|^2_{H^1(\mathbb{R}^n)} = \|f\|^2_{L_2(\mathbb{R}^n)} + \|\nabla f\|^2_{L_2(\mathbb{R}^n)^n} < \infty\Big\}, \\ H^1(\mathbb{R}^n)^n &:= \{\boldsymbol{f} = (f_1, \dots, f_n) : f_j \in H^1(\mathbb{R}^n),\ j = 1, \dots, n\}. \end{aligned}$$

The space $H^1(\mathbb{R}^n)$ can be equivalently described as

$$H^1(\mathbb{R}^n) = \Big\{f \in \mathcal{S}'(\mathbb{R}^n)\ :\ \big\|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{1}{2}}\mathcal{F}f]\big\|_{L^2(\mathbb{R}^n)} < \infty\Big\}\,,$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions and $\mathcal{F}$ is the Fourier transform (cf., e.g., [37, Theorem 3.18]). The dual of $H^1(\mathbb{R}^n)$ is the space $H^{-1}(\mathbb{R}^n)$.

Let Ω be a non-empty open subset of $\mathbb{R}^n$. Then similar to the definition of the space $\mathcal{D}(\mathbb{R}^n)$, let $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω, equipped with the inductive limit topology, and let $\mathcal{D}'(\Omega)$ be its topological dual. Let $\mathcal{D}(\overline{\Omega})$ denote the space of restrictions of functions from $\mathcal{D}(\mathbb{R}^n)$ onto $\overline{\Omega}$. Also, $L_2(\Omega)$ is the Lebesgue space of equivalence classes of measurable, square-integrable functions on Ω, and $L_\infty(\Omega)$ is the space of equivalence classes of essentially bounded measurable functions on Ω. Let also

$$H^1(\Omega) := \{f \in L_2(\Omega) : \nabla f \in L_2(\Omega)^n\}\,, \tag{2.1}$$

endowed with the norm

$$\|f\|_{H^1(\Omega)} = \sqrt{\|f\|^2_{L_2(\Omega)} + \|\nabla f\|^2_{L_2(\Omega)^n}}\,. \tag{2.2}$$

The space $\widetilde{H}^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\mathbb{R}^n)$. The dual of $\widetilde{H}^1(\Omega)$ is the space $H^{-1}(\Omega)$. On the other hand, since $\mathcal{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$ (see, e.g., [37, p. 77]), the dual of $H^1(\Omega)$, denoted by $\widetilde{H}^{-1}(\Omega)$, is a space of distributions.

Let $H^1(\Omega)^n$ and $\widetilde{H}^1(\Omega)^n$ the spaces of vector-valued functions whose components belong to the spaces $H^1(\Omega)$ and $\widetilde{H}^1(\Omega)$, respectively. Similar notations will be employed also for all other Sobolev spaces of vector-valued functions, as well as for the Sobolev spaces of matrix-valued functions, whose components belong to the above mentioned Sobolev spaces of scalar functions.

Let now Ω be a Lipschitz set with boundary $\partial\Omega$. Then the boundary Sobolev space, $H^s(\partial\Omega)$, $0 < s < 1$, can be defined by

$$H^s(\partial\Omega') = \left\{ f \in L_2(\partial\Omega') : \int_{\partial\Omega'}\int_{\partial\Omega} \frac{|f(\mathbf{x}) - f(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n-1+2s}} d\sigma_{\mathbf{x}} d\sigma_{\mathbf{y}} < \infty \right\},$$

where $\sigma_{\mathbf{y}}$ is the surface measure on $\partial\Omega$ (see, e.g., [41, Proposition 2.5.1]). The dual of $H^s(\partial\Omega)$ is the space $H^{-s}(\partial\Omega)$, and we set $H^0(\partial\Omega') = L_2(\partial\Omega)$.

All L_2-based Sobolev spaces mentioned above are Hilbert spaces.

We will further consider transmission problems and will often use the indices $+$ and $-$ to denote *adjacent Lipschitz sets*, such as e.g. for Ω_+ and Ω_-. Sometimes we will also use the notation Ω_+ for bounded Lipschitz sets, and Ω_- for unbounded sets (even if they are not necessarily adjacent sets). The well-known trace theorem for Lipschitz domains (see [18], [38, Lemma 2.6], [41, Theorem 2.5.2]) can be written for Lipschitz sets as follows.

Theorem 2.1. *Let Ω be a Lipschitz set in $\mathbb{R}^n$, $n \geq 2$, with the boundary $\partial\Omega$. Then there exists a linear bounded trace operator $\gamma_{\Omega} : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ such that $\gamma_{\Omega}\phi = \phi|_{\partial\Omega}$ for any $\phi \in C^\infty(\overline{\Omega})$. The operator γ_{Ω} is surjective and has a linear and bounded right inverse operator $\gamma_{\Omega}^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$. The trace operator $\gamma_{\mathbb{R}^n} : H^1(\mathbb{R}^n) \to H^{\frac{1}{2}}(\partial\Omega)$ is also bounded linear and surjective and there exists a linear and bounded right inverse operator $\gamma_{\mathbb{R}^n}^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\mathbb{R}^n)$[1]. In addition, any function $u \in H^1_{\rm loc}(\mathbb{R}^n)$ satisfies the condition $\gamma_{\mathbb{R}^n} u \in H^{\frac{1}{2}}(\partial\Omega)$.*

We will use the simplified notation γ instead of γ_{Ω} or $\gamma_{\mathbb{R}^n}$ whenever the set Ω or the space $\mathbb{R}^n$ is clear from the context and this simplification does not produce any confusion. In addition, if the set, over which the trace operator is considered, is labeled with the sign $\pm$, then this operator will be denoted by $\gamma_{\pm}$. For example, we adopt the notation $\gamma_{\pm}$ instead of $\gamma_{\Omega_{\pm}}$.

Note that the assumption that the set Ω has a compact Lipschitz boundary shows that $\widetilde{H}^1(\Omega)$ can be identified isomorphically with the space $\mathring{H}^1(\Omega)$ of all functions in $H^1(\Omega)$ with null traces on the boundary of Ω (cf., e.g., [37, Theorem 3.33]).

Further properties of Sobolev spaces can be found in [1, 37, 41].

2.2. **Weighted Sobolev spaces.** Let $|\mathbf{x}| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ denote the Euclidean distance of a point $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ to the origin of $\mathbb{R}^n$, $n \geq 3$. Let ρ be the

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[1]The trace operators defined on Sobolev spaces of *vector* fields on Ω or $\mathbb{R}^n$ are also denoted by γ_{Ω} and $\gamma_{\mathbb{R}^n}$, respectively.

weight function

$$\rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{\frac{1}{2}} . \tag{2.3}$$

2.2.1. *Weighted Sobolev spaces on* $\mathbb{R}^n$. The weighted Lebesgue space $L_2(\rho^{-1};\mathbb{R}^n)$ defined by

$$L_2(\rho^{-1};\mathbb{R}^n) := \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : \rho^{-1} f \in L_2(\mathbb{R}^n) \right\}, \tag{2.4}$$

is a Hilbert space with respect to the inner product and the associated norm

$$(f,g)_{L_2(\rho^{-1};\mathbb{R}^n)} := \int_{\mathbb{R}^n} fg\rho^{-2}dx,\ \|f\|^2_{L_2(\rho^{-1};\mathbb{R}^n)} := (f,f)_{L_2(\rho^{-1};\mathbb{R}^n)}. \tag{2.5}$$

We also consider the weighted Sobolev space

$$\mathcal{H}^1(\mathbb{R}^n) := \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : \rho^{-1} f \in L_2(\mathbb{R}^n),\ \nabla f \in L_2(\mathbb{R}^n)^n \right\} \tag{2.6}$$

(cf. [4, Definition 1.1], [26, Theorem I.1]), which is a Hilbert space with the norm

$$\|f\|^2_{\mathcal{H}^1(\mathbb{R}^n)} := \left\|\rho^{-1} f\right\|^2_{L_2(\mathbb{R}^n)} + \|\nabla f\|^2_{L_2(\mathbb{R}^n)^n} . \tag{2.7}$$

The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{H}^1(\mathbb{R}^n)$ (cf., e.g., [2, p. 727], and [26, Théorème I.1] and [45, Proposition 2.1] in the case $n = 3$), and, thus, the dual $\mathcal{H}^{-1}(\mathbb{R}^n)$ of $\mathcal{H}^1(\mathbb{R}^n)$ is a space of distributions. Let us consider the semi-norm

$$|f|_{\mathcal{H}^1(\mathbb{R}^n)} := \|\nabla f\|_{L_2(\mathbb{R}^n)^n}. \tag{2.8}$$

This semi-norm is a norm on the space $\mathcal{H}^1(\mathbb{R}^n)$ and is equivalent to the norm $\|\cdot\|_{\mathcal{H}^1(\mathbb{R}^n)}$, given by (2.7) (cf., e.g., [2, Theorem 1.1]).

2.2.2. *Weighted Sobolev spaces on exterior Lipschitz domains and sets.* Let $\Omega\subset\mathbb{R}^n$ be a non-empty open set, $n \geq 3$. The weighted Sobolev space $\mathcal{H}^1(\Omega)$ can be defined as in (2.6) with Ω in place of $\mathbb{R}^n$ (see [26, Definition I.1, p. 229-230]),

$$\mathcal{H}^1(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) : \rho^{-1} v \in L_2(\Omega),\ \nabla v \in L_2(\Omega)^n \right\} , \tag{2.9}$$

and is a Hilbert space with a norm given by

$$\|f\|^2_{\mathcal{H}^1(\Omega)} := \left\|\rho^{-1} f\right\|^2_{L_2(\Omega)} + \|\nabla f\|^2_{L_2(\Omega)^n} . \tag{2.10}$$

The space $\widetilde{\mathcal{H}}^{-1}(\Omega)$ denotes the dual of the space $\mathcal{H}^1(\Omega)$.

If Ω is a bounded open set, then $\mathcal{H}^1(\Omega)$ coincides with $H^1(\Omega)$ with equivalent norms. Moreover, the restriction of any function from $\mathcal{H}^1(\Omega)$ to any bounded open set Ω_0 contained in Ω belongs to $H^1(\Omega_0)$. The space $\mathring{\mathcal{H}}^1(\Omega)$ is defined as the closure of the space $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{\mathcal{H}^1(\Omega)}$ defined in (2.10) (cf., e.g., [6], [4, Definition 1.1]), and is a Hilbert space.

The space $\mathcal{D}(\Omega)$ is dense in $\mathring{\mathcal{H}}^1(\Omega)$. Hence, the dual of $\mathring{\mathcal{H}}^1(\Omega)$ denoted by $\mathcal{H}^{-1}(\Omega)$ is a subspace of $\mathcal{D}'(\Omega)$. We need also the space $\widetilde{\mathcal{H}}^1(\Omega) \subset \mathcal{H}^1(\mathbb{R}^n)$, defined as the closure of $\mathcal{D}(\Omega)$ in $\mathcal{H}^1(\mathbb{R}^n)$.

Let $\Omega_0 \subset \Omega$ be a bounded Lipschitz set containing the boundary of Ω. Since the restriction of any function from $\mathcal{H}^1(\Omega)$ to Ω_0 belongs to $H^1(\Omega_0)$, the statement of Theorem 2.1 extends also to the weighted Sobolev space $\mathcal{H}^1(\Omega)$. Particularly, for any unbounded Lipschitz set Ω with boundary $\partial\Omega$ there exists a bounded linear and surjective trace operator

$$\gamma_{_\Omega} : \mathcal{H}^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)\,, \tag{2.11}$$

which has a bounded linear right inverse $\gamma_{\Omega}^{-1}: H^{\frac{1}{2}}(\partial\Omega)\to\mathcal{H}^1(\Omega)$ (see [28, Lemma 2.2], [38, Theorem 2.3, Lemma 2.6], [45, p. 69]).

The trace operator $\gamma_{\mathbb{R}^n}: \mathcal{H}^1(\mathbb{R}^n)\to H^{\frac{1}{2}}(\partial\Omega)$ is bounded linear and surjective as well (cf., e.g., [38, Theorem 2.3, Lemma 2.6]). The space $\mathring{\mathcal{H}}^1(\Omega)$ can be also characterized as

$$\begin{aligned}\mathring{\mathcal{H}}^1(\Omega)&=\left\{v\in\mathcal{H}^1(\Omega):\gamma_{\Omega}v=0 \text{ on } \partial\Omega\right\}\\&=\{v|_{\Omega}:v\in\mathcal{H}^1(\mathbb{R}^n)\text{ and }\gamma v=0\text{ on }\partial\Omega\}\end{aligned}\tag{2.12}$$

(see, e.g., [6, (1.2)]), while the space $\widetilde{\mathcal{H}}^1(\Omega)$ can be characterized as

$$\widetilde{\mathcal{H}}^1(\Omega)=\left\{u\in\mathcal{H}^1(\mathbb{R}^n):\operatorname{supp}u\subseteq\overline{\Omega}\right\},\tag{2.13}$$

and can be identified isomorphically with $\mathring{\mathcal{H}}^1(\Omega)$ via the operator $\mathring{E}_{\Omega}$ of extension by zero outside Ω (cf., e.g., [37, Theorem 3.33]).

If Ω is an unbounded (exterior) Lipschitz domain, then the semi-norm

$$|f|_{\mathcal{H}^1(\Omega)}:=\|\nabla f\|_{L_2(\Omega)^n}\tag{2.14}$$

is a norm in $\mathcal{H}^1(\Omega)$ that is equivalent to the full norm $\|\cdot\|_{\mathcal{H}^1(\Omega)}$ given by (2.10) (cf., e.g., [6, Theorem 1.2], [4, Theorem 1.2 (ii)]).

2.3. **The conormal derivative for the anisotropic Stokes system.** Let Ω_+ and Ω_- be two adjacent Lipschitz sets in R^n, $n\geq 2$, and let $\partial\Omega_+$ and $\partial\Omega_-$ be the corresponding boundaries of them. Then $\partial\Omega_+\cap\partial\Omega_-$ is their interface. Let $\boldsymbol{\nu}^+=(\nu_1^+,\dots,\nu_n^+)^\top$ denote the *outward* unit normal to Ω_+, which is defined a.e. on $\partial\Omega_+$. To facilitate the analysis of transmission problems in further sections, we assume that the unit normal $\boldsymbol{\nu}^-$ to Ω_- that exists a.e. on $\partial\Omega_-$ is oriented *inward* to this set, which implies that $\boldsymbol{\nu}^-=\boldsymbol{\nu}^+=:\boldsymbol{\nu}$ on the interface $\partial\Omega_+\cap\partial\Omega_-$.

As before, $\mathbb{L}$ is the divergence form second-order elliptic differential operator given by (1.7), and the coefficients $A^{\alpha\beta}$ of the anisotropic tensor $\mathbb{A}=\left(A^{\alpha\beta}\right)_{1\leq\alpha,\beta\leq n}$ are $n\times n$ matrix-valued functions in $L_\infty(\mathbb{R}^n)^{n\times n}$, with bounded measurable, real-valued entries $a_{ij}^{\alpha\beta}$, satisfying the symmetry and ellipticity conditions (1.3) and (1.4). Moreover, $\boldsymbol{\mathcal{L}}$ is the Stokes operator given by (1.9).

In the special case when $a_{ij}^{\alpha\beta}\in C^0(\overline{\Omega}_\pm)$ and $(\mathbf{u},\pi)\in C^1(\overline{\Omega}_\pm)^n\times C^0(\overline{\Omega}_\pm)$, the *classical* conormal derivatives (i.e., the *boundary traction fields*) of the Stokes system

$$\boldsymbol{\mathcal{L}}(\mathbf{u},\pi)=\mathbb{L}\mathbf{u}-\nabla\pi=\mathbf{f},\quad\operatorname{div}\mathbf{u}=g\quad\text{in }\Omega_\pm,\tag{2.15}$$

where $\mathbf{f}\in L_2(\Omega_\pm)^n$, $g\in L_2(\Omega_\pm)$, are defined on $\partial\Omega_\pm$ by the formula

$$\mathbf{t}^{c\pm}(\mathbf{u},\pi):=-(\gamma_\pm\pi)\boldsymbol{\nu}^\pm+\mathrm{T}^{c\pm}\mathbf{u},\tag{2.16}$$

where $\mathrm{T}^{c\pm}\mathbf{u}$ are the conormal derivatives of $\mathbf{u}$ on $\partial\Omega_\pm$ associated with the operator $\mathbb{L}$ and defined by

$$\mathrm{T}^{c\pm}\mathbf{u}:=\gamma_\pm\big(A^{\alpha\beta}\partial_\beta\mathbf{u}\big)\nu_\alpha^\pm\tag{2.17}$$

(cf., e.g., [16]). In view of (1.3), we obtain (cf. [20]) that[2]

$$\big(\mathrm{T}^{c\pm}\mathbf{u}\big)_i=\gamma_\pm\big(a_{ij}^{\alpha\beta}\partial_\beta u_j\big)\nu_\alpha^\pm=\gamma_\pm\big(a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u})\big)\nu_\alpha^\pm\,,\tag{2.18}$$

where $E_{j\beta}(\mathbf{u}):=\frac{1}{2}\left(\partial_j u_\beta+\partial_\beta u_j\right)$.

[2]Here and in the sequel, the notation $\pm$ applies to the conormal derivatives from $\Omega_\pm$.

Note that in the isotropic case (1.10), the classical conormal derivatives $\mathbf{t}^{c\pm}(\mathbf{u},\pi)$ reduce to the well known formulas (cf., e.g., Appendix III, Part I, Section 1 in [48])

$$\left(\mathbf{t}^{c\pm}(\mathbf{u},\pi)\right)_i = -\gamma_\pm\pi\,\nu_i^\pm + \gamma_\pm\left(\lambda\operatorname{div}\mathbf{u}\right)\nu_i^\pm + 2\gamma_\pm\left(\mu E_{i\alpha}(\mathbf{u})\right)\nu_\alpha^\pm,\quad i=1,\dots,n. \tag{2.19}$$

For the classical conormal derivatives defined by (2.16)-(2.18), the *Green formula*

$$\pm\left\langle \mathbf{t}^{c\pm}(\mathbf{u},\pi),\boldsymbol{\varphi}\right\rangle_{\partial\Omega_\pm} = \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}),E_{i\alpha}(\boldsymbol{\varphi})\right\rangle_{\Omega_\pm} - \langle\pi,\operatorname{div}\boldsymbol{\varphi}\rangle_{\Omega_\pm} + \langle\boldsymbol{\mathcal{L}}(\mathbf{u},\pi),\boldsymbol{\varphi}\rangle_{\Omega_\pm}\quad\forall\,\boldsymbol{\varphi}\in\mathcal{D}(\mathbb{R}^n)^n \tag{2.20}$$

holds in Lipschitz sets and suggests the following definition of the *generalized conormal derivative* for the Stokes system with L_∞ viscosity tensor coefficient in the setting of weighted Sobolev spaces generalizing [37, Lemma 4.3], [28, Lemma 2.9], [38, Definition 3.1, Theorem 3.2], [41, Theorem 10.4.1], see also [31, Definition 2.4].

Definition 2.2. Let Ω_+ and Ω_- be two adjacent Lipschitz sets in R^n, $n\geq 2$, and let $\partial\Omega_+$ and $\partial\Omega_-$ be the corresponding boundaries of them. Let conditions (1.2), (1.3) hold. Then for any $(\mathbf{u}_\pm,\pi_\pm,\tilde{\mathbf{f}}_\pm)\in\mathcal{H}^1(\Omega_\pm)^n\times L_2(\Omega_\pm)\times\widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^n$, the *formal conormal derivatives* $\mathbf{t}^\pm(\mathbf{u}_\pm,\pi_\pm;\tilde{\mathbf{f}}_\pm)\in H^{-\frac{1}{2}}(\partial\Omega_\pm)^n$ are defined in the weak form by the formulas

$$\pm\left\langle \mathbf{t}^\pm(\mathbf{u}_\pm,\pi_\pm;\tilde{\mathbf{f}}_\pm),\mathbf{\Phi}\right\rangle_{\partial\Omega_\pm} := \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}_\pm),E_{i\alpha}(\gamma_\pm^{-1}\mathbf{\Phi})\right\rangle_{\Omega_\pm} - \left\langle\pi_\pm,\operatorname{div}(\gamma_\pm^{-1}\mathbf{\Phi})\right\rangle_{\Omega_\pm} + \left\langle\tilde{\mathbf{f}}_\pm,\gamma_\pm^{-1}\mathbf{\Phi}\right\rangle_{\Omega_\pm}, \tag{2.21}$$

for any $\mathbf{\Phi}\in H^{\frac{1}{2}}(\partial\Omega_\pm)^n$, where $\gamma_\pm^{-1}: H^{\frac{1}{2}}(\partial\Omega_\pm)^n\to\mathcal{H}^1(\Omega_\pm)^n$ are bounded right inverses to the trace operators $\gamma_\pm:\mathcal{H}^1(\Omega_\pm)^n\to H^{\frac{1}{2}}(\partial\Omega_\pm)^n$.

Moreover, if $(\mathbf{u}_\pm,\pi_\pm,\tilde{\mathbf{f}}_\pm)\in\boldsymbol{\mathcal{H}}^1(\Omega_\pm,\boldsymbol{\mathcal{L}})$, where

$$\boldsymbol{\mathcal{H}}^1(\Omega_\pm,\boldsymbol{\mathcal{L}}) := \Big\{(\mathbf{v}_\pm,q_\pm,\tilde{\boldsymbol{\phi}}_\pm)\in\mathcal{H}^1(\Omega_\pm)^n\times L_2(\Omega_\pm)\times\widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^n : \boldsymbol{\mathcal{L}}(\mathbf{v}_\pm,q_\pm)=\tilde{\boldsymbol{\phi}}_\pm|_{\Omega_\pm}\text{ in }\Omega_\pm\Big\}, \tag{2.22}$$

then (2.21) define the *generalized conormal derivatives* $\mathbf{t}^\pm(\mathbf{u}_\pm,\pi_\pm;\tilde{\mathbf{f}}_\pm)\in H^{-\frac{1}{2}}(\partial\Omega_\pm)^n$.

Some properties of the conormal derivatives are presented (cf. [37, Lemma 4.3], [38, Theorem 3.9], [39, Theorem 5.3], [28, Lemma 2.9], [41, Theorem 10.4.1]).

Lemma 2.3. *Let the assumptions of Definition 2.2 hold. Then the following properties hold.*

(*i*) *The formal conormal derivative operators*

$$\mathbf{t}^\pm:\mathcal{H}^1(\Omega_\pm)^n\times L_2(\Omega_\pm)\times\widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^n\to H^{-\frac{1}{2}}(\partial\Omega_\pm)^n$$

are linear and bounded.

(*ii*) *The generalized conormal derivative operators* $\mathbf{t}^\pm:\boldsymbol{\mathcal{H}}^1(\Omega_\pm,\boldsymbol{\mathcal{L}})\to H^{-\frac{1}{2}}(\partial\Omega_\pm)^n$ *with* $\boldsymbol{\mathcal{L}}$ *given by* (1.8)*, are linear and bounded, and do not depend on the choice of the right inverse operators* $\gamma_\pm^{-1}$ *in* (2.21)*. In addition, for all* $\mathbf{w}_\pm\in\mathcal{H}^1(\Omega_\pm)^n$ *and* $(\mathbf{u}_\pm,\pi_\pm,\tilde{\mathbf{f}}_\pm)\in\boldsymbol{\mathcal{H}}^1(\Omega_\pm,\boldsymbol{\mathcal{L}})$*, the following first Green identity holds*

$$\pm\left\langle\mathbf{t}^\pm(\mathbf{u}_\pm,\pi_\pm;\tilde{\mathbf{f}}_\pm),\gamma_\pm\mathbf{w}_\pm\right\rangle_{\partial\Omega_\pm} = \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}_\pm),E_{i\alpha}(\mathbf{w}_\pm)\right\rangle_{\Omega_\pm} - \langle\pi_\pm,\operatorname{div}\mathbf{w}_\pm\rangle_{\Omega_\pm} + \langle\tilde{\mathbf{f}}_\pm,\mathbf{w}_\pm\rangle_{\Omega_\pm}. \tag{2.23}$$

Proof. We use similar arguments to those in [30, Lemma 2.2] (see also [38, Definition 3.1, Theorem 3.2], [39], [41, Theorem 10.4.1]). First, we note that for $(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm) \in \mathcal{H}^1(\Omega_\pm)^n \times L_2(\Omega_\pm) \times \widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^n$, the right hand side in (2.21) defines a bounded linear functional acting on $\mathbf{\Phi} \in H^{\frac12}(\partial\Omega_\pm)^n$, and, hence, the left hand side determines the formal conormal derivatives $\mathbf{t}^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm)$ in $H^{-\frac12}(\partial\Omega_\pm)^n$ and the formal conormal derivative operators $\mathbf{t}^\pm : \mathcal{H}^1(\Omega_\pm)^n \times L_2(\Omega_\pm) \times \widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^n \to H^{-\frac12}(\partial\Omega_\pm)^n$ given by (2.21) are bounded. Therefore, the generalized conormal derivative operators $\mathbf{t}^\pm : \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}) \to H^{-\frac12}(\partial\Omega_\pm)^n$ are bounded as well.

Now, the property that the conormal derivative operators $\mathbf{t}^\pm : \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}) \to H^{-\frac12}(\partial\Omega_\pm)^n$ defined by (2.21) are invariant with respect to the choice of a right inverse of the trace operator $\gamma_\pm : \mathcal{H}^1(\Omega_\pm)^n \to H^{\frac12}(\partial\Omega)^n$ can be obtained with an argument similar to that for Theorem 3.2 in [38].

Going on, let $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}})$. In view of formula (2.21), we have that

$$\begin{aligned}
\pm \left\langle \mathbf{t}^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm), \gamma_\pm \mathbf{w}_\pm \right\rangle_{\partial\Omega_\pm} &= \left\langle A^{\alpha\beta}\partial_\beta(\mathbf{u}_\pm), \partial_\alpha\left(\gamma_\pm^{-1}(\gamma_\pm \mathbf{w}_\pm)\right)\right\rangle_{\Omega_\pm} \\
&- \left\langle \pi_\pm, \operatorname{div}\left(\gamma_\pm^{-1}(\gamma_\pm \mathbf{w}_\pm)\right)\right\rangle_{\Omega_\pm} + \left\langle \tilde{\mathbf{f}}_\pm, \gamma_\pm^{-1}(\gamma_\pm \mathbf{w}_\pm)\right\rangle_{\Omega_\pm} \\
= \left\langle A^{\alpha\beta}\partial_\beta(\mathbf{u}_\pm), \partial_\alpha(\mathbf{w}_\pm)\right\rangle_{\Omega_\pm} &- \left\langle \pi_\pm, \operatorname{div} \mathbf{w}_\pm\right\rangle_{\Omega_\pm} + \left\langle \tilde{\mathbf{f}}_\pm, \mathbf{w}_\pm\right\rangle_{\Omega_\pm} \\
&+ \left\langle A^{\alpha\beta}\partial_\beta(\mathbf{u}_\pm), \partial_\alpha\left(\gamma_\pm^{-1}(\gamma_\pm \mathbf{w}_\pm) - \mathbf{w}_\pm\right)\right\rangle_{\Omega_\pm} \\
&- \left\langle \pi_\pm, \operatorname{div}\left(\gamma_\pm^{-1}(\gamma_\pm \mathbf{w}_\pm) - \mathbf{w}_\pm\right)\right\rangle_{\Omega_\pm} + \left\langle \tilde{\mathbf{f}}_\pm, \gamma_\pm^{-1}(\gamma_\pm \mathbf{w}_\pm) - \mathbf{w}_\pm\right\rangle_{\Omega_\pm} \ \forall\, \mathbf{w} \in \mathcal{H}^1(\Omega_\pm)^n.
\end{aligned} \tag{2.24}$$

Because $\gamma_\pm^{-1}$ are right inverses of the trace operators $\gamma_\pm$, we have the equalities $\gamma_\pm\left(\gamma_\pm^{-1}(\gamma_\pm \mathbf{w}_\pm) - \mathbf{w}_\pm\right) = \mathbf{0}$ on $\partial\Omega_\pm$, and hence

$$\gamma_\pm^{-1}(\gamma_\pm \mathbf{w}_\pm) - \mathbf{w}_\pm \in \mathring{\mathcal{H}}^1(\Omega_\pm)^n\,, \tag{2.25}$$

where the spaces $\mathring{\mathcal{H}}^1(\Omega_\pm)^n$ are characterized as, cf. (2.12),

$$\mathring{\mathcal{H}}^1(\Omega_\pm)^n = \left\{\mathbf{v}_\pm \in \mathcal{H}^1(\Omega_\pm)^n : \gamma_\pm \mathbf{v}_\pm = \mathbf{0} \text{ on } \partial\Omega_\pm\right\}. \tag{2.26}$$

Therefore, the Green formula (2.23) follows from formula (2.24) if we show that

$$\left\langle A^{\alpha\beta}\partial_\beta(\mathbf{u}_\pm), \partial_\alpha(\mathbf{v}_\pm)\right\rangle_{\partial\Omega_\pm} - \langle \pi_\pm, \operatorname{div} \mathbf{v}_\pm\rangle_{\Omega_\pm} + \left\langle \tilde{\mathbf{f}}_\pm, \mathbf{v}_\pm\right\rangle_{\Omega_\pm} = 0 \ \ \forall\, \mathbf{v}_\pm \in \mathring{\mathcal{H}}^1(\Omega_\pm)^n. \tag{2.27}$$

Since the space $\mathcal{D}(\Omega_\pm)^n$ is dense in $\mathring{\mathcal{H}}^1(\Omega_\pm)^n$, we need to show identity (2.27) only for the test functions $\mathbf{v}_\pm$ in $\mathcal{D}(\Omega_\pm)^n$. Indeed, the membership of $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm)$ in $\boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}})$ implies the equality $\mathcal{L}(\mathbf{u}_\pm, \pi_\pm) = \tilde{\mathbf{f}}_\pm|_{\Omega_\pm}$ in the sense of distributions, and accordingly identity (2.27) holds for any $\mathbf{v}_\pm \in \mathcal{D}(\Omega_\pm)^n$.

Finally, we note that conditions (1.3) lead to the second equality in (2.23). □

In the sequel we use the simplified notation $\mathbf{t}^\pm(\mathbf{u}_\pm, \pi_\pm)$ for $\mathbf{t}^\pm(\mathbf{u}_\pm, \pi_\pm; \mathbf{0})$.

Remark 1. It is easy to verify that Definition 2.2 and all assertions of Section 2.3 can be stated also if the weighted Sobolev spaces are replaced by the standard Sobolev spaces.

2.4. **Abstract mixed variational formulations.** A major role in our analysis of mixed variational formulations is played by the following well-posedness result by Babuška [7] and Brezzi [11, Theorem 1.1] (see also [21, Theorem 2.34 and Remark 2.35(i)] and [12]).

Theorem 2.4. *Let X and $\mathcal{M}$ be two real Hilbert spaces. Let $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ and $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be bounded bilinear forms. Let $f \in X'$ and $g \in \mathcal{M}'$. Let V be the subspace of X defined by*

$$V := \{v \in X : b(v,q) = 0 \quad \forall\, q \in \mathcal{M}\}\,. \tag{2.28}$$

Assume that $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is coercive, which means that there exists a constant $C_a > 0$ such that

$$a(w,w) \geq C_a^{-1}\|w\|_X^2 \quad \forall\, w \in V, \tag{2.29}$$

and that $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ satisfies the Babuška-Brezzi condition

$$\inf_{q\in\mathcal{M}\setminus\{0\}} \sup_{v\in X\setminus\{0\}} \frac{b(v,q)}{\|v\|_X\|q\|_{\mathcal{M}}} \geq C_b^{-1}\,, \tag{2.30}$$

with some constant $C_b > 0$. Then the mixed variational formulation

$$\begin{cases} a(u,v) + b(v,p) &= f(v) \quad \forall\, v \in X, \\ b(u,q) &= g(q) \quad \forall\, q \in \mathcal{M}, \end{cases} \tag{2.31}$$

has a unique solution $(u,p) \in X \times \mathcal{M}$ and

$$\|u\|_X \leq C_a\|f\|_{X'} + C_b(1 + \|a\|C_a)\|g\|_{\mathcal{M}'}, \tag{2.32}$$

$$\|p\|_{\mathcal{M}} \leq C_b(1 + \|a\|C_a)\|f\|_{X'} + \|a\|C_b^2(1 + \|a\|C_a)\|g\|_{\mathcal{M}'}, \tag{2.33}$$

where $\|a\|$ is the norm of the bilinear form $a(\cdot,\cdot)$.

We need also the following extension (cf. [21, Lemma A.40]) of the Babuška-Brezzi result.

Lemma 2.5. *Let X and $\mathcal{M}$ be reflexive Banach spaces. Let $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be a bounded bilinear form. Let $B : X \to \mathcal{M}'$ and $B^* : \mathcal{M} \to X'$ be the linear bounded operator and its transpose operator defined by*

$$\langle Bv, q\rangle = b(v,q),\ \langle v, B^*q\rangle = \langle Bv, q\rangle \quad \forall\, v \in X,\ \forall\, q \in \mathcal{M}, \tag{2.34}$$

where $\langle\cdot,\cdot\rangle := {}_{X'}\langle\cdot,\cdot\rangle_X$ denotes the duality pairing between the dual spaces X' and X. The duality pairing between the spaces $\mathcal{M}'$ and $\mathcal{M}$ is also denoted by $\langle\cdot,\cdot\rangle$. Let $V := \operatorname{Ker} B$ and $V^\perp = X'\perp V := \{g \in X' : \langle g, v\rangle = 0\ \forall\, v \in V\}$. Then the following assertions are equivalent:

(i) *There exists a constant $C_b > 0$ such that $b(\cdot,\cdot)$ satisfies the inf-sup condition (2.30).*

(ii) *The operator $B : X/V \to \mathcal{M}'$ is an isomorphism and*

$$\|Bw\|_{\mathcal{M}'} \geq C_b^{-1}\|w\|_{X/V} \quad \forall\, w \in X/V. \tag{2.35}$$

(iii) *The operator $B^* : \mathcal{M} \to V^\perp$ is an isomorphism and*

$$\|B^*q\|_{X'} \geq C_b^{-1}\|q\|_{\mathcal{M}} \quad \forall\, q \in \mathcal{M}. \tag{2.36}$$

3. **Dirichlet and Dirichlet-transmission problems for the anisotropic Stokes system in a bounded Lipschitz domain.** In this section we show well-posedness of Dirichlet and Dirichlet-transmission problems for the anisotropic Stokes system in a bounded Lipschitz domain of $\mathbb{R}^n$, $n \geq 2$. This property will facilitate the analysis in Sections 5 and 6 of the same boundary problems for the anisotropic Navier-Stokes system in bounded and exterior Lipschitz domains. All along Section 3, Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega$ (not necessarily connected).

3.1. **Notations and auxiliary results.** Let us denote by $\mathring{H}^1(\Omega)^n$ the closed subspace of the Sobolev space $H^1(\Omega)$ consisting of functions with zero traces on $\partial\Omega$. The semi-norm

$$|\mathbf{u}|_{\mathring{H}^1(\Omega)^n} = |\mathbf{u}|_{H^1(\Omega)^n} = \|\nabla \mathbf{u}\|_{L_2(\Omega)^{n\times n}}\,. \tag{3.1}$$

is also a norm in $\mathring{H}^1(\Omega)$, which is equivalent to the norm

$$\|\mathbf{u}\|_{H^1(\Omega)^n} = \|\mathbf{u}\|_{L_2(\Omega)^n} + \|\nabla \mathbf{u}\|_{L_2(\Omega)^{n\times n}} \tag{3.2}$$

(cf., e.g., Theorem II.5.1 and Remark II.6.2 in [22]).

Since $H^{-1}(\Omega)^n$ can be identified with $\left(\mathring{H}^1(\Omega)^n\right)'$, let $\|\cdot\|_{H^{-1}(\Omega)^n}$ denote the corresponding norm in $H^{-1}(\Omega)^n$ generated by the seminorm $|\cdot|_{\mathring{H}^1(\Omega)^n}$ in (3.1), i.e.,

$$\|\mathbf{\Phi}\|_{H^{-1}(\Omega)^n} := \sup_{\mathbf{v}\in\mathring{H}^1(\Omega)^n,\ \|\nabla\mathbf{v}\|_{L_2(\Omega)^{n\times n}}=1} |\langle \mathbf{\Phi}, \mathbf{v}\rangle_\Omega| \quad \forall\ \mathbf{\Phi} \in H^{-1}(\Omega)^n. \tag{3.3}$$

Let us introduce the spaces

$$\mathcal{D}_{\mathrm{div}}(\Omega)^n := \{\mathbf{w} \in \mathcal{D}(\Omega)^n : \operatorname{div}\mathbf{w} = 0 \text{ in } \Omega\}, \tag{3.4}$$

$$\mathring{H}^1_{\mathrm{div}}(\Omega)^n := \{\mathbf{w} \in \mathring{H}^1(\Omega)^n : \operatorname{div}\mathbf{w} = 0 \text{ in } \Omega\}, \tag{3.5}$$

$$(\mathring{H}^1_{\mathrm{div}}(\Omega)^n)^\perp = H^{-1}(\Omega)^n \perp \mathring{H}^1_{\mathrm{div}}(\Omega)^n$$
$$:= \{\mathbf{\Phi} \in H^{-1}(\Omega)^n : \langle \mathbf{\Phi}, \mathbf{v}\rangle_\Omega = 0 \quad \forall\, \mathbf{v} \in \mathring{H}^1_{\mathrm{div}}(\Omega)^n\}, \tag{3.6}$$

$$L_{2;0}(\Omega) = L_2(\Omega)\perp\mathbb{R} := \{\phi \in L_2(\Omega) : \langle \phi, 1\rangle_\Omega = \int_\Omega \phi\, dx = 0\}. \tag{3.7}$$

The dual $\left(H^{-1}(\Omega)^n \perp \mathring{H}^1_{\mathrm{div}}(\Omega)^n\right)'$ can be identified with $\mathring{H}^1(\Omega)^n/\mathring{H}^1_{\mathrm{div}}(\Omega)^n$, and the dual $(L_{2;0}(\Omega))'$ with the space $L_2(\Omega)/\mathbb{R}$, (see, e.g., formula (5.118) in [41]).

For $\mathbf{v} \in \mathring{H}^1(\Omega)^n/\mathring{H}^1_{\mathrm{div}}(\Omega)^n$ we also introduce the following norm in this quotient space,

$$|\mathbf{v}|_{\mathring{H}^1(\Omega)^n/\mathring{H}^1_{\mathrm{div}}(\Omega)^n} := \inf_{\boldsymbol{\phi}\in\mathring{H}^1_{\mathrm{div}}(\Omega)^n} \|\nabla(\mathbf{v} - \boldsymbol{\phi})\|_{L_2(\Omega)^{n\times n}}. \tag{3.8}$$

The next result provides an isomorphism property of the divergence and gradient operators in bounded Lipschitz domains (cf., e.g., [47, Lemmas 7-9 in p. 30], [24, Corollary 2.4 and Theorem 2.3 in Chapter 1], [48, Proposition 1.2(i) and Remark 1.4 in Chapter 1], and [5, Theorem 3.1]).

Theorem 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then the operators*

$$\operatorname{div} : \mathring{H}^1(\Omega)^n/\mathring{H}^1_{\mathrm{div}}(\Omega)^n \to L_{2;0}(\Omega), \tag{3.9}$$

$$\operatorname{grad} : L_2(\Omega)/\mathbb{R} \to H^{-1}(\Omega)^n \perp \mathring{H}^1_{\mathrm{div}}(\Omega)^n\,. \tag{3.10}$$

are isomorphisms. The following norms of the operators inverse to operators (3.9) *and* (3.10) *are equal,*

$$\|\mathrm{div}^{-1}\|_{L_{2;0}(\Omega)\to\mathring{H}^1(\Omega)^n/\mathring{H}^1_{\mathrm{div}}(\Omega)^n} := \sup_{g\in L_{2;0}(\Omega),\ \|g\|_{L_2(\Omega)}=1} |\mathrm{div}^{-1}g|_{\mathring{H}^1(\Omega)^n/\mathring{H}^1_{\mathrm{div}}(\Omega)^n}$$

$$= \|\mathrm{grad}^{-1}\|_{H^{-1}(\Omega)^n\perp\mathring{H}^1_{\mathrm{div}}(\Omega)^n\to L_2(\Omega)/\mathbb{R}}$$

$$:= \sup_{\mathbf{f}\in H^{-1}(\Omega)^n\perp\mathring{H}^1_{\mathrm{div}}(\Omega)^n,\ \|\mathbf{f}\|_{H^{-1}(\Omega)^n}=1} \|\mathrm{grad}^{-1}\mathbf{f}\|_{L_2(\Omega)^n/\mathbb{R}} =: C_\Omega < \infty\,, \qquad (3.11)$$

and hence

$$|\mathbf{v}|_{\mathring{H}^1(\Omega)^n/\mathring{H}^1_{\mathrm{div}}(\Omega)^n} \le C_\Omega\|\mathrm{div}\,\mathbf{v}\|_{L_2(\Omega)} \quad \forall\,\mathbf{v}\in\mathring{H}^1(\Omega)^n/\mathring{H}^1_{\mathrm{div}}(\Omega)^n, \qquad (3.12)$$

$$\|q\|_{L_2(\Omega)/\mathbb{R}} \le C_\Omega\|\mathrm{grad}\,q\|_{H^{-1}(\Omega)^n} \quad \forall\,q\in L_2(\Omega)/\mathbb{R}\,. \qquad (3.13)$$

Moreover, the norm value, C_Ω, may depend on the shape of Ω but not on the domain scaling. Particularly, if Ω is a ball B, then $C_\Omega = C_B$ does not depend on its diameter.

Proof. In view of Lemma 2.5, norms of the operators inverse to (3.9) and (3.10) are equal. Moreover, the independence of C_Ω of the domain size follows by considering the simple scaling $\tilde{x} = \lambda x$ in all the operators and norms, e.g., in (3.9) and in the first two lines of (3.11), cf. [35, Corollary 2.1]. □

3.2. **Dirichlet problem for the Stokes system in a bounded Lipschitz domain.** Let $\boldsymbol{\mathcal{L}}$ be the anisotropic Stokes operator defined in (1.9). Let us consider the following Dirichlet problem for the anisotropic Stokes system

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u},\pi) = \mathbf{f}\,, \quad \mathrm{div}\,\mathbf{u} = g & \text{in } \Omega,\\ \gamma\mathbf{u} = \mathbf{0} & \text{on } \partial\Omega\,, \end{cases} \qquad (3.14)$$

with the unknowns $(\mathbf{u},\pi)\in H^1(\Omega)^n\times L_2(\Omega)/\mathbb{R}$ and the given data $(\mathbf{f},g)\in H^{-1}(\Omega)^n\times L_{2;0}(\Omega)$. A variational approach (as in the proof of Theorem 3.4 corresponding to the transmission problem) implies that the Dirichlet problem (3.14) has the following equivalent mixed variational formulation with $\mathbf{F} = -\mathbf{f}$.

Given $\mathbf{F}\in H^{-1}(\Omega)^n$ and $g\in L_{2;0}(\Omega)$, find $(\mathbf{u},\pi)\in\mathring{H}^1(\Omega)^n\times L_2(\Omega)/\mathbb{R}$ such that

$$\begin{cases} a_{\mathbb{A};\Omega}(\mathbf{u},\mathbf{v}) + b_\Omega(\mathbf{v},\pi) = \langle\mathbf{F},\mathbf{v}\rangle_\Omega & \forall\,\mathbf{v}\in\mathring{H}^1(\Omega)^n\,,\\ b_\Omega(\mathbf{u},q) = -\langle g,q\rangle_\Omega & \forall\,q\in L_2(\Omega)/\mathbb{R}\,, \end{cases} \qquad (3.15)$$

where $a_{\mathbb{A};\Omega}$: $\mathring{H}^1(\Omega)^n\times\mathring{H}^1(\Omega)^n\to\mathbb{R}$ and $b_\Omega : \mathring{H}^1(\Omega)^n\times L_2(\Omega)/\mathbb{R}\to\mathbb{R}$ are the bounded bilinear forms

$$a_{\mathbb{A};\Omega}(\mathbf{u},\mathbf{v}) := \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}),E_{i\alpha}(\mathbf{v})\right\rangle_\Omega \quad \forall\,\mathbf{u},\mathbf{v}\in\mathring{H}^1(\Omega)^n\,, \qquad (3.16)$$

$$b_\Omega(\mathbf{v},q) := -\langle\mathrm{div}\,\mathbf{v},q\rangle_\Omega \quad \forall\,\mathbf{v}\in\mathring{H}^1(\Omega)^n \quad \forall\,q\in L_2(\Omega)/\mathbb{R}\,. \qquad (3.17)$$

Then we obtain the following well-posedness result.

Theorem 3.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n\ge 2$. Let conditions* (1.2)-(1.4) *hold in Ω. Then for all $\mathbf{F}\in H^{-1}(\Omega)^n$ and $g\in L_{2;0}(\Omega)$, the linear variational problem* (3.15) *and the Dirichlet problem* (3.14) *with $\mathbf{f} = -\mathbf{F}$ have a unique solution $(\mathbf{u},\pi)\in\mathring{H}^1(\Omega)^n\times L_2(\Omega)/\mathbb{R}$ and the following estimates hold,*

$$\|\nabla\mathbf{u}\|_{L_2(\Omega)^{n\times n}} \le 2c_{\mathbb{A}}\|\mathbf{F}\|_{H^{-1}(\Omega)^n} + C'_\Omega\|g\|_{L_2(\Omega)}, \qquad (3.18)$$

$$\|\pi\|_{L_2(\Omega)/\mathbb{R}} \le C'_\Omega\|\mathbf{F}\|_{H^{-1}(\Omega)^n} + C^*_\Omega\|g\|_{L_2(\Omega)}, \qquad (3.19)$$

where the ellipticity constant $c_{\mathbb{A}}$ *is defined in* (1.4),

$$C'_\Omega := C_\Omega(1 + 2c_{\mathbb{A}} n^4 \|\mathbb{A}\|_{L_\infty(\Omega)^n}), \quad C^*_\Omega := n^4 \|\mathbb{A}\|_{L_\infty(\Omega)^n} C_\Omega C'_\Omega,$$

and the constant C_Ω *is defined in Theorem* 3.1.

Proof. The bilinear form $a_{\mathbb{A};\Omega} : \mathring{H}^1(\Omega)^n \times \mathring{H}^1(\Omega)^n \to \mathbb{R}$ is bounded on the Hilbert space $\mathring{H}^1(\Omega)^n$ and by (1.5), we have the following estimate

$$|a_{\mathbb{A};\Omega}(\mathbf{v}, \mathbf{w})| \leq n^4 \|\mathbb{A}\|_{L_\infty(\Omega)^n} \|\nabla \mathbf{v}\|_{L_2(\Omega)^{n\times n}} \|\nabla \mathbf{w}\|_{L_2(\Omega)^{n\times n}} \quad \forall\, \mathbf{v}, \mathbf{w} \in \mathring{H}^1(\Omega)^n. \tag{3.20}$$

The first Korn inequality in $\mathring{H}^1(\Omega)^n$ (see, e.g., [37, Theorem 10.1]) and the ellipticity condition (1.4) show that

$$\begin{aligned} \frac{1}{2} c_{\mathbb{A}}^{-1} \|\nabla \mathbf{v}\|^2_{L_2(\Omega)^{n\times n}} &= c_{\mathbb{A}}^{-1} \|\mathbb{E}(\mathbf{v})\|^2_{L_2(\Omega)^{n\times n}} \\ &\leq \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{v}), E_{i\alpha}(\mathbf{v}) \right\rangle_\Omega \\ &= a_{\mathbb{A};\Omega}(\mathbf{v}, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathring{H}^1_{\mathrm{div}}(\Omega)^n \,, \end{aligned} \tag{3.21}$$

implying that $a_{\mathbb{A};\Omega}$ is also coercive on $\mathring{H}^1_{\mathrm{div}}(\Omega)^n$.

By Theorem 3.1 and Lemma 2.5, the bilinear form $b_\Omega(\cdot,\cdot) : \mathring{H}^1(\Omega)^n \times L_2(\Omega) \to \mathbb{R}$ satisfies the inf-sup condition

$$\inf_{q \in L_2(\Omega)/\mathbb{R} \setminus \{0\}} \ \sup_{\mathbf{v} \in \mathring{H}^1(\Omega)^n \setminus \{\mathbf{0}\}} \frac{b_\Omega(\mathbf{v}, q)}{\|\nabla \mathbf{v}\|_{L_2(\Omega)^{n\times n}} \|q\|_{L_2(\Omega)/\mathbb{R}}} \geq C_\Omega^{-1}. \tag{3.22}$$

Then due to estimates (3.20), (3.21), (3.22), Theorem 2.4 with $X = \mathring{H}^1(\Omega)^n$, $V = \mathring{H}^1_{\mathrm{div}}(\Omega)^n$ and $\mathcal{M} = L_2(\Omega)/\mathbb{R}$ implies that for any $(\mathbf{F}, g) \in H^{-1}(\Omega)^n \times L_{2;0}(\Omega)$, there exists a unique solution $(\mathbf{u}, \pi) \in \mathring{H}^1(\Omega)^n \times L_2(\Omega)/\mathbb{R}$ of the variational problem (3.15) and inequalities (3.18), (3.19) hold. □

3.3. **Dirichlet-transmission problems for the Stokes system in a bounded domain.** Let $n \geq 2$. We make the following geometrical assumption.

Assumption 3.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, with (not necessarily connected) boundary $\partial\Omega$. Let Ω^0 be a bounded Lipschitz set such that $\overline{\Omega^0} \subset \Omega$. Let $\Omega^0_+ := \Omega^0$ and $\Omega^0_- := \Omega \setminus \overline{\Omega^0}$. Thus, the composite domain Ω can be written as $\Omega = \overline{\Omega^0_+} \cup \Omega^0_-$, and the boundary $\partial\Omega^0 = \partial\Omega^0_+$ of Ω^0_+ is also the interface between Ω^0_+ and Ω^0_- (see Figure 1).

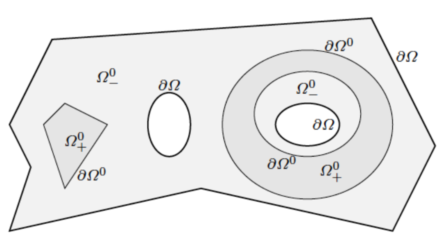

FIGURE 1. Bounded composite domain $\Omega = \overline{\Omega^0_+} \cup \Omega^0_-$

Let us introduce the following spaces

$$H^1(\Omega^0_-;\partial\Omega)^n := \left\{\mathbf{v}\in H^1(\Omega^0_-)^n : \gamma_+\mathbf{v}=\mathbf{0} \text{ on } \partial\Omega\right\}, \tag{3.23}$$

$$\widetilde{H}^{-1}(\Omega^0_-;\partial\Omega^0)^n := \left\{\boldsymbol{\varphi}\in H^{-1}(\Omega)^n : \boldsymbol{\varphi}=\mathbf{0} \text{ on } \Omega^0_+\right\}, \tag{3.24}$$

$$\mathfrak{X}_{\Omega^0_+,\Omega^0_-} := \left\{(\mathbf{v}_+,q_+,\mathbf{v}_-,q_-) : \mathbf{v}_+\in H^1(\Omega^0_+)^n,\ \mathbf{v}_-\in H^1(\Omega^0_-)^n,\right.$$
$$\left. q_+=q|_{\Omega^0_+},\ q_-=q|_{\Omega^0_-},\ q\in L_2(\Omega)/\mathbb{R}\right\}, \tag{3.25}$$

$$\mathfrak{Y}_{\Omega^0_+,\Omega^0_-} := \widetilde{H}^{-1}(\Omega^0_+)^n\times \widetilde{H}^{-1}(\Omega^0_-;\partial\Omega^0)^n\times L_{2;0}(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega^0)^n. \tag{3.26}$$

The space $\widetilde{H}^{-1}(\Omega^0_-;\partial\Omega^0)^n$ can be identified with the dual of $H^1(\Omega^0_-;\partial\Omega)^n$ (cf., e.g., arguments of Theorems 3.29 and 3.30 in [37]).

Next, we consider the Dirichlet-transmission problem for the anisotropic Stokes system

$$\boldsymbol{\mathcal{L}}(\mathbf{u}_+,\pi_+)=\tilde{\mathbf{f}}_+|_{\Omega^0_+},\quad \operatorname{div}\mathbf{u}_+=g|_{\Omega^0_+} \qquad \text{in } \Omega^0_+, \tag{3.27}$$

$$\boldsymbol{\mathcal{L}}(\mathbf{u}_-,\pi_-)=\tilde{\mathbf{f}}_-|_{\Omega^0_-},\quad \operatorname{div}\mathbf{u}_-=g|_{\Omega^0_-} \qquad \text{in } \Omega^0_-, \tag{3.28}$$

$$\gamma_+\mathbf{u}_+-\gamma_-\mathbf{u}_-=\mathbf{0} \qquad \text{on } \partial\Omega^0, \tag{3.29}$$

$$\mathbf{t}^+(\mathbf{u}_+,\pi_+;\tilde{\mathbf{f}}_+)-\mathbf{t}^-(\mathbf{u}_-,\pi_-;\tilde{\mathbf{f}}_-)=\boldsymbol{\psi} \qquad \text{on } \partial\Omega^0, \tag{3.30}$$

$$\gamma_-\mathbf{u}_-=\mathbf{0} \qquad \text{on } \partial\Omega, \tag{3.31}$$

with given data $(\tilde{\mathbf{f}}_+,\tilde{\mathbf{f}}_-,g,\boldsymbol{\psi})\in\mathfrak{Y}_{\Omega^0_+,\Omega^0_-}$ and unknown $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathfrak{X}_{\Omega^0_+,\Omega^0_-}$.

Let us consider $\mathbf{F}\in H^{-1}(\Omega)^n$ defined as

$$\begin{aligned}\langle\mathbf{F},\mathbf{v}\rangle_\Omega &= -\langle\tilde{\mathbf{f}}_+,\mathbf{v}|_{\Omega^0_+}\rangle_{\Omega^0_+}-\langle\tilde{\mathbf{f}}_-,\mathbf{v}|_{\Omega^0_-}\rangle_{\Omega^0_-}+\langle\boldsymbol{\psi},\gamma\mathbf{v}\rangle_{\partial\Omega^0}\\ &= -\langle\tilde{\mathbf{f}}_+,\mathbf{v}\rangle_\Omega-\langle\tilde{\mathbf{f}}_-,\mathbf{v}\rangle_\Omega+\langle\gamma^*\boldsymbol{\psi},\mathbf{v}\rangle_\Omega \quad \forall\,\mathbf{v}\in\mathring{H}^1(\Omega)^n,\end{aligned} \tag{3.32}$$

i.e.,

$$\mathbf{F}=-(\tilde{\mathbf{f}}_++\tilde{\mathbf{f}}_-)+\gamma^*\boldsymbol{\psi}, \tag{3.33}$$

where $\gamma^*: H^{-\frac{1}{2}}(\partial\Omega^0)^n\to H^{-1}(\mathbb{R}^n)^n$ is the adjoint of the trace operator $\gamma: H^1(\mathbb{R}^n)^n\to H^{\frac{1}{2}}(\partial\Omega^0)^n$, and the support of $\gamma^*\boldsymbol{\psi}$ is a subset of $\partial\Omega^0$.

Then we obtain the following well-posedness result.

Theorem 3.4. *Let $n\ge 2$ and Assumption 3.3 hold. Let conditions (1.2)-(1.4) hold in Ω. Given $(\tilde{\mathbf{f}}_+,\tilde{\mathbf{f}}_-,g,\boldsymbol{\psi})\in\mathfrak{Y}_{\Omega^0_+,\Omega^0_-}$, let $(\mathbf{u},\pi)\in\mathring{H}^1(\Omega)^n\times L_2(\Omega)/\mathbb{R}$ be the solution of variational problem (3.15) provided by Theorem 3.2 for the data $(\mathbf{F},g)$ with $\mathbf{F}$ given by (3.33). Then the linear Dirichlet-transmission problem (3.27)-(3.31) has a unique solution in $\mathfrak{X}_{\Omega^0_+,\Omega^0_-}$ given by*

$$\mathbf{u}_+=\mathbf{u}|_{\Omega^0_+},\ \mathbf{u}_-=\mathbf{u}|_{\Omega^0_-},\ \pi_+=\pi|_{\Omega^0_+},\ \pi_-=\pi|_{\Omega^0_-}, \tag{3.34}$$

and estimates (3.18) and (3.19) hold.

Proof. To show the equivalence between the transmission problem (3.27)-(3.31) and the variations problem (3.15), let us first assume that $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathfrak{X}_{\Omega^0_+,\Omega^0_-}$ satisfy transmission problem (3.27)-(3.31). Then transmission condition (3.29) and Lemma A.1(i,ii) along with boundary condition (3.31) imply that there exists a unique pair $(\mathbf{u},\pi)\in\mathring{H}^1(\Omega)^n\times L_2(\Omega)/\mathbb{R}$ satisfying conditions (3.34).

Due to the first equations in (3.27) and (3.28) along with Lemma 2.3 and Remark 1, the first Green identities (2.23) are valid. Employing transmission condition (3.30) they show that the pair $(\mathbf{u},\pi)$ satisfies

$$\left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}_+),E_{i\alpha}(\mathbf{v})\right\rangle_{\Omega_+^0}-\langle\pi_+,\operatorname{div}\mathbf{v}\rangle_{\Omega_+^0}+\langle\tilde{\mathbf{f}}_+,\mathbf{v}\rangle_{\Omega_+^0}$$
$$+\left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}_-),E_{i\alpha}(\mathbf{v})\right\rangle_{\Omega_-^0}-\langle\pi_-,\operatorname{div}\mathbf{v}\rangle_{\Omega_-^0}+\langle\tilde{\mathbf{f}}_-,\mathbf{v}\rangle_{\Omega_-^0}=\langle\boldsymbol{\psi},\gamma\mathbf{v}\rangle_{\partial\Omega^0}\,. \qquad (3.35)$$

for any $\mathbf{v}\in\mathring{H}^1(\Omega)^n$. Together with (3.32) this leads to the first equation in (3.15). The second equation in (3.15) follows from the second equations in (3.27) and (3.28).

Let us now show that $(\mathbf{u},\pi)\in\mathring{H}^1(\Omega)^n\times L_2(\Omega)/\mathbb{R}$ solving the variations problem (3.15) with $\mathbf{F}\in H^{-1}(\Omega)^n$ given by (3.33) provides also a solution given by (3.34) of the Dirichlet-transmission problem (3.27)-(3.31). The membership relation $(\mathbf{u},\pi)\in\mathring{H}^1(\Omega)^n\times L_2(\Omega)/\mathbb{R}$ yields $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathfrak{X}_{\Omega_+^0,\Omega_-^0}$.

Choosing $\mathbf{v}\in\mathcal{D}(\Omega_\pm^0)^n$ and $q\in\mathcal{D}(\Omega_\pm^0)$, equations in (3.15) imply that the couples $(\mathbf{u}_\pm,\pi_\pm)$ satisfy equations (3.27) and (3.28) in the sense of distributions. For any $\mathbf{v}\in\mathcal{D}(\mathbb{R}^n)^n$ in the first equation in (3.15), using again relation (3.33), we obtain that (3.35) is satisfied.

Then by Lemma 2.3 and Remark 1 the first Green identity (2.23) shows that equation (3.35) reduces to the equation

$$\left\langle\mathbf{t}^+\left(\mathbf{u}_+,\pi_+;\tilde{\mathbf{f}}_++(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+\right)-\mathbf{t}^-\left(\mathbf{u}_-,\pi_-;\tilde{\mathbf{f}}_-+(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-\right),\gamma\mathbf{v}\right\rangle_{\partial\Omega^0}$$
$$=\langle\boldsymbol{\psi},\gamma\mathbf{v}\rangle_{\partial\Omega^0}\ \ \forall\,\mathbf{v}\in\mathcal{D}(\Omega)^n,$$

or, equivalently, to the equation

$$\left\langle\mathbf{t}^+\big(\mathbf{u}_+,\pi_+;\tilde{\mathbf{f}}_++(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+\big)-\mathbf{t}^-\big(\mathbf{u}_-,\pi_-;\tilde{\mathbf{f}}_-+(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-\big),\boldsymbol{\phi}\right\rangle_{\partial\Omega^0}=\langle\boldsymbol{\psi},\boldsymbol{\phi}\rangle_{\partial\Omega^0}$$

for any $\boldsymbol{\phi}\in H^{\frac12}(\partial\Omega^0)^n$, due to the dense embedding of the space $\mathcal{D}(\Omega)^n$ in $\mathring{H}^1(\Omega)^n$ and the surjectivity of the trace operator γ from $\mathring{H}^1(\Omega)^n$ to $H^{\frac12}(\partial\Omega^0)^n$. Therefore, transmission condition (3.30) follows, as asserted. Transmission condition (3.29) and boundary condition (3.31) are obviously satisfied since $\mathbf{u}\in\mathring{H}(\Omega)^n$. □

4. The anisotropic Stokes system in $\mathbb{R}^n$ and in exterior Lipschitz domains.

Girault and Sequeira in [25, Theorem 3.4] used a variational approach to show the well-posedness in $\mathcal{H}^1(\Omega_-)^n\times L_2(\Omega_-)$ for the exterior Dirichlet problem for the constant coefficient isotropic Stokes system in an exterior Lipschitz domain Ω_- of $\mathbb{R}^n$, $n=2,3$, see also [4] for $n\geq 3$ and settings in a wider range of weighted spaces. We present here well-posedness of the corresponding problems for the anisotropic variable-coefficient case with $n\geq 3$.

4.1. The Stokes system in $\mathbb{R}^n$.

The spaces $\mathcal{H}^1(\mathbb{R}^n)^n$ and $\mathcal{H}^{-1}(\mathbb{R}^n)^n$ used in this section are described in Section 2.2.1 together with some their properties. Recall also that the semi-norm

$$|\mathbf{v}|_{\mathcal{H}^1(\mathbb{R}^n)^n}:=\|\nabla\mathbf{v}\|_{L_2(\mathbb{R}^n)^{n\times n}}\quad\forall\,\mathbf{v}\in\mathcal{H}^1(\mathbb{R}^n)^n,$$

is a norm in the space $\mathcal{H}^1(\mathbb{R}^n)^n$, which is equivalent to the norm $\|\cdot\|_{\mathcal{H}^1(\mathbb{R}^n)^n}$ given by (2.7). Hence the dual $\mathcal{H}^{-1}(\mathbb{R}^n)^n=\left(\mathring{\mathcal{H}}^1(\mathbb{R}^n)^n\right)'$ can be endowed with the norm

$$\|\boldsymbol{\Phi}\|_{\mathcal{H}^{-1}(\mathbb{R}^n)^n}:=\sup_{\mathbf{v}\in\mathcal{H}^1(\mathbb{R}^n)^n,\ \|\nabla\mathbf{v}\|_{L_2(\mathbb{R}^n)^{n\times n}}=1}|\langle\boldsymbol{\Phi},\mathbf{v}\rangle_{\mathbb{R}^n}|\quad\forall\ \boldsymbol{\Phi}\in\mathcal{H}^{-1}(\mathbb{R}^n)^n. \qquad (4.1)$$

Let us also denote

$$\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n := \left\{\mathbf{w}\in\mathcal{H}^1(\mathbb{R}^n)^n : {\rm div}\,\mathbf{w}=0 \text{ in } \mathbb{R}^n\right\}.$$

The subspace $\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n$ of $\mathcal{H}^1(\mathbb{R}^n)^n$ has also the characterization

$$\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n = \left\{\mathbf{w}\in\mathcal{H}^1(\mathbb{R}^n)^n : \langle {\rm div}\,\mathbf{w}, q\rangle_{\mathbb{R}^n} =0 \quad \forall\, q\in L_2(\mathbb{R}^n)\right\}. \tag{4.2}$$

We need the following assertion due to Proposition 2.1 and Theorem 2.5 in [2].

Theorem 4.1. *Let $n\geq 3$. Then the operators*

$$ {\rm div} : \mathcal{H}^1(\mathbb{R}^n)^n/\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n \to L_2(\mathbb{R}^n), \tag{4.3}$$

$$ {\rm grad} : L_2(\mathbb{R}^n) \to \mathcal{H}^{-1}(\mathbb{R}^n)^n \perp \mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n \tag{4.4}$$

are isomorphisms,

$$\begin{aligned}
\|{\rm div}^{-1}\|_{L_2(\mathbb{R}^n)\to\mathcal{H}^1(\mathbb{R}^n)^n/\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n} &:= \sup_{g\in L_2(\mathbb{R}^n),\ \|g\|_{L_2(\mathbb{R}^n)}=1} |{\rm div}^{-1} g|_{\mathcal{H}^1(\mathbb{R}^n)^n/\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n}\\
&= \|{\rm grad}^{-1}\|_{\mathcal{H}^{-1}(\mathbb{R}^n)^n\perp\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n\to L_2(\mathbb{R}^n)}\\
&:= \sup_{\mathbf{f}\in\mathcal{H}^{-1}(\mathbb{R}^n)^n\perp\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n,\ \|\mathbf{f}\|_{\mathcal{H}^{-1}(\mathbb{R}^n)^n}=1} \|{\rm grad}^{-1}\mathbf{f}\|_{L_2(\mathbb{R}^n)^n} =: C_{\mathbb{R}^n} < \infty,
\end{aligned} \tag{4.5}$$

and hence

$$|\mathbf{v}|_{\mathcal{H}^1(\mathbb{R}^n)^n/\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n} \leq C_{\mathbb{R}^n}\|{\rm div}\,\mathbf{v}\|_{L_2(\mathbb{R}^n)} \quad \forall\,\mathbf{v}\in\mathcal{H}^1(\mathbb{R}^n)^n/\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n, \tag{4.6}$$

$$\|q\|_{L_2(\mathbb{R}^n)} \leq C_{\mathbb{R}^n}\|{\rm grad}\,q\|_{\mathcal{H}^{-1}(\mathbb{R}^n)^n} \quad \forall\, q\in L_2(\mathbb{R}^n). \tag{4.7}$$

Proof. In view of Proposition 2.1 in [2], the operator ${\rm div} : \mathcal{H}^1(\mathbb{R}^n)^n/\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n \to L_2(\mathbb{R}^n)$ is an isomorphism. Then by Lemma 2.5 its adjoint, $-{\rm grad} : L_2(\mathbb{R}^n) \to \mathcal{H}^{-1}(\mathbb{R}^n)^n\perp\mathcal{H}^1_{\rm div}(\mathbb{R}^n)^n$ is an isomorphism as well (cf. Theorem 2.5 in [2]) and the corresponding norms of the operators ${\rm div}^{-1}$ and ${\rm grad}^{-1}$ as defined in (4.5) are equal. □

Let $\mathbb{A}$ satisfy conditions (1.2)-(1.4) and let $a_{\mathbb{A};\mathbb{R}^n} : \mathcal{H}^1(\mathbb{R}^n)^n\times\mathcal{H}^1(\mathbb{R}^n)^n\to\mathbb{R}$ and $b_{\mathbb{R}^n} : \mathcal{H}^1(\mathbb{R}^n)^n\times L_2(\mathbb{R}^n)\to\mathbb{R}$ be the bilinear forms given by

$$\begin{aligned}
a_{\mathbb{A};\mathbb{R}^n}(\mathbf{u},\mathbf{v}) :&= \left\langle A^{\alpha\beta}\partial_\beta\mathbf{u}, \partial_\alpha\mathbf{v}\right\rangle_{\mathbb{R}^n}\\
&= \left\langle a^{\alpha\beta}_{ij}E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v})\right\rangle_{\mathbb{R}^n} \quad \forall\ \mathbf{u}\in\mathcal{H}^1(\mathbb{R}^n)^n,\ \mathbf{v}\in\mathcal{H}^1(\mathbb{R}^n)^n,
\end{aligned} \tag{4.8}$$

$$b_{\mathbb{R}^n}(\mathbf{v},q) := -\langle{\rm div}\,\mathbf{v}, q\rangle_{\mathbb{R}^n} \quad \forall\ \mathbf{v}\in\mathcal{H}^1(\mathbb{R}^n)^n, \quad \forall\, q\in L_2(\mathbb{R}^n). \tag{4.9}$$

Now we can show the following well-posedness result for the anisotropic Stokes system in $\mathbb{R}^n$, $n\geq 3$, (cf. [31, Lemma 3.1] for the anisotropic case with the strong ellipticity condition).

Theorem 4.2. *Let conditions (1.2)-(1.4) hold in $\mathbb{R}^n$, $n\geq 3$. Let $a_{\mathbb{A};\mathbb{R}^n}$ and $b_{\mathbb{R}^n}$ be the bilinear forms defined in (4.8) and (4.9), respectively. Then for all given data $\mathbf{f}\in\mathcal{H}^{-1}(\mathbb{R}^n)^n$ and $g\in L_2(\mathbb{R}^n)$, the mixed variational problem*

$$\begin{cases} a_{\mathbb{A};\mathbb{R}^n}(\mathbf{u},\mathbf{v}) + b_{\mathbb{R}^n}(\mathbf{v},\pi) = -\langle\mathbf{f},\mathbf{v}\rangle_{\mathbb{R}^n} \quad \forall\,\mathbf{v}\in\mathcal{H}^1(\mathbb{R}^n)^n,\\ b_{\mathbb{R}^n}(\mathbf{u},q) = \langle g,q\rangle_{\mathbb{R}^n} \quad \forall\, q\in L_2(\mathbb{R}^n)\end{cases} \tag{4.10}$$

has a unique solution $(\mathbf{u},\pi)\in\mathcal{H}^1(\mathbb{R}^n)^n\times L_2(\mathbb{R}^n)$ and the following estimates hold

$$\|\nabla\mathbf{u}\|_{L_2(\mathbb{R}^n)^{n\times n}} \leq 2c_{\mathbb{A}}\|\mathbf{f}\|_{\mathcal{H}^{-1}(\mathbb{R}^n)^n} + C'_{\mathbb{R}^n}\|g\|_{L_2(\mathbb{R}^n)}, \tag{4.11}$$

$$\|\pi\|_{L_2(\mathbb{R}^n)} \leq C'_{\mathbb{R}^n}\|\mathbf{f}\|_{\mathcal{H}^{-1}(\mathbb{R}^n)^n} + C^*_{\mathbb{R}^n}\|g\|_{L_2(\mathbb{R}^n)}, \tag{4.12}$$

where the ellipticity constant $c_{\mathbb{A}}$ *is defined in* (1.4),

$$C'_{\mathbb{R}^n} := C_{\mathbb{R}^n}(1 + 2c_{\mathbb{A}} n^4 \|\mathbb{A}\|_{L_\infty(\mathbb{R}^n)^n}), \quad C^*_{\mathbb{R}^n} := n^4 \|\mathbb{A}\|_{L_\infty(\mathbb{R}^n)^n} C_{\mathbb{R}^n} C'_{\mathbb{R}^n},$$

and the constant $C_{\mathbb{R}^n}$ *is defined in* (4.5).

Moreover, $(\mathbf{u}, \pi)$ *is the unique solution of the Stokes system*

$$\boldsymbol{\mathcal{L}}(\mathbf{u}, \pi) = \mathbf{f}, \quad \text{div } \mathbf{u} = g \quad in \quad \mathbb{R}^n. \tag{4.13}$$

Proof. The theorem has been proved in [32, Lemma 3.1 and Theorem 3.2] except for the explicit expressions of the constants in estimates (4.11), (4.12), which can be obtained by using similar arguments to those in the proof of Theorem 4.7. □

4.2. **Transmission problem for the Stokes system in $\mathbb{R}^n$.**

Assumption 4.3. Let Ω^0 be a bounded Lipschitz set in $\mathbb{R}^n$, $n \geq 3$, with boundary $\partial\Omega^0$. Thus, the Lipschitz set $\Omega^0_- := \mathbb{R}^n \setminus \overline{\Omega^0}$ contains an unbounded (exterior) Lipschitz domain. Let $\Omega^0_+ := \Omega^0$. The boundary $\partial\Omega^0 = \partial\Omega^0_+ = \partial\Omega^0_-$ is the interface between Ω^0_+ and Ω^0_- (see Figure 2).

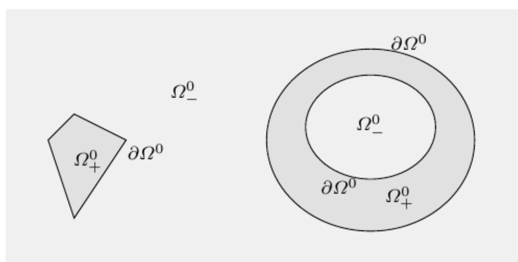

FIGURE 2. Composite space $\mathbb{R}^n = \overline{\Omega^0_+} \cup \Omega^0_-$

Let us consider the transmission problem

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u}_+, \pi_+) = \tilde{\mathbf{f}}_+|_{\Omega^0_+}, \quad \text{div}\, \mathbf{u}_+ = g_+ & \text{in } \Omega^0_+, \\ \boldsymbol{\mathcal{L}}(\mathbf{u}_-, \pi_-) = \tilde{\mathbf{f}}_-|_{\Omega_-}, \quad \text{div}\, \mathbf{u}_- = g_- & \text{in } \Omega^0_-, \\ \gamma_+ \mathbf{u}_+ - \gamma_- \mathbf{u}_- = \mathbf{0} & \text{on } \partial\Omega^0, \\ \mathbf{t}^+ \left(\mathbf{u}_+, \pi_+; \tilde{\mathbf{f}}_+\right) - \mathbf{t}^- \left(\mathbf{u}_-, \pi_-; \tilde{\mathbf{f}}_-\right) = \boldsymbol{\psi} & \text{on } \partial\Omega^0, \end{cases} \tag{4.14}$$

with given data $(\tilde{\mathbf{f}}_+, g_+, \tilde{\mathbf{f}}_-, g_-, \boldsymbol{\psi}) \in \mathcal{Y}_{\Omega^0_+, \Omega^0_-}$ and unknown $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathcal{X}_{\Omega^0_+, \Omega^0_-}$. Here,

$$\mathcal{X}_{\Omega^0_+, \Omega^0_-} := H^1(\Omega^0_+)^n \times L_2(\Omega^0_+) \times \mathcal{H}^1(\Omega^0_-)^n \times L_2(\Omega^0_-), \tag{4.15}$$

$$\mathcal{Y}_{\Omega^0_+, \Omega^0_-} := \widetilde{H}^{-1}(\Omega^0_+)^n \times L_2(\Omega^0_+) \times \widetilde{\mathcal{H}}^{-1}(\Omega^0_-)^n \times L_2(\Omega^0_-) \times H^{-\frac{1}{2}}(\partial\Omega^0)^n. \tag{4.16}$$

Following arguments similar to the ones for the well-posedness of transmission problem in a bounded domain (see Theorem 3.4), one can prove the well-posedness of the transmission problem (4.14) for the anisotropic Stokes system in $\mathbb{R}^n$.

Theorem 4.4. *Let Assumption* 4.3 *hold. Let conditions* (1.2)-(1.4) *be satisfied in* $\mathbb{R}^n$, $n \geq 3$. *Let* $\left(\tilde{\mathbf{f}}_+, g_+, \tilde{\mathbf{f}}_-, g_-, \boldsymbol{\psi}\right) \in \mathcal{Y}_{\Omega^0_+, \Omega^0_-}$ *be given and let* $(\mathbf{u}, \pi) \in \mathcal{H}^1(\mathbb{R}^n)^n \times L_2(\mathbb{R}^n)$ *be the solution of the variational problem* (4.10) *provided by Theorem* 4.2 *for* $\mathbf{F} = -(\tilde{\mathbf{f}}_+ + \tilde{\mathbf{f}}_-) + \gamma^* \boldsymbol{\psi} \in \mathcal{H}^{-1}(\mathbb{R}^n)^n$ *and* $g = \mathring{E}_{\Omega^0_+} g_+ + \mathring{E}_{\Omega^0_-} g_- \in L_2(\mathbb{E}^n)$, *where*

$\gamma^*: H^{-\frac{1}{2}}(\partial\Omega^0)^n \to \mathcal{H}^{-1}(\mathbb{R}^n)^n$ *is the adjoint of the trace operator* $\gamma: \mathcal{H}^1(\mathbb{R}^n)^n \to H^{\frac{1}{2}}(\partial\Omega^0)^n$. *Then the linear transmission problem* (4.14) *has a unique solution in* $\mathcal{X}_{\Omega^0_+,\Omega^0_-}$ *given by*

$$\mathbf{u}_+ = \mathbf{u}|_{\Omega^0_+},\ \mathbf{u}_- = \mathbf{u}|_{\Omega^0_-},\ \pi_+ = \pi|_{\Omega^0_+},\ \pi_- = \pi|_{\Omega^0_-}, \tag{4.17}$$

and estimates (4.11), (4.12) *hold.*

Proof. Lemma A.1 (iii) implies that the transmission problem (4.14) has the mixed variational formulation (4.10). Then the well-posedness result in Theorem 4.2 shows that the transmission problem (4.14) is also well-posed. Moreover, the connection between the solution $(\mathbf{u},\pi)\in\mathcal{H}^1(\mathbb{R}^n)^n\times L_2(\mathbb{R}^n)$ of problem (4.14) and the solution $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathcal{X}_{\Omega^0_+,\Omega^0_-}$ of problem (4.10) is given by relations (4.17), and each of them satisfies estimates (4.11) and (4.12). □

4.3. **Exterior Dirichlet problem for the Stokes system.** Let $n\geq 3$ and $\Omega_-\subset\mathbb{R}^n$ be an exterior Lipschitz domain with compact boundary $\partial\Omega$, not necessarily connected. Thus,

$$\Omega_- = \mathbb{R}^n \setminus \bigcup_{i=1}^m \overline{\Omega}_i, \tag{4.18}$$

where $m\geq 1$, Ω_i, $i=1,\ldots,m$, are bounded Lipschitz domains with connected boundaries $\partial\Omega_i$, such that $\overline{\Omega}_i\cap\overline{\Omega}_j=\emptyset$, $i\neq j$. Let $\Omega:=\bigcup_{i=1}^m\Omega_i$ and let $\partial\Omega:=\bigcup_{i=1}^m\partial\Omega_i$ denote the boundary of Ω. Moreover, $\partial\Omega$ is the boundary of Ω_- as well.

Recall that $\boldsymbol{\mathcal{L}}$ is the Stokes operator defined in (1.9), with the viscosity tensor coefficient $\mathbb{A}$ satisfying conditions (1.2)-(1.4).

The spaces $\mathcal{H}^1(\Omega_-)^n$, $\mathcal{H}^{-1}(\Omega_-)^n$, $\mathring{\mathcal{H}}^1(\Omega_-)^n$ and $\widetilde{\mathcal{H}}^{-1}(\Omega_-)^n$ used in this section are described in Section 2.2.2 together with some of their properties. Recall also that the semi-norm

$$|\mathbf{v}|_{\mathcal{H}^1(\Omega_-)^n} := \|\nabla\mathbf{v}\|_{L_2(\Omega_-)^{n\times n}} \quad \forall\,\mathbf{v}\in\mathcal{H}^1(\Omega_-)^n,$$

is a norm in the space $\mathcal{H}^1(\Omega_-)^n$ and in its subspace $\mathring{\mathcal{H}}^1(\Omega_-)^n$, which is equivalent to the norm $\|\cdot\|_{\mathcal{H}^1(\Omega_-)^n}$ given by (2.10). The closure of the space $\mathcal{D}(\Omega_-)^n$ with respect to the semi-norm $|\cdot|_{\mathcal{H}^1(\Omega_-)^n}$ coincides with $\mathring{\mathcal{H}}^1(\Omega)^n$. Hence the dual $\mathcal{H}^{-1}(\Omega_-)^n = \left(\mathring{\mathcal{H}}^1(\Omega_-)^n\right)'$ can be endowed with the norm

$$\|\boldsymbol{\Phi}\|_{\mathcal{H}^{-1}(\Omega_-)^n} := \sup_{\mathbf{v}\in\mathcal{H}^1(\Omega_-)^n,\ \|\nabla\mathbf{v}\|_{L_2(\Omega_-)^{n\times n}}=1} |\langle\boldsymbol{\Phi},\mathbf{v}\rangle_{\Omega_-}| \quad \forall\,\boldsymbol{\Phi}\in\mathcal{H}^{-1}(\Omega_-)^n. \tag{4.19}$$

Let also

$$\mathring{\mathcal{H}}^1_{\rm div}(\Omega_-)^n := \left\{\mathbf{v}\in\mathring{\mathcal{H}}^1(\Omega_-)^n : {\rm div}\,\mathbf{v}=0 \text{ in } \Omega_-\right\}. \tag{4.20}$$

4.3.1. *Properties of div and grad operators in an exterior domain.* The next result follows from Theorem 4 and Corollary 3 of Bogovskiĭ [8] (see also [9, Theorem 3.3, Theorem 4.2b and Corollary 4.5b]).

Theorem 4.5. *Let* Ω_- *be an exterior Lipschitz domain in* $\mathbb{R}^n$, $n\geq 3$. *Then the following operators are isomorphisms*

$$ {\rm div}: \mathring{\mathcal{H}}^1(\Omega_-)^n/\mathring{\mathcal{H}}^1_{\rm div}(\Omega_-)^n \to L_2(\Omega_-), \tag{4.21}$$

$$ {\rm grad}: L_2(\Omega_-) \to \mathcal{H}^{-1}(\Omega_-)^n \perp \mathring{\mathcal{H}}^1_{\rm div}(\Omega_-)^n. \tag{4.22}$$

Proof. In view of Theorem 4 and Corollary 3 of Bogovskiĭ [8], the operator in (4.21) is an isomorphism. Then by Lemma 2.5 its adjoint, the operator in (4.22), is an isomorphism as well. □

Theorem 4.6. *Let the conditions of Theorem 4.5 hold. Then the following norms of the operators inverse to operators (4.21) and (4.22) are equal,*

$$\|\mathrm{div}^{-1}\|_{L_2(\Omega_-)\to\mathring{\mathcal{H}}^1(\Omega_-)^n/\mathring{\mathcal{H}}^1_{\mathrm{div}}(\Omega_-)^n} := \sup_{g\in L_2(\Omega_-),\ \|g\|_{L_2(\Omega_-)}=1} |\mathrm{div}^{-1}g|_{\mathring{\mathcal{H}}^1(\Omega_-)^n/\mathring{\mathcal{H}}^1_{\mathrm{div}}(\Omega_-)^n}$$

$$= \|\mathrm{grad}^{-1}\|_{\mathcal{H}^{-1}(\Omega_-)^n\perp\mathring{\mathcal{H}}^1_{\mathrm{div}}(\Omega_-)^n\to L_2(\Omega_-)}$$

$$:= \sup_{\mathbf{f}\in\mathcal{H}^{-1}(\Omega_-)^n\perp\mathring{\mathcal{H}}^1_{\mathrm{div}}(\Omega_-)^n,\ \|\mathbf{f}\|_{\mathcal{H}^{-1}(\Omega_-)^n}=1} \|\mathrm{grad}^{-1}\mathbf{f}\|_{L_2(\Omega_-)^n} =: C_{\Omega_-} < \infty, \tag{4.23}$$

and hence

$$|\mathbf{v}|_{\mathring{\mathcal{H}}^1(\Omega_-)^n/\mathring{\mathcal{H}}^1_{\mathrm{div}}(\Omega_-)^n} \le C_{\Omega_-}\|\mathrm{div}\,\mathbf{v}\|_{L_2(\Omega_-)} \quad \forall\,\mathbf{v}\in\mathring{\mathcal{H}}^1(\Omega_-)^n/\mathring{\mathcal{H}}^1_{\mathrm{div}}(\Omega_-)^n, \tag{4.24}$$

$$\|q\|_{L_2(\Omega_-)} \le C_{\Omega_-}\|\mathrm{grad}\,q\|_{\mathcal{H}^{-1}(\Omega_-)^n} \quad \forall\,q\in L_2(\Omega_-). \tag{4.25}$$

Moreover, the norm value, C_{Ω_-}, may depend on the shape of Ω_- but not on the domain scaling. Particularly, if Ω_- is the exterior of a ball B, then $C_{\Omega_-} = C_{\mathbb{R}^n\setminus\overline{B}}$ does not depend on the ball diameter.

Proof. In view of Proposition 2.1 in [2], the norms of the operators inverse to operators (4.21) and (4.22) are equal. Moreover, the independence of C_{Ω_-} of the domain size follows by considering the simple scaling $\tilde{x} = \lambda x$ in all the operators and norms, e.g., in (4.21) and in the first two lines of (4.23) (cf. [35, Corollary 2.1] for bounded domains). □

4.3.2. *Well-posedness for the exterior Dirichlet problem for the Stokes system.* First, we consider the following problem for the anisotropic Stokes system with homogeneous Dirichlet condition in the exterior Lipschitz domain Ω_- with boundary $\partial\Omega$ (as described in (4.18)),

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u},\pi) = \mathbf{f}, \quad \mathrm{div}\,\mathbf{u} = g & \text{in } \Omega_-,\\ \gamma_-\mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{4.26}$$

for $(\mathbf{u},\pi)\in\mathcal{H}^1(\Omega_-)^n\times L_2(\Omega_-)$, which can be reformulated as the problem

$$\boldsymbol{\mathcal{L}}(\mathbf{u},\pi) = \mathbf{f}, \quad \mathrm{div}\,\mathbf{u} = g \quad \text{in } \Omega_- \tag{4.27}$$

for $(\mathbf{u},\pi)\in\mathring{\mathcal{H}}^1(\Omega_-)^n\times L_2(\Omega_-)$. We show that problem (4.27) has a unique solution whenever $\mathbf{f}\in\mathcal{H}^{-1}(\Omega_-)^n$, $g\in L_2(\Omega_-)$

Let $a_{\mathbb{A};\Omega_-} : \mathring{\mathcal{H}}^1(\Omega_-)^n\times\mathring{\mathcal{H}}^1(\Omega_-)^n\to\mathbb{R}$ and $b_{\Omega_-} : \mathring{\mathcal{H}}^1(\Omega_-)^n\times L_2(\Omega_-)\to\mathbb{R}$ be the bilinear forms

$$a_{\mathbb{A};\Omega_-}(\mathbf{v},\mathbf{w}) := \left\langle A^{\alpha\beta}\partial_\beta\mathbf{v},\partial_\alpha\mathbf{w}\right\rangle_{\Omega_-} = \left\langle a^{\alpha\beta}_{ij}E_{j\beta}(\mathbf{v}),E_{i\alpha}(\mathbf{w})\right\rangle_{\Omega_-} \quad \forall\,\mathbf{v},\mathbf{w}\in\mathring{\mathcal{H}}^1(\Omega_-)^n, \tag{4.28}$$

$$b_{\Omega_-}(\mathbf{v},q) := -\langle\mathrm{div}\,\mathbf{v},q\rangle_{\Omega_-} \quad \forall\,\mathbf{v}\in\mathring{\mathcal{H}}^1(\Omega_-)^n,\ q\in L_2(\Omega_-). \tag{4.29}$$

A standard variational argument (cf. the proof of Theorem 3.4 for the transmission problem) implies that the Dirichlet problem (4.26) is equivalent to the following mixed variational formulation for $\mathbf{F} = -\mathbf{f}$.

Given $\mathbf{F}\in\mathcal{H}^{-1}(\Omega_-)^n$, find $(\mathbf{u},\pi)\in\mathring{\mathcal{H}}^1(\Omega_-)^n\times L_2(\Omega_-)$ such that

$$\begin{cases} a_{\mathbb{A};\Omega_-}(\mathbf{u},\mathbf{w}) + b_{\Omega_-}(\mathbf{w},\pi) = \langle\mathbf{F},\mathbf{w}\rangle_{\Omega_-} & \forall\,\mathbf{w}\in\mathring{\mathcal{H}}^1(\Omega_-)^n,\\ b_{\Omega_-}(\mathbf{u},q) = -\langle g,q\rangle_{\Omega_-} & \forall\,q\in L_2(\Omega_-). \end{cases} \tag{4.30}$$

Then we are in the position to prove the following well-posedness result.

Theorem 4.7. *Let conditions (1.2)-(1.4) be satisfied in an exterior Lipschitz domain $\Omega_-\subset\mathbb{R}^n$, $n\geq 3$, with boundary $\partial\Omega$ not necessarily connected. Then for all $\mathbf{F}\in\mathcal{H}^{-1}(\Omega_-)^n$ and $g\in L_2(\Omega_-)$, the linear variational problem (4.30) and the Dirichlet problem (4.26) with $\mathbf{f}=-\mathbf{F}$ have a unique solution $(\mathbf{u},\pi)\in\mathring{\mathcal{H}}^1(\Omega_-)^n\times L_2(\Omega_-)$, and the following estimates hold*

$$\|\nabla\mathbf{u}\|_{L_2(\Omega_-)^{n\times n}}\leq 2c_{\mathbb{A}}\|\mathbf{F}\|_{\mathcal{H}^{-1}(\Omega_-)^n}+C'_{\Omega_-}\|g\|_{L_2(\Omega_-)}\,, \tag{4.31}$$

$$\|\pi\|_{L_2(\Omega_-)}\leq C'_{\Omega_-}\|\mathbf{F}\|_{\mathcal{H}^{-1}(\Omega_-)^n}+C^*_{\Omega_-}\|g\|_{L_2(\Omega_-)}, \tag{4.32}$$

where the ellipticity constant $c_{\mathbb{A}}$ is defined in (1.4),

$$C'_{\Omega_-}:=C_{\Omega_-}(1+2c_{\mathbb{A}}n^4\|\mathbb{A}\|_{L_\infty(\Omega_-)^n}),\quad C^*_{\Omega_-}:=n^4\|\mathbb{A}\|_{L_\infty(\Omega_-)^n}C_{\Omega_-}C'_{\Omega_-},$$

and the constant C_{Ω_-} is defined in Theorem 4.6.

Proof. In view of (1.2), (1.5) and the Hölder inequality, we obtain that

$$|a_{\mathbb{A};\Omega_-}(\mathbf{v},\mathbf{w})|\leq n^4\|\mathbb{A}\|_{L_\infty(\Omega_-)}\|\nabla\mathbf{v}\|_{L_2(\Omega_-)^{n\times n}}\|\nabla\mathbf{w}\|_{L_2(\Omega_-)^{n\times n}},\quad\forall\,\mathbf{v},\,\mathbf{w}\in\mathring{\mathcal{H}}^1(\Omega_-)^n. \tag{4.33}$$

Due to the ellipticity condition (1.4),

$$\begin{aligned}|\mathbf{u}|^2_{\mathcal{H}^1(\Omega_-)^n}&\leq\|\nabla\mathbf{u}\|^2_{L_2(\Omega_-)^{n\times n}}\\&\leq 2\|\mathbb{E}(\mathbf{u})\|^2_{L_2(\Omega_-)^{n\times n}}\\&\leq 2c_{\mathbb{A}}a_{\mathbb{A};\Omega_-}(\mathbf{u},\mathbf{u})\quad\forall\,\mathbf{u}\in\mathring{\mathcal{H}}^1_{\rm div}(\Omega_-)^n.\end{aligned} \tag{4.34}$$

Note that the second inequality in (4.34) is a version of the Korn type inequality in the weighted Sobolev space $\mathring{\mathcal{H}}^1(\Omega_-)^n$ (cf. [45, Eq.(2.2)] for $n=3$). It follows, e.g., from the proof of [37, Theorem 10.1], where it is shown that the Korn inequality is valid for any function in $\mathcal{D}(\mathbb{R}^n)^n$. Then the density of the space $\mathcal{D}(\Omega_-)^n\subset\mathcal{D}(\mathbb{R}^n)^n$ in $\mathring{\mathcal{H}}^1(\Omega_-)^n$ implies that it is valid also in $\mathring{\mathcal{H}}^1(\Omega_-)^n$.

Moreover, arguments similar to those for inequality (4.33) imply that the bilinear form $b_{\Omega_-}(\cdot,\cdot):\mathring{\mathcal{H}}^1(\Omega_-)^n\times L_2(\Omega_-)\to\mathbb{R}$ given by (4.29) is also bounded. Then Theorems 4.5 and 4.6 and Lemma 2.5 imply that it satisfies the inf-sup condition

$$\inf_{q\in L_2(\Omega_-)\setminus\{0\}}\ \sup_{\mathbf{w}\in\mathring{\mathcal{H}}^1(\Omega_-)^n\setminus\{\mathbf{0}\}}\frac{b_{\Omega_-}(\mathbf{w},q)}{\|\mathbf{w}\|_{\mathring{\mathcal{H}}^1(\Omega_-)^n}\|q\|_{L_2(\Omega_-)}}\geq C^{-1}_{\Omega_-} \tag{4.35}$$

(cf. [25, Theorem 3.3]).

Then due to estimates (4.33), (4.34), (4.35), Theorem 2.4 with $X=\mathring{\mathcal{H}}^1(\Omega_-)^n$, $V=\mathring{\mathcal{H}}^1_{\rm div}(\Omega_-)^n$ and $\mathcal{M}=L_2(\Omega_-)$ implies that for any $(\mathbf{f},g)\in H^{-1}(\Omega_-)^n\times L_2(\Omega_-)$, there exists a unique solution $(\mathbf{u},\pi)\in\mathring{\mathcal{H}}^1(\Omega_-)^n\times L_2(\Omega_-)$ of the variational problem (4.30) and hence of problems (4.27) and (4.26), while inequalities (4.31) and (4.32) hold. □

4.4. Exterior Dirichlet-transmission problem for the Stokes system.

Assumption 4.8. Let $n\geq 3$ and Ω_- be an exterior Lipschitz domain in $\mathbb{R}^n$ with compact (not necessarily connected) boundary $\partial\Omega$. Let $\Omega^0\subset\mathbb{R}^n$ be a bounded Lipschitz set such that $\overline{\Omega^0}\subset\Omega_-$. Let $\Omega^0_+:=\Omega^0$ and $\Omega^0_-:=\Omega_-\setminus\overline{\Omega^0_+}$. Thus, Ω^0_- is an unbounded Lipschitz set, and the boundary $\partial\Omega^0$ of Ω^0 is also the interface between Ω^0_+ and Ω^0_- (see Figure 1 where the outer boundary should be dropped).

Let us introduce the following spaces

$$\mathcal{H}^1(\Omega^0_-;\partial\Omega)^n := \left\{\mathbf{v}\in\mathcal{H}^1(\Omega^0_-)^n : \gamma_+\mathbf{v}=\mathbf{0} \text{ on } \partial\Omega\right\}, \tag{4.36}$$

$$\widetilde{\mathcal{H}}^{-1}(\Omega^0_-;\partial\Omega^0)^n := \left\{\boldsymbol{\varphi}\in\mathcal{H}^{-1}(\Omega)^n : \boldsymbol{\varphi}=\mathbf{0} \text{ on } \Omega^0_+\right\}, \tag{4.37}$$

$$\mathcal{X}_{\Omega^0_+,\Omega^0_-} := H^1(\Omega^0_+)^n\times L_2(\Omega^0_+)\times\mathcal{H}^1(\Omega^0_-)^n\times L_2(\Omega^0_-), \tag{4.38}$$

$$\mathcal{Y}^D_{\Omega^0_+,\Omega^0_-} := \widetilde{H}^{-1}(\Omega^0_+)^n\times L_2(\Omega^0_+)\times\widetilde{H}^{-1}(\Omega^0_-;\partial\Omega^0)^n\times L_2(\Omega^0_-)\times H^{-\frac{1}{2}}(\partial\Omega^0)^n. \tag{4.39}$$

The space $\widetilde{\mathcal{H}}^{-1}(\Omega^0_-;\partial\Omega^0)^n$ can be identified with the dual of $\mathcal{H}^1(\Omega^0_-;\partial\Omega)^n$ (cf., e.g., arguments of Theorems 3.29 and 3.30 in [37]).

Next, we consider the exterior Dirichlet-transmission problem for the Stokes system

$$\begin{aligned}
&\boldsymbol{\mathcal{L}}(\mathbf{u}_+,\pi_+)=\tilde{\mathbf{f}}_+|_{\Omega^0_+}, \quad \operatorname{div}\mathbf{u}_+=g_+ && \text{in } \Omega^0_+, && (4.40)\\
&\boldsymbol{\mathcal{L}}(\mathbf{u}_-,\pi_-)=\tilde{\mathbf{f}}_-|_{\Omega^0_-}, \quad \operatorname{div}\mathbf{u}_-=g_- && \text{in } \Omega^0_-, && (4.41)\\
&\gamma_+\mathbf{u}_+-\gamma_-\mathbf{u}_-=\mathbf{0} && \text{on } \partial\Omega^0, && (4.42)\\
&\mathbf{t}^+(\mathbf{u}_+,\pi_+;\tilde{\mathbf{f}}_+)-\mathbf{t}^-(\mathbf{u}_-,\pi_-;\tilde{\mathbf{f}}_-)=\boldsymbol{\psi} && \text{on } \partial\Omega^0, && (4.43)\\
&\gamma_-\mathbf{u}_-=\mathbf{0} && \text{on } \partial\Omega, && (4.44)
\end{aligned}$$

with data $(\tilde{\mathbf{f}}_+,g_+,\tilde{\mathbf{f}}_-,g_-,\boldsymbol{\psi})\in\mathcal{Y}^D_{\Omega^0_+,\Omega^0_-}$ and unknown $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathcal{X}_{\Omega^0_+,\Omega^0_-}$.

Following arguments similar to the ones for Theorem 3.4, which refers to the transmission problem in a bounded domain, one can prove the following assertion about the equivalence of the Dirichlet-transmission problem (4.40)-(4.44) and the variational problem (4.30), and hence the well-posedness of the exterior Dirichlet-transmission problem (4.40)-(4.44) for the anisotropic Stokes system.

Theorem 4.9. *Let Assumption 4.8 hold. Let conditions (1.2)-(1.4) hold in Ω_-. Let $\left(\tilde{\mathbf{f}}_+,g_+,\tilde{\mathbf{f}}_-,g_-,\boldsymbol{\psi}\right)\in\mathcal{Y}^D_{\Omega^0_+,\Omega^0_-}$ and $(\mathbf{u},\pi)\in\mathring{\mathcal{H}}^1(\Omega)^n\times L_2(\Omega)$ be the solution of the variational problem (4.30) provided by Theorem 4.7 for $\mathbf{F}=-(\tilde{\mathbf{f}}_++\tilde{\mathbf{f}}_-)+\gamma^*\boldsymbol{\psi}\in\mathcal{H}^{-1}(\Omega)^n$ and $g=\mathring{E}_{\Omega^0_+}g_++\mathring{E}_{\Omega^0_-}g_-$, where $\gamma^*:H^{-\frac{1}{2}}(\partial\Omega^0)^n\to\mathcal{H}^{-1}(\Omega)^n$ is the adjoint of the trace operator $\gamma:\mathcal{H}^1(\Omega)^n\to H^{\frac{1}{2}}(\partial\Omega^0)^n$. Then the linear exterior Dirichlet-transmission problem (4.40)-(4.44) has a unique solution in $\mathcal{X}_{\Omega^0_+,\Omega^0_-}$ given by the relations $\mathbf{u}_+=\mathbf{u}|_{\Omega^0_+}$, $\mathbf{u}_-=\mathbf{u}|_{\Omega^0_-}$, $\pi_+=\pi|_{\Omega^0_+}$, $\pi_-=\pi|_{\Omega^0_-}$, and estimates (4.31), (4.32) hold.*

5. **Dirichlet and transmission problems for the anisotropic Navier-Stokes system with general data in a bounded Lipschitz domain.** In this section we combine a well-posedness result for the Stokes system with the Leray-Schauder fixed point Theorem and show existence of solutions for the nonlinear Dirichlet and Dirichlet-transmission problems for the anisotropic Navier-Stokes system in a bounded Lipschitz composite domain (see Theorem 5.2). Further, in Section 6, we will use these results to construct solutions of the transmission and Dirichlet problems in the sense of distributions in $\mathbb{R}^3$ and exterior domains for the anisotropic Navier-Stokes system with general (including large) data.

We need the following version of the Leray-Schauder fixed point theorem (see, e.g., [23, Theorem 11.3]).

Theorem 5.1. *Let X be a Banach space. Let $T:X\to X$ be a continuous and compact operator. If there exists a constant $K>0$ such that $\|x\|_X\le K$ for each* *pair* $(x,\chi)\in X\times[0,1]$ *satisfying* $x=\chi Tx$*, then the operator* T *has a fixed point* x_0 *(and* $\|x_0\|_X\le K$*).*

5.1. **Dirichlet problem for the Navier-Stokes system in a bounded domain.** Let us consider the following mixed nonlinear variational problem in a bounded Lipschitz domain Ω of $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ (not necessarily connected).

Given $\mathbf{F}\in H^{-1}(\Omega)^n$*, find* $(\mathbf{u},\pi)\in \mathring{H}^1(\Omega)^3\times L_2(\Omega)/\mathbb{R}$ *such that*

$$\begin{cases} a_{\mathbb{A};\Omega}(\mathbf{u},\mathbf{v})+b_\Omega(\mathbf{v},\pi)=\langle\mathbf{F},\mathbf{v}\rangle_\Omega-\langle(\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}\rangle_\Omega & \forall\,\mathbf{v}\in\mathring{H}^1(\Omega)^3,\\ b_\Omega(\mathbf{u},q)=0 & \forall\,q\in L_2(\Omega)/\mathbb{R}\,, \end{cases}\tag{5.1}$$

where the bilinear forms $a_{\mathbb{A};\Omega}:\mathring{H}^1(\Omega)^3\times\mathring{H}^1(\Omega)^3\to\mathbb{R}$ *and* $b_\Omega:\mathring{H}^1(\Omega)^3\times L_2(\Omega)/\mathbb{R}\to\mathbb{R}$ *are defined in* (3.16) *and* (3.17)*.*

Let $\boldsymbol{\mathcal{L}}$ be the Stokes operator defined in (1.9). Then a standard variational argument (as in the proof of Theorem 3.4 for the linear transmission problem) implies that the variational problem (5.1) is equivalent to the following Dirichlet problem with $\mathbf{F}=-\mathbf{f}$.

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u},\pi)=\mathbf{f}+(\mathbf{u}\cdot\nabla)\mathbf{u}\,,\quad \operatorname{div}\mathbf{u}=0 & \text{in } \Omega,\\ \gamma\mathbf{u}=\mathbf{0} & \text{on } \partial\Omega, \end{cases}\tag{5.2}$$

with the unknown fields $(\mathbf{u},\pi)\in H^1(\Omega)^3\times L_2(\Omega)/\mathbb{R}$.

Now we can prove the existence result for the variational problem (5.1), and hence for the Dirichlet problem (5.2) corresponding to the anisotropic $L_\infty-$coefficient Navier-Stokes equation in a bounded Lipschitz domain (see also [46, Proposition 1.1] in the constant coefficient case).

Theorem 5.2. *Assume that* Ω *is a bounded Lipschitz domain in* $\mathbb{R}^3$*. Let conditions* (1.2)*-*(1.4) *hold in* Ω*. Then for any* $\mathbf{F}\in H^{-1}(\Omega)^3$ *there exists a pair* $(\mathbf{u},\pi)\in\mathring{H}^1(\Omega)^3\times L_2(\Omega)/\mathbb{R}$ *which satisfies the nonlinear variational problem* (5.1)*, as well as the nonlinear Dirichlet problem* (5.2) *with* $\mathbf{f}=-\mathbf{F}$*. Moreover, the estimates*

$$\|\nabla\mathbf{u}\|_{L_2(\Omega)^{n\times n}}\le 2c_{\mathbb{A}}\|\mathbf{F}\|_{H^{-1}(\Omega)^3}\tag{5.3}$$

$$\|\pi\|_{L_2(\Omega)/\mathbb{R}}\le C'_\Omega\|\mathbf{F}\|_{H^{-1}(\Omega)^3}+C''_\Omega|\Omega|^{1/6}\|\mathbf{F}\|^2_{H^{-1}(\Omega)^3}\tag{5.4}$$

hold, where $c_{\mathbb{A}}$ *is the constant defined in* (1.4)*,*

$$C'_\Omega:=C_\Omega(1+2c_{\mathbb{A}}3^4\|\mathbb{A}\|_{L_\infty(\Omega)^3}),\quad C''_\Omega:=\frac{16}{3}C_\Omega c_{\mathbb{A}}^2\,,\tag{5.5}$$

the constant C_Ω *is as in Theorem* 3.1 *and does not depend on the diameter of* Ω*. Moreover,* $|\Omega|=\int_\Omega dx$ *and the norm* $\|\cdot\|_{H^{-1}(\Omega)^3}$ *is defined in* (3.3)*.*

Proof. For any $\mathbf{F}\in H^{-1}(\Omega)^3$ and an arbitrary, fixed element $\mathbf{w}\in\mathring{H}^1_{\rm div}(\Omega)^3$, we define the functional

$$\mathbf{F_w}:\mathring{H}^1(\Omega)^3\to\mathbb{R},\ \langle\mathbf{F_w},\mathbf{v}\rangle_\Omega:=\langle\mathbf{F},\mathbf{v}\rangle_\Omega-\langle(\mathbf{w}\cdot\nabla)\mathbf{w},\mathbf{v}\rangle_\Omega\ \ \forall\,\mathbf{v}\in\mathring{H}^1(\Omega)^3.\tag{5.6}$$

Inequality (B.2) implies that $\mathbf{F_w}$ is well-defined, linear and continuous (cf. [46, Proposition 1.1]), and hence $\mathbf{F_w}\in H^{-1}(\Omega)^3$.

By Theorem 3.2, there exists a unique solution $(\mathbf{u_w},\pi_\mathbf{w})\in\mathring{H}^1(\Omega)^3\times L_2(\Omega)/\mathbb{R}$ of the variational problem

$$\begin{cases} a_{\mathbb{A};\Omega}(\mathbf{u_w},\mathbf{v})+b_\Omega(\mathbf{v},\pi_\mathbf{w})=\langle\mathbf{F_w},\mathbf{v}\rangle_\Omega & \forall\,\mathbf{v}\in\mathring{H}^1(\Omega)^3,\\ b_\Omega(\mathbf{u_w},q)=0 & \forall\,q\in L_2(\Omega)/\mathbb{R} \end{cases}\tag{5.7}$$

and, in view of estimate (3.18), $\mathbf{u_w}$ satisfies the inequality

$$\|\nabla \mathbf{u_w}\|_{L_2(\Omega)^{3\times 3}} \le 2c_{\mathbb{A}} \|\mathbf{F_w}\|_{H^{-1}(\Omega)^3}. \tag{5.8}$$

The second equation in (5.7) yields the membership relation $\mathbf{u_w} \in \mathring{H}^1_{\rm div}(\Omega)^3$, and the first equation in (5.7) implies that $\mathbf{u_w}$ satisfies the equation

$$\left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u_w}), E_{i\alpha}(\mathbf{v})\right\rangle_\Omega = \langle \mathbf{F_w}, \mathbf{v}\rangle_\Omega \quad \forall\ \mathbf{v} \in \mathring{H}^1_{\rm div}(\Omega)^3 . \tag{5.9}$$

Note that the coercivity condition (3.21) yields that $\mathbf{u_w}$ is the unique solution of equation (5.9).

Consequently, we can define the operator

$$\mathbf{U} : \mathring{H}^1_{\rm div}(\Omega)^3 \to \mathring{H}^1_{\rm div}(\Omega)^3,\ \mathbf{U}(\mathbf{w}) := \mathbf{u_w}, \tag{5.10}$$

which associates to any $\mathbf{w} \in \mathring{H}^1_{\rm div}(\Omega)^3$ the unique solution $\mathbf{u_w} \in \mathring{H}^1_{\rm div}(\Omega)^3$ of the variational problem (5.7), and, thus, of the variational equation (5.9).

Next we show that the nonlinear operator $\mathbf{U}$ has a fixed point $\mathbf{u} \in \mathring{H}^1_{\rm div}(\Omega)^3$ ($\mathbf{U}(\mathbf{u}) = \mathbf{u}$), and accordingly that $\mathbf{u}$ will be a weak solution of the nonlinear Dirichlet problem (5.2). We intend to use the variant of the Leray-Schauder fixed point Theorem given in Theorem 5.1.

First, we show that $\mathbf{U}$ is continuous. Let $\mathbf{w}, \mathbf{w}_0 \in \mathring{H}^1_{\rm div}(\Omega)^3$. Then by (5.8), (5.6) and (B.2) there exists a constant $C_1 = C_1(\Omega) > 0$ such that

$$\begin{aligned}\|\nabla(\mathbf{U}(\mathbf{w}) - \mathbf{U}(\mathbf{w}_0))\|_{L_2(\Omega)^{n\times n}} &\le 2c_{\mathbb{A}}\|\mathbf{F_w} - \mathbf{F}_{\mathbf{w}_0}\|_{H^{-1}(\Omega)^3}\\ &= 2c_{\mathbb{A}}\|(\mathbf{w}\cdot\nabla)\mathbf{w} - (\mathbf{w}_0\cdot\nabla)\mathbf{w}_0\|_{H^{-1}(\Omega)^3} \qquad (5.11)\\ &= 2c_{\mathbb{A}}\|((\mathbf{w}-\mathbf{w}_0)\cdot\nabla)\mathbf{w} + (\mathbf{w}_0\cdot\nabla)(\mathbf{w}-\mathbf{w}_0)\|_{H^{-1}(\Omega)^3}\\ &\le 2C_1c_{\mathbb{A}}\|\mathbf{w}-\mathbf{w}_0\|_{H^1(\Omega)^3}\left(\|\mathbf{w}\|_{H^1(\Omega)^3} + \|\mathbf{w}_0\|_{H^1(\Omega)^3}\right).\end{aligned}$$

Therefore, the operator $\mathbf{U}$ defined by (5.10) is continuous.

Let us now show that the operator $\mathbf{U}$ is compact. To this end, assume that $\{(\mathbf{w}_k)\}_{k\in\mathbb{N}}$ is a bounded sequence in the space $\mathring{H}^1_{\rm div}(\Omega)^3$. Thus, there exists a constant $M > 0$ such that $\|\nabla\mathbf{w}_k\|_{L_2(\Omega)^{3\times 3}} \le M$, for any $k \in \mathbb{N}$. We claim that the sequence $\{\mathbf{U}(\mathbf{w}_k)\}_{k\in\mathbb{N}}$ contains a convergent subsequence in $\mathring{H}^1_{\rm div}(\Omega)^3$.

Indeed, the Rellich compactness theorem (see, e.g., [1, Theorem 6.3, Part I]) implies the compactness of the embedding of the space $\mathring{H}^1_{\rm div}(\Omega)^3$ into the space $L_3(\Omega)^3$. This, in turn, implies that there exists a subsequence of $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$, labeled as the sequence, which converges in $L_3(\Omega)^3$, and, thus, is a Cauchy sequence in $L_3(\Omega)^3$. Then we show that the corresponding subsequence $\{\mathbf{u}_k\}_{k\in\mathbb{N}}$, $\mathbf{u}_k = \mathbf{U}(\mathbf{w}_k)$, is a Cauchy sequence in $\mathring{H}^1_{\rm div}(\Omega)^3$.

By (5.8), (5.6), (B.4) and (B.7), we obtain similar to (5.11),

$$\begin{aligned}\|\nabla(\mathbf{U}(\mathbf{w}_k) - \mathbf{U}(\mathbf{w}_\ell))\|_{L_2(\Omega)^{3\times 3}} &\le 2c_{\mathbb{A}}\|\mathbf{F}_{\mathbf{w}_k} - \mathbf{F}_{\mathbf{w}_\ell}\|_{H^{-1}(\Omega)^3}\\ &\le 2c_{\mathbb{A}}\|(\mathbf{w}_k\cdot\nabla)\mathbf{w}_k - (\mathbf{w}_\ell\cdot\nabla)\mathbf{w}_\ell\|_{H^{-1}(\Omega)^3}\\ &= 2c_{\mathbb{A}}\|((\mathbf{w}_k-\mathbf{w}_\ell)\cdot\nabla)\mathbf{w}_k + (\mathbf{w}_\ell\cdot\nabla)(\mathbf{w}_k-\mathbf{w}_\ell)\|_{H^{-1}(\Omega)^3}\\ &\le 2c_{\mathbb{A}}\|\mathbf{w}_k-\mathbf{w}_\ell\|_{L_3(\Omega)^3}\left(C_2\|\mathbf{w}_k\|_{H^1(\Omega)^3} + C_3\|\mathbf{w}_\ell\|_{H^1(\Omega)^3}\right)\\ &\le C_*\|\mathbf{w}_k-\mathbf{w}_\ell\|_{L_3(\Omega)^3}, \qquad (5.12)\end{aligned}$$

with some constants $C_2, C_3 > 0$ depending on Ω, and $C_* := 2c_{\mathbb{A}}(C_2 + C_3)M$. Therefore, the convergence of $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$ in $L_3(\Omega)^3$ implies that $\{\mathbf{U}(\mathbf{w}_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence and, thus, it converges in the space $\mathring{H}^1_{\rm div}(\Omega)^3$. Consequently, the operator $\mathbf{U}$ defined by (5.10) is compact.

In order to apply the statement of Theorem 5.1 to the operator $\mathbf{U}$, it remains to show that the following set is bounded

$$\mathcal{A} := \left\{\mathbf{w} \in \mathring{H}^1_{\rm div}(\Omega)^3 : \mathbf{w} = \lambda \mathbf{U}(\mathbf{w}),\ 0 \le \lambda \le 1\right\}, \tag{5.13}$$

i.e., there exists a constant $K > 0$ such that for each pair $(\mathbf{w}, \lambda) \in \mathring{H}^1_{\rm div}(\Omega)^3 \times [0, 1]$ satisfying $\mathbf{w} = \lambda \mathbf{U}(\mathbf{w})$, we have $\|\nabla \mathbf{w}\|_{L_2(\Omega)^{3\times 3}} \le K$. (We use the property that $\|\nabla(\cdot)\|_{L_2(\Omega)^{3\times 3}}$ is a norm on the space $\mathring{H}^1_{\rm div}(\Omega)^3$.)

Let $\lambda \in (0, 1]$ and $\mathbf{w} \in \mathring{H}^1_{\rm div}(\Omega)^3$ be such that $\mathbf{w} = \lambda \mathbf{U}(\mathbf{w})$. Thus, $\frac{1}{\lambda}\mathbf{w} = \mathbf{U}(\mathbf{w})$. Then taking $\mathbf{u}_{\mathbf{w}} = \frac{1}{\lambda}\mathbf{w}$ in (5.9), we obtain that

$$\begin{aligned}\left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{v})\right\rangle_\Omega &= \lambda \langle \mathbf{F}_{\mathbf{w}}, \mathbf{v}\rangle_\Omega \\ &= \lambda\langle \mathbf{F}, \mathbf{v}\rangle_\Omega - \lambda \left\langle (\mathbf{w}\cdot\nabla)\mathbf{w}, \mathbf{v}\right\rangle_\Omega \ \ \forall\, \mathbf{v} \in \mathring{H}^1_{\rm div}(\Omega)^3 .\end{aligned} \tag{5.14}$$

Let us take $\mathbf{v} = \mathbf{w}$ in (5.14). Then by relation (B.9) we obtain the equality

$$\left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{w})\right\rangle_\Omega = \lambda\langle \mathbf{F}, \mathbf{w}\rangle_\Omega . \tag{5.15}$$

Finally, as in (3.21), the first Korn inequality, condition (1.4), definition (3.3) of the norm $\|\cdot\|_{H^{-1}(\Omega)^3}$ of the space $H^{-1}(\Omega)^3$, and equality (5.15) imply that

$$\begin{aligned}\frac{1}{2}c_{\mathbb{A}}^{-1}\|\nabla \mathbf{w}\|^2_{L_2(\Omega)^{3\times 3}} &\le c_{\mathbb{A}}^{-1}\|\mathbb{E}(\mathbf{w})\|^2_{L_2(\Omega)^{3\times 3}} \\ &\le \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{w})\right\rangle_\Omega \\ &\le \lambda \|\mathbf{F}\|_{H^{-1}(\Omega)^3}\|\nabla \mathbf{w}\|_{L_2(\Omega)^{3\times 3}}.\end{aligned}$$

Therefore, for any $\lambda \in (0, 1]$, we have the inequality

$$\|\nabla \mathbf{w}\|_{L_2(\Omega)^{3\times 3}} \le 2\lambda c_{\mathbb{A}}\|\mathbf{F}\|_{H^{-1}(\Omega)^3} \le 2c_{\mathbb{A}}\|\mathbf{F}\|_{H^{-1}(\Omega)^3}, \tag{5.16}$$

and thus the set $\mathcal{A}$ is bounded (with respect to the norm $\|\nabla(\cdot)\|_{L_2(\Omega)}$ on $\mathring{H}^1(\Omega)^3$).

Consequently, the operator $\mathbf{U}$ in (5.10) satisfies the assumption of Theorem 5.1 (with $X = \mathring{H}^1_{\rm div}(\Omega)^3$ and $K = 2c_{\mathbb{A}}\|\mathbf{F}\|_{H^{-1}(\Omega)^3}$), and then there exists $\mathbf{u} \in \mathring{H}^1_{\rm div}(\Omega)^3$ such that $\mathbf{U}(\mathbf{u}) = \mathbf{u}$. Moreover, the couple $(\mathbf{u}, \pi)$ with some $\pi = \pi_{\mathbf{u}} \in L_2(\Omega)/\mathbb{R}$ provided by Theorem 3.2 satisfies the variational problem (5.7) (with $\mathbf{F}_{\mathbf{w}} = \mathbf{F}_{\mathbf{u}}$ given by (5.6)) and, thus, is a solution of the nonlinear Dirichlet problem (5.2), as asserted. In addition, estimate (5.16) holds with $\mathbf{w} = \mathbf{u}$ and implies estimate (5.3).

To show estimate (5.4) we proceed as follows. The first equation in (5.2) yields

$$\nabla\pi = \boldsymbol{\mathcal{F}} = \mathbf{F} + {\rm div}\,(\mathbb{A}\mathbb{E}(\mathbf{u})) - (\mathbf{u}\cdot\nabla)\mathbf{u} \text{ in } \Omega. \tag{5.17}$$

We already remarked that $\mathbf{F}_{\mathbf{u}} = \mathbf{F} - (\mathbf{u}\cdot\nabla)\mathbf{u} \in H^{-1}(\Omega)^3$, and since $\mathbf{u} \in \mathring{H}^1_{\rm div}(\Omega)^3$, we obtain that $\boldsymbol{\mathcal{F}} \in H^{-1}(\Omega)^3$. Moreover, (5.9) implies that $\boldsymbol{\mathcal{F}}$ is bi-orthogonal to $\mathring{H}^1_{\rm div}(\Omega)^3$ and hence $\boldsymbol{\mathcal{F}} \in H^{-1}(\Omega)^3 \perp \mathring{H}^1_{\rm div}(\Omega)^3$. Then by inequality (3.13) in Theorem 3.1 along with equation (5.17), we obtain that

$$\|\pi\|_{L_2(\Omega)/\mathbb{R}} \le C_\Omega\big(\|\mathbf{F}\|_{H^{-1}(\Omega)^3} + \|{\rm div}\,(\mathbb{A}\mathbb{E}(\mathbf{u}))\,\|_{H^{-1}(\Omega)^3} + \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{H^{-1}(\Omega)^3}\big), \tag{5.18}$$

By (1.5) and (5.3) we obtain that

$$\begin{aligned}\|\mathrm{div}\,(\mathbb{A}\mathbb{E}(\mathbf{u}))\,\|_{H^{-1}(\Omega)^3} &= \sup_{\mathbf{\Psi}\in\mathring{H}^1(\Omega)^3,\,\|\nabla\mathbf{\Psi}\|_{L_2(\Omega)^3}=1}\Big|\,\langle \mathrm{div}\,(\mathbb{A}\mathbb{E}(\mathbf{u}))\,,\mathbf{\Psi}\rangle_\Omega\,\Big|\\ &= \sup_{\mathbf{\Psi}\in\mathring{H}^1(\Omega)^3,\,\|\nabla\mathbf{\Psi}\|_{L_2(\Omega)^3}=1}\Big|\,\langle \mathbb{A}\mathbb{E}(\mathbf{u}),\nabla\mathbf{\Psi}\rangle_\Omega\,\Big|\\ &\le \|\mathbb{A}\mathbb{E}(\mathbf{u})\|_{L_2(\Omega)^{3\times3}} \le 3^4\|\mathbb{A}\|_{L_\infty(\Omega)}\|\nabla\,\mathbf{u}\|_{L_2(\Omega)^{3\times3}}\\ &\le 2c_{\mathbb{A}}3^4\|\mathbb{A}\|_{L_\infty(\Omega)}\|\mathbf{F}\|_{H^{-1}(\Omega)^3},\end{aligned} \tag{5.19}$$

From (B.13) and (5.16) we have

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{H^{-1}(\Omega)^3} \le \frac{4}{3}|\Omega|^{1/6}\|\nabla\mathbf{u}\|^2_{L_2(\Omega)^{3\times3}} \le \frac{16}{3}|\Omega|^{1/6}c^2_{\mathbb{A}}\|\mathbf{F}\|^2_{H^{-1}(\Omega)^3}\,. \tag{5.20}$$

By substituting (5.19) and (5.20) in (5.18), we obtain (5.4), as asserted. □

5.2. **Dirichlet-transmission problem for the Navier-Stokes system in a bounded composite domain.** Let the geometry be as described in Assumption 3.3. Let us consider the Dirichlet-transmission problem for the Navier-Stokes system

$$\begin{cases}\boldsymbol{\mathcal{L}}(\mathbf{u}_+,\pi_+)=\tilde{\mathbf{f}}_+|_{\Omega^0_+}+(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+\,,\ \ \mathrm{div}\,\mathbf{u}_+=0 & \text{in } \Omega^0_+,\\ \boldsymbol{\mathcal{L}}(\mathbf{u}_-,\pi_-)=\tilde{\mathbf{f}}_-|_{\Omega^0_-}+(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-\,,\ \ \mathrm{div}\,\mathbf{u}_-=0 & \text{in } \Omega^0_-,\\ \gamma_+\mathbf{u}_+-\gamma_-\mathbf{u}_-=\mathbf{0} & \text{on } \partial\Omega^0,\\ \mathbf{t}^+\Big(\mathbf{u}_+,\pi_+;\tilde{\mathbf{f}}_++\mathring{E}_{\Omega^0_+}\left((\mathbf{u}_+\cdot\nabla)\mathbf{u}_+\right)\Big) & \\ \quad -\mathbf{t}^-\Big(\mathbf{u}_-,\pi_-;\tilde{\mathbf{f}}_-+\mathring{E}_{\Omega^0_-}\left((\mathbf{u}_-\cdot\nabla)\mathbf{u}_-\right)\Big)=\boldsymbol{\psi} & \text{on } \partial\Omega^0,\\ \gamma_-\mathbf{u}_-=\mathbf{0} & \text{on } \partial\Omega\end{cases} \tag{5.21}$$

with given data $\left(\tilde{\mathbf{f}}_+,\tilde{\mathbf{f}}_-,\boldsymbol{\psi}\right)\in\widetilde{H}^{-1}(\Omega^0_+)^3\times\widetilde{H}^{-1}(\Omega^0_-;\partial\Omega^0)^3\times H^{-\frac12}(\partial\Omega^0)^3$ and unknown $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathfrak{X}_{\Omega^0_+,\Omega^0_-}$, where the spaces $\widetilde{H}^{-1}(\Omega^0_-;\partial\Omega^0)^3$ and $\mathfrak{X}_{\Omega^0_+,\Omega^0_-}$ are defined in (3.24) and (3.25). Here Ω, Ω^0_+ and Ω^0_-, $\partial\Omega$ and $\partial\Omega^0$ are the Lipschitz sets and boundaries satisfying Assumption 3.3, $\boldsymbol{\mathcal{L}}$ is the Stokes operator defined in (1.9), and $\mathring{E}_{\Omega^0_+}$ and $\mathring{E}_{\Omega^0_-}$ are the operators of extension by zero outside Ω^0_+ and Ω^0_-, respectively.

A variational argument similar to the one in the proof of Theorem 3.4 for the linear transmission problem and based on Lemma A.1 shows that the nonlinear transmission problem (5.21) is equivalent to the mixed variational formulation (5.1). This result and Theorem 5.2 bring us to the following existence result for the nonlinear Dirichlet-transmission problem (5.21).

Theorem 5.3. *Let the geometry be as in Assumption 3.3. Let conditions (1.2)-(1.4) hold in Ω. Given $\left(\tilde{\mathbf{f}}_+,\tilde{\mathbf{f}}_-,\boldsymbol{\psi}\right)$ in $\widetilde{H}^{-1}(\Omega^0_+)^3\times\widetilde{H}^{-1}(\Omega^0_-;\partial\Omega^0)^3\times H^{-\frac12}(\partial\Omega^0)^3$ let $(\mathbf{u},\pi)\in\mathring{H}^1(\Omega)^3\times L_2(\Omega)/\mathbb{R}$ be the solution of variational problem (5.1) provided by Theorem 5.2 for the data $(\mathbf{F},g)$ with $\mathbf{F}=-(\tilde{\mathbf{f}}_++\tilde{\mathbf{f}}_-)+\gamma^*\boldsymbol{\psi}$. Then there exists a solution $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathfrak{X}_{\Omega^0_+,\Omega^0_-}$ of the nonlinear Dirichlet-transmission problem (5.21) given by the relations $\mathbf{u}_+=\mathbf{u}|_{\Omega^0_+}$, $\mathbf{u}_-=\mathbf{u}|_{\Omega^0_-}$, $\pi_+=\pi|_{\Omega^0_+}$, $\pi_-=\pi|_{\Omega^0_-}$, and estimates (5.3), (5.4) hold.*

5.3. **Uniqueness for the Navier-Stokes problems with small data in a bounded Lipschitz domain.** Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain. Then the space $\mathring{H}^1(\Omega)^3$ is continuously embedded in $L_4(\Omega)^3$. Moreover, the semi-norm

$\|\nabla \mathbf{v}\|_{L_2(\Omega)^{3\times 3}}$ is a norm on $\mathring{H}^1(\Omega)^3$, which is equivalent to the norm $\|\mathbf{v}\|_{H^1(\Omega)^3}$ given by (2.2). Then, in view of (B.11) and (B.12), there exists a positive constant $\mathring{C}_\Omega$ independent of $\mathbf{v}$ such that

$$\|\mathbf{v}\|_{L_4(\Omega)^3} \leq \mathring{C}_\Omega \|\nabla \mathbf{v}\|_{L_2(\Omega)^{3\times 3}} \quad \forall\, \mathbf{v} \in \mathring{H}^1(\Omega)^3. \tag{5.22}$$

This inequality and an additional constraint to the given data of the nonlinear problem (5.21) imply the following uniqueness result (see also [46, Lemma 3.1] and [43, Corollary 1] in the isotropic case (1.10) with $\mu = 1$, $\lambda = 0$ and homogeneous Dirichlet condition, and [31, Theorem 4.2] for a pseudostress approach).

Theorem 5.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Let conditions (1.2)-(1.4) hold on Ω. Let $\mathbf{F} \in H^{-1}(\Omega)^3$ be such that*

$$4c_{\mathbb{A}}^2 \mathring{C}_\Omega^2 \|\mathbf{F}\|_{H^{-1}(\Omega)^3} < 1, \tag{5.23}$$

with the constants $c_{\mathbb{A}}$ and $\mathring{C}_\Omega$ given in (1.4) and (5.22), respectively. Then the nonlinear Dirichlet problem (5.2) has a unique solution $(\mathbf{u}, \pi) \in H^1(\Omega)^3 \times L_2(\Omega)/\mathbb{R}$.

In addition, if Ω is a composite domain satisfying Assumption 3.3 and the given data $\left(\tilde{\mathbf{f}}_+, \tilde{\mathbf{f}}_-, \boldsymbol{\psi}\right)$ belong to the space $\widetilde{H}^{-1}(\Omega_+^0)^3 \times \widetilde{H}^{-1}(\Omega_-^0; \partial\Omega^0)^3 \times H^{-\frac{1}{2}}(\partial\Omega^0)^3$, and $\mathbf{F} = -(\tilde{\mathbf{f}}_+ + \tilde{\mathbf{f}}_-) + \gamma^* \boldsymbol{\psi}$, then the nonlinear Dirichlet-transmission problem (5.21) has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)$ in the space $\mathfrak{X}_{\Omega_+^0, \Omega_-^0}$ defined in (3.25).

Proof. Assume that the variational problem (5.1) has two solutions $(\mathbf{u}_1, \pi_1)$ and $(\mathbf{u}_2, \pi_2)$ in the space $\mathring{H}^1(\Omega)^3 \times L_2(\Omega)/\mathbb{R}$. Then we obtain the following equality

$$\begin{aligned}\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}_1 - \mathbf{u}_2), E_{i\alpha}(\mathbf{v})\right\rangle_\Omega + \langle (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2, \mathbf{v}\rangle_\Omega \\ - \langle \operatorname{div} \mathbf{v}, \pi_1 - \pi_2\rangle_\Omega = 0 \quad \forall\, \mathbf{v} \in \mathcal{D}(\Omega)^3.\end{aligned} \tag{5.24}$$

Moreover, the dense embedding of the space $\mathcal{D}_{\rm div}(\Omega)^3$ in $\mathring{H}^1_{\rm div}(\Omega)^3$ (see, e.g., [47, p. 32, Lemma 10]) shows that relation (5.24) is satisfied also for any $\mathbf{v} \in \mathring{H}^1_{\rm div}(\Omega)^3$. Then by choosing $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ in (5.24), we obtain that

$$\begin{aligned}\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}_1 - \mathbf{u}_2), E_{i\alpha}(\mathbf{u}_1 - \mathbf{u}_2)\right\rangle_\Omega = - \langle ((\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla))\, \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2\rangle_\Omega \\ - \langle (\mathbf{u}_2 \cdot \nabla)(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2\rangle_\Omega\,.\end{aligned} \tag{5.25}$$

Due to the membership of $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathring{H}^1_{\rm div}(\Omega)^3$, relation (B.9) yields the equality

$$\langle (\mathbf{u}_2 \cdot \nabla)(\mathbf{u}_1 - \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2\rangle_\Omega = 0\,, \tag{5.26}$$

which shows that equation (5.25) reduces to

$$\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}_1 - \mathbf{u}_2), E_{i\alpha}(\mathbf{u}_1 - \mathbf{u}_2)\right\rangle_\Omega = - \left\langle \left((\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla\right)\mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2\right\rangle_\Omega\,. \tag{5.27}$$

On the other hand, in view of condition (1.4) and the first Korn inequality,

$$\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2_{L_2(\Omega)^{3\times 3}} \leq 2c_{\mathbb{A}} \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}_1 - \mathbf{u}_2), E_{i\alpha}(\mathbf{u}_1 - \mathbf{u}_2)\right\rangle_\Omega\,. \tag{5.28}$$

Note that inequality (5.16) implies that any solution of (5.1) satisfies inequality (5.3). Thus, by the Hölder inequality and inequalities (5.22) and (5.3), we obtain

$$\begin{aligned}\langle ((\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla))\, \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2\rangle_\Omega &\leq \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{L_4(\Omega)^3} \|\nabla \mathbf{u}_1\|_{L_2(\Omega)^{3\times 3}} \\ &\leq \mathring{C}_\Omega^2 \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2_{L_2(\Omega)^{3\times 3}} \|\nabla \mathbf{u}_1\|_{L_2(\Omega)^{3\times 3}} \\ &\leq 2c_{\mathbb{A}} \mathring{C}_\Omega^2 \|\mathbf{F}\|_{H^{-1}(\Omega)^3} \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2_{L_2(\Omega)^{3\times 3}}.\end{aligned} \tag{5.29}$$

Then equality (5.27) and inequalities (5.28) and (5.29) imply that

$$\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|^2_{L_2(\Omega)^{3\times 3}} \le 4c_{\mathbb{A}}^2 \mathring{C}_\Omega^2 \|\mathbf{F}\|_{H^{-1}(\Omega)^3} \|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|^2_{L_2(\Omega)^{3\times 3}} . \quad (5.30)$$

Assumption (5.23) shows that estimate (5.30) is possible only if $\mathbf{u}_1-\mathbf{u}_2=\mathbf{0}$.

Hence, equation (5.24) reduces to $\langle \operatorname{div}\mathbf{v}, \pi_1-\pi_2\rangle_\Omega = 0$ for any $\mathbf{v}\in\mathcal{D}(\Omega)^3$, and thus $\nabla(\pi_1-\pi_2)=0$ in Ω. Then $\pi_1-\pi_2$ is a constant, i.e., $\pi_1=\pi_2$ in $L_2(\Omega)/\mathbb{R}$.

Finally, the equivalence of the nonlinear Dirichlet problem (5.2) and of the nonlinear Dirichlet-transmission problem (5.21) with the mixed variational formulation (5.1) completes the proof. □

6. **The anisotropic Navier-Stokes system and the transmission problem with general data in $\mathbb{R}^3$ and in an exterior Lipschitz domain.** In this section we construct a sequence of weak solutions for the homogeneous Dirichlet problem for the incompressible anisotropic Navier-Stokes system in a sequence of increasing bounded Lipschitz domains which approximate $\mathbb{R}^3$, or an exterior Lipschitz domain. From such a sequence we extract a convergent subsequence to a weak solution of the nonlinear transmission problem in $\mathbb{R}^3$, or of the exterior Dirichlet problem for the anisotropic Navier-Stokes system with general (including large) data in weighted Sobolev spaces.

In the isotropic case (1.10) with $\mu=1$ and $\lambda=0$, we refer to [3, Theorem 1.3] for the existence of a weak solution of the exterior Dirichlet problem for Navier-Stokes system, [43] for the Dirichlet problem for the Navier-Stokes system in a bounded Lipschitz domain in $\mathbb{R}^2$, under singular sources, and to [28, Theorem 5.2]. Existence in the case of the anisotropic tensor $\mathbb{A}$ satisfying a more restrictive ellipticity condition than in (1.4) was analyzed in [31, Theorem 4.2] in a pseudostress setting, assuming small given data.

6.1. **The Navier-Stokes system in $\mathbb{R}^3$.** First, we show the existence of a solution $(\mathbf{u},\pi)\in\mathcal{H}^1(\mathbb{R}^3)^3\times L_{2,\mathrm{loc}}(\mathbb{R}^3)$ of the Navier-Stokes system

$$\boldsymbol{\mathcal{L}}(\mathbf{u},\pi)=\mathbf{f}+(\mathbf{u}\cdot\nabla)\mathbf{u},\quad \operatorname{div}\mathbf{u}=0 \text{ in } \mathbb{R}^3 \quad (6.1)$$

in the sense of distributions. This result will imply that the pair $(\mathbf{u},\pi)\in\mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^3)^3\times L_{2,\mathrm{loc}}(\mathbb{R}^3)$ satisfies the equivalent mixed variational formulation with $\mathbf{F}=-\mathbf{f}$,

$$\begin{cases} \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v})\right\rangle_{\mathbb{R}^3} + \langle(\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}\rangle_{\mathbb{R}^3} - \langle\operatorname{div}\mathbf{v},\pi\rangle_{\mathbb{R}^3} \\ \qquad\qquad = \langle\mathbf{F},\mathbf{v}\rangle_{\mathbb{R}^3} \quad \forall\,\mathbf{v}\in\mathcal{D}(\mathbb{R}^3)^3, \\ \langle\operatorname{div}\mathbf{u},q\rangle_{\mathbb{R}^3}=0 \quad \forall\, q\in\mathcal{D}(\mathbb{R}^3). \end{cases} \quad (6.2)$$

In particular, we will obtain some estimates of the pressure norm growth, which seem to be new even in the simpler isotropic constant-coefficient case.

Theorem 6.1. *Let conditions (1.2)-(1.4) hold in $\mathbb{R}^3$ and let $\boldsymbol{\mathcal{L}}$ denote the Stokes operator defined in (1.9). Then for any $\mathbf{F}\in\mathcal{H}^{-1}(\mathbb{R}^3)^3$ there exists a pair $(\mathbf{u},\pi)\in\mathcal{H}^1(\mathbb{R}^3)^3\times L_{2,\mathrm{loc}}(\mathbb{R}^3)$, which satisfies the nonlinear variational problem (6.2) as well as the Navier-Stokes system (6.1) with $\mathbf{f}=-\mathbf{F}$ in the sense of distributions. In addition,*

$$\|\nabla\mathbf{u}\|_{L_2(\mathbb{R}^3)^3} \le 2c_{\mathbb{A}}\|\mathbf{F}\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}, \quad (6.3)$$

$$\|\pi\|_{L_2(\widehat{\Omega})/\mathbb{R}} \le C'_{\mathbb{R}^3}\|\mathbf{F}\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3} + C''_{\mathbb{R}^3}|\widehat{\Omega}|^{1/6}\|\mathbf{F}\|^2_{\mathcal{H}^{-1}(\mathbb{R}^3)^3} \quad (6.4)$$

for any bounded domain $\widehat{\Omega}$ in $\mathbb{R}^3$. Here $c_{\mathbb{A}}$ is the ellipticity constant introduced in (1.4),

$$C'_{\mathbb{R}^3} := C_{\widehat{\Omega}}(1 + 2c_{\mathbb{A}}3^4\|\mathbb{A}\|_{L_\infty(\mathbb{R}^3)^3}), \quad C''_{\mathbb{R}^3} := \frac{16}{3}C_{\widehat{\Omega}}c_{\mathbb{A}}^2\,,$$

and the constant $C_{\widehat{\Omega}}$ is as in Theorem 3.1, while $|\widehat{\Omega}| = \int_{\widehat{\Omega}} dx$. The norm $\|\!|\cdot|\!\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}$ is defined in (4.1). The constant $C_{\widehat{\Omega}}$ and hence the constants $C'_{\mathbb{R}^3}$ and $C''_{\mathbb{R}^3}$ in (6.4) do not depend on $\widehat{\Omega}$ if $\widehat{\Omega}$ is a ball.

Proof. We follow the steps similar to those in the proof of [3, Theorem 1.3]. To this end, we consider an increasing sequence of real numbers $\{R_k\}_{k\geq 0}$ with $R_0 > 0$, and $R_k \to \infty$ as $k \to \infty$. Let $\Omega_k := B_{R_k}$ be the ball of radius R_k and center 0 in $\mathbb{R}^3$.

If $\mathbf{F}\in\mathcal{H}^{-1}(\mathbb{R}^3)^3$, then $\mathbf{F}|_{\Omega_k} \in H^{-1}(\Omega_k)^3$, and by Theorem 5.2 there exists a pair $(\mathbf{u}_k, \pi_k) \in \mathring{H}^1_{\rm div}(\Omega_k)^3 \times L_2(\Omega_k)$, which satisfies the anisotropic Navier-Stokes system

$$\boldsymbol{\mathcal{L}}(\mathbf{u}_k, \pi_k) = -\mathbf{F}|_{\Omega_k} + (\mathbf{u}_k \cdot \nabla)\mathbf{u}_k\,, \quad {\rm div}\, \mathbf{u}_k = 0 \text{ in } \Omega_k, \tag{6.5}$$

and the inequality

$$\|\nabla \mathbf{u}_k\|_{L_2(\Omega_k)^{3\times 3}} \leq 2c_{\mathbb{A}}\|\mathbf{F}|_{\Omega_k}\|_{H^{-1}(\Omega_k)^3}\,. \tag{6.6}$$

Equations (6.5) imply that

$$\left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}_k), E_{i\alpha}(\mathbf{v})\right\rangle_{\Omega_k} + \langle(\mathbf{u}_k\cdot\nabla)\mathbf{u}_k, \mathbf{v}\rangle_{\Omega_k} = \langle \mathbf{F}|_{\Omega_k}, \mathbf{v}\rangle_{\Omega_k}\ \forall\, \mathbf{v} \in \mathcal{D}_{\rm div}(\Omega_k)^3\,. \tag{6.7}$$

Note that $\mathbf{F} \in \mathcal{H}^{-1}(\mathbb{R}^3)^3$ satisfies the relations

$$\begin{aligned}\|\mathbf{F}|_{\Omega_k}\|_{H^{-1}(\Omega_k)^3} &= \sup_{\boldsymbol{\Psi}\in\mathring{H}^1(\Omega_k)^3,\ \|\nabla\boldsymbol{\Psi}\|_{L_2(\Omega_k)^3}=1} \left| \langle \mathbf{F}|_{\Omega_k}, \boldsymbol{\Psi}\rangle_{\Omega_k}\right| \\ &= \sup_{\boldsymbol{\Psi}\in\mathring{H}^1(\Omega_k)^3,\ \|\nabla\mathring{E}_{\Omega_k}\boldsymbol{\Psi}\|_{L_2(\mathbb{R}^3)^3}=1} \left|\left\langle \mathbf{F}, \mathring{E}_{\Omega_k}\boldsymbol{\Psi}\right\rangle_{\mathbb{R}^3}\right| \\ &\leq \sup_{\boldsymbol{\phi}\in\mathcal{H}^1(\mathbb{R}^3)^3,\ \|\nabla\boldsymbol{\phi}\|_{L_2(\mathbb{R}^3)^{3\times 3}}=1} |\langle \mathbf{F}, \boldsymbol{\phi}\rangle_{\mathbb{R}^3}| = \|\!|\mathbf{F}|\!\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}\,.\end{aligned} \tag{6.8}$$

Further, let us denote by $\mathring{\mathbf{u}}_k$ the extension of $\mathbf{u}_k$ by zero in $\mathbb{R}^3 \setminus \overline{\Omega_k}$. Hence $\mathring{\mathbf{u}}_k \in \mathcal{H}^1_{\rm div}(\mathbb{R}^3)^3$ (since $\mathring{\mathbf{u}}_k$ does not have jump across $\partial\Omega_k$, and then ${\rm div}\,\mathring{\mathbf{u}}_k = 0$ in $\mathbb{R}^3$). Moreover, by inequalities (6.6) and (6.8), we have

$$\|\nabla\mathring{\mathbf{u}}_k\|_{L_2(\mathbb{R}^3)^{3\times 3}} = \|\nabla\mathbf{u}_k\|_{L_2(\Omega_k)^{3\times 3}} \leq 2c_{\mathbb{A}}\|\!|\mathbf{F}|\!\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}. \tag{6.9}$$

By inequality (6.9), the sequence $\{\mathring{\mathbf{u}}_k\}_{k\in\mathbb{N}}$ is bounded in the Hilbert space $\mathcal{H}^1(\mathbb{R}^3)^3$ and also in its closed subspace $\mathcal{H}^1_{\rm div}(\mathbb{R}^3)^3$. Hence $\{\mathring{\mathbf{u}}_k\}_{k\in\mathbb{N}}$ contains a subsequence (still labeled as the sequence) weakly convergent to an element $\mathbf{u} \in \mathcal{H}^1_{\rm div}(\mathbb{R}^3)^3$. This particularly implies that

$$\begin{aligned}\left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathring{\mathbf{u}}_k), E_{i\alpha}(\boldsymbol{\phi})\right\rangle_{\mathbb{R}^3} &= \left\langle E_{j\beta}(\mathring{\mathbf{u}}_k), a_{ij}^{\alpha\beta}E_{i\alpha}(\boldsymbol{\phi})\right\rangle_{\mathbb{R}^3} \to \left\langle E_{j\beta}(\mathbf{u}), a_{ij}^{\alpha\beta}E_{i\alpha}(\boldsymbol{\phi})\right\rangle_{\mathbb{R}^3} \\ &= \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}), E_{i\alpha}(\boldsymbol{\phi})\right\rangle_{\mathbb{R}^3} \text{ as } k\to\infty\,, \quad \forall\, \boldsymbol{\phi} \in \mathcal{H}^1(\mathbb{R}^3)^3.\end{aligned} \tag{6.10}$$

According to the property that $\|\nabla(\cdot)\|_{L_2(\mathbb{R}^3)^{3\times 3}}$ is a norm in $\mathcal{H}^1(\mathbb{R}^3)^3$ and by using [22, Theorem II.1.3(i)] and (6.9), we obtain that

$$\|\nabla\mathbf{u}\|_{L_2(\mathbb{R}^3)^{3\times 3}} \leq \liminf_{k\to\infty}\|\nabla\mathring{\mathbf{u}}_k\|_{L_2(\mathbb{R}^3)^{3\times 3}} \leq 2c_{\mathbb{A}}\|\!|\mathbf{F}|\!\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}\,, \tag{6.11}$$

i.e., $\mathbf{u}$ satisfies estimate (6.3), as asserted.

Next we show that $\mathbf{u}$ satisfies the equation

$$\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v})\right\rangle_{\mathbb{R}^3} + \langle(\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}\rangle_{\mathbb{R}^3} = \left\langle \mathbf{F},\mathbf{v}\right\rangle_{\mathbb{R}^3} \quad \forall\, \mathbf{v}\in\mathcal{D}_{\rm div}(\mathbb{R}^3)^3. \tag{6.12}$$

To this end, let $\boldsymbol{\phi}\in\mathcal{D}_{\rm div}(\mathbb{R}^3)^3$ and let $k_0\in\mathbb{N}$ be such that $\operatorname{supp}\boldsymbol{\phi}\subset\Omega_{k_0}\subseteq\Omega_k$ for any $k\geq k_0$. Then $\boldsymbol{\phi}\in\mathcal{D}_{\rm div}(\Omega_k)^3$ for any $k\geq k_0$ and by (6.7),

$$\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathring{\mathbf{u}}_k), E_{i\alpha}(\boldsymbol{\phi})\right\rangle_{\Omega_k} + \langle(\mathring{\mathbf{u}}_k\cdot\nabla)\mathring{\mathbf{u}}_k,\boldsymbol{\phi}\rangle_{\mathbb{R}^3} = \langle\mathbf{F},\boldsymbol{\phi}\rangle_{\mathbb{R}^3} \quad \forall\, k\geq k_0. \tag{6.13}$$

Moreover, the compactness of the embedding $H^1(\Omega_{k_0})^3\hookrightarrow L_2(\Omega_{k_0})^3$ yields that there exists a subsequence of the sequence $\{\mathbf{u}_k\}_{k\in\mathbb{N}}$, labeled again as the sequence, such that $\{\mathbf{u}_k\}_{k\in\mathbb{N}}$ converges strongly to $\mathbf{u}$ in $L_2(\Omega_{k_0})^3$. Let us prove that

$$\langle(\mathring{\mathbf{u}}_k\cdot\nabla)\mathring{\mathbf{u}}_k,\boldsymbol{\phi}\rangle_{\mathbb{R}^3}\to\langle(\mathbf{u}\cdot\nabla)\mathbf{u},\boldsymbol{\phi}\rangle_{\mathbb{R}^3} \text{ as } k\to\infty \quad \forall\boldsymbol{\phi}\in\mathcal{D}(\mathbb{R}^3)^3. \tag{6.14}$$

Indeed, the Hölder inequality, (6.9) and the limiting relation $\|\mathring{\mathbf{u}}_k-\mathbf{u}\|_{L_2(\Omega_{k_0})}\to 0$ as $k\to\infty$ yield that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\left(\left((\mathring{\mathbf{u}}_k-\mathbf{u})\cdot\nabla\right)\mathring{\mathbf{u}}_k\right)\cdot\boldsymbol{\phi}dx\Big| &= \Big|\int_{\Omega_{k_0}}\left(\left((\mathring{\mathbf{u}}_k-\mathbf{u})\cdot\nabla\right)\mathring{\mathbf{u}}_k\right)\cdot\boldsymbol{\phi}dx\Big|\\ &\leq\|\mathring{\mathbf{u}}_k-\mathbf{u}\|_{L_2(\Omega_{k_0})^3}\|\nabla\mathring{\mathbf{u}}_k\|_{L_2(\mathbb{R}^3)^{3\times 3}}\|\boldsymbol{\phi}\|_{L_\infty(\Omega_{k_0})^3}\\ &\leq 2c_{\mathbb{A}}\|\mathbf{F}\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}\|\boldsymbol{\phi}\|_{L_\infty(\Omega_{k_0})^3}\|\mathring{\mathbf{u}}_k-\mathbf{u}\|_{L_2(\Omega_{k_0})^3}\\ &\to 0 \text{ as } k\to\infty.\end{aligned} \tag{6.15}$$

In addition, by using the assumption that $\operatorname{supp}\boldsymbol{\phi}\subset\Omega_{k_0}$, identity (B.8), and again the strong convergence property of $\{\mathring{\mathbf{u}}_k\}_{k\in\mathbb{N}}$ to $\mathbf{u}$ in $L_2(\Omega_{k_0})^3$, we obtain that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^3}\left((\mathbf{u}\cdot\nabla)(\mathring{\mathbf{u}}_k-\mathbf{u})\right)\cdot\boldsymbol{\phi}\,dx\Big| = \Big|\int_{\Omega_{k_0}}\left((\mathbf{u}\cdot\nabla)\boldsymbol{\phi}\right)\cdot(\mathring{\mathbf{u}}_k-\mathbf{u})dx\Big|\\ &\leq\|\nabla\boldsymbol{\phi}\|_{L_\infty(\Omega_{k_0})}\|\mathbf{u}\|_{L_2(\Omega_{k_0})}\|\mathring{\mathbf{u}}_k-\mathbf{u}\|_{L_2(\Omega_{k_0})}\to 0 \text{ as } k\to\infty.\end{aligned} \tag{6.16}$$

Then relations (6.15) and (6.16) lead to relation (6.14). Finally, passing to the limit in formula (6.13) and using relations (6.10) and (6.14), we conclude that $\mathbf{u}$ satisfies equation (6.12), and accordingly that $\mathbf{u}$ is a weak solution of the Navier-Stokes equation (in the Leray sense).

Note that $\operatorname{div}(\mathbb{A}\mathbb{E}(\mathbf{u}))\in\mathcal{H}^{-1}(\mathbb{R}^3)^3\hookrightarrow\mathcal{D}'(\mathbb{R}^3)^3$ (∂_α continuously maps the space $L_2(\mathbb{R}^3)$ to $\mathcal{H}^{-1}(\mathbb{R}^3)$). In addition, the embedding $\mathcal{H}^1(\mathbb{R}^3)\hookrightarrow L_6(\mathbb{R}^3)$ (see, e.g., Remark 3.8(i) in [2]) and the Hölder inequality imply for $\mathbf{u}\in\mathcal{H}^1(\mathbb{R}^3)^3$ that

$$(\mathbf{u}\cdot\nabla)\mathbf{u}\in L_{\frac{3}{2}}(\mathbb{R}^3)^3\hookrightarrow L_{\frac{3}{2};\rm loc}(\mathbb{R}^3)^3\hookrightarrow H^{-1}_{\rm loc}(\mathbb{R}^3)^3\hookrightarrow\mathcal{D}'(\mathbb{R}^3)^3.$$

Thus, for given $\mathbf{F}\in\mathcal{H}^{-1}(\mathbb{R}^3)^3\hookrightarrow\mathcal{D}'(\mathbb{R}^3)^3$, we have

$$\widetilde{\boldsymbol{\mathcal{F}}} := \mathbf{F}+\operatorname{div}(\mathbb{A}\mathbb{E}(\mathbf{u}))-(\mathbf{u}\cdot\nabla)\mathbf{u}\in H^{-1}_{\rm loc}(\mathbb{R}^3)^3\hookrightarrow\mathcal{D}'(\mathbb{R}^3)^3, \tag{6.17}$$

and, by (6.12),

$$\langle\widetilde{\boldsymbol{\mathcal{F}}},\boldsymbol{\phi}\rangle_{\mathbb{R}^3}=0 \quad \forall\,\boldsymbol{\phi}\in\mathcal{D}_{\rm div}(\mathbb{R}^3)^3. \tag{6.18}$$

Then due to the De Rham Theorem (cf., e.g., Proposition 1.1 in [48, Chapter 1], see also Theorem 4.1), there exists $\pi\in\mathcal{D}'(\mathbb{R}^3)$ such that $\nabla\pi=\widetilde{\boldsymbol{\mathcal{F}}}$ in $\mathcal{D}'(\mathbb{R}^3)^3$, i.e.,

$$\langle\nabla\pi,\mathbf{v}\rangle_{\mathbb{R}^3}=\langle\mathbf{F}+\operatorname{div}(\mathbb{A}\mathbb{E}(\mathbf{u}))-(\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}\rangle_{\mathbb{R}^3} \quad \forall\,\mathbf{v}\in\mathcal{D}(\mathbb{R}^3)^3, \tag{6.19}$$

and hence, the first equation in (6.2) is satisfied, while the second equation is satisfied since $\mathbf{u}\in\mathcal{H}^1_{\rm div}(\mathbb{R}^3)^3$.

Moreover, by (6.18) $\widetilde{\boldsymbol{\mathcal{F}}}$ defined by (6.17) belongs locally to $H^{-1}(\mathbb{R}^3)^3 \perp \mathring{H}^1_{\rm div}(\Omega)^3$. Then equation $\nabla\pi = \widetilde{\boldsymbol{\mathcal{F}}}$ implies that $\pi \in L_{2;\rm loc}(\mathbb{R}^3)$ (cf. Proposition 1.2 (ii) and Remark 1.4 in [48, Chapter 1], [47, Lemma 9], [22, Lemma X.1.1]) and in addition, by Theorem 3.1, we obtain for any bounded domain $\widehat{\Omega} \subset \mathbb{R}^3$ that

$$\|\pi\|_{L_2(\widehat{\Omega})/\mathbb{R}} \le C_{\widehat{\Omega}} \|\widetilde{\boldsymbol{\mathcal{F}}}\|_{H^{-1}(\widehat{\Omega})^3}. \tag{6.20}$$

In addition, by (6.17) we obtain that

$$\|\widetilde{\boldsymbol{\mathcal{F}}}\|_{H^{-1}(\widehat{\Omega})^3} \le \|\mathbf{F}\|_{H^{-1}(\widehat{\Omega})^3} + \|{\rm div}\,(\mathbb{A}\mathbb{E}(\mathbf{u}))\,\|_{H^{-1}(\widehat{\Omega})^3} + \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{H^{-1}(\widehat{\Omega})^3}, \tag{6.21}$$

Similar to (6.8), we have

$$\|\mathbf{F}\|_{H^{-1}(\widehat{\Omega})^3} \le \|\mathbf{F}\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3} \tag{6.22}$$

By (1.5), and (6.3),

$$\begin{aligned}
\|{\rm div}\,(\mathbb{A}\mathbb{E}(\mathbf{u}))\,\|_{H^{-1}(\widehat{\Omega})^3} &= \sup_{\boldsymbol{\Psi}\in\mathring{H}^1(\widehat{\Omega})^3,\, \|\nabla\boldsymbol{\Psi}\|_{L_2(\widehat{\Omega})^3}=1} \Big|\, \langle {\rm div}\,(\mathbb{A}\mathbb{E}(\mathbf{u}))\,, \boldsymbol{\Psi}\rangle_{\widehat{\Omega}}\,\Big| \\
&= \sup_{\boldsymbol{\Psi}\in\mathring{H}^1(\widehat{\Omega})^3,\, \|\nabla\boldsymbol{\Psi}\|_{L_2(\widehat{\Omega})^3}=1} \Big|\, \langle \mathbb{A}\mathbb{E}(\mathbf{u}), \nabla\boldsymbol{\Psi}\rangle_{\widehat{\Omega}}\,\Big| \\
&\le \|\mathbb{A}\mathbb{E}(\mathbf{u})\|_{L_2(\widehat{\Omega})^{3\times 3}} \le 3^4\|\mathbb{A}\|_{L_\infty(\widehat{\Omega})}\|\nabla\,\mathbf{u}\|_{L_2(\widehat{\Omega})^{3\times 3}} \\
&\le 3^4\|\mathbb{A}\|_{L_\infty(\widehat{\Omega})}\|\nabla\,\mathbf{u}\|_{L_2(\mathbb{R}^3)^{3\times 3}} \\
&\le 2c_{\mathbb{A}}3^4\|\mathbb{A}\|_{L_\infty(\widehat{\Omega})}\|\mathbf{F}\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3},
\end{aligned} \tag{6.23}$$

From (B.14) and (6.3) we have,

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{H^{-1}(\widehat{\Omega})^3} \le \frac{4}{3}|\widehat{\Omega}|^{1/6}\|\nabla\mathbf{u}\|^2_{L_2(\mathbb{R}^3)^{3\times 3}} \le \frac{4}{3}|\widehat{\Omega}|^{1/6}4c_{\mathbb{A}}^2\|\mathbf{F}\|^2_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}. \tag{6.24}$$

Hence, substituting (6.22), (6.23) and (6.24) in (6.21) and then (6.21) in (6.20), we obtain (6.4). Finally, the independence of the constant $C_{\widehat{\Omega}}$ of $\widehat{\Omega}$ whenever the domain $\widehat{\Omega}$ is a ball follows from Theorem 3.1. □

6.2. **Transmission problem for the Navier-Stokes system in $\mathbb{R}^3$.** Let $n = 3$ and Assumption 4.3 about the geometry hold. Let us consider the transmission problem for the anisotropic Navier-Stokes system

$$\begin{cases}
\boldsymbol{\mathcal{L}}(\mathbf{u}_+,\pi_+) = \tilde{\mathbf{f}}_+|_{\Omega^0_+} + (\mathbf{u}_+\cdot\nabla)\mathbf{u}_+\,, \quad {\rm div}\,\mathbf{u}_+ = 0 & \text{in } \Omega^0_+,\\
\boldsymbol{\mathcal{L}}(\mathbf{u}_-,\pi_-) = \tilde{\mathbf{f}}_-|_{\Omega^0_-} + (\mathbf{u}_-\cdot\nabla)\mathbf{u}_-\,, \quad {\rm div}\,\mathbf{u}_- = 0 & \text{in } \Omega^0_-,\\
\gamma_+\mathbf{u}_+ - \gamma_-\mathbf{u}_- = \mathbf{0} & \text{on } \partial\Omega^0,\\
\mathbf{t}^+\Big(\mathbf{u}_+,\pi_+;\tilde{\mathbf{f}}_+ + \mathring{E}_+\left((\mathbf{u}_+\cdot\nabla)\mathbf{u}_+\right)\Big) & \\
\quad -\mathbf{t}^-\Big(\mathbf{u}_-,\pi_-;\tilde{\mathbf{f}}_- + \mathring{E}_-\left((\mathbf{u}_-\cdot\nabla)\mathbf{u}_-\right)\Big) = \boldsymbol{\psi} & \text{on } \partial\Omega^0,
\end{cases} \tag{6.25}$$

with given data $(\tilde{\mathbf{f}}_+, \tilde{\mathbf{f}}_-, \boldsymbol{\psi}) \in \mathcal{Y}^N_{\Omega^0_+,\Omega^0_-}$ and unknown $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-) \in \mathcal{X}^N_{\Omega^0_+,\Omega^0_-}$. Here,

$$\mathcal{X}^N_{\Omega^0_+,\Omega^0_-} := H^1(\Omega^0_+)^3 \times L_2(\Omega^0_+) \times \mathcal{H}^1(\Omega^0_-)^3 \times L_{2,\rm loc}(\Omega^0_-)\,, \tag{6.26}$$

$$\mathcal{Y}^N_{\Omega^0_+,\Omega^0_-} := \widetilde{H}^{-1}(\Omega^0_+)^3 \times \widetilde{\mathcal{H}}^{-1}(\Omega^0_-)^3 \times H^{-\frac{1}{2}}(\partial\Omega^0)^3\,. \tag{6.27}$$

Next we show the existence of a solution of the nonlinear transmission problem (6.25) with general (including large) given data.

Theorem 6.2. *Let $n=3$ and Ω^0_+ and Ω^0_- be as in Assumption 4.3. Let conditions (1.2)-(1.4) hold in $\mathbb{R}^3$. Let $\big(\tilde{\mathbf{f}}_+,\tilde{\mathbf{f}}_-,\boldsymbol{\psi}\big)\in\mathcal{Y}^N_{\Omega^0_+,\Omega^0_-}$ and $(\mathbf{u},\pi)\in\mathcal{H}^1_{\rm div}(\mathbb{R}^3)^3\times L_{2,\rm loc}(\mathbb{R}^n)$ be the solution of the variational problem (6.2) provided by Theorem 6.1 for $\mathbf{F}\in\mathcal{H}^{-1}(\mathbb{R}^3)^3$ given by*

$$\mathbf{F}=-(\tilde{\mathbf{f}}_++\tilde{\mathbf{f}}_-)+\gamma^*\boldsymbol{\psi}\,, \tag{6.28}$$

where $\gamma^*:H^{-\frac12}(\partial\Omega^0)^3\to\mathcal{H}^{-1}(\mathbb{R}^3)^3$ is the adjoint of the trace map $\gamma:\mathcal{H}^1(\mathbb{R}^3)^3\to H^{\frac12}(\partial\Omega^0)^3$. Then there exists a solution $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathcal{X}^N_{\Omega^0_+,\Omega^0_-}$ of the nonlinear transmission problem (6.25) in the sense of distributions, given by the relations $\mathbf{u}_+=\mathbf{u}|_{\Omega^0_+}$, $\mathbf{u}_-=\mathbf{u}|_{\Omega^0_-}$, $\pi_+=\pi|_{\Omega^0_+}$, $\pi_-=\pi|_{\Omega^0_-}$, and estimates (6.3), (6.4) hold.

Proof. We have to show that $(\mathbf{u},\pi)\in\mathcal{H}^1_{\rm div}(\mathbb{R}^3)^3\times L_{2,\rm loc}(\mathbb{R}^3)$ solving system (6.1) with $\mathbf{F}\in\mathcal{H}^{-1}(\mathbb{R}^3)^3$ given by (6.28) provides also a solution $\mathbf{u}_\pm:=\mathbf{u}|_{\Omega^0_\pm}$, $\pi_\pm:=\pi|_{\Omega^0_\pm}$ of the transmission problem (6.25) in the sense of distributions. Since $(\mathbf{u},\pi)\in\mathcal{H}^1_{\rm div}(\mathbb{R}^3)^3\times L_{2,\rm loc}(\mathbb{R}^3)$, we have $\mathbf{u}_+\in H^1_{\rm div}(\Omega^0_+)^3$, $\mathbf{u}_-\in\mathcal{H}^1_{\rm div}(\Omega^0_-)^3$, $\pi_+\in L_2(\Omega^0_+)$, $\pi_-\in L_{2,\rm loc}(\Omega^0_-)$. System (6.1) implies that the couples $(\mathbf{u}_\pm,\pi_\pm)$ satisfy the Navier-Stokes equations

$$\boldsymbol{\mathcal{L}}(\mathbf{u}_\pm,\pi_\pm)=\tilde{\mathbf{f}}_\pm|_{\Omega_\pm}+(\mathbf{u}_\pm\cdot\nabla)\mathbf{u}_\pm\ \text{ in } \Omega^0_\pm. \tag{6.29}$$

For any $\mathbf{v}\in\mathcal{D}(\mathbb{R}^3)^3$ in (6.2), using again relation (6.28), we obtain that

$$\begin{aligned}\Big\langle a^{\alpha\beta}_{ij}E_{j\beta}(\mathbf{u}_+),E_{i\alpha}(\mathbf{v})\Big\rangle_{\Omega^0_+}&+\langle(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+,\mathbf{v}\rangle_{\Omega^0_+}-\langle\pi_+,{\rm div}\,\mathbf{v}\rangle_{\Omega^0_+}+\langle\tilde{\mathbf{f}}_+,\mathbf{v}\rangle_{\Omega^0_+}\\ +\Big\langle a^{\alpha\beta}_{ij}E_{j\beta}(\mathbf{u}_-),E_{i\alpha}(\mathbf{v})\Big\rangle_{\Omega^0_-}&+\langle(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-,\mathbf{v}\rangle_{\Omega^0_-}-\langle\pi_-,{\rm div}\,\mathbf{v}\rangle_{\Omega^0_-}+\langle\tilde{\mathbf{f}}_-,\mathbf{v}\rangle_{\Omega^0_-}\\ &=\langle\boldsymbol{\psi},\gamma\mathbf{v}\rangle_{\partial\Omega^0}\,. \end{aligned}\tag{6.30}$$

Now let Ω^* be a bounded Lipschitz domain (e.g., a ball) such that $\overline{\Omega^0_+}\subset\Omega^*$ and let $\Omega^*_-:=\Omega^0_-\cap\Omega^*$. By choosing $\mathbf{v}\in\mathcal{D}(\Omega^*)^3$, Ω^0_- in (6.30) can be replaced by Ω^*_-. Then the first Green identity (2.23) shows that equation (6.30) reduces to

$$\begin{aligned}\Big\langle\mathbf{t}^+\Big(\mathbf{u}_+,\pi_+;\tilde{\mathbf{f}}_++(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+\Big)-\mathbf{t}^-\Big(\mathbf{u}_-,\pi_-;\tilde{\mathbf{f}}_-+(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-\Big),\gamma\mathbf{v}\Big\rangle_{\partial\Omega^0}&\\ =\langle\boldsymbol{\psi},\gamma\mathbf{v}\rangle_{\partial\Omega^0}\ \ \forall\,\mathbf{v}\in\mathcal{D}(\Omega^*)^3,&\end{aligned}$$

or, equivalently, to the equation

$$\big\langle\mathbf{t}^+\big(\mathbf{u}_+,\pi_+;\tilde{\mathbf{f}}_++(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+\big)-\mathbf{t}^-\big(\mathbf{u}_-,\pi_-;\tilde{\mathbf{f}}_-+(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-\big),\boldsymbol{\phi}\big\rangle_{\partial\Omega^0}=\langle\boldsymbol{\psi},\boldsymbol{\phi}\rangle_{\partial\Omega^0}$$

for any $\boldsymbol{\phi}\in H^{\frac12}(\partial\Omega^0)^3$, due to the dense embedding of the space $\mathcal{D}(\Omega^*)^3$ in $\mathring{H}^1(\Omega^*)^3$ and the surjectivity of the trace operator γ from $\mathring{H}^1(\Omega^*)^3$ to $H^{\frac12}(\partial\Omega^0)^3$. Therefore, the second transmission condition in (6.25) follows, as asserted. The first transmission condition is obviously satisfied since $\mathbf{u}\in\mathcal{H}^1(\mathbb{R}^3)^3$. $\square$

Theorem 6.2 as proved is a corollary of Theorem 6.1. Note that it could be also proved directly, modifying the proof of Theorem 6.1 so that one considers there weak solutions of the Dirichlet-transmission problem in an increasing sequence of bounded composite Lipschitz domains Ω_k, covering the entire space $\mathbb{R}^3$, and then employs the results of Theorem 5.3.

6.3. **Exterior Dirichlet problem for the Navier-Stokes system.** As in Section 4.3, let Ω_- be an exterior Lipschitz domain with a compact (not necessarily connected) boundary $\partial\Omega$.

Let us consider the exterior Dirichlet problem for the Navier-Stokes system

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u},\pi) = \mathbf{f} + (\mathbf{u}\cdot\nabla)\mathbf{u}\,, \quad \operatorname{div}\mathbf{u}_- = 0 & \text{in } \Omega_-\,,\\ \gamma_-(\mathbf{u}) = \mathbf{0} & \text{on } \partial\Omega\,, \end{cases} \tag{6.31}$$

with the given datum $\mathbf{f}\in\mathcal{H}^{-1}(\Omega_-)^3$ and unknown $(\mathbf{u},\pi)\in\mathcal{H}^1(\Omega_-)^3\times L_{2,\mathrm{loc}}(\Omega_-)$. Let us also consider the mixed variational problem

$$\begin{cases} \left\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v})\right\rangle_{\Omega_-} + \langle(\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}\rangle_{\Omega_-} - \langle\operatorname{div}\mathbf{v},\pi\rangle_{\Omega_-}\\ \qquad = \langle\mathbf{F},\mathbf{v}\rangle_{\Omega_-}\ \forall\,\mathbf{v}\in\mathcal{D}(\Omega_-)^3,\\ \langle\operatorname{div}\mathbf{u},q\rangle_{\mathbb{R}^3} = 0 \quad \forall\, q\in\mathcal{D}({\Omega_-}^3)\,. \end{cases} \tag{6.32}$$

with given $\mathbf{F}\in\mathcal{H}^{-1}(\Omega_-)^3$ and unknown $(\mathbf{u},\pi)\in\mathring{\mathcal{H}}^1(\Omega_-)^3\times L_{2,\mathrm{loc}}(\Omega_-)$.

Now we prove the following existence result for problem (6.31) (cf. [3, Theorem 1.3] for the exterior Dirichlet problem for the Navier-Stokes system with constant coefficients).

Theorem 6.3. *Let Ω_- be an exterior Lipschitz domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. Let conditions (1.2)-(1.4) hold in Ω_-. Then for any $\mathbf{F}\in\mathcal{H}^{-1}(\Omega_-)^3$ there exists a pair $(\mathbf{u},\pi)\in\mathring{\mathcal{H}}^1(\Omega_-)^3\times L_{2,\mathrm{loc}}(\Omega_-)$, which satisfies the nonlinear variational problem (6.32) as well as the exterior Dirichlet problem (6.31) with $\mathbf{f}=-\mathbf{F}$ in the sense of distributions. In addition,*

$$\|\nabla\mathbf{u}\|_{L_2(\Omega_-)^3} \le 2c_{\mathbb{A}}\|\mathbf{F}\|_{\mathcal{H}^{-1}(\Omega_-)^3}\,, \tag{6.33}$$

$$\|\pi\|_{L_2(\widehat{\Omega})/\mathbb{R}} \le C'_{\Omega_-}\|\mathbf{F}\|_{\mathcal{H}^{-1}(\Omega_-)^3} + C''_{\Omega_-}|\widehat{\Omega}|^{1/6}\|\mathbf{F}\|^2_{\mathcal{H}^{-1}(\Omega_-)^3} \tag{6.34}$$

for any bounded domain $\widehat{\Omega}\subset\Omega_-$. Here $c_{\mathbb{A}}$ is the ellipticity constant in (1.4),

$$C'_{\Omega_-} := C_{\widehat{\Omega}}(1+2c_{\mathbb{A}}3^4\|\mathbb{A}\|_{L_\infty(\Omega_-)^3}),\quad C''_{\Omega_-} := \frac{16}{3}C_{\widehat{\Omega}}c_{\mathbb{A}}^2,$$

and the constant $C_{\widehat{\Omega}}$ is as in Theorem 3.1, while $|\widehat{\Omega}| = \int_{\widehat{\Omega}}dx$. The norm $\|\cdot\|_{\mathcal{H}^{-1}(\Omega_-)^3}$ is defined in (4.19).

Proof. Let $B_R\subseteq\mathbb{R}^3$ denote an open ball of radius R and center 0. We consider an increasing sequence $\{R_k\}_{k\ge0}\subset\mathbb{R}$ such that $\lim_{k\to\infty}R_k=\infty$ and $B_{R_0}\supset\mathbb{R}^3\setminus\Omega_-$. Let us define the bounded domains $\Omega_k := \Omega_-\cap B_{R_k}$. Thus, $\partial\Omega_k = \partial\Omega\cup\partial B_{R_k}$.

The rest of the proof is omitted as it is very similar to the proof of Theorem 6.1 after replacing there $\mathcal{H}^1(\mathbb{R}^3)^3$ by $\mathring{\mathcal{H}}^1(\Omega_-)^3$ and also $\mathbb{R}^3$ by Ω_-. □

6.4. **Exterior Dirichlet-transmission problem for the Navier-Stokes system.** Let Assumption 4.8 about the geometry holds and let us introduce the spaces

$$\mathcal{X}^N_{\Omega^0_+,\Omega^0_-} := H^1(\Omega^0_+)^3\times L_2(\Omega^0_+)\times\mathcal{H}^1(\Omega^0_-)^3\times L_{2,\mathrm{loc}}(\Omega^0_-)\,, \tag{6.35}$$

$$\mathcal{Y}^{ND}_{\Omega^0_+,\Omega^0_-} := \widetilde{H}^{-1}(\Omega^0_+)^3\times\widetilde{H}^{-1}(\Omega^0_-;\partial\Omega^0)^3\times H^{-\frac12}(\partial\Omega^0)^3\,, \tag{6.36}$$

where the space $\widetilde{\mathcal{H}}^{-1}(\Omega^0_-;\partial\Omega^0)^3$ is defined in (4.37).

Next, we consider the exterior Dirichlet-transmission problem

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{u}_+,\pi_+)=\tilde{\mathbf{f}}_+|_{\Omega^0_+}+(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+\,, \quad \operatorname{div}\mathbf{u}_+=0 & \text{in } \Omega^0_+,\\ \boldsymbol{\mathcal{L}}(\mathbf{u}_-,\pi_-)=\tilde{\mathbf{f}}_-|_{\Omega_-}+(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-\,, \quad \operatorname{div}\mathbf{u}_-=0 & \text{in } \Omega^0_-,\\ \gamma_+\mathbf{u}_+-\gamma_-\mathbf{u}_-=\mathbf{0} & \text{on } \partial\Omega^0,\\ \mathbf{t}^+\left(\mathbf{u}_+,\pi_+;\tilde{\mathbf{f}}_++\mathring{E}_+\left((\mathbf{u}_+\cdot\nabla)\mathbf{u}_+\right)\right) & \\ \quad -\mathbf{t}^-\left(\mathbf{u}_-,\pi_-;\tilde{\mathbf{f}}_-+\mathring{E}_-\left((\mathbf{u}_-\cdot\nabla)\mathbf{u}_-\right)\right)=\boldsymbol{\psi} & \text{on } \partial\Omega^0,\\ \gamma_-\mathbf{u}_-=\mathbf{0} & \text{on } \partial\Omega\,. \end{cases} \tag{6.37}$$

with given data $(\tilde{\mathbf{f}}_+,\tilde{\mathbf{f}}_-,\boldsymbol{\psi})\in\mathcal{Y}^D_{\Omega^0_+,\Omega^0_-}$ and unknown $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathcal{X}_{\Omega^0_+,\Omega^0_-}$.

Following arguments similar to the ones for Theorem 6.2, one can prove

Theorem 6.4. *Let $n=3$ and the geometry be as in Assumption 4.8. Let conditions (1.2)-(1.4) hold on Ω_-. Let $\left(\tilde{\mathbf{f}}_+,\tilde{\mathbf{f}}_-,\boldsymbol{\psi}\right)\in\mathcal{Y}^{ND}_{\Omega^0_+,\Omega^0_-}$ and let $(\mathbf{u},\pi)\in\mathring{\mathcal{H}}^1(\Omega_-)^3\times L_2(\Omega_-)$ be the solution of the distributional system (6.32) provided by Theorem 6.3 for $\mathbf{F}=-(\tilde{\mathbf{f}}_++\tilde{\mathbf{f}}_-)+\gamma^*\boldsymbol{\psi}$, where $\gamma^*:H^{-\frac12}(\partial\Omega^0)^3\to\mathcal{H}^{-1}(\Omega_-)^3$ is the adjoint of the trace operator $\gamma:\mathcal{H}^1(\Omega_-)^3\to H^{\frac12}(\partial\Omega^0)^3$. Then there exists a solution $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathfrak{X}^N_{\Omega^0_+,\Omega^0_-}$ of the nonlinear exterior Dirichlet-transmission problem (6.37) given by the relations $\mathbf{u}_+=\mathbf{u}|_{\Omega^0_+}$, $\mathbf{u}_-=\mathbf{u}|_{\Omega^0_-}$, $\pi_+=\pi|_{\Omega^0_+}$, $\pi_-=\pi|_{\Omega^0_-}$, and estimate (6.33) holds.*

Remark 2. (i) In the case of small data, the existence results in Theorems 6.2, 6.3 and can be supplemented with uniqueness results, as in Theorem 5.4, see also [31, Theorem 4.2].

(ii) The well-posedness results obtained in this paper can be extended, similar to [31] and [34], to the setting of L_p-based Sobolev spaces with p in an open interval containing 2.

Appendix A. Extension result in weighted Sobolev spaces.

Lemma A.1. *Let Ω be $\mathbb{R}^n$ or a (bounded or unbounded) Lipschitz domain in $\mathbb{R}^n$. Let Ω^0 be a bounded Lipschitz set such that $\overline{\Omega^0}\subset\Omega$. Let $\Omega^0_+:=\Omega^0$, $\Omega^0_-:=\Omega\setminus\overline{\Omega^0}$ and $\partial\Omega^0$ denote the interface between Ω^0_+ and Ω^0_-.*

- (i) *Let $q_+\in L_2(\Omega^0_+)$ and $q_-\in L_2(\Omega^0_-)$. Then there exists a unique function $q\in L_2(\Omega)$ such that $q|_{\Omega^0_\pm}=q_\pm$. Moreover, $\|q\|^2_{L_2(\Omega)}=\|q_+\|^2_{L_2(\Omega^0_+)}+\|q_-\|^2_{L_2(\Omega^0_-)}$.*
- (ii) *Let $u_+\in H^1(\Omega^0_+)$ and $u_-\in H^1(\Omega^0_-)$ be such that $\gamma_+u_+=\gamma_-u_-$ on $\partial\Omega^0$. Then there exists a unique function $u\in H^1(\Omega)$ such that $u|_{\Omega^0_\pm}=u_\pm$. Moreover, there exists a constant $C'=C'(n,\Omega^0_+,\Omega^0_-)$, such that $\|u\|_{H^1(\Omega)}\le C'\left(\|u_+\|_{H^1(\Omega^0_+)}+\|u_-\|_{H^1(\Omega^0_-)}\right)$.*
- (iii) *Let Ω be $\mathbb{R}^n$ or an exterior Lipschitz domain in $\mathbb{R}^n$. Let $u_+\in H^1(\Omega^0_+)$ and $u_-\in\mathcal{H}^1(\Omega^0_-)$ be such that $\gamma_+u_+=\gamma_-u_-$ on $\partial\Omega^0$. Then there exists a unique $u\in\mathcal{H}^1(\Omega)$ such that $u|_{\Omega^0_\pm}=u_\pm$. Moreover, there exists $C=C(n,\Omega^0_+,\Omega^0_-)>0$, such that $\|u\|_{\mathcal{H}^1(\Omega)}\le C\left(\|u_+\|_{H^1(\Omega^0_+)}+\|u_-\|_{\mathcal{H}^1(\Omega^0_-)}\right)$.*
- (iv) *If $u\in H^1(\Omega)$ or $u\in\mathcal{H}^1(\Omega)$ then $[\gamma u]=0$ on $\partial\Omega^0$, where $[\gamma u]=\gamma_+(u|_{\Omega^0_+})-\gamma_-(u|_{\Omega^0_-})$.*

Proof. (i) We can take $q=\mathring{E}_{\Omega^0_+}q_++\mathring{E}_{\Omega^0_-}q_-\in L_2(\Omega)$, where $\mathring{E}_{\Omega^0_\pm}$ are the operators of extension by zero to $\mathbb{R}^n\setminus\Omega^0_\pm$. Then $q|_{\Omega^0_\pm}=q_\pm$. To show uniqueness, assume that

q_1 and q_2 are two such functions. Then $q_0 := q_1 - q_2 \in L_2(\Omega)$ and $q_0|_{\Omega^0_\pm} = 0$. Hence $q_0 = 0$ a.e. in Ω.

The proof of item *(ii)* follows the same arguments as those in the proof for item *(iii)*, with the obvious replacement of the weighted Sobolev space $\mathcal{H}^1$ by the standard Sobolev space H^1.

(iii) We follow arguments similar to those for Theorem 5.13 in [10]. First, we show that there exists a bounded linear extension operator $\mathcal{E}_{\Omega^0_+}$ from $H^1(\Omega^0_+)$ to $\mathcal{H}^1(\Omega)$.

To this end, we consider the bounded Lipschitz set Ω^0_+ (with $\overline{\Omega^0_+} \subset \Omega$) as $\Omega^0_+ = \bigcup_{i=1}^m \Omega_i$, $m \geq 1$, where $\Omega_i \subset \mathbb{R}^n$ are bounded Lipschitz domains with disjoint closures. Then there exist bounded linear extension operators $\mathcal{E}_{\Omega_i}$ from $H^1(\Omega_i)$ to $H^1(\mathbb{R}^n)$ (see, e.g., [1, Theorem 5.24]). Now let $v^0 \in H^1(\Omega^0_+)$, $v^0_i := r_{\Omega_i} v^0 \in H^1(\Omega_i)$, and thus $\mathcal{E}_{\Omega_i} v^0_i \in H^1(\mathbb{R}^n)$, $i = 1, \dots, m$. Let $\chi_i \in \mathcal{D}(\mathbb{R}^n)$ be a cut-off function such that $\chi_i = 1$ in Ω_i and $\operatorname{supp} \chi \cap \overline{\Omega}_j = \emptyset$ if $i \neq j$. Then the function $v := \sum_{i=1}^m \chi_i \mathcal{E}_{\Omega_i} v^0_i$ is an extension of $v^0 \in H^1(\Omega^0_+)$ to $H^1(\mathbb{R}^n)$. Therefore, $E_{\Omega^0_+} := \sum_{i=1}^m \chi_i \mathcal{E}_{\Omega_i} r_{\Omega_i}$ is a bounded linear extension operator from $H^1(\Omega^0_+)$ to $H^1(\mathbb{R}^n)$ and hence from $H^1(\Omega^0_+)$ to $\mathcal{H}^1(\Omega)$, as asserted.

Assume now that $u_+ \in H^1(\Omega^0_+)$ and $u_- \in \mathcal{H}^1(\Omega^0_-)$ satisfy $\gamma_+ u_+ = \gamma_- u_-$ on $\partial\Omega^0$. Let

$$u^*_- := (\mathcal{E}_{\Omega^0_+} u_+)|_{\Omega^0_-} \text{ in } \Omega^0_-. \tag{A.1}$$

Then $u^*_- \in \mathcal{H}^1(\Omega^0_-)$, and there exists $c = c(n, \Omega^0_+, \Omega^0_-)$, such that

$$\|u^*_-\|_{\mathcal{H}^1(\Omega^0_-)} \leq c \|u_+\|_{H^1(\Omega^0_+)}.$$

In addition, in view of (A.1) we have $\gamma_- u^*_- = \gamma_-(\mathcal{E}_{\Omega^0_+} u_+) = \gamma_+ u_+ = \gamma_- u_-$ on $\partial\Omega^0_-$ and hence $\gamma_-(u_- - u^*_-) = 0$ on $\partial\Omega^0_-$. Thus, $\mathring{E}_{\Omega^0_-}(u_- - u^*_-)$ belongs to $\mathcal{H}^1(\Omega)$ and there exists $c_1 = c_1(n, \Omega^0_\pm) > 0$, such that

$$\|\mathring{E}_{\Omega^0_-}(u_- - u^*_-)\|_{\mathcal{H}^1(\Omega)} \leq c_1 \left(\|u_+\|_{H^1(\Omega^0_+)} + \|u_-\|_{\mathcal{H}^1(\Omega^0_-)} \right), \tag{A.2}$$

where $\mathring{E}_{\Omega^0_-}$ is the operator of extension by zero outside Ω^0_-. Let us now define the function

$$u := \mathring{E}_{\Omega^0_-}(u_- - u^*_-) + \mathcal{E}_{\Omega^0_+} u_+. \tag{A.3}$$

It belongs to $\mathcal{H}^1(\Omega)$. According to (A.1) and (A.3) we have also the following relations

$$u|_{\Omega^0_+} = 0 + (\mathcal{E}_{\Omega^0_+} u_+)|_{\Omega^0_+} = u_+ \text{ a.e. in } \Omega^0_+,$$
$$u|_{\Omega^0_-} = u_- - u^*_- + (\mathcal{E}_{\Omega^0_+} u_+)|_{\Omega^0_-} = u_- - u^*_- + u^*_- = u_- \text{ a.e. in } \Omega^0_-,$$

and thus the existence of a gluing function u is proved.

To show that the function u is unique, assume that there are two such functions, u_1 and u_2. Then $u_0 := u_1 - u_2$ belongs to $\mathcal{H}^1(\Omega)$ and $u_0|_{\Omega^0_\pm} = 0$. Thus, $u_0 \in H^1(\Omega) \subset L_2(\Omega)$ and its support is a subset of $\partial\Omega^0 = \partial\Omega^0_\pm$. Hence $u_0 = 0$ a.e in Ω (cf. also [38, Theorem 2.10(i)]).

(iv). For $u \in H^1(\Omega)$ the result is well known. Let $u \in \mathcal{H}^1(\Omega)$. Consequently, $u \in H^1_{\rm loc}(\Omega)$, and then $\gamma_+ u = \gamma_- u$, i.e., $[\gamma u] = 0$. □

Appendix B. **Several norm estimates.** In this appendix we provide some estimates used in the analysis of the Navier-Stokes problems. Let $n = 3$ or $n = 4$ and Ω be a bounded domain in $\mathbb{R}^n$.

• By the Sobolev embedding theorem (see, e.g., [44, Section 2.2.4, Corollary 2]), the space $H^1(\Omega)^n$ is continuously embedded in $L_{\frac{2n}{n-2}}(\Omega)^n$ and hence in $L_n(\Omega)^n$. Thus by the Hölder inequality there exists a constant $c_1 > 0$ such that

$$\begin{aligned}\|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\|_{L_{\frac{n}{n-1}}(\Omega)^n} &\le \|\mathbf{v}_1\|_{L_{\frac{2n}{n-2}}(\Omega)^n}\|\nabla\mathbf{v}_2\|_{L_2(\Omega)^{n\times n}}\\ &\le c_1\|\mathbf{v}_1\|_{H^1(\Omega)^n}\|\mathbf{v}_2\|_{H^1(\Omega)^n}\quad \forall\,\mathbf{v}_1,\mathbf{v}_2\in H^1(\Omega)^n.\end{aligned}$$

Consequently, $(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2 \in L_{\frac{n}{n-1}}(\Omega)^n$ for all $\mathbf{v}_1,\mathbf{v}_2\in H^1(\Omega)^n$, and, thus, $(\mathbf{v}\cdot\nabla)\mathbf{v}\in L_{\frac{n}{n-1}}(\Omega)^n$ for any $\mathbf{v}\in H^1(\Omega)^n$. Then by the Hölder inequality, we obtain for all $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\in H^1(\Omega)^n$,

$$\begin{aligned}|\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_\Omega| &\le \|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\|_{L_{\frac{n}{n-1}}(\Omega)^n}\|\mathbf{v}_3\|_{L_n(\Omega)^n}\\ &\le c_2\|\mathbf{v}_1\|_{L_{\frac{2n}{n-2}}(\Omega)^n}\|\nabla\mathbf{v}_2\|_{L_2(\Omega)^{n\times n}}\|\mathbf{v}_3\|_{H^1(\Omega)^n},\end{aligned}\tag{B.1}$$

with some constant $c_2 > 0$. Taking $\mathbf{v}_3 \in \mathring{H}^1(\Omega)^n$ in (B.1), we deduce that the term $(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2$ belongs to the dual of the space $\mathring{H}^1(\Omega)^n$, i.e., to the space $H^{-1}(\Omega)^n$. Moreover, there exists a constant $c_3 > 0$ such that, for all $\mathbf{v}_1,\mathbf{v}_2\in H^1(\Omega)^n$,

$$\begin{aligned}\|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\|_{H^{-1}(\Omega)^n} &\le c_2\|\mathbf{v}_1\|_{L_{\frac{2n}{n-2}}(\Omega)^n}\|\nabla\mathbf{v}_2\|_{L_2(\Omega)^{n\times n}}\\ &\le c_3\|\mathbf{v}_1\|_{H^1(\Omega)^n}\|\mathbf{v}_2\|_{H^1(\Omega)^n}.\end{aligned}\tag{B.2}$$

Similar to (B.1), we have for all $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\in H^1(\Omega)^n$,

$$\begin{aligned}|\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_\Omega| &\le \|\mathbf{v}_3\|_{L_{\frac{2n}{n-2}}(\Omega)^n}\|\nabla\mathbf{v}_2\|_{L_2(\Omega)^{n\times n}}\|\mathbf{v}_1\|_{L_n(\Omega)^n}\\ &\le c_4\|\mathbf{v}_1\|_{L_n(\Omega)^n}\|\nabla\mathbf{v}_2\|_{L_2(\Omega)^{n\times n}}\|\mathbf{v}_3\|_{H^1(\Omega)^n},\end{aligned}\tag{B.3}$$

with some constant $c_4 > 0$. Taking $\mathbf{v}_3 \in \mathring{H}^1(\Omega)^n$ and $\mathbf{v}_1,\mathbf{v}_2\in H^1(\Omega)^n$ in (B.3) we obtain

$$\begin{aligned}\|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\|_{H^{-1}(\Omega)^3} &\le c_4\|\mathbf{v}_1\|_{L_n(\Omega)^n}\|\nabla\mathbf{v}_2\|_{L_2(\Omega)^{n\times n}}\\ &\le c_5\|\mathbf{v}_1\|_{L_n(\Omega)^n}\|\mathbf{v}_2\|_{H^1(\Omega)^n}.\end{aligned}\tag{B.4}$$

with some constant $c_5 > 0$.

• Let now $\mathbf{v}_1,\mathbf{v}_2\in H^1(\Omega)^n$, $\mathbf{v}_3\in\mathring{H}^1(\Omega)^n$. The density of the space $\mathcal{D}(\Omega)^n$ in $\mathring{H}^1(\Omega)^n$ along with the divergence theorem and estimate (B.2) lead to the following identity

$$\begin{aligned}\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_\Omega &= \int_\Omega \nabla\cdot(\mathbf{v}_1(\mathbf{v}_2\cdot\mathbf{v}_3))\,d\mathbf{x} - \langle(\nabla\cdot\mathbf{v}_1)\mathbf{v}_3+(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3,\mathbf{v}_2\rangle_\Omega\\ &= -\langle(\nabla\cdot\mathbf{v}_1)\mathbf{v}_3+(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3,\mathbf{v}_2\rangle_\Omega\\ &\qquad\qquad \forall\,\mathbf{v}_1,\mathbf{v}_2\in H^1(\Omega)^n,\ \mathbf{v}_3\in\mathring{H}^1(\Omega)^n.\end{aligned}\tag{B.5}$$

Then in view of identity (B.5) and estimate (B.1), there exists a constant $c_6 > 0$ such that

$$|\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_\Omega| \le \|(\nabla\cdot\mathbf{v}_1)\mathbf{v}_3+(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3\|_{L_{\frac{n}{n-1}}(\Omega)^n}\|\mathbf{v}_2\|_{L_n(\Omega)^n} \le c_6\|\mathbf{v}_1\|_{H^1(\Omega)^n}\|\mathbf{v}_2\|_{L_n(\Omega)^n}\|\mathbf{v}_3\|_{H^1(\Omega)^n}, \tag{B.6}$$

and accordingly that

$$\|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\|_{H^{-1}(\Omega)^3} \le c_6\|\mathbf{v}_1\|_{H^1(\Omega)^n}\|\mathbf{v}_2\|_{L_n(\Omega)^n} \quad \forall\ \mathbf{v}_1,\mathbf{v}_2\in H^1(\Omega)^n. \tag{B.7}$$

- From (B.5) we also have

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_\Omega = -\ \langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3,\mathbf{v}_2\rangle_\Omega \qquad \forall\ \mathbf{v}_1\in H^1_{\mathrm{div}}(\Omega)^n,\ \mathbf{v}_2\in H^1(\Omega)^n,\ \mathbf{v}_3\in \mathring{H}^1(\Omega)^n\,, \tag{B.8}$$

implying the well known formula

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_2\rangle_\Omega = 0 \quad \forall\ \mathbf{v}_1\in H^1_{\mathrm{div}}(\Omega)^n,\ \mathbf{v}_2\in \mathring{H}^1(\Omega)^n. \tag{B.9}$$

Identity (B.8) also implies for $\mathbf{v}_1\in H^1_{\mathrm{div}}(\Omega)^n$ and $\mathbf{v}_2\in H^1(\Omega)^n$,

$$\begin{aligned}\|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\|_{H^{-1}(\Omega)^n} &= \sup_{\mathbf{v}_3\in\mathring{H}^1(\Omega)^n,\,\|\nabla\mathbf{v}_3\|_{L_2(\Omega)^{n\times n}}=1}\Big|\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_\Omega\Big| \\ &= \sup_{\mathbf{v}_3\in\mathring{H}^1(\Omega)^n,\,\|\nabla\mathbf{v}_3\|_{L_2(\Omega)^{n\times n}}=1}\Big|\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3,\mathbf{v}_2\rangle_\Omega\Big| \\ &\le \sup_{\mathbf{v}_3\in\mathring{H}^1(\Omega)^n,\,\|\nabla\mathbf{v}_3\|_{L_2(\Omega)^{n\times n}}=1}\|\mathbf{v}_1\otimes\mathbf{v}_2\|_{L_2(\Omega)^{n\times n}}\|\nabla\mathbf{v}_3\|_{L_2(\Omega)^{n\times n}}\end{aligned}$$

and hence

$$\|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\|_{H^{-1}(\Omega)^n} \le \|\mathbf{v}_1\otimes\mathbf{v}_2\|_{L_2(\Omega)^{n\times n}} \quad \forall\ \mathbf{v}_1\in H^1_{\mathrm{div}}(\Omega)^n,\ \mathbf{v}_2\in H^1(\Omega)^n. \tag{B.10}$$

- By the Hölder inequality,

$$\|v\|^4_{L_4(\Omega)} \le \|v\|^4_{L_6(\Omega)}|\Omega|^{1/3}, \quad |\Omega| := \int_\Omega dx \quad \forall\ v\in L_6(\Omega), \tag{B.11}$$

and the Sobolev inequality (see, e.g., Eq. (II.3.7) in [22]) gives

$$\|v\|_{L_6(\mathbb{R}^n)} \le \frac{(n-1)}{\sqrt{n}(n-2)}\|\nabla v\|_{L_2(\mathbb{R}^n)} \quad \forall\ v\in\mathcal{D}(\mathbb{R}^n). \tag{B.12}$$

Since $\mathcal{D}_{\mathrm{div}}(\Omega)\subset\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathring{H}^1_{\mathrm{div}}(\Omega)^3$ (see, e.g., [47, p. 32, Lemma 10]), (B.10) and (B.12) imply

$$\begin{aligned}|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1\|_{H^{-1}(\Omega)^3} &\le \|\mathbf{v}_1\otimes\mathbf{v}_1\|_{L_2(\Omega)^{3\times3}} \\ &\le \|\mathbf{v}_1\|^2_{L_4(\Omega)^3} \\ &\le |\Omega|^{1/6}\|\mathbf{v}_1\|^2_{L_6(\Omega)^3} \\ &\le \frac{4}{3}|\Omega|^{1/6}\|\nabla\mathbf{v}_1\|^2_{L_2(\Omega_+)^{3\times3}}\ \forall\ \mathbf{v}_1\in\mathring{H}^1_{\mathrm{div}}(\Omega)^3,\end{aligned} \tag{B.13}$$

cf. Lemma IX.1.1 in [22], where a similar estimate was obtained with $\frac{2\sqrt{2}}{3}$ instead of $\frac{4}{3}$.

Similarly, since the space $\mathcal{D}_{\rm div}(\mathbb{R}^n) \subset \mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{H}^1_{\rm div}(\Omega)^3$ (see, e.g., [4, Proposition 2.2]), inequalities (B.10) and (B.12) imply for all $\mathbf{v}_1 \in \mathcal{H}^1_{\rm div}(\mathbb{R}^3)^3$ that

$$\begin{aligned}|(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1\|_{H^{-1}(\Omega)^3} &\leq |\Omega|^{1/6}\|\mathbf{v}_1\|^2_{L_6(\Omega)^3}\\ &\leq |\Omega|^{1/6}\|\mathbf{v}_1\|^2_{L_6(\mathbb{R}^3)^3}\\ &\leq \frac{4}{3}|\Omega|^{1/6}\|\nabla\mathbf{v}_1\|^2_{L_2(\mathbb{R}^3)^{3\times 3}}\,. \qquad \text{(B.14)}\end{aligned}$$

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REFERENCES

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, Academic Press, 2003.
[2] F. Alliot and C. Amrouche, The Stokes problem in $\mathbb{R}^n$: An approach in weighted Sobolev spaces, *Math. Models Meth. Appl. Sci.*, **9** (1999), 723–754.
[3] F. Alliot and C. Amrouche, On the regularity and decay of the weak solutions to the steady-state Navier-Stokes equations in exterior domains, in *Applied Nonlinear Analysis* (eds. A. Sequeira, H. Beirao da Veiga and J. H. Videman), Kluwer Academic, Dordrecht, (1999), 1–18.
[4] F. Alliot and C. Amrouche, Weak solutions for the exterior Stokes problem in weighted Sobolev spaces, *Math. Meth. Appl. Sci.*, **23** (2000), 575–600.
[5] C. Amrouche, P. G. Ciarlet and C. Mardare, On a Lemma of Jacques-Louis Lions and its relation to other fundamental results, *J. Math. Pures Appl.*, **104** (2015), 207–226.
[6] C. Amrouche, V. Girault and J. Giroire, Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator. An approach in weighted Sobolev spaces, *J. Math. Pures Appl.*, **76** (1997), 55–81.
[7] I. Babuška, The finite element method with Lagrangian multipliers, *Numer. Math.*, **20** (1973), 179–192.
[8] M. E. Bogovskiĭ, Solution of some problems of vector analysis related with operators *div* and *grad*, in *Teoriya Kubaturnyh Formul I Prilozheniya Funktsional'nogo Analiza k Zadacham Matematicheskoi Fiziki. Trudy Seminara S.L.Soboleva, No.1*, Siberian brunch of the Academy of Sci. of USSR, Institute of Mathematics, Novosibirsk, (1980), 5–40 (in Russian).
[9] W. Borchers and H. Sohr, The equations $\operatorname{rot} v = g$ and $\operatorname{div} u = f$ with zero boundary condition, *Hokkaido Math. J.*, **19** (1990), 67–87.
[10] K. Brewster, D. Mitrea, I. Mitrea and M. Mitrea, Extending Sobolev functions with partially vanishing traces from locally (ϵ, δ)-domains and applications to mixed boundary problems, *J. Funct. Anal.*, **266** (2014), 4314–4421.
[11] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from lagrangian multipliers, *R.A.I.R.O. Anal. Numer.*, **8** (1974), 129–151.
[12] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Series in Comput. Math., **15**, Springer-Verlag, New York, 1991.
[13] O. Chkadua, S. E. Mikhailov and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. I. Equivalence and invertibility, *J. Int. Equ. Appl.*, **21** (2009), 499–542.
[14] O. Chkadua, S. E. Mikhailov and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. II. Solution regularity and asymptotics, *J. Int. Equ. Appl.*, **22** (2010), 19–37.
[15] J. Choi, H. Dong and D. Kim, Conormal derivative problems for stationary Stokes system in Sobolev spaces, *Discrete Contin. Dyn. Syst.*, **38** (2018), 2349–2374.
[16] J. Choi, H. Dong and D. Kim, Green functions of conormal derivative problems for stationary Stokes system, *J. Math. Fluid Mech.*, **20** (2018), 1745–1769.

[17] J. Choi and M. Yang, Fundamental solutions for stationary Stokes systems with measurable coefficients, *J. Diff. Equ.*, **263** (2017), 3854–3893.

[18] M. Costabel, Boundary integral operators on Lipschitz domains: Elementary results, *SIAM J. Math. Anal.*, **19** (1988), 613–626.

[19] M. Dindoš and M. Mitrea, The stationary Navier-Stokes system in nonsmooth manifolds: The Poisson problem in Lipschitz and C^1 domains, *Arch. Rational Mech. Anal.*, **174** (2004), 1–47.

[20] B. R. Duffy, Flow of a liquid with an anisotropic viscosity tensor, *J. Nonnewton. Fluid Mech.*, **4** (1978), 177–193.

[21] A. Ern and J. L. Guermond, *Theory and Practice of Finite Elements*, Springer, New York, 2004.

[22] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Steady-State Problems*, Second Edition, Springer, New York, 2011.

[23] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.

[24] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Theory and Algorithms, Springer, Berlin, 1986.

[25] V. Girault and A. Sequeira, A well-posed problem for the exterior Stokes equations in two and three dimensions, *Arch. Rational Mech. Anal.*, **114** (1991), 313–333.

[26] B. Hanouzet, Espaces de Sobolev avec poids – application au probléme de Dirichlet dans un demi-espace, *Rend. Sere. Mat. Univ. Padova*, **46** (1971), 227–272.

[27] G. C. Hsiao and W. L. Wendland, *Boundary Integral Equations*, Springer-Verlag, Heidelberg 2008.

[28] M. Kohr, M. Lanza de Cristoforis, S. E. Mikhailov and W. L. Wendland, Integral potential method for transmission problem with Lipschitz interface in $\mathbb{R}^3$ for the Stokes and Darcy-Forchheimer-Brinkman PDE systems, *Z. Angew. Math. Phys.*, **67** (2016), Art. 116, 30 pp.

[29] M. Kohr, M. Lanza de Cristoforis and W. L. Wendland, Nonlinear Neumann-transmission problems for Stokes and Brinkman equations on Euclidean Lipschitz domains, *Potential Anal.*, **38** (2013), 1123–1171.

[30] M. Kohr, M. Lanza de Cristoforis and W. L. Wendland, Poisson problems for semilinear Brinkman systems on Lipschitz domains in $\mathbb{R}^3$, *Z. Angew. Math. Phys.*, **66** (2015), 833–864.

[31] M. Kohr, S. E. Mikhailov and W. L. Wendland, Potentials and transmission problems in weighted Sobolev spaces for anisotropic Stokes and Navier-Stokes systems with L_∞ strongly elliptic coefficient tensor, *Complex Var. Elliptic Equ.*, **65** (2020), 109–140.

[32] M. Kohr, S. E. Mikhailov and W. L. Wendland, Layer potential theory for the anisotropic Stokes system with variable L_∞ symmetrically elliptic tensor coefficient, *Math. Meth. Appl. Sci.*, to appear.

[33] M. Kohr and W. L. Wendland, Variational approach for the Stokes and Navier-Stokes systems with nonsmooth coefficients in Lipschitz domains on compact Riemannian manifolds, *Calc. Var. Partial Differ. Equ.*, **57** (2018), Paper No. 165, 41 pp.

[34] M. Kohr and W. L. Wendland, Layer potentials and Poisson problems for the nonsmooth coefficient Brinkman system in Sobolev and Besov spaces, *J. Math. Fluid Mech.*, **20** (2018), 1921–1965.

[35] O. A. Ladyzhenskaya and V. A. Solonnikov, Some problems of vector analysis and generalized formulations of boundary value problems for Navier-Stokes equations, *Zap. Nauchn. Sem. LOMI. Leningrad. Otdel. Mat. Inst. Steklov*, **59** (1976), 81–116.

[36] A. L. Mazzucato and V. Nistor, Well-posedness and regularity for the elasticity equation with mixed boundary conditions on polyhedral domains and domains with cracks, *Arch. Rational Mech. Anal.*, **195** (2010), 25–73.

[37] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, UK, 2000.

[38] S. E. Mikhailov, Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains, *J. Math. Anal. Appl.*, **378** (2011), 324–342.

[39] S. E. Mikhailov, Solution regularity and co-normal derivatives for elliptic systems with non-smooth coefficients on Lipschitz domains, *J. Math. Anal. Appl.*, **400** (2013), 48–67.

[40] S. E. Mikhailov and C. Fresneda-Portillo, Boundary-domain integral equations equivalent to an exterior mixed BVP for the variable-viscosity compressible Stokes PDEs, *Comm. Pure and Applied Analysis*, **18** (2019), 3059–3088.

[41] M. Mitrea and M. Wright, Boundary value problems for the Stokes system in arbitrary Lipschitz domains. *Astérisque*, **344** (2012), viii+241 pp.

[42] O. A. Oleinik, A. S. Shamaev and G. A. Yosifian, *Mathematical Problems in Elasticity and Homogenization*, North-Holland, Amsterdam, 1992.

[43] E. Otárola and A. J. Salgado, A weighted setting for the stationary Navier-Stokes equations under singular forcing, *Appl. Math. Letters*, **99** (2020), 105933, 7pp.

[44] T. Runst and W. Sickel, *Sobolev Spaces of Fractional order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, De Gruyter, Berlin, 1996.

[45] F.-J. Sayas and V. Selgas, Variational views of Stokeslets and stresslets, *SeMA*, **63** (2014), 65–90.

[46] G. Seregin, *Lecture Notes on Regularity Theory for the Navier-Stokes Equations*, World Scientific, London, 2015.

[47] L. Tartar, *Topics in Nonlinear Analysis*, Publications Mathéatiques d'Orsay, 1978.

[48] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, AMS Chelsea Publishing, Providence, 2001.

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