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Calculus of Variations

Non-homogeneous Dirichlet-transmission problems for the anisotropic Stokes and Navier-Stokes systems in Lipschitz domains with transversal interfaces

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Abstract
This paper is build around the stationary anisotropic Stokes and Navier-Stokes systems with an L^∞-tensor coefficient satisfying an ellipticity condition in terms of symmetric matrices in $\mathbb{R}^{n\times n}$ with zero matrix traces. We analyze, in L^2-based Sobolev spaces, the non-homogeneous boundary value problems of Dirichlet-transmission type for the anisotropic Stokes and Navier-Stokes systems in a compressible framework in a bounded Lipschitz domain with a transversal Lipschitz interface in $\mathbb{R}^n$, $n \geq 2$ ($n = 2, 3$ for the nonlinear problems). Thus, the interface intersects transversally the boundary of the Lipschitz domain and divides the domain into two Lipschitz sub-domains. First, we use a mixed variational approach to prove the well-posedness of linear problems related to the anisotropic Stokes system. Then we show the existence of a weak solution to the Dirichlet and Dirichlet-transmission problems for the nonlinear anisotropic Navier-Stokes system. This is done by implementing the Leray-Schauder fixed point theorem and using various results and estimates from the linear case, as well as the Leray-Hopf and some other norm inequalities. Explicit conditions for uniqueness of solutions to the nonlinear problems are also provided.

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1 Introduction

Variational methods have been intensively used in the analysis of elliptic boundary problems, in particular, boundary value problems for the Stokes and Navier-Stokes equations (see, e.g., [17, 26, 55]). Employing variational methods, Angot [4, 5] analyzed a well-posedness of some Stokes/Brinkman problems with constant isotropic viscosity and a family of embedded jump conditions on an immersed (transversal) interface with weak regularity assumptions.

The authors in [32] combined a layer potential approach with the Leray-Schauder fixed point theorem and proved existence results for a nonlinear Neumann-transmission problem for the Stokes and Brinkman systems in L^p, Sobolev, and Besov spaces.

Dong and Kim [19] obtained regularity results for the Stokes system with measurable coefficients in one direction (see also [15]). Korobkov, Pileckas and Russo [39] analyzed the flux problem in the theory of steady Navier-Stokes equations with constant coefficients and non-homogeneous boundary conditions. Amrouche and Rodríguez-Bellido [2] proved the existence of a very weak solution for the non-homogeneous Dirichlet problem for the compressible Navier-Stokes system in a bounded domain of the class $C^{1,1}$ in $\mathbb{R}^3$.

An alternative integral approach, which reduces boundary value problems for the Stokes system with variable coefficients and a large spectrum of other variable-coefficient elliptic partial differential equations to boundary-domain integral equations (BDIEs), by employing explicit parametrix-based integral potentials, was developed in [12–14, 24, 48].

Mazzucato and Nistor [44] obtained well-posedness and regularity results in weighted Sobolev spaces for the anisotropic linear elasticity equations with mixed boundary conditions on polyhedral domains. Brewster et al. [9] used a variational approach to show well-posedness of Dirichlet, Neumann and mixed boundary problems for higher order divergence-form elliptic equations with L^∞ coefficients in locally (ϵ, δ)-domains and in Besov and Bessel potential spaces.

The coupling of fluid flows with porous media flows involving the stationary incompressible Navier-Stokes equations with constant viscosity and the Darcy equations with a permeability given in terms of a uniformly elliptic matrix-valued function with L^∞ coefficients has been recently investigated in [27] (see also [26]).

The authors in [34] developed a variational analysis in the pseudostress setting for transmission problems with internal interfaces in weighted Sobolev spaces for the anisotropic Stokes and Navier-Stokes systems with L^∞ strongly elliptic coefficient tensor, see also [19]. Note that in [34] and [19] it was assumed that the coefficients of the viscosity tensor satisfy a stronger ellipticity condition than in (2.4), for all matrices in $\mathbb{R}^{n\times n}$ (not only for symmetric and with zero-trace, see [34, Eqs. (2)–(3)]). Such a condition allowed to explore the associated non-symmetric pseudostress setting (see also [33], and [37, 38] for the Stokes and Navier-Stokes systems with non-smooth coefficients in compact Riemannian setting). The authors extended in [35] and [36] their variational analysis to other transmission and exterior boundary problems with internal interfaces for the anisotropic Stokes and Navier-Stokes systems by assuming that the corresponding L^∞ viscosity tensor coefficient satisfies the ellipticity condition only in terms of symmetric matrices in $\mathbb{R}^{n\times n}$ with zero traces, that is, the relaxed ellipticity condition (2.4). Only homogeneous Dirichlet conditions and zero velocity jumps were considered in the (nonlinear) Navier-Stokes problems in [33–38].

In this paper we investigate *non-homogeneous* Dirichlet-transmission problems for the anisotropic Stokes and Navier-Stokes systems in a bounded Lipschitz domain of $\mathbb{R}^n$ ($n = 2, 3$ for the nonlinear problems) with *a transversal Lipschitz interface that intersects the boundary of the domain.* As in [35] and [36], we impose the ellipticity condition (2.4), which is less restrictive than in [34] and [19]. We show well-posedness results for the linear problems, as well as existence results for the nonlinear problems in L^2-based Sobolev spaces. First, we explore equivalent mixed variational formulations and prove the well-posedness of linear Dirichlet-transmission problems for the anisotropic Stokes system in a compressible framework in bounded Lipschitz domains of $\mathbb{R}^n$ with transversal Lipschitz interfaces and given data in L^2-based Sobolev spaces. Next, we use well-posedness results in the linear case and the Leray-Schauder fixed point theorem and show the existence of a weak solution of the Dirichlet problem for the anisotropic Navier-Stokes system with general non-homogeneous data in L^2-based Sobolev spaces in a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$. Finally, we prove the existence of weak solutions $\mathbf{u}$ of the Dirichlet-transmission problems for the anisotropic Navier-Stokes system in a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, with transversal Lipschitz interface and data in L^2-based Sobolev spaces.

In addition to their mathematical interest, the anisotropic Stokes and Navier-Stokes interface problems analyzed in this paper describe *multiphase flows* of immiscible fluids with variable anisotropic viscosity tensors and compressibility influenced, e.g., by varying temperature of the fluids (cf., e.g., [20], [43, Chapter 3]) and transmission conditions prescribed on the interfaces. They are motivated by various industrial, biological, medical and environmental applications (see, e.g., [8, Section 1.1] and the references therein). Note that mathematically the interface Stokes problems are close to the interface problems of elasticity encountered in modelling composite materials, see, e.g., [59] and the references therein, or in unilateral contact problems, cf. [21, 22].

2 Anisotropic Stokes system with elliptic L^∞ viscosity tensor coefficient

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an open set, and let $\mathfrak{L}$ denote a second order differential operator in the component-wise divergence form[1],

$$(\mathfrak{L}\mathbf{u})_i := \partial_\alpha\big(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u})\big), \quad i = 1, \ldots, n, \tag{2.1}$$

where $\mathbf{u} = (u_1, \ldots, u_n)^\top$, $E_{j\beta}(\mathbf{u}) := \frac{1}{2}(\partial_j u_\beta + \partial_\beta u_j)$ are the entries of the symmetric part $\mathbb{E}(\mathbf{u})$ of $\nabla\mathbf{u}$ (the gradient of $\mathbf{u}$), and $a_{ij}^{\alpha\beta}$ are essentially bounded, measurable, real-valued components of the tensor viscosity coefficient $\mathbb{A}$, that is,

$$\mathbb{A} := \left(a_{ij}^{\alpha\beta}\right)_{1\leq i,j,\alpha,\beta\leq n}, \ a_{ij}^{\alpha\beta} \in L^\infty(\Omega), \ 1 \leq i, j, \alpha, \beta \leq n, \tag{2.2}$$

and satisfy the following symmetry conditions

$$a_{ij}^{\alpha\beta}(x) = a_{\alpha j}^{i\beta}(x) = a_{i\beta}^{\alpha j}(x), \quad x \in \Omega \tag{2.3}$$

(see [52, Eqs (3.1),(3.3)]). In addition, we require that $\mathbb{A}$ satisfies the ellipticity condition only in terms of all *symmetric* matrices in $\mathbb{R}^{n\times n}$ with *zero matrix trace*. This ellipticity condition

[1] The standard notation ∂_β for the first order partial derivative $\frac{\partial}{\partial x_\beta}$, $\beta = 1, \ldots, n$, and the Einstein summation rule on repeated indices are used all along the paper.

has been first used in [35, 36]. Thus, we assume that there exists a constant $C_{\mathbb{A}} > 0$ such that, for almost all $x \in \Omega$,

$$a_{ij}^{\alpha\beta}(x)\xi_{i\alpha}\xi_{j\beta} \geq C_{\mathbb{A}}^{-1}|\boldsymbol{\xi}|^2\,, \quad \forall\, \boldsymbol{\xi} = (\xi_{i\alpha})_{i,\alpha=1,\ldots,n} \in \mathbb{R}^{n\times n}$$
$$\text{such that } \boldsymbol{\xi} = \boldsymbol{\xi}^{\top} \text{ and } \sum_{i=1}^{n}\xi_{ii} = 0, \tag{2.4}$$

where $|\boldsymbol{\xi}|^2 = \xi_{i\alpha}\xi_{i\alpha}$, and the superscript $\top$ denotes the transpose of a matrix.

The tensor coefficient $\mathbb{A}$ is endowed with the norm

$$\|\mathbb{A}\| := \max\left\{\|a_{ij}^{\alpha\beta}\|_{L^\infty(\Omega)} : i, j, \alpha, \beta = 1\ldots, n\right\}. \tag{2.5}$$

The tensor $\mathbb{A}$ can be also considered as consisting of $n \times n$ matrix valued functions $A^{\alpha\beta}$, that is,

$$\mathbb{A} = \left(A^{\alpha\beta}\right)_{1\leq\alpha,\beta\leq n}\,,\ A^{\alpha\beta} := \left(a_{ij}^{\alpha\beta}\right)_{1\leq i,j\leq n}\,,\ 1 \leq \alpha, \beta \leq n\,. \tag{2.6}$$

and the symmetry conditions (2.3) lead to the following equivalent forms of the operator $\mathfrak{L}$

$$(\mathfrak{L}\mathbf{u})_i = \partial_\alpha\big(a_{ij}^{\alpha\beta}\partial_\beta u_j\big),\ \ i = 1, \ldots, n; \quad \mathfrak{L}\mathbf{u} = \partial_\alpha\left(A^{\alpha\beta}\partial_\beta\mathbf{u}\right), \tag{2.7}$$

Let $\mathbf{u}$ be an unknown vector field, π be an unknown scalar field, $\mathbf{f}$ be a given vector field and g be a given scalar field defined in Ω. Then the equations

$$\boldsymbol{\mathcal{L}}(\mathbf{u}, \pi) := \mathfrak{L}\mathbf{u} - \nabla\pi = \mathbf{f},\ \operatorname{div}\mathbf{u} = g \text{ in } \Omega \tag{2.8}$$

determine the *anisotropic Stokes system with variable viscosity tensor coefficient* $\mathbb{A} = \left(A^{\alpha\beta}\right)_{1\leq\alpha,\beta\leq n}$ *in a compressible framework*.

According to (2.7) and (2.1), the Stokes operator $\boldsymbol{\mathcal{L}}$ can be written in any of the following equivalent forms

$$\boldsymbol{\mathcal{L}}(\mathbf{u}, \pi) = \partial_\alpha\left(A^{\alpha\beta}\partial_\beta\mathbf{u}\right) - \nabla\pi, \tag{2.9}$$
$$(\boldsymbol{\mathcal{L}}(\mathbf{u}, \pi))_i = \partial_\alpha\big(a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u})\big) - \partial_i\pi,\ i = 1, \ldots, n\,. \tag{2.10}$$

In addition, the following nonlinear system

$$\boldsymbol{\mathcal{L}}(\mathbf{u}, \pi) - (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}\,,\ \ \operatorname{div}\mathbf{u} = g \text{ in } \Omega \tag{2.11}$$

is called the *anisotropic Navier-Stokes system with variable viscosity tensor coefficient* $\mathbb{A} = \left(A^{\alpha\beta}\right)_{1\leq\alpha,\beta\leq n}$ *in a compressible framework*.

If $g = 0$ in (2.8) and (2.11) one obtains the *anisotropic Stokes and Navier-Stokes systems in the incompressible case*.

In the *isotropic case*, the tensor $\mathbb{A}$ in (2.2) has the following entries

$$a_{ij}^{\alpha\beta}(x) = \lambda(x)\delta_{i\alpha}\delta_{j\beta} + \mu(x)\left(\delta_{\alpha j}\delta_{\beta i} + \delta_{\alpha\beta}\delta_{ij}\right),\ 1 \leq i, j, \alpha, \beta \leq n\,, \tag{2.12}$$

where $\lambda, \mu \in L^\infty(\Omega)$, and $c_\mu^{-1} \leq \mu(x) \leq c_\mu$ for a.e. $x \in \Omega$, with some constant $c_\mu > 0$ (cf., e.g., Appendix III, Part I, Sect. 1 in [58]). Then it is immediate that condition (2.4) is fulfilled (see also [36]) and thus our results apply also to the Stokes system in the *isotropic case*.

The relaxed ellipticity condition (2.4) [for symmetric matrices with zero matrix trace] is essentially weaker than, say, the corresponding strong ellipticity condition [for all matrices].

For example, the isotropic viscosity tensor (2.12) satisfies the relaxed ellipticity condition (2.4) for any constant λ and any constant $\mu > 0$. However, if $\lambda < -2\mu/n$, the tensor fails to satisfy the strong ellipticity condition on the identity matrix, i.e., $\xi_{i\alpha} = \delta_{i\alpha}, i, \alpha \in \{1, \dots, n\}$, (having the non-zero trace $\xi_{ii} = n$). Note also that the isotropic viscosity tensor (2.12) fails to satisfy the strong ellipticity condition on the antisymmetric matrices, $\xi_{i\alpha} = -\xi_{\alpha i}$, $i, \alpha \in \{1, \dots, n\}$, but still satisfies the relaxed ellipticity condition (2.4) . Imposing the ellipticity condition only on the symmetric matrices is also intrinsically related to the fact that the Stokes and Navier-Stokes equations can be formulated in terms of the symmetric strain rate tensor $\mathbb{E}$ rather than the velocity gradient.

It appears that the relaxed ellipticity condition (2.4) is still sufficient to deduce the well-posedness of the problems considered in the paper.

3 Functional framework and preliminaries

Given a Banach space $\mathcal{X}$, its topological dual is denoted by $\mathcal{X}'$, and the notation $\langle \cdot, \cdot \rangle_X$ means the duality pairing of two dual spaces defined on a set $X \subseteq \mathbb{R}^n$.

3.1 Sobolev spaces on Lipschitz domains in $\mathbb{R}^n$

Let $n \geq 2$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with connected boundary $\partial\Omega$. Let $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω, equipped with the inductive limit topology. Let $\mathcal{D}'(\Omega)$ denote the corresponding space of distributions on Ω, i.e., the dual of the space $\mathcal{D}(\Omega)$. Let $L^2(\Omega)$ be the Lebesgue space of square-integrable functions on Ω, and $L^\infty(\Omega)$ be the space of (equivalence classes of) essentially bounded measurable functions on Ω. Let also

$$L_0^2(\Omega) := \{f \in L^2(\Omega) : \langle f, 1\rangle_\Omega = 0\}. \tag{3.1}$$

The dual of $L_0^2(\Omega)$ is the space $L^2(\Omega)/\mathbb{R}$. The Sobolev space $H^1(\Omega)$ is defined as

$$H^1(\Omega) := \left\{f \in L^2(\Omega) : \nabla f \in L^2(\Omega)^n\right\}, \tag{3.2}$$

and is endowed with the norm

$$\|f\|_{H^1(\Omega)}^2 = \|f\|_{L^2(\Omega)}^2 + \|\nabla f\|_{L^2(\Omega)^n}^2. \tag{3.3}$$

The space $\widetilde{H}^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\mathbb{R}^n)$, and can be also described as

$$\widetilde{H}^1(\Omega) := \left\{\widetilde{f} \in H^1(\mathbb{R}^n) : \operatorname{supp} \widetilde{f} \subseteq \overline{\Omega}\right\}, \tag{3.4}$$

where $\operatorname{supp} f := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$. The dual of $\widetilde{H}^1(\Omega)$ is the space $H^{-1}(\Omega)$. Since $\mathcal{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$ (see, e.g., [45, p. 77]), the dual of $H^1(\Omega)$, denoted by $\widetilde{H}^{-1}(\Omega)$, is a space of distributions. Then the following equivalent characterization of the spaces $H^{\pm 1}(\Omega)$ holds

$$H^{\pm 1}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists\, F \in H^{\pm 1}(\mathbb{R}^n) \text{ such that } F|_\Omega = f\}, \tag{3.5}$$

where $|_X = r_X$ is the restriction operator of functions or distributions to a set X.

The closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ is denoted by $\mathring{H}^1(\Omega)$ and can be equivalently described as the space of all functions in $H^1(\Omega)$ with null traces on $\partial\Omega$, that is,

$$\mathring{H}^1(\Omega) := \{f \in H^1(\Omega) : \gamma_\Omega f = 0 \text{ on } \partial\Omega\}, \tag{3.6}$$

where $\gamma_\Omega : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is the trace operator. Recall that this is a linear, bounded and onto operator (cf. [17], [46, Lemma 2.6], [51, Theorem 2.5.2]). We will use the same notation γ_Ω for the trace operator acting on vector-valued functions.

Note that the spaces $\widetilde{H}^1(\Omega)$ and $\mathring{H}^1(\Omega)$ can be identified isomorphically via the operator $\mathring{E}_\Omega$ of extension by zero outside Ω (see, e.g., [45, Theorem 3.33]). The dual of $H^1(\Omega)$ is denoted by $\widetilde{H}^{-1}(\Omega)$, and is a space of distributions. (Note that $\widetilde{H}^{-1}(\mathbb{R}^n) = H^{-1}(\mathbb{R}^n)$.) Moreover, the following spaces can be isomorphically identified (cf., e.g., [45, Theorem 3.14])

$$\big(H^1(\Omega)\big)' = \widetilde{H}^{-1}(\Omega), \quad H^{-1}(\Omega) = \big(\widetilde{H}^1(\Omega)\big)'. \tag{3.7}$$

Let $s \in (0, 1)$. Then the boundary Sobolev space $H^s(\partial\Omega)$ is defined by

$$H^s(\partial\Omega) := \left\{ f \in L^2(\partial\Omega) : \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(\mathbf{x}) - f(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{n-1+2s}} d\sigma_{\mathbf{x}} d\sigma_{\mathbf{y}} < \infty \right\}, \tag{3.8}$$

where $\sigma_{\mathbf{y}}$ is the surface measure on $\partial\Omega$ (see, e.g., [51, Proposition 2.5.1]). The dual of $H^s(\partial\Omega)$ is the space $H^{-s}(\partial\Omega)$, and $H^0(\partial\Omega) = L^2(\partial\Omega)$.

By $H^1(\Omega)^n$, $\widetilde{H}^1(\Omega)^n$, $H^s(\partial\Omega)^n$ we denote the spaces of vector-valued functions whose components belong to the spaces $H^1(\Omega)$, $\widetilde{H}^1(\Omega)$, and $H^s(\partial\Omega)$, respectively. For further properties of Sobolev spaces we refer the reader to [31, 45, 51].

We will need the following well known result (see, e.g., [40, Lemma 2.5], [7], [3, Theorem 3.1]), for which we will provide several generalizations further on.

Proposition 3.1 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, with connected boundary. Then the divergence operator* $\mathrm{div} : \mathring{H}^1(\Omega)^n \to L^2_0(\Omega)$ *is bounded, linear and surjective. It has a bounded, linear right inverse $\mathcal{R}_\Omega : L^2_0(\Omega) \to \mathring{H}^1(\Omega)^n$. Thus, there exists a constant $C = C(\Omega, n) > 0$ such that*

$$\mathrm{div}(\mathcal{R}_\Omega f) = f, \ \|\mathcal{R}_\Omega f\|_{H^1(\Omega)^n} \leq C\|f\|_{L^2(\Omega)}, \ \forall\, f \in L^2_0(\Omega). \tag{3.9}$$

4 Dirichlet problems for the anisotropic compressible Stokes system in bounded Lipschitz domains

Dindoš and Mitrea [18] obtained well-posedness results in Sobolev and Besov spaces for the Dirichlet problem for the Stokes and Navier-Stokes systems with smooth coefficients in Lipschitz domains on compact Riemannian manifolds. Mitrea and Wright [51] obtained well-posedness results in Sobolev and Besov spaces for Dirichlet problems for the Stokes system with constant coefficients in Lipschitz domains in $\mathbb{R}^n$ (see also the references therein, and [2] for Dirichlet problems for the Stokes, Oseen and Navier-Stokes systems with constant coefficients in a non-solenoidal framework). Dirichlet problems for the anisotropic Stokes system in exterior Lipschitz domains and in $\mathbb{R}^n$, $n \geq 3$, have been studied in [34] by using both variational and potential approaches (see also [16, 36] and [35]).

4.1 Mixed variational formulation for the anisotropic Stokes system in a bounded Lipschitz domain with homogeneous Dirichlet condition

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with connected boundary $\partial\Omega$. Recall that $\mathring{H}^1(\Omega)^n$ is the closure of the space $\mathcal{D}(\Omega)^n$ in $H^1(\Omega)^n$ and that

$$|\mathbf{u}|_{H^1(\Omega)^n} := \|\nabla\mathbf{u}\|_{L^2(\Omega)^{n\times n}} \tag{4.1}$$

is a norm on the space $\mathring{H}^1(\Omega)$, equivalent to the norm

$$\|\mathbf{u}\|_{H^1(\Omega)^n} = \|\mathbf{u}\|_{L^2(\Omega)^n} + \|\nabla \mathbf{u}\|_{L^2(\Omega)^{n\times n}} \tag{4.2}$$

(cf., e.g., [25, Theorem II.5.1 and Remark II.6.2]), that is,

$$\|\mathbf{u}\|_{H^1(\Omega)^n} \leq \mathring{C}\|\nabla \mathbf{u}\|_{L^2(\Omega)^{n\times n}} \quad \forall\, \mathbf{u} \in \mathring{H}^1(\Omega)^n \tag{4.3}$$

for some constant $\mathring{C} = \mathring{C}(\Omega, n) > 0$. Let also $H^{-1}(\Omega)^n = \big(\mathring{H}^1(\Omega)^n\big)'$ endowed with the norm

$$\|\mathbf{g}\|_{H^{-1}(\Omega)^n} := \sup_{\mathbf{v}\in\mathring{H}^1(\Omega)^n,\ \|\mathbf{v}\|_{H^1(\Omega)^n}=1} |\langle \mathbf{g}, \mathbf{v}\rangle_\Omega|\,, \quad \forall\, \mathbf{g} \in H^{-1}(\Omega)^n,$$

and let $|||\cdot|||_{H^{-1}(\Omega)^n}$ denote the corresponding norm on $H^{-1}(\Omega)^n$ generated by the semi-norm (4.1), i.e.,

$$|||\mathbf{g}|||_{H^{-1}(\Omega)^n} := \sup_{\mathbf{v}\in\mathring{H}^1(\Omega)^n,\ \|\nabla\mathbf{v}\|_{L^2(\Omega)^{n\times n}}=1} |\langle \mathbf{g}, \mathbf{v}\rangle_\Omega|\,, \quad \forall\, \mathbf{g} \in H^{-1}(\Omega)^n. \tag{4.4}$$

This implies that

$$|\langle \mathbf{g}, \mathbf{v}\rangle_\Omega| \leq |||\mathbf{g}|||_{H^{-1}(\Omega)^n}\, \|\nabla\mathbf{v}\|_{L^2(\Omega)^{n\times n}}\,, \quad \forall\, \mathbf{g} \in H^{-1}(\Omega)^n,\ \forall\, \mathbf{v} \in \mathring{H}^1(\Omega)^n,$$

and

$$\|\mathbf{g}\|_{H^{-1}(\Omega)^n} \leq |||\mathbf{g}|||_{H^{-1}(\Omega)^n}\,, \quad \forall\, \mathbf{g} \in H^{-1}(\Omega)^n. \tag{4.5}$$

Let $a_{\mathbb{A};\Omega} : \mathring{H}^1(\Omega)^n \times \mathring{H}^1(\Omega)^n \to \mathbb{R}$ and $b_\Omega : \mathring{H}^1(\Omega)^n \times L^2(\Omega)/\mathbb{R} \to \mathbb{R}$ be the bilinear forms given by

$$a_{\mathbb{A};\Omega}(\mathbf{u}, \mathbf{v}) := \Big\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v})\Big\rangle_\Omega\,, \ \forall\, \mathbf{u}, \mathbf{v} \in \mathring{H}^1(\Omega)^n\,, \tag{4.6}$$

$$b_\Omega(\mathbf{v}, q) := -\langle \operatorname{div}\mathbf{v}, q\rangle_\Omega,\ \forall\, \mathbf{v} \in \mathring{H}^1(\Omega)^n,\ \forall\, q \in L^2(\Omega)/\mathbb{R}\,. \tag{4.7}$$

Let us also introduce the following spaces of divergence-free vector fields

$$H^1_{\operatorname{div}}(\Omega)^n := \{\mathbf{w} \in H^1(\Omega)^n : \operatorname{div}\mathbf{w} = 0 \text{ in } \Omega\}\,,$$
$$\mathring{H}^1_{\operatorname{div}}(\Omega)^n := \{\mathbf{w} \in \mathring{H}^1(\Omega)^n : \operatorname{div}\mathbf{w} = 0 \text{ in } \Omega\}$$

We also have the characterization

$$\mathring{H}^1_{\operatorname{div}}(\Omega)^n = \Big\{\mathbf{w} \in \mathring{H}^1(\Omega)^n : b_\Omega(\mathbf{w}, q) = 0,\ \forall\, q \in L^2(\Omega)/\mathbb{R}\Big\}\,.$$

Indeed, if $\operatorname{div}\mathbf{w} = 0$, then obviously $b_\Omega(\mathbf{w}, q) = 0\ \forall\, q \in L^2(\Omega)/\mathbb{R}$. On the other hand, $\operatorname{div}\mathbf{w} \in L^2_0(\Omega)$ for any $\mathbf{w} \in \mathring{H}^1(\Omega)^n$ (see Proposition 3.1). Since the space $L^2(\Omega)/\mathbb{R}$ is dual to the space $L^2_0(\Omega)$, the condition $b_\Omega(\mathbf{w}, q) = 0\ \forall\, q \in L^2(\Omega)/\mathbb{R}$ implies $\operatorname{div}\mathbf{w} = 0$.

The Hölder inequality implies that there exists a constant $\mathcal{C} > 0$, such that

$$|a_{\mathbb{A};\Omega}(\mathbf{u}, \mathbf{v})| \leq \mathcal{C}\|\nabla\mathbf{u}\|_{L^2(\Omega)^{n\times n}}\|\nabla\mathbf{v}\|_{L^2(\Omega)^{n\times n}},\ \forall\, \mathbf{u}, \mathbf{v} \in \mathring{H}^1(\Omega)^n. \tag{4.8}$$

Thus, the bilinear form $a_{\mathbb{A};\Omega}(\cdot,\cdot) : \mathring{H}^1(\Omega)^n \times \mathring{H}^1(\Omega)^n \to \mathbb{R}$ is bounded. Moreover, the Korn first inequality applied to functions in $\mathring{H}^1(\Omega)^n$,

$$\|\nabla\mathbf{v}\|_{L^2(\Omega)^{n\times n}} \leq 2^{\frac{1}{2}}\|\mathbb{E}(\mathbf{v})\|_{L^2(\Omega)^{n\times n}} \tag{4.9}$$

(cf. [45, Theorem 10.1]) combined with the ellipticity condition (2.4) and the property that the semi-norm $\|\nabla(\cdot)\|_{L^2(\Omega)^{n\times n}}$ is a norm in $\mathring{H}^1(\Omega)^n$, shows that the bilinear form $a_{\mathbb{A};\Omega}(\cdot,\cdot) : \mathring{H}^1_{\rm div}(\Omega)^n \times \mathring{H}^1_{\rm div}(\Omega)^n \to \mathbb{R}$ is coercive, that is,

$$a_{\mathbb{A};\Omega}(\mathbf{v},\mathbf{v}) \geq \frac{1}{2}C_{\mathbb{A}}^{-1}\|\nabla \mathbf{u}\|^2_{L^2(\Omega)^{n\times n}}\,, \quad \forall\, \mathbf{v} \in \mathring{H}^1_{\rm div}(\Omega)^n\,. \tag{4.10}$$

On the other hand, the surjectivity of the operator div : $\mathring{H}^1(\Omega)^n \to L^2_0(\Omega)$ (see Proposition 3.1) shows that the bilinear and bounded form $b_\Omega : \mathring{H}^1(\Omega)^n \times L^2_0(\Omega) \to \mathbb{R}$ satisfies the inf-sup condition

$$\inf_{q\in L^2(\Omega)/\mathbb{R}\setminus\{0\}} \sup_{v\in \mathring{H}^1(\Omega)^n\setminus\{0\}} \frac{b_\Omega(v,q)}{\|v\|_{\mathring{H}^1(\Omega)^n}\|q\|_{L^2(\Omega)/\mathbb{R}}} \geq c_b\,,$$

with some constant $c_b > 0$ (see [23, Theorem A.56, Remark 2.7]). Then Theorem 2.34 in [23] leads to the following well-posedness result, whose detailed proof can be consulted in [35, Theorem 3.2] (see also [34, Lemma 3.1]).

Theorem 4.1 *Let conditions (2.2)–(2.4) hold. Let $a_{\mathbb{A};\Omega}$ and b_Ω be the bilinear forms defined in (4.6) and (4.7). Then the following properties hold.*

(i) *For all given data $\mathfrak{F} \in H^{-1}(\Omega)^n$ and $g \in L^2_0(\Omega)$, the variational problem*

$$\begin{cases} a_{\mathbb{A};\Omega}(\mathbf{u},\mathbf{v}) + b_\Omega(\mathbf{v},\pi) = \langle \mathfrak{F},\mathbf{v}\rangle_\Omega\,, & \forall\, \mathbf{v} \in \mathring{H}^1(\Omega)^n,\\ b_\Omega(\mathbf{u},q) = -\langle g,q\rangle_\Omega\,, & \forall\, q \in L^2(\Omega)/\mathbb{R} \end{cases} \tag{4.11}$$

for $(\mathbf{u},\pi) \in \mathring{H}^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ is well-posed, that is, (4.11) has a unique solution and there exists a constant $C > 0$ depending only on $\|\mathbb{A}\|$, $C_{\mathbb{A}}$, Ω and n, such that

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C\left(\|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + \|g\|_{L^2(\Omega)}\right). \tag{4.12}$$

(ii) *The pair $(\mathbf{u},\pi)$ is the unique solution in $H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ of the Dirichlet problem for the anisotropic Stokes system*

$$\begin{cases} \mathcal{L}(\mathbf{u},\pi) = -\mathfrak{F}, \quad \operatorname{div}\mathbf{u} = g & \text{in } \Omega\,,\\ \gamma_\Omega \mathbf{u} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{4.13}$$

(iii) *The solution can be represented in the form $(\mathbf{u},\pi) = \mathfrak{U}(\mathfrak{F},g)$, where $\mathfrak{U} : H^{-1}(\Omega)^n \times L^2_0(\Omega) \to H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ is a linear continuous operator.*

4.2 Non-homogeneous Dirichlet problem for the anisotropic Stokes system

Let us consider the following non-homogeneous Dirichlet problem

$$\begin{cases} \mathcal{L}(\mathbf{u},\pi) = -\mathfrak{F}, \quad \operatorname{div}\mathbf{u} = g & \text{in } \Omega\,,\\ \gamma_\Omega \mathbf{u} = \boldsymbol{\varphi} & \text{on } \partial\Omega\,, \end{cases} \tag{4.14}$$

for the unknowns $(\mathbf{u},\pi) \in H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$, with the given data $(\mathfrak{F}, g, \boldsymbol{\varphi}) \in H^{-1}(\Omega)^n \times L^2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)^n$, which satisfy the compatibility condition

$$\int_\Omega g(x)dx = \int_{\partial\Omega} \boldsymbol{\varphi}\cdot\boldsymbol{\nu} d\sigma\,, \tag{4.15}$$

where $\boldsymbol{\nu}$ is the exterior unit normal to $\partial\Omega$.

To analyse the Dirichlet problem (4.14), we need the following well-known Bogovskii-type result (see, e.g., [7, 28], and the proof of Theorem 3.2 in [2]).

Lemma 4.2 *For any* $(g, \boldsymbol{\varphi}) \in L^2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)^n$ *satisfying condition* (4.15)*, there exists* $\mathbf{v} \in H^1(\Omega)^n$ *such that*

$$\begin{cases} \operatorname{div} \mathbf{v} = g \text{ in } \Omega \\ \gamma_\Omega \mathbf{v} = \boldsymbol{\varphi} \text{ on } \partial\Omega\,, \end{cases} \tag{4.16}$$

and there exists a constant $c = c(\Omega, n) > 0$ *such that*

$$\|\mathbf{v}\|_{H^1(\Omega)^n} \leq c\Big(\|g\|_{L^2(\Omega)} + \|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n}\Big)\,. \tag{4.17}$$

Theorem 4.3 *Let conditions* (2.2)–(2.4) *hold. Then for all given data* $(\mathfrak{F}, g, \boldsymbol{\varphi}) \in H^{-1}(\Omega)^n \times L^2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)^n$ *satisfying condition* (4.15)*, the Dirichlet problem* (4.14) *has a unique solution* $(\mathbf{u}, \pi) \in H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ *and there exists a constant* $C = C(\Omega, C_{\mathbb{A}}, \|\mathbb{A}\|, n) > 0$ *such that*

$$\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C\left(\|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + \|g\|_{L^2(\Omega)} + \|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n}\right). \tag{4.18}$$

Proof Let $\mathbf{v} \in H^1(\Omega)^n$ be the function given by Lemma 4.2. For the velocity-pressure couple $(\mathbf{v}, 0)$, let us also define

$$\check{\mathfrak{F}} := -\boldsymbol{\mathcal{L}}(\mathbf{v}, 0) = -\mathfrak{L}\mathbf{v} \in H^{-1}(\Omega)^n \tag{4.19}$$

(cf. notations for $\mathfrak{L}\mathbf{v}$ in (2.7) and (2.1)). Then the fully non-homogeneous Dirichlet problem (4.14) reduces to the following Dirichlet problem with homogeneous Dirichlet condition, for the new function $\mathbf{w} := \mathbf{u} - \mathbf{v}$,

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{w}, \pi) = -(\mathfrak{F} - \check{\mathfrak{F}}), \quad \operatorname{div} \mathbf{w} = 0 \ \text{ in } \Omega\,, \\ \gamma_\Omega \mathbf{w} = 0 \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega\,. \end{cases} \tag{4.20}$$

Theorem 4.1 implies that the Dirichlet problem (4.20) has a unique solution $(\mathbf{w}, \pi)$ in the space $(\mathbf{u}, \pi) \in H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ and depends continuously on the given data of this problem. Finally, the well-posedness of problem (4.20) implies that the couple $(\mathbf{u} = \mathbf{w}+\mathbf{v},\ \pi)$ determines a solution of the full non-homogeneous Dirichlet problem (4.14) in the space $H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$, and estimate (4.18) holds. This solution is unique by the uniqueness statement in Theorem 4.1. □

5 Dirichlet-transmission problems for the anisotropic Stokes system in bounded Lipschitz domains with transversal interfaces

Mitrea and Wright [51] obtained well-posedness results in Sobolev and Besov spaces for transmission problems for the Stokes system with constant coefficients in Lipschitz domains in $\mathbb{R}^n$ (see also the references therein). Various transmission problems for the anisotropic Stokes system in Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, with internal interface and homogeneous conditions for traces, have been studied in [34] by using both variational and potential approaches (see also [35] and [36]).

In this section we show the well-posedness of boundary value problems of Dirichlet-transmission type for the anisotropic Stokes system in a compressible framework in bounded Lipschitz domains with transversal Lipschitz interfaces satisfying the following assumption.

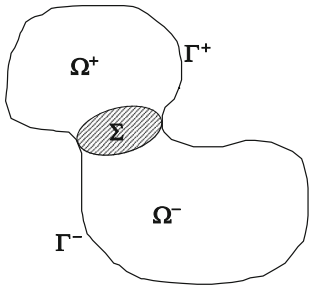

Fig. 1 Bounded composite domain $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$ with the interface Σ, for $n = 3$

Assumption 5.1 Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary $\partial\Omega$. The domain Ω is divided into two disjoint Lipschitz sub-domains Ω^+ and Ω^- by an $(n-1)$-dimensional Lipschitz open interface Σ, such that $\partial\Sigma = \overline{\Sigma} \cap \partial\Omega$ is a non-empty $(n-2)$-dimensional Lipschitz manifold if $n > 2$, and two distinct points if $n = 2$. In this case $\overline{\Sigma}$ intersects $\partial\Omega$ transversally and $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$, see Fig. 1.

Thus, the remaining boundaries $\Gamma^+ = \partial\Omega^+ \setminus \overline{\Sigma}$ and $\Gamma^- = \partial\Omega^- \setminus \overline{\Sigma}$ of $\partial\Omega^+$ and $\partial\Omega^-$, respectively, are non-empty relatively open subsets of $\partial\Omega$.

5.1 Sobolev spaces on bounded domains with partially vanishing traces

Let $\Omega' \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain with connected boundary $\partial\Omega'$. Let D and N be relatively open subsets of $\partial\Omega'$, such that D has positive $(n-1)$-Hausdorff measure, $D \cap N = \emptyset$, $\overline{D} \cup \overline{N} = \partial\Omega'$, and $\overline{D} \cap \overline{N} = \partial D = \partial N$ is an $(n-2)$-dimensional closed Lipschitz submanifold of $\partial\Omega'$.

We need the following space defined on the Lipschitz domains Ω'

$$C_D^\infty(\Omega')^n := \left\{ \boldsymbol{\varphi}|_{\Omega'} : \boldsymbol{\varphi} \in C^\infty(\mathbb{R}^n)^n,\ \operatorname{supp}(\boldsymbol{\varphi}) \cap \overline{D} = \emptyset \right\}, \tag{5.1}$$

and let $H_D^1(\Omega')^n$ be the closure of $C_D^\infty(\Omega')^n$ in $H^1(\Omega')^n$. The space $H_D^1(\Omega')^n$ can be equivalently characterized as

$$H_D^1(\Omega')^n = \left\{ \mathbf{v} \in H^1(\Omega')^n : \left(\gamma_{\Omega'}\mathbf{v}\right)|_D = \mathbf{0} \right\} \tag{5.2}$$

(cf. [9, Corollary 3.11]). Let also

$$H_{D;\mathrm{div}}^1(\Omega')^n := \left\{ \mathbf{w} \in H_D^1(\Omega')^n : \operatorname{div}\mathbf{w} = 0 \right\}. \tag{5.3}$$

Let Ξ be a relatively open $(n-1)$-dimensional subset of $\partial\Omega'$, e.g., D or N. Let r_Ξ denote the operator of restriction of distributions from $\partial\Omega'$ to Ξ. Then the boundary Sobolev spaces on N are defined by

$$H^{\frac{1}{2}}(\Xi)^n := \left\{ \boldsymbol{\phi}|_\Xi : \boldsymbol{\phi} \in H^{\frac{1}{2}}(\partial\Omega')^n \right\}, \tag{5.4}$$

$$\widetilde{H}^{\frac{1}{2}}(\Xi)^n := \left\{ \boldsymbol{\phi} \in H^{\frac{1}{2}}(\partial\Omega')^n : \boldsymbol{\phi} = \mathbf{0} \text{ on } \partial\Omega' \setminus \overline{\Xi} \right\}, \tag{5.5}$$

$$H^{-\frac{1}{2}}(\Xi)^n := \left(\widetilde{H}^{\frac{1}{2}}(\Xi)^n\right)',\ \widetilde{H}^{-\frac{1}{2}}(\Xi)^n := \left(H^{\frac{1}{2}}(\Xi)^n\right)' \tag{5.6}$$

(cf., e.g., [45], [9, Definition 4.8, Theorem 5.1]).

Lemma 5.2 *The trace operator $\gamma_{\Omega'} : H_D^1(\Omega')^n \to \widetilde{H}^{\frac{1}{2}}(N)^n$ is bounded, linear and surjective, with a (non-unique) bounded, linear right inverse $\gamma_{\Omega'}^{-1} : \widetilde{H}^{\frac{1}{2}}(N)^n \to H_D^1(\Omega')^n$.*

Proof Let $\gamma_{\Omega'}^{-1} : H^{\frac{1}{2}}(\partial\Omega')^n \to H^1(\Omega')^n$ be a bounded right inverse of the trace operator $\gamma_{\Omega'} : H^1(\Omega')^n \to H^{\frac{1}{2}}(\partial\Omega')^n$ (cf. [17], [46, Lemma 2.6], [51, Theorem 2.5.2]). Consequently, we have $\gamma_{\Omega'}\gamma_{\Omega'}^{-1}\boldsymbol{\phi} = \boldsymbol{\phi}$, for any $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\partial\Omega')^n$. Therefore, if $\boldsymbol{\phi} \in \widetilde{H}^{\frac{1}{2}}(N)^n$, that is, $\boldsymbol{\phi} = 0$ on D, then $\gamma_{\Omega'}\gamma_{\Omega'}^{-1}\boldsymbol{\phi} = 0$ on D, and, thus, $\gamma_{\Omega'}^{-1}\boldsymbol{\phi} \in H_D^1(\Omega')^n$. Hence, the existence of a right inverse $\gamma_{\Omega'}^{-1} : H^{\frac{1}{2}}(\partial\Omega')^n \to H^1(\Omega')^n$ of the trace operator $\gamma_{\Omega'} : H^1(\Omega')^n \to H^{\frac{1}{2}}(\partial\Omega')^n$ assures the existence of a bounded right inverse $\gamma_{\Omega'}^{-1} : \widetilde{H}^{\frac{1}{2}}(N)^n \to H_D^1(\Omega')^n$ of the operator $\gamma_{\Omega'} : H_D^1(\Omega')^n \to \widetilde{H}^{\frac{1}{2}}(N)^n$. □

5.2 Sobolev spaces, conormal derivatives and Green's identity in a bounded Lipschitz domain with a transversal Lipschitz interface

In the sequel, Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, satisfying Assumption 5.1. We need the following spaces defined on the domains Ω, Ω^+ and Ω^-,

$$H_{\Gamma^\pm}^1(\Omega)^n := \left\{\mathbf{v} \in H^1(\Omega)^n : (\gamma_\Omega \mathbf{v})|_{\Gamma^\pm} = \mathbf{0}\right\}, \tag{5.7}$$

$$H_{\Gamma^\pm}^1(\Omega^\pm)^n := \left\{\mathbf{v}^\pm \in H^1(\Omega^\pm)^n : (\gamma_{\Omega^\pm}\mathbf{v}^\pm)|_{\Gamma^\pm} = \mathbf{0}\right\}, \tag{5.8}$$

where $\gamma_{\Omega^\pm} : H^1(\Omega^\pm) \to H^{\frac{1}{2}}(\partial\Omega^\pm)$ are the trace operators acting on functions defined on the domains $\Omega^\pm$. The spaces $H_{\Gamma^\pm}^1(\Omega^\pm)^n$ can be equivalently described as

$$H_{\Gamma^\pm}^1(\Omega^\pm)^n := \left\{\mathbf{v}|_{\Omega^\pm} : \mathbf{v} \in H_{\Gamma^\pm}^1(\Omega)^n\right\}, \tag{5.9}$$

and the space $\mathring{H}^1(\Omega)^n = \left\{\mathbf{w} \in H^1(\Omega^\pm)^n : \gamma_\Omega \mathbf{w} = 0 \text{ on } \partial\Omega\right\}$ can be identified with the space

$$\left\{\left(\mathbf{v}^+, \mathbf{v}^-\right) \in H_{\Gamma^+}^1(\Omega^+)^n \times H_{\Gamma^-}^1(\Omega^-)^n : \left(\gamma_{\Omega^+}\mathbf{v}^+\right)\big|_\Sigma = \left(\gamma_{\Omega^-}\mathbf{v}^-\right)\big|_\Sigma\right\}. \tag{5.10}$$

This property is an immediate consequence of Lemma B.1.

Let us next introduce the Sobolev spaces on the interface Σ (cf., e.g., [9, 45]). First, define the space

$$H^{\frac{1}{2}}(\Sigma)^n := \left\{\boldsymbol{\phi} \in L^2(\Sigma)^n : \exists\, \boldsymbol{\phi}^+ \in H^{\frac{1}{2}}(\partial\Omega^+)^n \text{ such that } \boldsymbol{\phi} = \boldsymbol{\phi}^+\big|_\Sigma\right\}, \tag{5.11}$$

which can be identified with the space

$$\left\{\boldsymbol{\phi} \in L^2(\Sigma)^n : \exists\, \boldsymbol{\phi}^- \in H^{\frac{1}{2}}(\partial\Omega^-)^n \text{ such that } \boldsymbol{\phi} = \boldsymbol{\phi}^-\big|_\Sigma\right\} \tag{5.12}$$

due to the equivalence of both of them to the space defined as in (3.8), with Σ instead of $\partial\Omega$ (see also Lemma B.2).

Let us also consider the space

$$\widetilde{H}^{\frac{1}{2}}(\Sigma; \partial\Omega^+)^n := \left\{\widetilde{\boldsymbol{\phi}}^+ \in H^{\frac{1}{2}}(\partial\Omega^+)^n : \operatorname{supp}\widetilde{\boldsymbol{\phi}}^+ \subseteq \overline{\Sigma}\right\}, \tag{5.13}$$

which, by Lemma B.2(ii), can be identified with the space

$$\widetilde{H}^{\frac{1}{2}}(\Sigma; \partial\Omega^-)^n := \left\{\widetilde{\boldsymbol{\phi}}^- \in H^{\frac{1}{2}}(\partial\Omega^-)^n : \operatorname{supp}\widetilde{\boldsymbol{\phi}}^- \subseteq \overline{\Sigma}\right\}, \tag{5.14}$$

and both spaces inherit their norms from the spaces $H^{\frac{1}{2}}(\partial\Omega^{\pm})^n$, respectively. Let us denote by $H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$ the space consisting of all functions $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Sigma)^n$ such that their extensions by zero on $\partial\Omega^+$, $\mathring{E}_{\Sigma\to\partial\Omega^+}\boldsymbol{\phi}$, belong to $\widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^+)^n$, i.e.,

$$H_{\bullet}^{\frac{1}{2}}(\Sigma)^n := \{\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Sigma)^n : \mathring{E}_{\Sigma\to\partial\Omega^+}\boldsymbol{\phi} \in \widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^+)^n\}.$$

Lemma B.2(ii) shows that the space $H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$ can be also defined as

$$H_{\bullet}^{\frac{1}{2}}(\Sigma)^n := \{\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Sigma)^n : \mathring{E}_{\Sigma\to\partial\Omega^-}\boldsymbol{\phi} \in \widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^-)^n\}.$$

The space $H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$ can be endowed with the norm

$$\|\boldsymbol{\phi}\|_{H_{\bullet}^{\frac{1}{2}}(\Sigma)^n} = \max\left\{\|\mathring{E}_{\Sigma\to\partial\Omega^+}\boldsymbol{\phi}\|_{H^{\frac{1}{2}}(\partial\Omega^+)^n}, \|\mathring{E}_{\Sigma\to\partial\Omega^-}\boldsymbol{\phi}\|_{H^{\frac{1}{2}}(\partial\Omega^-)^n}\right\}.$$

Note that the operators of extension by zero

$$\mathring{E}_{\Sigma\to\partial\Omega^{\pm}} : H_{\bullet}^{\frac{1}{2}}(\Sigma)^n \to \widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^{\pm})^n \tag{5.15}$$

are continuous, and due to, e.g., [46, Theorem 2.10(i)] also surjective, implying that the space $H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$ can be identified with the spaces $\widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^{\pm})^n$. In addition, by Theorem B.3 the space $H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$ can be characterized as the weighted space $H_{00}^{\frac{1}{2}}(\Sigma)$ consisting of functions $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Sigma)^n$, such that $\delta^{-\frac{1}{2}}\boldsymbol{\phi} \in L^2(\Sigma)^n$, where $\delta(x)$ is the distance from $x \in \Sigma$ to the boundary $\partial\Sigma$. The counterpart of $H_{00}^{\frac{1}{2}}(\cdot)$ on smooth domains in $\mathbb{R}^n$ has been considered in [42, Chapter 1, Theorem 11.7] and on Lipschitz domains in [30, Corollary 1.4.4.10]. Note also that the space $H_{\bullet}^{\frac{1}{2}}(\cdot)$ is similar to the space $L^p_{s,z}(\cdot)$ in [50, Eq. (2.212)], cf. also [48, p.3].

Lemma 5.3 *The operator* $\gamma_{\Sigma} : \mathring{H}^1(\Omega)^n \to H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$ *given by*

$$\gamma_{\Sigma}\mathbf{v} := \left(\gamma_{\Omega^+}\left(\mathbf{v}|_{\Omega^+}\right)\right)\big|_{\Sigma} = \left(\gamma_{\Omega^-}\left(\mathbf{v}|_{\Omega^-}\right)\right)\big|_{\Sigma}, \quad \forall\, \mathbf{v} \in \mathring{H}^1(\Omega)^n, \tag{5.16}$$

is linear, bounded and surjective.

Proof The linearity and boundedness of the operator γ_{Σ} are immediate consequences of the linearity and boundedness of the trace operators

$$\gamma_{\Omega^{\pm}} : H^1_{\Gamma^{\pm}}(\Omega^{\pm})^n \to \widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^{\pm})^n, \tag{5.17}$$

cf. Lemma 5.2. The equality of restrictions to Σ of the traces from Ω^+ and Ω^- in (5.16) follows from the inclusion $\mathbf{v} \in \mathring{H}^1(\Omega)^n$, see Lemma B.1(ii). The surjectivity of operators (5.17) implies the surjectivity of the operator $\gamma_{\Sigma} : \mathring{H}^1(\Omega)^n \to H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$. To this end, let $\boldsymbol{\phi} \in H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$. Then $\mathring{E}_{\Sigma\to\partial\Omega^{\pm}}\boldsymbol{\phi} \in \widetilde{H}^{\frac{1}{2}}(\Sigma;\partial\Omega^{\pm})^n$ and by Lemma 5.2, there exist $\mathbf{v}^{\pm} \in H^1_{\Gamma^{\pm}}(\Omega^{\pm})^n$ such that $\gamma_{\Omega^{\pm}}\mathbf{v}^{\pm} = \mathring{E}_{\Sigma\to\partial\Omega^{\pm}}\boldsymbol{\phi}$ on $\partial\Omega^{\pm}$. Hence, we obtain that $\gamma_{\Sigma}\mathbf{v}^+ = \gamma_{\Sigma}\mathbf{v}^-$ on Σ. According to Lemma B.1(i) there exists a unique function $\mathbf{v} \in H^1(\Omega)^n$ such that $\mathbf{v}|_{\Omega^{\pm}} = \mathbf{v}^{\pm}$. Moreover, since $\gamma_{\Omega^{\pm}}\mathbf{v}^{\pm} = 0$ on $\Gamma^{\pm}$, we deduce that $\gamma_{\Omega}\mathbf{v} = 0$ (a.e.) on $\partial\Omega$, and hence $\mathbf{v} \in \mathring{H}^1(\Omega)^n$. □

Lemma 5.3 implies that the space $H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$ can be also characterised as

$$H_{\bullet}^{\frac{1}{2}}(\Sigma)^n = \Big\{ \boldsymbol{\phi} \in L^2(\Sigma) : \exists\, \mathbf{v} \in \mathring{H}^1(\Omega)^n \text{ such that } \boldsymbol{\phi} = \big(\gamma_{\Omega^+}\big(\mathbf{v}|_{\Omega^+}\big)\big)\big|_{\Sigma} = \big(\gamma_{\Omega^-}\big(\mathbf{v}|_{\Omega^-}\big)\big)\big|_{\Sigma} \Big\}. \tag{5.18}$$

In addition, we consider the spaces

$$H^{-\frac{1}{2}}(\Sigma)^n := \big(H_{\bullet}^{\frac{1}{2}}(\Sigma)^n\big)', \ \widetilde{H}^{-\frac{1}{2}}(\Sigma)^n := \big(H^{\frac{1}{2}}(\Sigma)^n\big)'. \tag{5.19}$$

In Appendix A we provide a definition of the generalized conormal derivative, associated with anisotropic Stokes operator $\boldsymbol{\mathcal{L}}$, on the entire boundary of the domain. If we need the conormal derivative only on a part of the boundary of the domain, we do not not need the extension of the PDE right hand side to the 'tilde-space' on the rest of the boundary. To this end, we consider the following counterpart of Definition A.2 in the case of $\tilde{\mathbf{f}}^{\pm} \in \big(H^1_{\Gamma^\pm}(\Omega^\pm)^n\big)'$; some characterisations of such spaces are available from Lemma B.6.

Definition 5.4 Let Assumption 5.1 and condition (2.2) hold. Let

$$\boldsymbol{H}^1_{\Gamma^\pm}(\Omega^\pm, \boldsymbol{\mathcal{L}}) := \Big\{ (\mathbf{u}^\pm, \pi^\pm, \tilde{\mathbf{f}}^\pm) \in H^1(\Omega^\pm)^n \times L^2(\Omega^\pm) \times \big(H^1_{\Gamma^\pm}(\Omega^\pm)^n\big)' : \boldsymbol{\mathcal{L}}(\mathbf{u}^\pm, \pi^\pm) = \tilde{\mathbf{f}}^\pm|_{\Omega^\pm} \text{ in } \Omega^\pm \Big\}. \tag{5.20}$$

If $(\mathbf{u}^\pm, \pi^\pm, \tilde{\mathbf{f}}^\pm) \in \boldsymbol{H}^1_{\Gamma^\pm}(\Omega^\pm, \boldsymbol{\mathcal{L}})$, then the formula

$$\begin{aligned} \Big\langle \big(\mathbf{t}_{\Omega^\pm}(\mathbf{u}^\pm, \pi^\pm; \tilde{\mathbf{f}}^\pm)\big)|_{\Sigma}, \boldsymbol{\Phi}^\pm \Big\rangle_{\Sigma} &:= \Big\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}^\pm), E_{i\alpha}(\gamma_{\Omega^\pm}^{-1}\boldsymbol{\Phi}^\pm) \Big\rangle_{\Omega^\pm} \\ &- \Big\langle \pi^\pm, \operatorname{div}(\gamma_{\Omega^\pm}^{-1}\boldsymbol{\Phi}^\pm) \Big\rangle_{\Omega^\pm} + \Big\langle \tilde{\mathbf{f}}^\pm, \gamma_{\Omega^\pm}^{-1}\boldsymbol{\Phi}^\pm \Big\rangle_{\Omega^\pm}, \ \forall\, \boldsymbol{\Phi}^\pm \in H_{\bullet}^{\frac{1}{2}}(\Sigma)^n, \end{aligned} \tag{5.21}$$

defines the generalized conormal derivatives $\big(\mathbf{t}_{\Omega^\pm}(\mathbf{u}^\pm, \pi^\pm; \tilde{\mathbf{f}}^\pm)\big)|_{\Sigma} \in H^{-\frac{1}{2}}(\Sigma)^n$, where $\gamma_{\Omega^\pm}^{-1} : H_{\bullet}^{\frac{1}{2}}(\Sigma)^n \to H^1_{\Gamma^\pm}(\Omega^\pm)^n$ are bounded right inverses of the trace operators $\gamma_{\Omega^\pm} : H^1_{\Gamma^\pm}(\Omega^\pm)^n \to H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$.

Note that, in view of Lemma 5.2, all duality pairings in formula (5.21) are well-defined. Moreover, we have the following result, whose proof is omitted for the sake of brevity (cf. [49, Proposition 8.1] for the Laplace operator, [38, Lemma 7.6] for extensions to compact Riemannian manifolds, and [9, Definition 7.1] in the case of higher order elliptic operators, see also [45, Lemma 4.3], [35, Lemma 2.3], [36, Lemma 1], [46, Definition 3.1, Theorem 3.2], [47], [51, Theorem 10.4.1]).

Lemma 5.5 *Let* Assumption 5.1 *and conditions* (2.2) *and* (2.3) *hold.*

(i) *The generalized conormal derivative operators* $\mathbf{t}_{\Omega^\pm} : \boldsymbol{H}^1_{\Gamma^\pm}(\Omega^\pm, \boldsymbol{\mathcal{L}}) \to H^{-\frac{1}{2}}(\Sigma)^n$ *are linear and bounded, and definition* (5.21) *does not depend on the particular choice of right inverses* $\gamma_{\Omega^\pm}^{-1} : H_{\bullet}^{\frac{1}{2}}(\Sigma)^n \to H^1_{\Gamma^\pm}(\Omega^\pm)^n$ *of the trace operators* $\gamma_{\Omega^\pm} : H^1_{\Gamma^\pm}(\Omega^\pm)^n \to H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$.

(ii) *Let* $(\mathbf{u}^\pm, \pi^\pm, \tilde{\mathbf{f}}^\pm) \in \boldsymbol{H}^1_{\Gamma^\pm}(\Omega^\pm, \boldsymbol{\mathcal{L}})$. *Let* $\pi \in L^2(\Omega)$ *and* $\tilde{\mathbf{f}} \in H^{-1}(\Omega)$ *be such that*

$$\pi|_{\Omega^\pm} = \pi^\pm, \ \ \tilde{\mathbf{f}} = \tilde{\mathbf{f}}^+ + \tilde{\mathbf{f}}^- . \tag{5.22}$$

Then the following first Green identities hold

$$\begin{aligned}\big\langle\big(\mathbf{t}_{\Omega^\pm}(\mathbf{u}^\pm,\pi^\pm;\tilde{\mathbf{f}}^\pm)\big)|_\Sigma\,,\,\gamma_\Sigma\mathbf{w}^\pm\big\rangle_\Sigma &= \Big\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}^\pm),E_{i\alpha}(\mathbf{w}^\pm)\Big\rangle_{\Omega^\pm}\\ &-\langle\pi,\operatorname{div}\mathbf{w}^\pm\rangle_{\Omega^\pm}+\langle\tilde{\mathbf{f}},\mathbf{w}^\pm\rangle_{\Omega^\pm}\,,\quad\forall\,\mathbf{w}^\pm\in H^1_{\Gamma^\pm}(\Omega^\pm)^n\,,\end{aligned}\tag{5.23}$$

and hence

$$\begin{aligned}&\big\langle\big(\mathbf{t}_{\Omega^+}(\mathbf{u}^+,\pi^+;\tilde{\mathbf{f}}^+)\big)|_\Sigma+\big(\mathbf{t}_{\Omega^-}(\mathbf{u}^-,\pi^-;\tilde{\mathbf{f}}^-)\big)|_\Sigma\,,\,\gamma_\Sigma\mathbf{w}\big\rangle_\Sigma\\ &=\Big\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}^+),E_{i\alpha}(\mathbf{w})\Big\rangle_{\Omega^+}+\Big\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}^-),E_{i\alpha}(\mathbf{w})\Big\rangle_{\Omega^-}\\ &\quad-\langle\pi,\operatorname{div}\mathbf{w}\rangle_\Omega+\langle\tilde{\mathbf{f}},\mathbf{w}\rangle_\Omega\,,\quad\forall\,\mathbf{w}\in\mathring{H}^1(\Omega)^n\,.\end{aligned}\tag{5.24}$$

Note that the existence of a function $\pi\in L^2(\Omega)$ as in (5.22) follows from Lemma B.1, while Lemma B.6 shows that $\tilde{\mathbf{f}}$ defined in (5.22) belongs to the space $H^{-1}(\Omega)^n$.

5.3 Dirichlet-transmission problem with homogeneous Dirichlet conditions

First, we analyze the following Dirichlet-transmission problem for the anisotropic Stokes system in Ω with homogeneous Dirichlet conditions

$$\begin{cases}\mathcal{L}(\mathbf{u}^+,\pi^+)=\widetilde{\mathbf{f}}^+|_{\Omega^+},\ \operatorname{div}\mathbf{u}^+=g|_{\Omega^+} & \text{in }\Omega^+,\\ \mathcal{L}(\mathbf{u}^-,\pi^-)=\widetilde{\mathbf{f}}^-|_{\Omega^-},\ \operatorname{div}\mathbf{u}^-=g|_{\Omega^-} & \text{in }\Omega^-,\\ (\gamma_{\Omega^+}\mathbf{u}^+)|_\Sigma=(\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma & \text{on }\Sigma,\\ (\mathbf{t}_{\Omega^+}(\mathbf{u}^+,\pi^+;\widetilde{\mathbf{f}}^+))|_\Sigma+(\mathbf{t}_{\Omega^-}(\mathbf{u}^-,\pi^-;\widetilde{\mathbf{f}}^-))|_\Sigma=\boldsymbol{\psi}_\Sigma & \text{on }\Sigma,\\ (\gamma_{\Omega^+}\mathbf{u}^+)|_{\Gamma^+}=\mathbf{0} & \text{on }\Gamma^+,\\ (\gamma_{\Omega^-}\mathbf{u}^-)|_{\Gamma^-}=\mathbf{0} & \text{on }\Gamma^-\end{cases}\tag{5.25}$$

and the given data $(\widetilde{\mathbf{f}}^+,\widetilde{\mathbf{f}}^-,g,\boldsymbol{\psi}_\Sigma)\in\mathfrak{Y}^0$. The space

$$\mathfrak{Y}^0:=\big(H^1_{\Gamma^+}(\Omega^+)^n\big)'\times\big(H^1_{\Gamma^-}(\Omega^-)^n\big)'\times L^2_0(\Omega)\times H^{-\frac{1}{2}}(\Sigma)^n\tag{5.26}$$

is endowed with the norm

$$\begin{aligned}\|(\widetilde{\mathbf{f}}^+,\widetilde{\mathbf{f}}^-,g,\boldsymbol{\psi}_\Sigma)\|_{\mathfrak{Y}^0}&:=\|\widetilde{\mathbf{f}}^+\|_{\left(H^1_{\Gamma^+}(\Omega^+)^n\right)'}+\|\widetilde{\mathbf{f}}^-\|_{\left(H^1_{\Gamma^-}(\Omega^-)^n\right)'}\\ &+\|g\|_{L^2(\Omega)}+\|\boldsymbol{\psi}_\Sigma\|_{H^{-\frac{1}{2}}(\Sigma)^n}.\end{aligned}$$

The conormal derivative operators $\mathbf{t}_{\Omega^+}$ and $\mathbf{t}_{\Omega^-}$, as introduced in Definition A.2, correspond to the outward unit normal vectors to Ω^+ and Ω^-, respectively, that have opposite directions on Σ. However, if one would consider the conormal derivatives with respect to unit normal vectors of the same direction on Σ, then the sum in the corresponding transmission condition in (5.25) would be replaced by the difference, leading to the *jump* of the conormal derivatives as, e.g., in [34–36].

We show that (5.25) has a unique solution $(\mathbf{u}^+,\pi^+,\mathbf{u}^-,\pi^-)$ in the space

$$\begin{aligned}\mathfrak{X}_{\Omega^+,\Omega^-}:=\big\{&(\mathbf{v}^+,q^+,\mathbf{v}^-,q^-):\ \mathbf{v}^+\in H^1(\Omega^+)^n,\ \mathbf{v}^-\in H^1(\Omega^-)^n,\\ &q^+=q|_{\Omega^+},\ q^-=q|_{\Omega^-},\ q\in L^2(\Omega)/\mathbb{R}\big\},\end{aligned}\tag{5.27}$$

endowed with the norm

$$\|(\mathbf{v}^+,q^+,\mathbf{v}^-,q^-)\|_{\mathfrak{X}_{\Omega^+,\Omega^-}}=\|\mathbf{v}^+\|_{H^1(\Omega^+)^n}+\|\mathbf{v}^-\|_{H^1(\Omega^-)^n}+\|q\|_{L^2(\Omega)/\mathbb{R}}.$$

(The choice of the space $L^2(\Omega)/\mathbb{R}$ for the pressure is only for convenience, and one may consider the space $L^2_0(\Omega)$ as well.)

Let $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$ and $\mathbf{u}^+$ and $\mathbf{u}^-$ satisfy the homogeneous interface condition for traces in (5.25), $(\gamma_{\Omega^+}\mathbf{u}^+)|_\Sigma - (\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma = \mathbf{0}$ on Σ. Then Lemma B.1 implies that there exists a unique pair $(\mathbf{u}, \pi) \in H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ such that

$$\mathbf{u}|_{\Omega^+} = \mathbf{u}^+, \quad \mathbf{u}|_{\Omega^-} = \mathbf{u}^-, \quad \pi|_{\Omega^+} = \pi^+, \quad \pi|_{\Omega^-} = \pi^-. \tag{5.28}$$

Assuming that $\mathbf{u}^+$ and $\mathbf{u}^-$ also satisfy the homogeneous Dirichlet condition in (5.25) we have that $\mathbf{u} \in \mathring{H}^1(\Omega)^n$. Therefore, $(\mathbf{u}, \pi) \in \mathring{H}^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$.

Note that the membership of $\widetilde{\mathbf{f}}^+$ to $\big(H^1_{\Gamma^+}(\Omega^+)^n\big)'$ and the identification of this space with the space defined by (B.13) in Lemma B.6 imply that $\widetilde{\mathbf{f}}^+$ can be considered also as an element from $H^{-1}(\Omega)^n$. Similarly, the assumption $\widetilde{\mathbf{f}}^- \in \big(H^1_{\Gamma^-}(\Omega^-)^n\big)'$ implies that $\widetilde{\mathbf{f}}^-$ can be considered as an element from $H^{-1}(\Omega)^n$.

Let also $\mathfrak{F} \in H^{-1}(\Omega)^n$ be such that

$$\begin{aligned}\langle \mathfrak{F}, \mathbf{v}\rangle_\Omega &:= -\big\langle \tilde{\mathbf{f}}^+, \mathbf{v}\big\rangle_{\Omega^+} - \big\langle \tilde{\mathbf{f}}^-, \mathbf{v}\big\rangle_{\Omega^-} + \langle \boldsymbol{\psi}_\Sigma, \gamma_\Sigma \mathbf{v}\rangle_\Sigma \\ &= -\big\langle \tilde{\mathbf{f}}^+ + \tilde{\mathbf{f}}^-, \mathbf{v}\big\rangle_\Omega + \langle \gamma_\Sigma^* \boldsymbol{\psi}_\Sigma, \mathbf{v}\rangle_\Omega, \quad \forall\, \mathbf{v} \in \mathring{H}^1(\Omega)^n, \end{aligned} \tag{5.29}$$

that is, $\mathfrak{F} = -(\tilde{\mathbf{f}}^+ + \tilde{\mathbf{f}}^-) + \gamma_\Sigma^* \boldsymbol{\psi}_\Sigma$. Note that $\gamma_\Sigma^* : H^{-\frac{1}{2}}(\Sigma)^n \to H^{-1}(\Omega)^n$ is the adjoint of the trace operator $\gamma_\Sigma : \mathring{H}^1(\Omega)^n \to H^{\frac{1}{2}}_\bullet(\Sigma)^n$ defined by (5.16), and the support of $\gamma_\Sigma^* \boldsymbol{\psi}_\Sigma$ is a subset of $\overline{\Sigma}$.

Now, we can show the well-posedness of the Dirichlet-transmission problem (5.25) (see also [4, Theorem 1.2], [5, Corollary 3.1] for interface problems involving the Stokes and Brinkman systems in Lipschitz domains with transversal interfaces and jump conditions in the isotropic case (2.12)).

Theorem 5.6 *Let* Assumption 5.1 *and conditions* (2.2)–(2.4) *hold.*

(i) *Then for all* $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\psi}_\Sigma) \in \mathfrak{Y}^0$ *the Dirichlet-transmission problem* (5.25) *has a unique solution* $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-)$ *in the space* $\mathfrak{X}_{\Omega^+,\Omega^-}$*, and there exists a positive constant* $C = C(\Omega^+, \Omega^-, C_{\mathbb{A}}, \|\mathbb{A}\|, n)$ *such that*

$$\|(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-)\|_{\mathfrak{X}_{\Omega^+,\Omega^-}} \le C \|(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\psi}_\Sigma)\|_{\mathfrak{Y}^0} \tag{5.30}$$

(ii) *The solution can be represented in the form* $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) = \mathfrak{U}^0(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g, \boldsymbol{\psi}_\Sigma)$*, where* $\mathfrak{U}^0 : \mathfrak{Y}^0 \to \mathfrak{X}_{\Omega^+,\Omega^-}$ *is a linear continuous operator.*

Proof Let us prove that the Dirichlet-transmission problem (5.25) with the unknowns $\big(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-\big) \in \mathfrak{X}_{\Omega^+,\Omega^-}$ is equivalent, in the sense of relations (5.28), to the variational problem (4.11) with the unknowns $(\mathbf{u}, \pi) \in \mathring{H}^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$, and with $\mathfrak{F} \in H^{-1}(\Omega)^n$ given by (5.29).

First, assume that $\big(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-\big) \in \mathfrak{X}_{\Omega^+,\Omega^-}$ satisfies the Dirichlet-transmission problem (5.25). Let $(\mathbf{u}, \pi) \in \mathring{H}^1(\Omega)^n \times L^2_0(\Omega)$ be the pair defined by formula (5.28) (cf. Lemma B.1). Then the first equation of variational problem (4.11) follows from the Green identity (5.24) and relation (5.29) for $\mathfrak{F}$. The second equation in (4.11) follows from the equations $\operatorname{div} \mathbf{u}^\pm = g|_{\Omega^\pm}$ in $\Omega^\pm$ and the inclusion $\operatorname{div} \mathbf{u} \in L^2(\Omega)$.

Conversely, assume that $(\mathbf{u},\pi)\in \mathring{H}^1(\Omega)^n\times L^2(\Omega)/\mathbb{R}$ satisfies the variational problem (4.11) and let $(\mathbf{u}^\pm,\pi^\pm)=(\mathbf{u}|_{\Omega^\pm},\pi|_{\Omega^\pm})$. Then the first equation in (4.11) can be written as

$$\begin{aligned}&\big\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}^+),E_{i\alpha}(\mathbf{w}^+)\big\rangle_{\Omega^+}-\big\langle\pi^+,\operatorname{div}\mathbf{w}^+\big\rangle_{\Omega^+}\\&\quad+\big\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}^-),E_{i\alpha}(\mathbf{w}^-)\big\rangle_{\Omega^-}-\big\langle\pi^-,\operatorname{div}\mathbf{w}^-\big\rangle_{\Omega^-}\\&\quad-\big\langle\tilde{\mathbf{f}}^+,\mathbf{w}\big\rangle_{\Omega^+}-\big\langle\tilde{\mathbf{f}}^-,\mathbf{w}\big\rangle_{\Omega^-}+\langle\boldsymbol{\psi}_\Sigma,\gamma_\Sigma\mathbf{w}\rangle_\Sigma=0\,,\ \forall\,\mathbf{w}\in\mathring{H}^1(\Omega)^n\,.\end{aligned}\tag{5.31}$$

Since the spaces $\mathcal{D}(\Omega^\pm)^n$ are subspaces of $\mathring{H}^1(\Omega)^n$, (the distributional form of) the anisotropic Stokes equation in (5.25), in each of the domains Ω^+ and Ω^-, follows from equation (5.31) written for all $\mathbf{w}\in\mathcal{D}(\Omega^+)^n$ and $\mathbf{w}\in\mathcal{D}(\Omega^-)^n$, respectively. A similar argument yields that the second variational equation in (4.11) implies the divergence equation $\operatorname{div}\mathbf{u}^\pm=g^\pm$ in $\Omega^\pm$. Thus, $\big(\mathbf{u}^+,\pi^+,\mathbf{u}^-,\pi^-\big)$ satisfies the anisotropic Stokes system in $\Omega^+\cup\Omega^-$, the Dirichlet boundary condition $(\gamma_{\Omega^\pm}\mathbf{u}^\pm)|_{\Gamma^\pm}=\mathbf{0}$ on $\Gamma^\pm$, and the interface condition $(\gamma_{\Omega^+}\mathbf{u}^+)|_\Sigma=(\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma$ on Σ. Then substituting (5.31) into the Green identity (5.24), we obtain the equation

$$\Big\langle\big(\mathbf{t}_{\Omega^+}(\mathbf{u}^+,\pi^+;\tilde{\mathbf{f}}^+)+\mathbf{t}_{\Omega^-}(\mathbf{u}^-,\pi^-;\tilde{\mathbf{f}}^-)\big)\big|_\Sigma,(\gamma_\Omega\mathbf{w})|_\Sigma\Big\rangle_\Sigma=\big\langle\boldsymbol{\psi}_\Sigma,\gamma_\Sigma\mathbf{w}\big\rangle_\Sigma\,.\tag{5.32}$$

In view of Lemma 5.3, formula (5.32) implies

$$\Big\langle\big(\mathbf{t}_{\Omega^+}(\mathbf{u}^+,\pi^+;\tilde{\mathbf{f}}^+)+\mathbf{t}_{\Omega^-}(\mathbf{u}^-,\pi^-;\tilde{\mathbf{f}}^-)\big)\big|_\Sigma,\boldsymbol{\phi}\Big\rangle_\Sigma=\big\langle\boldsymbol{\psi}_\Sigma,\boldsymbol{\phi}\big\rangle_\Sigma\,,\ \forall\,\boldsymbol{\phi}\in H_\bullet^{\frac12}(\Sigma)^n.\tag{5.33}$$

Therefore, $\big(\mathbf{t}_{\Omega^+}(\mathbf{u}^+,\pi^+;\tilde{\mathbf{f}}^+)+\mathbf{t}_{\Omega^-}(\mathbf{u}^-,\pi^-;\tilde{\mathbf{f}}^-)\big)\big|_\Sigma=\boldsymbol{\psi}_\Sigma$ on Σ.

Consequently, the Dirichlet-transmission problem (5.25) and the variational problem (4.11) are equivalent, as asserted. By Theorem 4.1, the variational problem (4.11) in the space $\mathring{H}^1(\Omega)^n\times L^2(\Omega)/\mathbb{R}$ is well-posed. Hence the proved equivalence implies the well-posedness of problem (5.25) in the space $\mathfrak{X}_{\Omega^+,\Omega^-}$, and estimate (5.30) follows from (4.12) and (5.29). Together with Theorem 4.1(iii) this also implies the representation of item (ii). □

5.4 Dirichlet-transmission problem with non-homogeneous interface and Dirichlet conditions

Let the space $\mathfrak{Y}_\bullet$ consists of all elements

$$\begin{aligned}(\widetilde{\mathbf{f}}^+,\widetilde{\mathbf{f}}^-,g^+,g^-,\boldsymbol{\varphi}_\Sigma,\boldsymbol{\psi}_\Sigma,\boldsymbol{\varphi})\in&\big(H^1_{\Gamma^+}(\Omega^+)^n\big)'\times\big(H^1_{\Gamma^-}(\Omega^-)^n\big)'\times L^2(\Omega^+)\times L^2(\Omega^-)\\&\times H_\bullet^{\frac12}(\Sigma)^n\times H^{-\frac12}(\Sigma)^n\times H^{\frac12}(\partial\Omega)^n\end{aligned}$$

such that g^+, g^-, $\boldsymbol{\varphi}$, and $\boldsymbol{\varphi}_\Sigma$ satisfy the compatibility condition

$$\int_{\Omega^+}g^+dx+\int_{\Omega^-}g^-dx=\int_{\partial\Omega}\boldsymbol{\varphi}\cdot\boldsymbol{\nu}d\sigma+\int_\Sigma\boldsymbol{\varphi}_\Sigma\cdot\boldsymbol{\nu}_\Sigma d\sigma\,,\tag{5.34}$$

where $\boldsymbol{\nu}_\Sigma$ is the unit normal to Σ oriented from Ω^+ to Ω^-. The space $\mathfrak{Y}_\bullet$ is endowed with the norm

$$\begin{aligned}\|(\widetilde{\mathbf{f}}^+,\widetilde{\mathbf{f}}^-,g^+,g^-,\boldsymbol{\varphi}_\Sigma,\boldsymbol{\psi}_\Sigma,\boldsymbol{\varphi})\|_{\mathfrak{Y}_\bullet}:=&\|\widetilde{\mathbf{f}}^+\|_{\left(H^1_{\Gamma^+}(\Omega^+)^n\right)'}+\|\widetilde{\mathbf{f}}^-\|_{\left(H^1_{\Gamma^-}(\Omega^-)^n\right)'}\\&+\|g^+\|_{L^2(\Omega^+)}+\|g^-\|_{L^2(\Omega^-)}+\|\boldsymbol{\varphi}_\Sigma\|_{H_\bullet^{\frac12}(\Sigma)^n}+\|\boldsymbol{\psi}_\Sigma\|_{H^{-\frac12}(\Sigma)^n}+\|\boldsymbol{\varphi}\|_{H^{\frac12}(\partial\Omega^n)}.\end{aligned}$$

Let us also define the space $\mathcal{M}_\bullet$ consisting of all elements

$$(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi}) \in L^2(\Omega^+) \times L^2(\Omega^-) \times H_\bullet^{\frac{1}{2}}(\Sigma)^n \times H^{\frac{1}{2}}(\partial\Omega)^n$$

satisfying the compatibility condition (5.34), and endowed with the norm

$$\|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet} := \|g^+\|_{L^2(\Omega^+)} + \|g^-\|_{L^2(\Omega^-)} + \|\boldsymbol{\varphi}_\Sigma\|_{H_\bullet^{\frac{1}{2}}(\Sigma)^n} + \|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n}. \tag{5.35}$$

Let us consider the following non-homogeneous Dirichlet-transmission problem

$$\begin{cases} \mathcal{L}(\mathbf{u}^+, \pi^+) = \widetilde{\mathbf{f}}^+|_{\Omega^+}, \quad \operatorname{div}\mathbf{u}^+ = g^+ & \text{in } \Omega^+, \\ \mathcal{L}(\mathbf{u}^-, \pi^-) = \widetilde{\mathbf{f}}^-|_{\Omega^-}, \quad \operatorname{div}\mathbf{u}^- = g^- & \text{in } \Omega^-, \\ (\gamma_{\Omega^+}\mathbf{u}^+)|_\Sigma - (\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma = \boldsymbol{\varphi}_\Sigma & \text{on } \Sigma, \\ \big(\mathbf{t}_{\Omega^+}(\mathbf{u}^+, \pi^+; \widetilde{\mathbf{f}}^+)\big)|_\Sigma + \big(\mathbf{t}_{\Omega^-}(\mathbf{u}^-, \pi^-; \widetilde{\mathbf{f}}^-)\big)|_\Sigma = \boldsymbol{\psi}_\Sigma & \text{on } \Sigma, \\ (\gamma_{\Omega^+}\mathbf{u}^+)|_{\Gamma^+} = \boldsymbol{\varphi}|_{\Gamma^+} & \text{on } \Gamma^+, \\ (\gamma_{\Omega^-}\mathbf{u}^-)|_{\Gamma^-} = \boldsymbol{\varphi}|_{\Gamma^-} & \text{on } \Gamma^-, \end{cases} \tag{5.36}$$

with the unknown functions $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-)$ in the space $\mathfrak{X}_{\Omega^+,\Omega^-}$ defined in (5.27), and with the given data $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}) \in \mathfrak{Y}_\bullet$.

In order to analyze the non-homogeneous Dirichlet-transmission problem, we need the following Bogovskii-type transmission result.

Lemma 5.7 *Let* Assumption 5.1 *hold. Then for all given data* $(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi}) \in \mathcal{M}_\bullet$ *there exist* $\mathbf{v}^\pm \in H^1(\Omega^\pm)^n$ *such that*

$$\operatorname{div}\mathbf{v}^+ = g^+ \ \text{ in } \Omega^+, \tag{5.37}$$

$$\operatorname{div}\mathbf{v}^- = g^- \ \text{ in } \Omega^-, \tag{5.38}$$

$$(\gamma_{\Omega^+}\mathbf{v}^+)|_\Sigma - (\gamma_{\Omega^-}\mathbf{v}^-)|_\Sigma = \boldsymbol{\varphi}_\Sigma \ \text{ on } \Sigma, \tag{5.39}$$

$$(\gamma_{\Omega^+}\mathbf{v}^+)|_{\Gamma^+} = \boldsymbol{\varphi}|_{\Gamma^+} \ \text{ on } \Gamma^+, \tag{5.40}$$

$$(\gamma_{\Omega^-}\mathbf{v}^-)|_{\Gamma^-} = \boldsymbol{\varphi}|_{\Gamma^-} \ \text{ on } \Gamma^-, \tag{5.41}$$

and there exists a constant $C_\Sigma = C_\Sigma(\Omega^+, \Omega^-, n) > 0$ *such that*

$$\|\mathbf{v}^\pm\|_{H^1(\Omega^\pm)^n} \le C_\Sigma \|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}. \tag{5.42}$$

Proof Let us introduce the functions

$$\mathbf{v}_1^+ := r_{\Omega^+}\gamma_\Omega^{-1}\boldsymbol{\varphi} + \frac{1}{2}\gamma_{\Omega^+}^{-1}\mathring{E}_{\Sigma\to\partial\Omega^+}\boldsymbol{\varphi}_\Sigma, \tag{5.43}$$

$$\mathbf{v}_1^- := r_{\Omega^-}\gamma_\Omega^{-1}\boldsymbol{\varphi} - \frac{1}{2}\gamma_{\Omega^-}^{-1}\mathring{E}_{\Sigma\to\partial\Omega^-}\boldsymbol{\varphi}_\Sigma, \tag{5.44}$$

where $\gamma_\Omega^{-1} : H^{\frac{1}{2}}(\partial\Omega)^n \to H^1(\Omega)^n$ and $\gamma_{\Omega^\pm}^{-1} : H^{\frac{1}{2}}(\partial\Omega^\pm)^n \to H^1(\Omega^\pm)^n$ are continuous right inverses to the corresponding trace operators, $\mathring{E}_{\Sigma\to\partial\Omega^\pm} : H_\bullet^{\frac{1}{2}}(\Sigma)^n \to \widetilde{H}^{\frac{1}{2}}(\Sigma; \partial\Omega^\pm)^n$ are operators of extension by zero, cf. (5.15), while $r_{\Omega^\pm} : H^1(\Omega)^n \to H^1(\Omega^\pm)^n$ are restriction operators to the corresponding domains, and each of these operators is continuous. Then $\mathbf{v}_1^\pm$ belong to $H^1(\Omega^\pm)^n$, respectively, and satisfy transmission and boundary conditions (5.39)–(5.41). Let us now define

$$g_1^\pm := \operatorname{div}\mathbf{v}_1^\pm \in L^2(\Omega^\pm) \tag{5.45}$$

Then by the divergence Theorem and condition (5.34) we obtain

$$\int_{\Omega^+} g_1^+(x)dx + \int_{\Omega^-} g_1^-(x)dx = \int_{\Omega^+} g^+ dx + \int_{\Omega_-} g^- dx. \tag{5.46}$$

Let $g_2 \in L^2(\Omega)$ be such that $g_2|_{\Omega_\pm} = g^\pm - g_1^\pm$. Hence g_2 belongs to the space $L^2_0(\Omega)$ defined in (3.1). Then by Proposition 3.1 there exists $\mathbf{v}_2 \in \mathring{H}^1(\Omega)$ satisfying

$$\operatorname{div} \mathbf{v}_2 = g_2 \text{ in } \Omega\,, \tag{5.47}$$

and a constant $C = C(\Omega, n) > 0$ such that

$$\|\mathbf{v}_2\|^2_{H^1(\Omega)^n} \leq C\|g_2\|^2_{L^2(\Omega)} = C(\|g^+ - g_1^+\|^2_{L^2(\Omega^+)} + \|g^- - g_1^-\|^2_{L^2(\Omega^-)}). \tag{5.48}$$

Finally, choosing $\mathbf{v}^\pm := \mathbf{v}_1^\pm + r_{\Omega^\pm}\mathbf{v}_2$ and using inequality (5.48) and continuity of the operators involved in (5.43)–(5.45), we obtain the desired result. □

Let us define the operators $\check{\mathfrak{L}}^\pm : H^1(\Omega^\pm)^n \to \widetilde{H}^{-1}(\Omega^\pm)^n$, cf. (2.1), as

$$(\check{\mathfrak{L}}^\pm \mathbf{v}^\pm)_i := \partial_\alpha \mathring{E}_{\Omega^\pm}\big(a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{v}^\pm)\big), \quad i = 1, \ldots, n, \quad \forall\, \mathbf{v}^\pm \in H^1(\Omega^\pm)^n\,, \tag{5.49}$$

where $\mathring{E}_{\Omega^\pm}$ are the operators of zero extensions from $\Omega^\pm$ onto $\mathbb{R}^n$. Then by (2.9) we have that $\big(\check{\mathfrak{L}}^\pm \mathbf{v}^\pm\big)|_{\Omega^\pm} = \boldsymbol{\mathcal{L}}(\mathbf{v}^\pm, 0)$ in $\Omega^\pm$.

Theorem 5.8 *Let* Assumption 5.1 *and conditions* (2.2)–(2.4) *hold. Then for all* $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}) \in \mathfrak{Y}_\bullet$ *the Dirichlet-transmission problem* (5.36) *has a unique solution* $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-)$ *in the space* $\mathfrak{X}_{\Omega^+,\Omega^-}$ *and there exists a constant* $C = C(\Omega^+, \Omega^-, C_{\mathbb{A}}, \|\mathbb{A}\|, n) > 0$ *such that*

$$\|(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-)\|_{\mathfrak{X}_{\Omega^+,\Omega^-}} \leq C\|(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi})\|_{\mathfrak{Y}_\bullet}$$

Proof Let $\mathbf{v}^\pm \in H^1(\Omega^\pm)^n$ be the functions given by Lemma 5.7. Defining $\check{\mathbf{f}}^\pm := \check{\mathfrak{L}}^\pm \mathbf{v}^\pm \in \widetilde{H}^{-1}(\Omega^\pm)^n$, we obtain that $\check{\mathbf{f}}^\pm|_{\Omega^\pm} = \boldsymbol{\mathcal{L}}(\mathbf{v}^\pm, 0)$ in $\Omega^\pm$ and $\mathbf{t}_{\Omega^\pm}(\mathbf{v}^\pm, 0; \check{\mathbf{f}}^\pm) = \mathbf{0}$ by Definition A.2.

Then the fully non-homogeneous Dirichlet-transmission problem (5.36) reduces to the following Dirichlet-transmission problem with homogeneous Dirichlet conditions on $\Gamma^\pm$ and homogeneous interface condition for the traces across Σ, for the new functions $\mathbf{w}^\pm := \mathbf{u}^\pm - \mathbf{v}^\pm$.

$$\begin{cases} \boldsymbol{\mathcal{L}}(\mathbf{w}^+, \pi^+) = (\widetilde{\mathbf{f}}^+ - \check{\mathbf{f}}^+)|_{\Omega^+},\ \operatorname{div} \mathbf{w}^+ = 0 & \text{in } \Omega^+, \\ \boldsymbol{\mathcal{L}}(\mathbf{w}^-, \pi^-) = (\widetilde{\mathbf{f}}^- - \check{\mathbf{f}}^-)|_{\Omega^-},\ \operatorname{div} \mathbf{w}^- = 0 & \text{in } \Omega^-, \\ (\gamma_{\Omega^+}\mathbf{w}^+)|_\Sigma = (\gamma_{\Omega^-}\mathbf{w}^-)|_\Sigma & \text{on } \Sigma, \\ \big(\mathbf{t}_{\Omega^+}(\mathbf{w}^+, \pi^+; \widetilde{\mathbf{f}}^+ - \check{\mathbf{f}}^+)\big)|_\Sigma + \big(\mathbf{t}_{\Omega^-}(\mathbf{w}^-, \pi^-; \widetilde{\mathbf{f}}^- - \check{\mathbf{f}}^-)\big)|_\Sigma = \boldsymbol{\psi}_\Sigma & \text{on } \Sigma, \\ (\gamma_{\Omega^+}\mathbf{w}^+)|_{\Gamma^+} = \mathbf{0} & \text{on } \Gamma^+, \\ (\gamma_{\Omega^-}\mathbf{w}^-)|_{\Gamma^-} = \mathbf{0} & \text{on } \Gamma^-. \end{cases} \tag{5.50}$$

Theorem 5.6 implies that the Dirichlet-transmission problem (5.50) has a unique solution $(\mathbf{w}^+, \pi^+, \mathbf{w}^-, \pi^-)$ in the space $\mathfrak{X}_{\Omega^+,\Omega^-}$ and depends continuously on the given data of this problem. Finally, the well-posedness of problem (5.50) implies that the functions $(\mathbf{u}^\pm = \mathbf{w}^\pm + \mathbf{v}^\pm,\ \pi^\pm)$ determine a solution of the full non-homogeneous Dirichlet-transmission problem (5.36) in the space $\mathfrak{X}_{\Omega^+,\Omega^-}$, and depends continuously on the given data $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}) \in \mathfrak{Y}_\bullet$. This solution is unique by the uniqueness statement in Theorem 5.6. □

5.5 Dirichlet-transmission problem with fully non-homogeneous interface and Dirichlet conditions

Let us now consider the non-homogeneous Dirichlet-transmission problem

$$\begin{cases} \mathcal{L}(\mathbf{u}^+,\pi^+)=\widetilde{\mathbf{f}}^+|_{\Omega^+}, \quad \operatorname{div}\mathbf{u}^+=g^+ & \text{in } \Omega^+,\\ \mathcal{L}(\mathbf{u}^-,\pi^-)=\widetilde{\mathbf{f}}^-|_{\Omega^-}, \quad \operatorname{div}\mathbf{u}^-=g^- & \text{in } \Omega^-,\\ (\gamma_{\Omega^+}\mathbf{u}^+)|_\Sigma-(\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma=\boldsymbol{\varphi}_\Sigma & \text{on } \Sigma,\\ \big(\mathbf{t}_{\Omega^+}(\mathbf{u}^+,\pi^+;\widetilde{\mathbf{f}}^+)\big)|_\Sigma+\big(\mathbf{t}_{\Omega^-}(\mathbf{u}^-,\pi^-;\widetilde{\mathbf{f}}^-)\big)|_\Sigma=\boldsymbol{\psi}_\Sigma & \text{on } \Sigma,\\ (\gamma_{\Omega^+}\mathbf{u}^+)|_{\Gamma^+}=\boldsymbol{\varphi}^+ & \text{on } \Gamma^+,\\ (\gamma_{\Omega^-}\mathbf{u}^-)|_{\Gamma^-}=\boldsymbol{\varphi}^- & \text{on } \Gamma^-, \end{cases} \tag{5.51}$$

with more general data $(\widetilde{\mathbf{f}}^+,\widetilde{\mathbf{f}}^-,g^+,g^-,\boldsymbol{\varphi}_\Sigma,\boldsymbol{\psi}_\Sigma,\boldsymbol{\varphi}^+,\boldsymbol{\varphi}^-)\in\mathfrak{Y}$, where $\mathfrak{Y}$ consists of

$$\begin{aligned}(\widetilde{\mathbf{f}}^+,\widetilde{\mathbf{f}}^-,g^+,g^-,\boldsymbol{\varphi}_\Sigma,\boldsymbol{\psi}_\Sigma,\boldsymbol{\varphi}^+,\boldsymbol{\varphi}^-)&\in\big(H^1_{\Gamma^+}(\Omega^+)^n\big)'\times\big(H^1_{\Gamma^-}(\Omega^-)^n\big)'\\ &\times L^2(\Omega^+)\times L^2(\Omega^-)\times H^{\frac12}(\Sigma)^n\times H^{-\frac12}(\Sigma)^n\times H^{\frac12}(\Gamma^+)^n\times H^{\frac12}(\Gamma^-)^n,\end{aligned}$$

such that $g^+,g^-,\boldsymbol{\varphi}_\Sigma,\boldsymbol{\varphi}^+,\boldsymbol{\varphi}^-$ satisfy the compatibility condition

$$\int_{\Omega^+}g^+dx+\int_{\Omega^-}g^-dx=\int_{\Gamma^+}\boldsymbol{\varphi}^+\cdot\boldsymbol{\nu}d\sigma+\int_{\Gamma^-}\boldsymbol{\varphi}^-\cdot\boldsymbol{\nu}d\sigma+\int_\Sigma\boldsymbol{\varphi}_\Sigma\cdot\boldsymbol{\nu}_\Sigma d\sigma, \tag{5.52}$$

and the condition

$$\boldsymbol{\varphi}_\Sigma-r_\Sigma\boldsymbol{\Phi}^++r_\Sigma\boldsymbol{\Phi}^-\in H^{\frac12}_\bullet(\Sigma)^n \tag{5.53}$$

for some extensions $\boldsymbol{\Phi}^\pm\in H^{\frac12}(\partial\Omega^\pm)^n$ such that $r_{\Gamma^\pm}\boldsymbol{\Phi}^\pm=\boldsymbol{\varphi}^\pm$.

Remark 5.9 (i) Condition (5.53) is particularly satisfied if $\boldsymbol{\varphi}_\Sigma\in H^{\frac12}_\bullet(\Sigma)^n$ and $\boldsymbol{\varphi}^\pm=r_{\Gamma^\pm}\boldsymbol{\varphi}$ for some $\boldsymbol{\varphi}\in H^{\frac12}(\partial\Omega)^n$, as in the case considered in Sect. 5.4. Indeed, we can choose $\boldsymbol{\Phi}^\pm=\gamma_{\Omega^\pm}\gamma_\Omega^{-1}\boldsymbol{\varphi}$ and obtain that $\boldsymbol{\Phi}^\pm\in H^{\frac12}(\partial\Omega^\pm)^n$ and

$$r_{\Gamma^\pm}\boldsymbol{\Phi}^\pm=r_{\Gamma^\pm}\gamma_{\Omega^\pm}\gamma_\Omega^{-1}\boldsymbol{\varphi}=r_{\Gamma^\pm}\gamma_\Omega\gamma_\Omega^{-1}\boldsymbol{\varphi}=r_{\Gamma^\pm}\boldsymbol{\varphi}=\boldsymbol{\varphi}^\pm,$$

which implies that $\boldsymbol{\Phi}^\pm$ are extensions of $\boldsymbol{\varphi}^\pm$ from $\Gamma^\pm$ to $\partial\Omega^\pm$. Moreover, the property $\gamma_\Omega^{-1}\boldsymbol{\varphi}\in H^1(\Omega)^n$ implies that

$$r_\Sigma\boldsymbol{\Phi}^+-r_\Sigma\boldsymbol{\Phi}^-=r_\Sigma\gamma_{\Omega^+}\gamma_\Omega^{-1}\boldsymbol{\varphi}-r_\Sigma\gamma_{\Omega^-}\gamma_\Omega^{-1}\boldsymbol{\varphi}=\mathbf{0}.$$

(ii) If condition (5.53) is satisfied for some functions $\boldsymbol{\Phi}^\pm\in H^{\frac12}(\partial\Omega^\pm)^n$ such that $\boldsymbol{\varphi}^\pm=r_{\Gamma^\pm}\boldsymbol{\Phi}^\pm$ then it is also satisfied for all functions $\boldsymbol{\Phi}^\pm_*\in H^{\frac12}(\partial\Omega^\pm)^n$ such that $\boldsymbol{\varphi}^\pm=r_{\Gamma^\pm}\boldsymbol{\Phi}^\pm_*$ because $\boldsymbol{\Phi}^\pm-\boldsymbol{\Phi}^\pm_*=\mathbf{0}$ on $\Gamma^\pm$ and hence $r_\Sigma(\boldsymbol{\Phi}^\pm-\boldsymbol{\Phi}^\pm_*)\in H^{\frac12}_\bullet(\Sigma)^n$.

In order to analyze the non-homogeneous Dirichlet-transmission problem, we need the following generalized Bogoskii-type transmission result.

Lemma 5.10 *Let* Assumption 5.1 *hold and let*

$$(g^+,g^-,\boldsymbol{\varphi}_\Sigma,\boldsymbol{\varphi}^+,\boldsymbol{\varphi}^-)\in L^2(\Omega^+)\times L^2(\Omega^-)\times H^{\frac12}(\Sigma)^n\times H^{\frac12}(\Gamma^+)^n\times H^{\frac12}(\Gamma^-)^n$$

satisfy conditions (5.52) *and* (5.53). *Then there exist* $\mathbf{v}^{\pm} \in H^1(\Omega^{\pm})^n$ *such that*

$$\operatorname{div} \mathbf{v}^+ = g^+ \ \text{ in } \Omega^+, \tag{5.54}$$

$$\operatorname{div} \mathbf{v}^- = g^- \ \text{ in } \Omega^-, \tag{5.55}$$

$$(\gamma_{\Omega^+}\mathbf{v}^+)|_{\Sigma} - (\gamma_{\Omega^-}\mathbf{v}^-)|_{\Sigma} = \boldsymbol{\varphi}_{\Sigma} \ \text{ on } \Sigma, \tag{5.56}$$

$$(\gamma_{\Omega^+}\mathbf{v}^+)|_{\Gamma^+} = \boldsymbol{\varphi}^+ \ \text{ on } \Gamma^+, \tag{5.57}$$

$$(\gamma_{\Omega^-}\mathbf{v}^-)|_{\Gamma^-} = \boldsymbol{\varphi}^- \ \text{ on } \Gamma^-, \tag{5.58}$$

and, moreover,

$$\begin{aligned}\|\mathbf{v}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{v}^-\|_{H^1(\Omega^-)^n} \le C_{\Sigma}\Big(&\|g^+\|_{L^2(\Omega^+)} + \|g^-\|_{L^2(\Omega^-)} \\ &+ \|\boldsymbol{\varphi}_{\Sigma}\|_{H^{\frac{1}{2}}(\Sigma)^n} + \|\boldsymbol{\varphi}^+\|_{H^{\frac{1}{2}}(\Gamma^+)^n} + \|\boldsymbol{\varphi}^-\|_{H^{\frac{1}{2}}(\Gamma^-)^n}\Big)\end{aligned} \tag{5.59}$$

with some constant $C_{\Sigma} = C_{\Sigma}(\Omega^+, \Omega^-, n) > 0$.

Proof We will prove this lemma by modifying the proof of Lemma 5.7 appropriately. Let $\gamma_{\Omega^{\pm}}^{-1} : H^{\frac{1}{2}}(\partial\Omega^{\pm})^n \to H^1(\Omega^{\pm})^n$ be some continuous right inverses to the corresponding trace operators.

Let $\boldsymbol{\Phi}^{\pm} \in H^{\frac{1}{2}}(\partial\Omega^{\pm})^n$ denote some extensions of the functions $\boldsymbol{\varphi}^{\pm}$ from $\Gamma^{\pm}$ to $\partial\Omega^{\pm}$, that is, $r_{\Gamma^{\pm}}\boldsymbol{\Phi}^{\pm} = \boldsymbol{\varphi}^{\pm}$. Let us introduce the functions

$$\boldsymbol{\Phi}_{\Sigma} := \boldsymbol{\varphi}_{\Sigma} - r_{\Sigma}\boldsymbol{\Phi}^+ + r_{\Sigma}\boldsymbol{\Phi}^-, \tag{5.60}$$

$$\mathbf{v}_1^+ := \gamma_{\Omega^+}^{-1}(\boldsymbol{\Phi}^+ + \frac{1}{2}\mathring{E}_{\Sigma\to\partial\Omega^+}\boldsymbol{\Phi}_{\Sigma}), \tag{5.61}$$

$$\mathbf{v}_1^- := \gamma_{\Omega^-}^{-1}(\boldsymbol{\Phi}^- - \frac{1}{2}\mathring{E}_{\Sigma\to\partial\Omega^-}\boldsymbol{\Phi}_{\Sigma}). \tag{5.62}$$

Due to condition (5.53) and Remark 5.9(ii), $\boldsymbol{\Phi}_{\Sigma} \in H_{\bullet}^{\frac{1}{2}}(\Sigma)^n$ and hence $\mathring{E}_{\Sigma\to\partial\Omega^{\pm}}\boldsymbol{\Phi}_{\Sigma} \in H^{\frac{1}{2}}(\partial\Omega^{\pm})^n$. Then $\mathbf{v}_1^{\pm} \in H^1(\Omega^{\pm})^n$ and satisfy transmission and boundary conditions (5.56)–(5.58). Let us now define

$$g_1^{\pm} := \operatorname{div} \mathbf{v}_1^{\pm} \in L^2(\Omega^{\pm}) \tag{5.63}$$

Then by the divergence Theorem and condition (5.52) we obtain

$$\int_{\Omega^+} g_1^+ dx + \int_{\Omega^-} g_1^- dx = \int_{\Omega^+} g^+ dx + \int_{\Omega_-} g^- dx. \tag{5.64}$$

Let $g_2 \in L^2(\Omega)$ be such that $g_2|_{\Omega_{\pm}} = g^{\pm} - g_1^{\pm}$. Hence g_2 belongs to the space $L_0^2(\Omega)$ defined in (3.1). Then by Proposition 3.1 there exists $\mathbf{v}_2 \in \mathring{H}^1(\Omega)$ such that

$$\operatorname{div} \mathbf{v}_2 = g_2 \ \text{ in } \Omega, \quad \|\mathbf{v}_2\|_{H^1(\Omega)^n} \le C\|g_2\|_{L^2(\Omega)}. \tag{5.65}$$

Finally, choosing $\mathbf{v}^{\pm} := \mathbf{v}_1^{\pm} + r_{\Omega^{\pm}}\mathbf{v}_2$ and using the inequality in (5.65) and the continuity of the operators involved in (5.60)–(5.63), we get the assertion. □

Theorem 5.11 *Let* Assumption 5.1 *and conditions* (2.2)–(2.4) *hold. Then for all* $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\psi}_{\Sigma}, \boldsymbol{\varphi}^+, \boldsymbol{\varphi}^-) \in \mathfrak{Y}$ *the Dirichlet-transmission problem* (5.51) *has a unique solution*

$(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-)$ in the space $\mathfrak{X}_{\Omega^+,\Omega^-}$ defined in (5.27), and there exists a constant $C = C(\Omega^+, \Omega^-, C_{\mathbb{A}}, \|\mathbb{A}\|, n) > 0$ such that

$$\|(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-)\|_{\mathfrak{X}_{\Omega^+,\Omega^-}} \leq C\|(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}^+, \boldsymbol{\varphi}^-)\|_{\mathfrak{Y}}.$$

Proof We use arguments similar to those in the proof of Theorem 5.8. Let $\mathbf{v}^\pm \in H^1(\Omega^\pm)^n$ be given by Lemma 5.10. For the velocity-pressure couples $(\mathbf{v}^\pm, 0)$, let $\check{\boldsymbol{f}}^\pm := \check{\mathfrak{L}}^\pm \mathbf{v}^\pm \in \widetilde{H}^{-1}(\Omega^\pm)^n$, where operators $\check{\mathfrak{L}}^\pm$ are defined in (5.49). Hence $\check{\boldsymbol{f}}^\pm \in \widetilde{H}^{-1}(\Omega^\pm)^n$, $\check{\boldsymbol{f}}^\pm|_{\Omega^\pm} = \mathcal{L}(\mathbf{v}^\pm, 0)$ in $\Omega^\pm$, cf. (2.9), and $\mathbf{t}_{\Omega^\pm}(\mathbf{v}^\pm, 0; \check{\mathbf{f}}^\pm) = \mathbf{0}$ by Definition A.2.

Then for the new functions $\mathbf{w}^\pm := \mathbf{u}^\pm - \mathbf{v}^\pm$, the fully non-homogeneous Dirichlet-transmission problem (5.51) reduces to the following Dirichlet-transmission problem with homogeneous Dirichlet conditions on $\Gamma^\pm$ and homogeneous interface condition for the traces across Σ.

$$\begin{cases} \mathcal{L}(\mathbf{w}^+, \pi^+) = (\widetilde{\mathbf{f}}^+ - \check{\mathbf{f}}^+)|_{\Omega^+}, \ \operatorname{div}\mathbf{w}^+ = 0 & \text{in } \Omega^+, \\ \mathcal{L}(\mathbf{w}^-, \pi^-) = (\widetilde{\mathbf{f}}^- - \check{\mathbf{f}}^-)|_{\Omega^-}, \ \operatorname{div}\mathbf{w}^- = 0 & \text{in } \Omega^-, \\ (\gamma_{\Omega^+}\mathbf{w}^+)|_\Sigma = (\gamma_{\Omega^-}\mathbf{w}^-)|_\Sigma & \text{on } \Sigma, \\ \big(\mathbf{t}_{\Omega^+}(\mathbf{w}^+, \pi^+; \widetilde{\mathbf{f}}^+ - \check{\mathbf{f}}^+)\big)|_\Sigma + \big(\mathbf{t}_{\Omega^-}(\mathbf{w}^-, \pi^-; \widetilde{\mathbf{f}}^- - \check{\mathbf{f}}^-)\big)|_\Sigma = \boldsymbol{\psi}_\Sigma & \text{on } \Sigma, \\ (\gamma_{\Omega^+}\mathbf{w}^+)|_{\Gamma^+} = \mathbf{0} & \text{on } \Gamma^+, \\ (\gamma_{\Omega^-}\mathbf{w}^-)|_{\Gamma^-} = \mathbf{0} & \text{on } \Gamma^-. \end{cases} \tag{5.66}$$

Theorem 5.6 implies that the Dirichlet-transmission problem (5.66) has a unique solution $(\mathbf{w}^+, \pi^+, \mathbf{w}^-, \pi^-)$ in the space $\mathfrak{X}_{\Omega^+,\Omega^-}$ and depends continuously on the given data of this problem. Finally, the well-posedness of problem (5.66) implies that the functions $(\mathbf{u}^\pm = \mathbf{w}^\pm + \mathbf{v}^\pm, \ \pi^\pm)$ determine a solution of the fully non-homogeneous Dirichlet-transmission problem (5.51) in the space $\mathfrak{X}_{\Omega^+,\Omega^-}$, and the solution depends continuously on the given data $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}^+, \boldsymbol{\varphi}^-) \in \mathfrak{Y}$. This solution is unique by the uniqueness statement in Theorem 5.6. □

6 Dirichlet problem for the incompressible anisotropic Navier-Stokes system with general data in a bounded Lipschitz domain

In this section, we show the existence of a weak solution of a fully non-homogeneous Dirichlet problem for the anisotropic Navier-Stokes system in the incompressible case with general data in L^2-based Sobolev spaces in a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$. We use the well-posedness result established in Theorem 4.1 for the Dirichlet problem for the Stokes system and the following variant of the *Leray-Schauder fixed point theorem* (see [29, Theorem 11.3]).

Theorem 6.1 *Let $\mathcal{X}$ denote a Banach space and $T : \mathcal{X} \to \mathcal{X}$ be a continuous and compact operator. If there exists a constant $M_0 > 0$ such that $\|x\|_{\mathcal{X}} \leq M_0$ for every pair $(x, \lambda) \in \mathcal{X} \times [0, 1]$ satisfying $x = \lambda T x$, then the operator T has a fixed point x_0 (with $\|x_0\|_{\mathcal{X}} \leq M_0$).*

Recall that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and denote

$$H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^n := \left\{ \boldsymbol{\varphi} \in H^{\frac{1}{2}}(\partial\Omega)^n : \langle \boldsymbol{\varphi}, \boldsymbol{\nu} \rangle_{\partial\Omega} = 0 \right\}. \tag{6.1}$$

Restricting our analysis to the case $n \in \{2, 3\}$ will allow to use some compact embedding results. Consider the following Dirichlet problem,

$$\begin{cases} \mathcal{L}(\mathbf{u}, \pi) = -\mathfrak{F} + (\mathbf{u} \cdot \nabla)\mathbf{u}\,, \quad \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \gamma_\Omega \mathbf{u} = \boldsymbol{\varphi} & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

for the couple of unknowns $(\mathbf{u}, \pi) \in H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ and the given data $(\mathfrak{F}, \boldsymbol{\varphi}) \in H^{-1}(\Omega)^n \times H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^n$.

The main tool for our next arguments is the following assertion (see, e.g., [39, (1.4)], [41]).

Lemma 6.2 *Let $\boldsymbol{\varphi} \in H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^n$. Then for any $\varepsilon > 0$ there exists $\mathbf{v}_\varepsilon = \mathbf{v}_\varepsilon(\boldsymbol{\varphi}; \Omega) \in H^1_{\operatorname{div}}(\Omega)^n$ such that*

$$\gamma_\Omega \mathbf{v}_\varepsilon = \boldsymbol{\varphi} \text{ on } \partial\Omega \tag{6.3}$$

and the following Leray-Hopf inequality holds

$$\left|\left\langle (\mathbf{v} \cdot \nabla)\mathbf{v}_\varepsilon, \mathbf{v} \right\rangle_\Omega\right| \leq \varepsilon \|\nabla \mathbf{v}\|^2_{L^2(\Omega)^{n\times n}}, \ \forall\, \mathbf{v} \in \mathring{H}^1_{\operatorname{div}}(\Omega)^n\,. \tag{6.4}$$

Next we show the following existence result (see also [56, Proposition 1.1] in the isotropic incompressible case (2.12) with $\mu = 1$).

Theorem 6.3 *Let $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, be a bounded Lipschitz domain. Let conditions (2.2)–(2.4) hold. Then for all given data $(\mathfrak{F}, \boldsymbol{\varphi}) \in H^{-1}(\Omega)^n \times H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^n$, the Dirichlet problem for the anisotropic Navier-Stokes system (6.2) has a solution $(\mathbf{u}, \pi) \in H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$.*

Proof We reduce the analysis of the nonlinear problem (6.2) to the analysis of a nonlinear operator in the Hilbert space $\mathring{H}^1_{\operatorname{div}}(\Omega)^n$ and show that this operator has a fixed-point due to the Leray-Schauder Theorem (cf. [41], see also [39]).

To this end, we represent a solution of problem (6.2) in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}_\varepsilon\,, \tag{6.5}$$

where $\mathbf{v}_\varepsilon \in H^1_{\operatorname{div}}(\Omega)^n$ satisfies relations (6.3) and (6.4) with an ε that will be specified later, while $\mathbf{u}_0 \in \mathring{H}^1_{\operatorname{div}}(\Omega)^n$.

Then the Dirichlet problem (6.2) reduces to the nonlinear equation

$$\mathcal{L}(\mathbf{u}_0, \pi) = \mathbf{F}_\varepsilon \mathbf{u}_0 \tag{6.6}$$

for the couple of unknowns $(\mathbf{u}_0, \pi) \in \mathring{H}^1_{\operatorname{div}}(\Omega)^n \times L^2(\Omega)/\mathbb{R}$, where

$$\mathbf{F}_\varepsilon \mathbf{w} := -\mathfrak{F} - \mathfrak{L}\mathbf{v}_\varepsilon + ((\mathbf{w} + \mathbf{v}_\varepsilon) \cdot \nabla)(\mathbf{w} + \mathbf{v}_\varepsilon), \quad \forall\, \mathbf{w} \in \mathring{H}^1_{\operatorname{div}}(\Omega)^n \tag{6.7}$$

(cf. notations in (2.7), (2.1) and (2.8)).

For a fixed $\mathbf{v}_\varepsilon \in H^1_{\operatorname{div}}(\Omega)^n$, formula (6.7) defines a nonlinear mapping $\mathbf{w} \mapsto \mathbf{F}_\varepsilon \mathbf{w}$, and the nonlinear operator $\mathbf{F}_\varepsilon$ acts from $\mathring{H}^1_{\operatorname{div}}(\Omega)^n$ to $H^{-1}(\Omega)^n$ due to the inclusion $\mathfrak{L}\mathbf{v}_\varepsilon \in H^{-1}(\Omega)^n$ (provided by (2.5)) and estimate (C.6).

By Theorem 4.1, the linear operator

$$\mathcal{L} : \mathring{H}^1_{\operatorname{div}}(\Omega)^n \times L^2(\Omega)/\mathbb{R} \to H^{-1}(\Omega)^n \tag{6.8}$$

is an isomorphism. Its inverse operator can be split into two operator components,

$$\mathcal{L}^{-1} = (\mathcal{U}, \mathcal{P}),$$

where $\boldsymbol{\mathcal{U}} : H^{-1}(\Omega)^n \to \mathring{H}^1_{\mathrm{div}}(\Omega)^n$ and $\mathcal{P} : H^{-1}(\Omega)^n \to L^2(\Omega)/\mathbb{R}$ are linear continuous operators such that $\boldsymbol{\mathcal{L}}(\boldsymbol{\mathcal{U}}\mathbf{F}, \mathcal{P}\mathbf{F}) = \mathbf{F}$ for any $\mathbf{F} \in H^{-1}(\Omega)^n$. By applying the operator $\boldsymbol{\mathcal{L}}^{-1}$ to equation (6.6) we obtain the equivalent nonlinear system

$$\mathbf{u}_0 = \mathbf{U}_\varepsilon \mathbf{u}_0, \tag{6.9}$$

$$\pi = P_\varepsilon \mathbf{u}_0, \tag{6.10}$$

where $\mathbf{U}_\varepsilon : \mathring{H}^1_{\mathrm{div}}(\Omega)^n \to \mathring{H}^1_{\mathrm{div}}(\Omega)^n$ and $P_\varepsilon : \mathring{H}^1_{\mathrm{div}}(\Omega)^n \to L^2(\Omega)/\mathbb{R}$ are the nonlinear operators defined as

$$\mathbf{U}_\varepsilon \mathbf{w} := \boldsymbol{\mathcal{U}}\, \mathbf{F}_\varepsilon \mathbf{w}, \tag{6.11}$$

$$P_\varepsilon \mathbf{w} := \mathcal{P}\, \mathbf{F}_\varepsilon \mathbf{w} \tag{6.12}$$

(cf. also [41] for $\mu = 1$ in the isotropic incompressible case (2.12)).

Since π is not involved in (6.9), we will first prove the existence of a solution $\mathbf{u}_0 \in \mathring{H}^1_{\mathrm{div}}(\Omega)^n$ to this equation and then use (6.10) as a representation formula. This formula provides the existence of the pressure field $\pi \in L^2(\Omega)/\mathbb{R}$.

In order to show the existence of a fixed point of the operator $\mathbf{U}_\varepsilon$ and, thus, the existence of a solution of equation (6.9), we employ Theorem 6.1.

We show first that $\mathbf{U}_\varepsilon$ is continuous. Let $\mathbf{w}, \mathbf{w}' \in \mathring{H}^1_{\mathrm{div}}(\Omega)^n$. Then by (6.7) and (C.6) there exists a constant $c_1 > 0$ such that

$$\begin{aligned}
\left\|\mathbf{F}_\varepsilon \mathbf{w} - \mathbf{F}_\varepsilon \mathbf{w}'\right\|_{H^{-1}(\Omega)^n} &\le \left\|(\mathbf{w}\cdot\nabla)\mathbf{w} - (\mathbf{w}'\cdot\nabla)\mathbf{w}'\right\|_{H^{-1}(\Omega)^n} \\
&\quad + \left\|(\mathbf{v}_\varepsilon\cdot\nabla)(\mathbf{w}-\mathbf{w}') + ((\mathbf{w}-\mathbf{w}')\cdot\nabla)\mathbf{v}_\varepsilon\right\|_{H^{-1}(\Omega)^n} \\
&\le \left\|((\mathbf{w}-\mathbf{w}')\cdot\nabla)\mathbf{w} + (\mathbf{w}'\cdot\nabla)(\mathbf{w}-\mathbf{w}')\right\|_{H^{-1}(\Omega)^n} \\
&\quad + 2c_1^2 \left\|\mathbf{w}-\mathbf{w}'\right\|_{H^1(\Omega)^n} \|\mathbf{v}_\varepsilon\|_{H^1(\Omega)^n} \\
&\le c_1^2 \left\|\mathbf{w}-\mathbf{w}'\right\|_{H^1(\Omega)^n} \left(\|\mathbf{w}\|_{H^1(\Omega)^n} + \|\mathbf{w}'\|_{H^1(\Omega)^n} + 2\|\mathbf{v}_\varepsilon\|_{H^1(\Omega)^n}\right).
\end{aligned} \tag{6.13}$$

This estimate shows that the operator $\mathbf{F}_\varepsilon : \mathring{H}^1_{\mathrm{div}}(\Omega)^n \to H^{-1}(\Omega)^n$ is continuous. Consequently, the operator $\mathbf{U}_\varepsilon = \boldsymbol{\mathcal{U}}\,\mathbf{F}_\varepsilon : \mathring{H}^1_{\mathrm{div}}(\Omega)^n \to \mathring{H}^1_{\mathrm{div}}(\Omega)^n$ is also continuous, as asserted.

Next we show that the operator $\mathbf{U}_\varepsilon$ is compact. To this end, we assume that $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$ is a bounded sequence in the space $\mathring{H}^1_{\mathrm{div}}(\Omega)^n$ endowed with the norm, coinciding with seminorm (4.1), and prove that the sequence $\{\mathbf{F}_\varepsilon \mathbf{w}_k\}_{k\in\mathbb{N}}$ contains a convergent subsequence in $H^{-1}(\Omega)^n$.

Let $M > 0$ be such that $\|\mathbf{w}_k\|_{\mathring{H}^1_{\mathrm{div}}(\Omega)^n} \le M$ for all $k \in \mathbb{N}$. Since the embedding of the space $\mathring{H}^1_{\mathrm{div}}(\Omega)^n$ into the space $L^4(\Omega)^n$ is compact (see, e.g., [1, Theorem 6.3]), there exists a subsequence of $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$, labeled as the sequence for the sake of brevity, which converges in $L^4(\Omega)^n$, and, hence, is a Cauchy sequence in $L^4(\Omega)^n$. From (6.7), (C.6) and (C.15), we obtain

$$\begin{aligned}
\left\|\mathbf{F}_\varepsilon \mathbf{w}_k - \mathbf{F}_\varepsilon \mathbf{w}_\ell\right\|_{H^{-1}(\Omega)^n} &\le \|((\mathbf{w}_k-\mathbf{w}_\ell)\cdot\nabla)\mathbf{w}_k + (\mathbf{w}_\ell\cdot\nabla)(\mathbf{w}_k-\mathbf{w}_\ell)\|_{H^{-1}(\Omega)^n} \\
&\quad + \|(\mathbf{v}_\varepsilon\cdot\nabla)(\mathbf{w}_k-\mathbf{w}_\ell) + ((\mathbf{w}_k-\mathbf{w}_\ell)\cdot\nabla)\mathbf{v}_\varepsilon\|_{H^{-1}(\Omega)^n} \\
&\le c_1 \left(\|\mathbf{w}_k\|_{H^1(\Omega)^n} + \|\mathbf{w}_\ell\|_{H^1(\Omega)^n} + 2\|\mathbf{v}_\varepsilon\|_{H^1(\Omega)^n}\right) \|\mathbf{w}_k - \mathbf{w}_\ell\|_{L^4(\Omega)^n} \\
&\le 2c_1 \left(M + \|\mathbf{v}_\varepsilon\|_{H^1(\Omega)^n}\right) \|\mathbf{w}_k - \mathbf{w}_\ell\|_{L^4(\Omega)^n}.
\end{aligned} \tag{6.14}$$

This inequality, combined with the property that $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in the space $L^4(\Omega)^n$, implies that $\{\mathbf{F}_\varepsilon \mathbf{w}_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in the space $H^{-1}(\Omega)^n$. Therefore,

$\mathbf{F}_\varepsilon : \mathring{H}^1_{\rm div}(\Omega)^n \to H^{-1}(\Omega)^n$ is a compact operator. Hence, the operator $\mathbf{U}_\varepsilon = \boldsymbol{\mathcal{U}}\mathbf{F}_\varepsilon : \mathring{H}^1_{\rm div}(\Omega)^n \to \mathring{H}^1_{\rm div}(\Omega)^n$ is also compact, as asserted.

Next, we show that there exists a constant $M_0 > 0$ such that if $\mathbf{w} \in \mathring{H}^1_{\rm div}(\Omega)^n$ satisfies the equation

$$\mathbf{w} = \lambda \mathbf{U}_\varepsilon \mathbf{w} \tag{6.15}$$

for some $\lambda \in [0, 1]$, then $\|\mathbf{w}\|_{H^1(\Omega)^n} \leq M_0$. Let us also introduce the function

$$q := \lambda P_\varepsilon \mathbf{w}. \tag{6.16}$$

By applying the operator $\boldsymbol{\mathcal{L}}$ to equations (6.15)–(6.16) and by using relations (6.11) and (6.12), we deduce that whenever the pair $(\mathbf{w}, \lambda) \in \mathring{H}^1_{\rm div}(\Omega)^n \times \mathbb{R}$ satisfies equation (6.15), then the equation

$$\boldsymbol{\mathcal{L}}(\mathbf{w}, q) = \lambda \mathbf{F}_\varepsilon \mathbf{w}, \tag{6.17}$$

is also satisfied. (Recall the isomorphism property of operator (6.8).) Then the first Green identity (A.1) implies the equation

$$\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{w}) \right\rangle_\Omega = -\langle \lambda \mathbf{F}_\varepsilon \mathbf{w}, \mathbf{w} \rangle_\Omega \,, \tag{6.18}$$

which, in view of relation (6.7), takes the form

$$\begin{aligned} \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{w}) \right\rangle_\Omega &= \lambda \langle \mathfrak{F}, \mathbf{w} \rangle_\Omega - \lambda \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{v}_\varepsilon), E_{i\alpha}(\mathbf{w}) \right\rangle_\Omega \\ &\quad - \lambda \left\langle ((\mathbf{w} + \mathbf{v}_\varepsilon) \cdot \nabla)\mathbf{w}, \mathbf{w} \right\rangle_\Omega - \lambda \left\langle (\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon, \mathbf{w} \right\rangle_\Omega - \lambda \left\langle (\mathbf{w} \cdot \nabla)\mathbf{v}_\varepsilon, \mathbf{w} \right\rangle_\Omega . \end{aligned} \tag{6.19}$$

Relation (C.13) implies that $\left\langle ((\mathbf{w}+\mathbf{v}_\varepsilon)\cdot\nabla)\mathbf{w}, \mathbf{w} \right\rangle_\Omega = 0$. Then by using the Korn first inequality (4.9), the ellipticity condition (2.4), equation (6.19), the Hölder inequality, relation (C.14), and the Leray-Hopf inequality (6.4), we obtain for $\lambda \geq 0$ that

$$\begin{aligned} &\frac{1}{2} C_{\mathbb{A}}^{-1} \|\nabla \mathbf{w}\|^2_{L^2(\Omega)^{n\times n}} \leq C_{\mathbb{A}}^{-1} \|\mathbb{E}(\mathbf{w})\|^2_{L^2(\Omega)^{n\times n}} \leq \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{w}) \right\rangle_\Omega \\ &\quad \leq \lambda |||\mathfrak{F}|||_{H^{-1}(\Omega)^n} \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} + \lambda \|\mathbb{A}\| \, \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} \|\nabla \mathbf{v}_\varepsilon\|_{L^2(\Omega)^{n\times n}} \\ &\qquad + \lambda \|\mathbf{v}_\varepsilon\|^2_{L^4(\Omega)^n} \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} + \lambda\varepsilon \|\nabla \mathbf{w}\|^2_{L^2(\Omega)^{n\times n}}, \end{aligned} \tag{6.20}$$

where the norm $|||\cdot|||_{H^{-1}(\Omega)^n}$ is defined in (4.4) and $\|\mathbb{A}\|$ is the norm of the viscosity tensor coefficient given by (2.5). Hence, for $\lambda \in [0, 1]$,

$$\left(\frac{1}{2} C_{\mathbb{A}}^{-1} - \varepsilon\right) \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} \leq |||\mathfrak{F}|||_{H^{-1}(\Omega)^n} + \|\mathbb{A}\| \, \|\nabla \mathbf{v}_\varepsilon\|_{L^2(\Omega)^{n\times n}} + \|\mathbf{v}_\varepsilon\|^2_{L^4(\Omega)^n} .$$

Choosing $\varepsilon < \frac{1}{2} C_{\mathbb{A}}^{-1}$ in the Leray-Hopf's inequality (6.4), we obtain the estimate

$$\|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} \leq \frac{2}{C_{\mathbb{A}}^{-1} - 2\varepsilon} \left(|||\mathfrak{F}|||_{H^{-1}(\Omega)^n} + \|\mathbb{A}\| \, \|\nabla \mathbf{v}_\varepsilon\|_{L^2(\Omega)^{n\times n}} + \|\mathbf{v}_\varepsilon\|^2_{L^4(\Omega)^n} \right), \tag{6.21}$$

that is, $\|\mathbf{w}\|_{H^1(\Omega)^n} \leq M_0$, where M_0 is given by the right hand side of (6.21) multiplied by the equivalence constant between the norm and semi-norm in $\mathring{H}^1(\Omega)^n$.

Therefore, the operator $\mathbf{U}_\varepsilon : \mathring{H}^1_{\rm div}(\Omega)^n \to \mathring{H}^1_{\rm div}(\Omega)^n$ satisfies the hypothesis of Theorem 6.1 (with $\mathcal{X} = \mathring{H}^1_{\rm div}(\Omega)^n$), and hence it has a fixed point $\mathbf{u}_0 \in \mathring{H}^1_{\rm div}(\Omega)^n$, that is, $\mathbf{u}_0 = \mathbf{U}_\varepsilon \mathbf{u}_0$. Then with $\pi \in L^2(\Omega)/\mathbb{R}$ as in (6.10), we obtain that the couple $(\mathbf{u}_0, \pi) \in \mathring{H}^1_{\rm div}(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ satisfies the nonlinear equation (6.6). Consequently, the couple $(\mathbf{u}, \pi) = (\mathbf{v}_\varepsilon + \mathbf{u}_0, \pi) \in H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ is a solution of the nonlinear Dirichlet problem (6.2). (Recall

that $\mathbf{v}_\varepsilon$ is an extension to $H^1_{\mathrm{div}}(\Omega)^n$ of the function $\boldsymbol{\varphi} \in H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^n$, and, thus, it satisfies the Dirichlet condition (6.3).) □

7 Dirichlet-transmission problem for the anisotropic Navier-Stokes system in a bounded Lipschitz domain with a transversal Lipschitz interface

In this section we show the existence of weak solutions of Dirichlet-transmission problems for the anisotropic Navier-Stokes system with data in L^2-based Sobolev spaces in a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, satisfying Assumption 5.1. First, we analyze a Dirichlet-transmission problem for the incompressible Navier-Stokes system with general PDE right hand sides and a jump of conormal derivatives on the transversal Lipschitz interface. We reduce this nonlinear problem to a Dirichlet problem for the Navier-Stokes system whose analysis is based on the Leray-Hopf inequality and the Leray-Schauder fixed point theorem. Then, we study a Dirichlet-transmission problem for the anisotropic Navier-Stokes system in a compressible framework with non-homogeneous Dirichlet condition and trace and conormal derivative jumps across the internal Lipschitz interface. We use a Bogovskii-type result established in Lemma 5.7, some useful estimates and the Leray-Schauder fixed point theorem to show the existence of a weak solution to this nonlinear problem. In the case of small data, the uniqueness of the weak solution is also established.

7.1 Dirichlet-transmission problem in a bounded Lipschitz domain with conormal derivative jump on the transversal Lipschitz interface

Let us consider the following *Dirichlet-transmission problem for the incompressible anisotropic Navier-Stokes system with a prescribed conormal derivative jump but without velocity jump on the interface*,

$$\begin{cases} \mathcal{L}(\mathbf{u}^+, \pi^+) = \widetilde{\mathbf{f}}^+|_{\Omega^+} + (\mathbf{u}^+ \cdot \nabla)\mathbf{u}^+, \ \operatorname{div} \mathbf{u}^+ = 0 & \text{in } \Omega^+, \\ \mathcal{L}(\mathbf{u}^-, \pi^-) = \widetilde{\mathbf{f}}^-|_{\Omega^-} + (\mathbf{u}^- \cdot \nabla)\mathbf{u}^-, \ \operatorname{div} \mathbf{u}^- = 0 & \text{in } \Omega^-, \\ (\gamma_{\Omega^+}\mathbf{u}^+)|_\Sigma = (\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma & \text{on } \Sigma, \\ \big(\mathbf{t}_{\Omega^+}\big(\mathbf{u}^+, \pi^+; \widetilde{\mathbf{f}}^+ + \mathring{E}_{\Omega^+\to\Omega}(\mathbf{u}^+ \cdot \nabla)\mathbf{u}^+\big)\big)\big|_\Sigma & \\ \quad + \big(\mathbf{t}_{\Omega^-}\big(\mathbf{u}^-, \pi^-; \widetilde{\mathbf{f}}^- + \mathring{E}_{\Omega^-\to\Omega}(\mathbf{u}^- \cdot \nabla)\mathbf{u}^-\big)\big)\big|_\Sigma = \boldsymbol{\psi}_\Sigma & \text{on } \Sigma, \\ (\gamma_{\Omega^+}\mathbf{u}^+)|_{\Gamma^+} = \boldsymbol{\varphi}|_{\Gamma^+} & \text{on } \Gamma^+, \\ (\gamma_{\Omega^-}\mathbf{u}^-)|_{\Gamma^-} = \boldsymbol{\varphi}|_{\Gamma^-} & \text{on } \Gamma^- \end{cases} \tag{7.1}$$

with the unknown $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$ and the given data $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}) \in \big(H^1_{\Gamma^+}(\Omega^+)^n\big)' \times \big(H^1_{\Gamma^-}(\Omega^-)^n\big)' \times H^{-\frac{1}{2}}(\Sigma)^n \times H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^n$. Here $\mathfrak{X}_{\Omega^+,\Omega^-}$ is the space defined in (5.27), and $\mathring{E}_{\Omega^\pm\to\Omega}$ are the zero extension operators from $\Omega^\pm$ to Ω.

Existence of a weak solution

Let $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$. Assume that $\mathbf{u}^+$ and $\mathbf{u}^-$ satisfy the homogeneous interface condition $(\gamma_{\Omega^+}\mathbf{u}^+)|_\Sigma - (\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma = \mathbf{0}$ on Σ. Then by Lemma B.1, there exists a unique

pair $(\mathbf{u},\pi)\in H^1(\Omega)^n\times L^2(\Omega)/\mathbb{R}$ such that

$$\mathbf{u}|_{\Omega^+}=\mathbf{u}^+,\quad \mathbf{u}|_{\Omega^-}=\mathbf{u}^-,\quad \pi|_{\Omega^+}=\pi^+,\quad \pi|_{\Omega^-}=\pi^-. \tag{7.2}$$

Let also $\mathfrak{F}\in H^{-1}(\Omega)^n=\big(\mathring{H}^1(\Omega)^n\big)'\subset\big(\mathring{H}^1_{\rm div}(\Omega)^n\big)'$ be such that

$$\langle\mathfrak{F},\mathbf{v}\rangle_\Omega:=-\big\langle\tilde{\mathbf{f}}^+ +\tilde{\mathbf{f}}^-,\mathbf{v}\big\rangle_\Omega+\langle\boldsymbol{\psi}_\Sigma,\gamma_\Sigma\mathbf{v}\rangle_\Sigma\,,\quad\forall\,\mathbf{v}\in\mathring{H}^1(\Omega)^n\,, \tag{7.3}$$

that is, $\mathfrak{F}=-(\tilde{\mathbf{f}}^+ +\tilde{\mathbf{f}}^-)+\gamma_\Sigma^*\boldsymbol{\psi}_\Sigma$. Here $\gamma_\Sigma^*:H^{-\frac12}(\Sigma)^n\to H^{-1}(\Omega)^n$ denotes the adjoint of the trace operator $\gamma_\Sigma:\mathring{H}^1(\Omega)^n\to\widetilde{H}^{\frac12}(\Sigma)^n$ defined by (5.16), and the support of $\gamma_\Sigma^*\boldsymbol{\psi}_\Sigma$ is a subset of $\overline{\Sigma}$.

An argument similar to that for problem (5.25) implies the following result.

Lemma 7.1 *The nonlinear Dirichlet-transmission problem* (7.1) *is equivalent, in the sense of relations* (7.2)*, to the nonlinear Dirichlet problem* (6.2) *with* $\mathfrak{F}=-(\tilde{\mathbf{f}}^+ +\tilde{\mathbf{f}}^-)+\gamma_\Sigma^*\boldsymbol{\psi}_\Sigma$.

Proof Assume that $\big(\mathbf{u}^+,\pi^+,\mathbf{u}^-,\pi^-\big)\in\mathfrak{X}_{\Omega^+,\Omega^-}$ satisfy the nonlinear Dirichlet-transmission problem (7.1). Then the Green identity (5.24) and the divergence theorem give the following weak equations:

$$\left\{\begin{array}{l}\big\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}^+),E_{i\alpha}(\mathbf{w})\big\rangle_{\Omega^+}-\big\langle\pi^+,\operatorname{div}\mathbf{w}\big\rangle_{\Omega^+}-\big\langle(\mathbf{u}^+\cdot\nabla)\mathbf{u}^+,\mathbf{w}\big\rangle_{\Omega^+}\\ \quad+\big\langle a_{ij}^{\alpha\beta}E_{j\beta}(\mathbf{u}^-),E_{i\alpha}(\mathbf{w}^-)\big\rangle_{\Omega^-}-\big\langle\pi^-,\operatorname{div}\mathbf{w}^-\big\rangle_{\Omega^-}-\big\langle(\mathbf{u}^-\cdot\nabla)\mathbf{u}^-,\mathbf{w}^-\big\rangle_{\Omega^-}\\ \qquad-\big\langle\tilde{\mathbf{f}}^+,\mathbf{w}\big\rangle_{\Omega^+}-\big\langle\tilde{\mathbf{f}}^-,\mathbf{w}\big\rangle_{\Omega^-}+\langle\boldsymbol{\psi}_\Sigma,\gamma_\Sigma\mathbf{w}\rangle_\Sigma=0\,,\ \forall\,\mathbf{w}\in\mathring{H}^1(\Omega)^n\,,\\ \langle\operatorname{div}\mathbf{u}^+,q\rangle_{\Omega^+}+\langle\operatorname{div}\mathbf{u}^-,q\rangle_{\Omega^-}=0,\quad\forall\,q\in L^2(\Omega)/\mathbb{R}.\end{array}\right. \tag{7.4}$$

Let us also complement these variational equations by the equations for traces from (7.1),

$$\left\{\begin{array}{ll}(\gamma_{\Omega^+}\mathbf{u}^+)|_\Sigma=(\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma & \text{on }\Sigma\,,\\ (\gamma_{\Omega^+}\mathbf{u}^+)|_{\Gamma^+}=\boldsymbol{\varphi}|_{\Gamma^+} & \text{on }\Gamma^+\,,\\ (\gamma_{\Omega^-}\mathbf{u}^-)|_{\Gamma^-}=\boldsymbol{\varphi}|_{\Gamma^-} & \text{on }\Gamma^-\,.\end{array}\right. \tag{7.5}$$

Hence if $\big(\mathbf{u}^+,\pi^+,\mathbf{u}^-,\pi^-\big)\in\mathfrak{X}_{\Omega^+,\Omega^-}$ satisfy the Dirichlet-transmission problem (7.1), they satisfy equations (7.4)–(7.5).

Conversely, assume that $\big(\mathbf{u}^+,\pi^+,\mathbf{u}^-,\pi^-\big)\in\mathfrak{X}_{\Omega^+,\Omega^-}$ satisfy equations (7.4)–(7.5). Since the spaces $\mathcal{D}(\Omega^\pm)^n$ are subspaces of $\mathring{H}^1(\Omega)^n$, the (distributional form of the) anisotropic Navier-Stokes equations in (7.1), in each of the domains Ω^+ and Ω^-, follows from the first equation in (7.4) written for all $\mathbf{w}\in\mathcal{D}(\Omega^+)^n$ and $\mathbf{w}\in\mathcal{D}(\Omega^-)^n$, respectively. Similarly, the second equation in (7.4) implies the equations $\operatorname{div}\mathbf{u}^\pm=0$ in $\Omega^\pm$. Thus, $\big(\mathbf{u}^+,\pi^+,\mathbf{u}^-,\pi^-\big)$ satisfies the anisotropic Navier-Stokes system in $\Omega^+\cup\Omega^-$ and the trace conditions (7.5). Then substituting the first equation in (7.4) into the Green identity (5.24), we obtain the equation

$$\begin{aligned}&\big\langle\big(\mathbf{t}_{\Omega^+}\big(\mathbf{u}^+,\pi^+;\tilde{\mathbf{f}}^+ +\mathring{E}_{\Omega^+\to\Omega}(\mathbf{u}^+\cdot\nabla)\mathbf{u}^+\big)\big)\big|_\Sigma\\ &\quad+\big(\mathbf{t}_{\Omega^-}\big(\mathbf{u}^-,\pi^-;\tilde{\mathbf{f}}^- +\mathring{E}_{\Omega^-\to\Omega}(\mathbf{u}^-\cdot\nabla)\mathbf{u}^-\big)\big)\big|_\Sigma,(\gamma_\Omega\mathbf{w})|_\Sigma\big\rangle_\Sigma=\big\langle\boldsymbol{\psi}_\Sigma,\gamma_\Sigma\mathbf{w}\big\rangle_\Sigma\,.\end{aligned}$$

In view of Lemma 5.3, this formula can be written in the equivalent form

$$\begin{aligned}&\big\langle\big(\mathbf{t}_{\Omega^+}\big(\mathbf{u}^+,\pi^+;\tilde{\mathbf{f}}^+ +\mathring{E}_{\Omega^+\to\Omega}(\mathbf{u}^+\cdot\nabla)\mathbf{u}^+\big)\big|_\Sigma\\ &\quad+\mathbf{t}_{\Omega^-}\big(\mathbf{u}^-,\pi^-;\tilde{\mathbf{f}}^- +\mathring{E}_{\Omega^-\to\Omega}(\mathbf{u}^-\cdot\nabla)\mathbf{u}^-\big)\big)\big|_\Sigma,\boldsymbol{\phi}\big\rangle_\Sigma=\big\langle\boldsymbol{\psi}_\Sigma,\boldsymbol{\phi}\big\rangle_\Sigma\,,\ \forall\,\boldsymbol{\phi}\in H^{\frac12}_\bullet(\Sigma)^n.\end{aligned}$$

Therefore, $\big(\mathbf{t}_{\Omega^+}(\mathbf{u}^+,\pi^+;\tilde{\mathbf{f}}^+)+\mathbf{t}_{\Omega^-}(\mathbf{u}^-,\pi^-;\tilde{\mathbf{f}}^-)\big)\big|_\Sigma=\boldsymbol{\psi}_\Sigma$ on Σ. Hence if $\big(\mathbf{u}^+,\pi^+,\mathbf{u}^-,\pi^-\big)\in\mathfrak{X}_{\Omega^+,\Omega^-}$ satisfy equations (7.4)–(7.5), they solve the Dirichlet-transmission problem (7.1).

Consequently, problems (7.1) and the variational formulation (7.4)–(7.5) are equivalent.

Let again $\left(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-\right) \in \mathfrak{X}_{\Omega^+,\Omega^-}$ and the homogeneous interface condition in (7.5) be satisfied. Then there exists a unique pair $(\mathbf{u}, \pi) \in \mathring{H}^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ defined by relations (7.2) (cf. Lemma B.1). Then taking into account relation (7.3), the variational formulation (7.4)–(7.5) reduces to

$$\begin{cases} \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{w})\right\rangle_{\Omega} + \langle (\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{w}\rangle_{\Omega} - \langle \pi, \operatorname{div}\mathbf{w}\rangle_{\Omega} \\ \quad = \langle \mathfrak{F}, \mathbf{w}\rangle_{\Omega}, \ \forall\, \mathbf{w} \in \mathring{H}^1(\Omega)^n\,, \\ \langle \operatorname{div}\mathbf{u}, q\rangle_{\Omega} = 0, \quad \forall\, q \in L^2(\Omega)/\mathbb{R}, \\ \gamma_{\Omega}\mathbf{u} = \boldsymbol{\varphi} \text{ on } \partial\Omega\,. \end{cases} \tag{7.6}$$

Repeating the arguments similar the above in this proof, one can see that the variational formulation (7.6) is also equivalent to the nonlinear Dirichlet problem (6.2).

Consequently, problems (7.1) and (6.2) are equivalent, as asserted. □

Theorem 7.2 *Let $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, be a bounded Lipschitz domain satisfying* Assumption 5.1. *Let conditions* (2.2)–(2.4) *hold. Then for all given data $(\tilde{\mathbf{f}}^+, \tilde{\mathbf{f}}^-, \boldsymbol{\psi}_{\Sigma}, \boldsymbol{\varphi})$ in the space $\left(H^1_{\Gamma^+}(\Omega^+)^n\right)' \times \left(H^1_{\Gamma^-}(\Omega^-)^n\right)' \times H^{-\frac{1}{2}}(\Sigma)^n \times H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^n$, the Dirichlet-transmission problem* (7.1) *for the anisotropic Navier-Stokes system has a weak solution $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$ defined by relations* (7.2) *in terms of the solution $(\mathbf{u}, \pi)$ of the nonlinear Dirichlet problem* (6.2) *with $\mathfrak{F} = -(\tilde{\mathbf{f}}^+ + \tilde{\mathbf{f}}^-) + \gamma^*_{\Sigma}\boldsymbol{\psi}_{\Sigma}$.*

Proof By Theorem 6.3, the Dirichlet problem (6.2) has a solution $(\mathbf{u}, \pi) \in H^1_{\operatorname{div}}(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ and then, in view of Lemma 7.1, the functions $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$ defined by (7.2) satisfy the nonlinear problem (7.1) in the distribution sense. □

Uniqueness result for the Dirichlet-transmission problem (7.1)

Next we show that an additional constraint to the given data of the nonlinear Dirichlet-transmission problem (7.1) leads to the uniqueness of the weak solution of this problem.

Recall that $\gamma^*_{\Sigma} : H^{-\frac{1}{2}}(\Sigma)^n \to H^{-1}(\Omega)^n$ is the adjoint of the trace operator $\gamma_{\Sigma} : \mathring{H}^1(\Omega)^n \to H^{\frac{1}{2}}_{\bullet}(\Sigma)^n$ defined by (5.16), and that $C_{\mathbb{A}}$ is the ellipticity constant in (2.4). On the other hand, in view of Lemma 4.2, there exists an extension $\mathbf{v}_{\boldsymbol{\varphi}}$ of $\boldsymbol{\varphi} \in H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^n$ to $H^1_{\operatorname{div}}(\Omega)$, that is, $\gamma_{\Omega}\mathbf{v}_{\boldsymbol{\varphi}} = \boldsymbol{\varphi}$ on $\partial\Omega$, and

$$\|\nabla\mathbf{v}_{\boldsymbol{\varphi}}\|_{L^2(\Omega)^{n\times n}} \le \|\mathbf{v}_{\boldsymbol{\varphi}}\|_{H^1(\Omega)^n} \le C\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n} \tag{7.7}$$

with some constant $C = C(\Omega, n) > 0$.

Then we prove the following uniqueness result (see also [56, Lemma 3.1] in the isotropic case (2.12) with $\mu = 1$ and homogeneous Dirichlet condition, and [34, Theorem 4.2] for a nonlinear transmission problem in a pseudostress approach).

Theorem 7.3 *Let $n = 2, 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain satisfying* Assumption 5.1. *Let conditions* (2.2)–(2.4) *be satisfied. Let $(\tilde{\mathbf{f}}^+, \tilde{\mathbf{f}}^-, \boldsymbol{\varphi}, \boldsymbol{\psi}_{\Sigma})$ be given in the space $\left(H^1_{\Gamma^+}(\Omega^+)^n\right)' \times \left(H^1_{\Gamma^-}(\Omega^-)^n\right)' \times H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^n \times H^{-\frac{1}{2}}(\Sigma)^n$. Let*

$$c_0^2\|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + (c_0^2 C\|\mathbb{A}\| + C_{\mathbb{A}}^{-1} C c_0 c_1)\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n} < \frac{1}{4}C_{\mathbb{A}}^{-2}, \tag{7.8}$$

where $C_{\mathbb{A}}$, C, c_0, and c_1 are the constants in (2.4), (7.7), (C.2), and (C.1), respectively, while $\mathfrak{F} = -(\tilde{\mathbf{f}}^+ + \tilde{\mathbf{f}}^-) + \gamma_\Sigma^*(\boldsymbol{\psi}_\Sigma)$. Then the nonlinear problem (7.1) has only one solution $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$.

Proof The solution existence is implied by Theorem 7.2. To prove that it is unique under the theorem conditions, let us assume that $\mathbf{u}^{(1)}, \mathbf{u}^{(2)} \in H^1_{\rm div}(\Omega)^n$ are the velocities in two solutions of the nonlinear problem (7.1) in the sense of (7.2). Let us write them in the form

$$\mathbf{u}^{(i)} = \mathbf{v}_{\boldsymbol{\varphi}} + \mathbf{u}_0^{(i)}, \quad i = 1, 2, \tag{7.9}$$

where $\mathbf{v}_{\boldsymbol{\varphi}} \in H^1_{\rm div}(\Omega)^n$ satisfies the relation $\gamma_\Omega \mathbf{v}_{\boldsymbol{\varphi}} = \boldsymbol{\varphi}$ on $\partial\Omega$ and estimate (7.7), while $\mathbf{u}_0^{(1)}, \mathbf{u}_0^{(2)} \in \mathring{H}^1_{\rm div}(\Omega)^n$ satisfy equations (6.6)–(6.7) with $\mathbf{v}_{\boldsymbol{\varphi}}$ instead of $\mathbf{v}_\varepsilon$.

Let us denote $\overline{\mathbf{u}} := \mathbf{u}^{(1)} - \mathbf{u}^{(2)} = \mathbf{u}_0^{(1)} - \mathbf{u}_0^{(2)}$, $\overline{\pi} := \pi^{(1)} - \pi^{(2)}$, where $\pi^{(i)}$ is the pressure term corresponding to $\mathbf{u}^{(i)}$, $i = 1, 2$. Using the first equations in the two upper lines of (7.1), we obtain

$$\boldsymbol{\mathcal{L}}(\overline{\mathbf{u}}, \overline{\pi}) = (\mathbf{u}^{(1)} \cdot \nabla)\mathbf{u}^{(1)} - (\mathbf{u}^{(2)} \cdot \nabla)\mathbf{u}^{(2)}. \tag{7.10}$$

This implies that

$$\big\langle a_{ij}^{\alpha\beta} E_{j\beta}(\overline{\mathbf{u}}), E_{i\alpha}(\overline{\mathbf{u}})\big\rangle_\Omega = -\big\langle (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}^{(1)}, \overline{\mathbf{u}}\big\rangle_\Omega - \big\langle (\mathbf{u}^{(2)} \cdot \nabla)\overline{\mathbf{u}}, \overline{\mathbf{u}}\big\rangle_\Omega. \tag{7.11}$$

Moreover, identity (C.13) and the inclusion $\overline{\mathbf{u}} \in \mathring{H}^1_{\rm div}(\Omega)^n$ show that the last term in the right-hand side of (7.11) equals zero. Therefore, equation (7.11) reduces to

$$\big\langle a_{ij}^{\alpha\beta} E_{j\beta}(\overline{\mathbf{u}}), E_{i\alpha}(\overline{\mathbf{u}})\big\rangle_\Omega = -\big\langle (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}^{(1)}, \overline{\mathbf{u}}\big\rangle_\Omega. \tag{7.12}$$

On the other hand, estimate (4.10) implies that

$$\|\nabla\overline{\mathbf{u}}\|^2_{L^2(\Omega)^{n\times n}} \le 2C_{\mathbb{A}}\big\langle a_{ij}^{\alpha\beta} E_{j\beta}(\overline{\mathbf{u}}), E_{i\alpha}(\overline{\mathbf{u}})\big\rangle_\Omega, \tag{7.13}$$

and by the Hölder inequality and inequalities (C.2) and (7.7) together with (C.12) and (C.1), we obtain

$$\begin{aligned}
\big|\big\langle (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}^{(1)}, \overline{\mathbf{u}}\big\rangle_\Omega\big| &= \big|\big\langle (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}}, \mathbf{u}^{(1)}\big\rangle_\Omega\big| \le \|\overline{\mathbf{u}}\|_{L^4(\Omega)^n} \|\nabla\overline{\mathbf{u}}\|_{L^2(\Omega)^{n\times n}} \|\mathbf{u}^{(1)}\|_{L^4(\Omega)^n} \\
&\le \|\overline{\mathbf{u}}\|_{L^4(\Omega)^n} \|\nabla\overline{\mathbf{u}}\|_{L^2(\Omega)^{n\times n}} (\|\mathbf{u}_0^{(1)}\|_{L^4(\Omega)^n} + \|\mathbf{v}_{\boldsymbol{\varphi}}\|_{L^4(\Omega)^n}) \\
&\le c_0 \|\nabla\overline{\mathbf{u}}\|^2_{L^2(\Omega)^{n\times n}} \left(c_0 \|\nabla\mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}} + Cc_1 \|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n} \right).
\end{aligned} \tag{7.14}$$

Hence

$$\|\nabla\overline{\mathbf{u}}\|^2_{L^2(\Omega)^{n\times n}} \le 2C_{\mathbb{A}} c_0 \|\nabla\overline{\mathbf{u}}\|^2_{L^2(\Omega)^{n\times n}} \left(c_0 \|\nabla\mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}} + Cc_1 \|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n} \right). \tag{7.15}$$

Moreover, an estimate similar to (6.20) with $\lambda = 1$, combined with estimates (C.2), (C.1), (C.12) and (7.7) together with relation (6.19) imply that

$$\begin{aligned}
&\frac{1}{2}C_{\mathbb{A}}^{-1}\|\nabla \mathbf{u}_0^{(1)}\|^2_{L^2(\Omega)^{n\times n}} \le C_{\mathbb{A}}^{-1}\|\mathbb{E}(\mathbf{u}_0^{(1)})\|^2_{L^2(\Omega)^{n\times n}} \le \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}_0^{(1)}),\, E_{i\alpha}(\mathbf{u}_0^{(1)})\right\rangle_{\Omega}\\
&\quad\le \|\mathfrak{F}\|_{H^{-1}(\Omega)^n}\|\nabla \mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}} + \|\mathbb{A}\|\,\|\nabla \mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}}\|\nabla \mathbf{v}_{\boldsymbol{\varphi}}\|_{L^2(\Omega)^{n\times n}}\\
&\qquad + \|\mathbf{v}_{\boldsymbol{\varphi}}\|^2_{L^4(\Omega)^n}\|\nabla \mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}} + \|\nabla \mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}}\|\mathbf{u}_0^{(1)}\|_{L^4(\Omega)^n}\|\mathbf{v}_{\boldsymbol{\varphi}}\|_{L^4(\Omega)^n}\\
&\quad\le \|\mathfrak{F}\|_{H^{-1}(\Omega)^n}\|\nabla \mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}} + C\|\mathbb{A}\|\,\|\nabla \mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}}\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n}\\
&\qquad + C^2c_1^2\|\boldsymbol{\varphi}\|^2_{H^{\frac{1}{2}}(\partial\Omega)^n}\|\nabla \mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}} + Cc_1c_0\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n}\|\nabla \mathbf{u}_0^{(1)}\|^2_{L^2(\Omega)^{n\times n}}.
\end{aligned}$$

Thus, we obtain the estimate

$$\begin{aligned}
&\left(\frac{1}{2}C_{\mathbb{A}}^{-1} - Cc_1c_0\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n}\right)\|\nabla \mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}}\\
&\quad\le \|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + C\|\mathbb{A}\|\,\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n} + C^2c_1^2\|\boldsymbol{\varphi}\|^2_{H^{\frac{1}{2}}(\partial\Omega)^n}.
\end{aligned}\tag{7.16}$$

From (7.8) we have that

$$C_{\mathbb{A}}^{-1}Cc_0c_1\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n} < \frac{1}{4}C_{\mathbb{A}}^{-2} - c_0^2\|\mathfrak{F}\|_{H^{-1}(\Omega)^n} - c_0^2C\|\mathbb{A}\|\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n} \le \frac{1}{2}C_{\mathbb{A}}^{-2}.$$

This implies that the term $\left(\frac{1}{2}C_{\mathbb{A}}^{-1} - Cc_1c_0\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n}\right)$ is positive. Dividing (7.16) by this term and combining the resulting inequality with (7.15), we obtain

$$\begin{aligned}
&\|\nabla \overline{\mathbf{u}}\|^2_{L^2(\Omega)^{n\times n}} \le 2C_{\mathbb{A}}c_0\|\nabla \overline{\mathbf{u}}\|^2_{L^2(\Omega)^{n\times n}}\\
&\quad\times\left[c_0\,\frac{\|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + C\|\mathbb{A}\|\,\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n} + C^2c_1^2\|\boldsymbol{\varphi}\|^2_{H^{\frac{1}{2}}(\partial\Omega)^n}}{\frac{1}{2}C_{\mathbb{A}}^{-1} - Cc_1c_0\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n}} + Cc_1\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n}\right].
\end{aligned}$$

This finally reduces to the estimate

$$\begin{aligned}
&\frac{1}{4}C_{\mathbb{A}}^{-2}\|\nabla \overline{\mathbf{u}}\|^2_{L^2(\Omega)^{n\times n}}\\
&\quad\le\left[c_0^2\|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + c_0^2C\|\mathbb{A}\|\,\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n} + C_{\mathbb{A}}^{-1}Cc_0c_1\|\boldsymbol{\varphi}\|_{H^{\frac{1}{2}}(\partial\Omega)^n}\right]\|\nabla \overline{\mathbf{u}}\|^2_{L^2(\Omega)^{n\times n}}.
\end{aligned}$$

In view of assumption (7.8), this is possible only if $\nabla \overline{\mathbf{u}} = 0$, and since $\overline{\mathbf{u}} \in \mathring{H}^1(\Omega)^n$, we obtain that $\mathbf{u}^{(1)} = \mathbf{u}^{(2)}$ in Ω. Moreover, equation (7.10) reduces to the equation $\nabla \overline{\pi} = 0$, that is, $\pi^{(1)} = \pi^{(2)}$ in $L^2(\Omega)/\mathbb{R}$. □

In the special case of zero Dirichlet datum on $\partial\Omega$, we obtain the following result.

Corollary 7.4 *Let $n = 2, 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain satisfying* Assumption 5.1. *Let $\mathbb{A}$ satisfy conditions (2.2)–(2.4). Let $\left(\widetilde{\mathbf{f}}, \boldsymbol{\psi}_{\Sigma}\right) \in H^{-1}(\Omega)^n \times H^{-\frac{1}{2}}(\Sigma)^n$ and $\mathfrak{F} \in H^{-1}(\Omega)^n$ be given by $\mathfrak{F} = -(\tilde{\mathbf{f}}^+ + \tilde{\mathbf{f}}^-) + \gamma_{\Sigma}^*(\boldsymbol{\psi}_{\Sigma})$. If*

$$c_0^2\|\mathfrak{F}\|_{H^{-1}(\Omega)^n} < \frac{1}{4}C_{\mathbb{A}}^{-2},\tag{7.17}$$

where c_0 is the constant given in (C.2), then the nonlinear problem (7.1) with $\boldsymbol{\varphi} = \mathbf{0}$ has a unique weak solution $\mathbf{u} \in \mathring{H}^1_{\mathrm{div}}(\Omega)^n$.

Note that for the isotropic case (2.12) with a constant parameter $\mu > 0$, it is easy to see that $C_{\mathbb{A}} = 1/(2\mu)$. Then the uniqueness condition (7.17) reduces to the known uniqueness condition

$$c_0^2 \|\mathfrak{F}\|_{H^{-1}(\Omega)^n} < \mu^2.$$

for the isotropic Navier-Stokes equation, cf. [56, Lemma 3.1].

7.2 Dirichlet-transmission problem for the compressible anisotropic Navier-Stokes system with trace and conormal derivative jumps on a transversal Lipschitz interface in a bounded Lipschitz domain

Let us consider the Dirichlet-transmission problem

$$\begin{cases} \mathcal{L}(\mathbf{u}^+, \pi^+) = \widetilde{\mathbf{f}}^+|_{\Omega^+} + (\mathbf{u}^+ \cdot \nabla)\mathbf{u}^+, \ \operatorname{div} \mathbf{u}^+ = g^+ & \text{in } \Omega^+, \\ \mathcal{L}(\mathbf{u}^-, \pi^-) = \widetilde{\mathbf{f}}^-|_{\Omega^-} + (\mathbf{u}^- \cdot \nabla)\mathbf{u}^-, \ \operatorname{div} \mathbf{u}^- = g^- & \text{in } \Omega^-, \\ (\gamma_{\Omega^+}\mathbf{u}^+)|_\Sigma - (\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma = \boldsymbol{\varphi}_\Sigma & \text{on } \Sigma, \\ \big(\mathbf{t}_{\Omega^+}\big(\mathbf{u}^+, \pi^+; \widetilde{\mathbf{f}}^+ + \mathring{E}_{\Omega^+\to\Omega}(\mathbf{u}^+ \cdot \nabla)\mathbf{u}^+\big)\big)\big|_\Sigma & \\ \quad + \big(\mathbf{t}_{\Omega^-}\big(\mathbf{u}^-, \pi^-; \widetilde{\mathbf{f}}^- + \mathring{E}_{\Omega^-\to\Omega}(\mathbf{u}^- \cdot \nabla)\mathbf{u}^-\big)\big)\big|_\Sigma = \boldsymbol{\psi}_\Sigma & \text{on } \Sigma, \\ (\gamma_{\Omega^+}\mathbf{u}^+)|_{\Gamma^+} = \boldsymbol{\varphi}|_{\Gamma^+} & \text{on } \Gamma^+, \\ (\gamma_{\Omega^-}\mathbf{u}^-)|_{\Gamma^-} = \boldsymbol{\varphi}|_{\Gamma^-} & \text{on } \Gamma^- \end{cases} \tag{7.18}$$

with the unknowns $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$ and the given data $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}) \in \mathfrak{Y}_\bullet$. Recall that $\mathfrak{X}_{\Omega^+,\Omega^-}$ is the space defined in (5.27), and $\mathfrak{Y}_\bullet$ is the space defined in the beginning of Sect. 5.4.

Note that by Lemma 5.7 there exist some functions $\mathbf{v}_{\boldsymbol{\varphi}}^\pm \in H^1(\Omega^\pm)^n$ such that

$$\begin{cases} \operatorname{div} \mathbf{v}_{\boldsymbol{\varphi}}^\pm = g^\pm & \text{in } \Omega^\pm, \\ \big(\gamma_{\Omega^+}\mathbf{v}_{\boldsymbol{\varphi}}^+\big)|_\Sigma - \big(\gamma_{\Omega^-}\mathbf{v}_{\boldsymbol{\varphi}}^-\big)|_\Sigma = \boldsymbol{\varphi}_\Sigma & \text{on } \Sigma, \\ \big(\gamma_{\Omega^\pm}\mathbf{v}_{\boldsymbol{\varphi}}^\pm\big)|_{\Gamma^\pm} = \boldsymbol{\varphi}|_{\Gamma^\pm} & \text{on } \Gamma^\pm, \end{cases} \tag{7.19}$$

and some constant $C_\Sigma = C_\Sigma(\Omega^+, \Omega^-, n) > 0$ such that

$$\|\mathbf{v}_{\boldsymbol{\varphi}}^\pm\|_{H^1(\Omega^\pm)^n} \leq C_\Sigma \|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}, \tag{7.20}$$

where $\|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}$ is defined by (5.35).

By estimate (C.1) there exist some constants $c_1^\pm = c_1^\pm(\Omega^\pm, n) > 0$, such that

$$\|\mathbf{v}\|_{L^4(\Omega^\pm)^n} \leq c_1^\pm \|\mathbf{v}\|_{H^1(\Omega^\pm)^n} \leq c_1^* \|\mathbf{v}\|_{H^1(\Omega^\pm)^n}, \quad \forall\, \mathbf{v} \in H^1(\Omega^\pm)^n, \tag{7.21}$$

where $c_1^* = \max(c_1^+, c_1^-)$. In addition, inequality (C.2) holds on Ω.

Let us prove the existence of a solution to problem (7.18) by employing arguments similar to those in the proof of Theorem 6.3.

Theorem 7.5 *Let* Assumption 5.1 *and conditions* (2.2)–(2.4) *hold with* $n \in \{2, 3\}$. *Let* $(\widetilde{\mathbf{f}}^+, \widetilde{\mathbf{f}}^-, g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\psi}_\Sigma, \boldsymbol{\varphi}) \in \mathfrak{Y}_\bullet$.

(i) *If*

$$\|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet} < \frac{1}{4} C_{\mathbb{A}}^{-1} c_0^{-1} (c_1^* + c_0)^{-1} C_\Sigma^{-1}, \tag{7.22}$$

where $C_{\mathbb{A}}$, C_Σ, c_1^* *and* c_0, *are the constants in* (2.4), (7.20), (7.21) *and* (C.2), *respectively, then the Dirichlet-transmission problem* (7.18) *for the Navier-Stokes system has a solution* $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$.

(ii) *If*

$$c_0^2 \|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + \Big(c_0^2 C_\Sigma \|\mathbb{A}\| + 2C_{\mathbb{A}}^{-1} C_\Sigma c_0 (c_1^* + c_0)\Big) \|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet} < \frac{1}{4} C_{\mathbb{A}}^{-2}, \tag{7.23}$$

where $\mathfrak{F} := (\widetilde{\mathbf{f}}^+ + \widetilde{\mathbf{f}}^-) - \gamma_\Sigma^* \boldsymbol{\psi}_\Sigma$, *then the nonlinear Dirichlet-transmission problem* (7.18) *has a unique solution* $(\mathbf{u}^+, \pi^+, \mathbf{u}^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$.

Proof (i) Let us represent the unknowns $\mathbf{u}^\pm$ of problem (7.18) in the form

$$\mathbf{u}^\pm = \mathbf{u}_0^\pm + \mathbf{v}_{\boldsymbol{\varphi}}^\pm, \tag{7.24}$$

where $(\mathbf{v}_{\boldsymbol{\varphi}}^+, \mathbf{v}_{\boldsymbol{\varphi}}^-) \in H^1(\Omega^+)^n \times H^1(\Omega^-)^n$ satisfy relations (7.19) and estimate (7.20). Then the nonlinear Dirichlet-transmission problem (7.18) reduces to the problem

$$\begin{cases} \mathcal{L}(\mathbf{u}_0^+, \pi^+) = (\widetilde{\mathbf{F}}_{\boldsymbol{\varphi}}^+ \mathbf{u}_0^+)|_{\Omega^+}, \ \operatorname{div} \mathbf{u}_0^+ = 0 & \text{in } \Omega^+, \\ \mathcal{L}(\mathbf{u}_0^-, \pi_0^-) = (\widetilde{\mathbf{F}}_{\boldsymbol{\varphi}}^- \mathbf{u}_0^-)|_{\Omega^-}, \ \operatorname{div} \mathbf{u}_0^- = 0 & \text{in } \Omega^-, \\ (\gamma_{\Omega^+} \mathbf{u}_0^+)|_\Sigma = (\gamma_{\Omega^-} \mathbf{u}_0^-)|_\Sigma & \text{on } \Sigma, \\ \big(\mathbf{t}_{\Omega^+}(\mathbf{u}_0^+, \pi^+; \widetilde{\mathbf{F}}_{\boldsymbol{\varphi}}^+ \mathbf{u}_0^+)\big)|_\Sigma + \big(\mathbf{t}_{\Omega^-}(\mathbf{u}_0^-, \pi^-; \widetilde{\mathbf{F}}_{\boldsymbol{\varphi}}^- \mathbf{u}_0^-)\big)|_\Sigma = \boldsymbol{\psi}_\Sigma & \text{on } \Sigma, \\ (\gamma_{\Omega^+} \mathbf{u}_0^+)|_{\Gamma^+} = \mathbf{0} & \text{on } \Gamma^+, \\ (\gamma_{\Omega^-} \mathbf{u}_0^-)|_{\Gamma^-} = \mathbf{0} & \text{on } \Gamma^-, \end{cases} \tag{7.25}$$

for the unknowns $(\mathbf{u}_0^+, \pi^+, \mathbf{u}_0^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$, where

$$\widetilde{\mathbf{F}}_{\boldsymbol{\varphi}}^\pm \mathbf{w}^\pm := \widetilde{\mathbf{f}}^\pm - \check{\mathfrak{L}}^\pm \mathbf{v}_{\boldsymbol{\varphi}}^\pm + \mathring{E}_{\Omega^\pm}\big[((\mathbf{w}^\pm + \mathbf{v}_{\boldsymbol{\varphi}}^\pm) \cdot \nabla)(\mathbf{w}^\pm + \mathbf{v}_{\boldsymbol{\varphi}}^\pm)\big] \tag{7.26}$$

and $\mathring{E}_{\Omega^\pm}$ is used here as a shorter notation for $\mathring{E}_{\Omega^\pm \to \Omega}$. Recall that $\check{\mathfrak{L}}^\pm : H^1(\Omega^\pm)^n \to \widetilde{H}^{-1}(\Omega^\pm)^n$ are the operators defined in (5.49) and $\big(\check{\mathfrak{L}}^\pm \mathbf{v}^\pm\big)|_{\Omega^\pm} = \mathcal{L}(\mathbf{v}^\pm, 0)$ in $\Omega^\pm$. For fixed $\mathbf{v}_{\boldsymbol{\varphi}}^\pm \in H^1(\Omega)^n$, formula (7.26) defines nonlinear operators $\mathbf{w}^\pm \mapsto \widetilde{\mathbf{F}}_{\boldsymbol{\varphi}}^\pm \mathbf{w}^\pm$ from the space $H^1(\Omega^\pm)^n$ to the space $\big(H^1_{\Gamma^\pm}(\Omega^\pm)^n\big)'$ due to estimate (C.17), Lemma B.6, and the inclusion $\check{\mathfrak{L}}^\pm \mathbf{v}_{\boldsymbol{\varphi}}^\pm \in \widetilde{H}^{-1}(\Omega^\pm)^n \hookrightarrow \big(H^1_{\Gamma^\pm}(\Omega^\pm)^n\big)'$.

In addition, the inclusion $(\mathbf{u}_0^+, \pi^+, \mathbf{u}_0^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$, the homogeneous interface condition $(\gamma_{\Omega^+} \mathbf{u}_0^+)|_\Sigma - (\gamma_{\Omega^-} \mathbf{u}_0^-)|_\Sigma = \mathbf{0}$ in (7.25), and Lemma B.1 imply that there exists a unique pair $(\mathbf{u}_0, \pi) \in H^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ such that

$$\mathbf{u}_0|_{\Omega^+} = \mathbf{u}_0^+, \quad \mathbf{u}_0|_{\Omega^-} = \mathbf{u}_0^-, \quad \pi|_{\Omega^+} = \pi^+, \quad \pi|_{\Omega^-} = \pi^-. \tag{7.27}$$

Since $\mathbf{u}_0^+$ and $\mathbf{u}_0^-$ also satisfy the homogeneous Dirichlet condition in (7.25) and are divergence-free, we have that $\mathbf{u}_0 \in \mathring{H}^1_{\operatorname{div}}(\Omega)^n$. Hence, $(\mathbf{u}_0, \pi) \in \mathring{H}^1_{\operatorname{div}}(\Omega)^n \times L^2(\Omega)/\mathbb{R}$.

For any $\mathbf{w} \in \mathring{H}^1_{\operatorname{div}}(\Omega)^n$ let $\mathbf{F}_{\boldsymbol{\varphi}} \mathbf{w}$ be defined by

$$\begin{aligned} \mathbf{F}_{\boldsymbol{\varphi}} \mathbf{w} &= \widetilde{\mathbf{F}}_{\boldsymbol{\varphi}}^+ r_{\Omega^+} \mathbf{w} + \widetilde{\mathbf{F}}_{\boldsymbol{\varphi}}^- r_{\Omega^-} \mathbf{w} - \gamma_\Sigma^* \boldsymbol{\psi}_\Sigma \\ &= \mathfrak{F} - \big(\check{\mathfrak{L}}^+ \mathbf{v}_{\boldsymbol{\varphi}}^+ + \check{\mathfrak{L}}^- \mathbf{v}_{\boldsymbol{\varphi}}^-\big) + \mathring{E}_{\Omega^+}\big[(\mathbf{v}_{\boldsymbol{\varphi}}^+ \cdot \nabla)\mathbf{v}_{\boldsymbol{\varphi}}^+\big] + \mathring{E}_{\Omega^-}\big[(\mathbf{v}_{\boldsymbol{\varphi}}^- \cdot \nabla)\mathbf{v}_{\boldsymbol{\varphi}}^-\big] + (\mathbf{w} \cdot \nabla)\mathbf{w} \\ &\quad + ((\mathring{E}_{\Omega^+} \mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-} \mathbf{v}_{\boldsymbol{\varphi}}^-) \cdot \nabla)\mathbf{w} + \mathbf{w} \cdot (\mathring{E}_{\Omega^+} \nabla \mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-} \nabla \mathbf{v}_{\boldsymbol{\varphi}}^-), \end{aligned} \tag{7.28}$$

where $\mathfrak{F} := (\widetilde{\mathbf{f}}^+ + \widetilde{\mathbf{f}}^-) - \gamma_\Sigma^* \boldsymbol{\psi}_\Sigma$, $\gamma_\Sigma^* : H^{-\frac{1}{2}}(\Sigma)^n \to H^{-1}(\Omega)^n$ is the adjoint of the trace operator $\gamma_\Sigma : \mathring{H}^1(\Omega)^n \to H^{\frac{1}{2}}_\bullet(\Sigma)^n$ defined by (5.16), and hence $\mathfrak{F} \in H^{-1}(\Omega)^n$ due

to Lemma B.6. For fixed $\mathbf{v}_{\boldsymbol{\varphi}}^{\pm} \in H^1(\Omega^{\pm})^n$, formula (7.28) defines a nonlinear operator $\mathbf{w} \mapsto \mathbf{F}_{\boldsymbol{\varphi}}\mathbf{w}$ from $\mathring{H}^1_{\rm div}(\Omega)^n$ to $H^{-1}(\Omega)^n$ due to Lemma B.6 and estimates (C.5) and (C.6).

Now, arguing as in the proof of Lemma 7.1 (cf. also Theorem 5.6), we obtain that the nonlinear Dirichlet-transmission problem (7.25) with the unknowns $\left(\mathbf{u}_0^+, \pi^+, \mathbf{u}_0^-, \pi^-\right) \in \mathfrak{X}_{\Omega^+,\Omega^-}$ is equivalent to the nonlinear equation

$$\boldsymbol{\mathcal{L}}(\mathbf{u}_0, \pi) = \mathbf{F}_{\boldsymbol{\varphi}}\mathbf{u}_0 \quad \text{in } \Omega \tag{7.29}$$

for the unknowns $(\mathbf{u}_0, \pi) \in \mathring{H}^1_{\rm div}(\Omega)^n \times L^2(\Omega)/\mathbb{R}$, with $\mathbf{F}_{\boldsymbol{\varphi}}$ given by (7.28).

The following arguments are similar to those in the proof of Theorem 6.3. By Theorem 4.1, the linear operator

$$\boldsymbol{\mathcal{L}} : \mathring{H}^1_{\rm div}(\Omega)^n \times L^2(\Omega)/\mathbb{R} \to H^{-1}(\Omega)^n \tag{7.30}$$

is an isomorphism. Its inverse operator can be split into two operator components,

$$\boldsymbol{\mathcal{L}}^{-1} = (\boldsymbol{\mathcal{U}}, \mathcal{P}),$$

where $\boldsymbol{\mathcal{U}} : H^{-1}(\Omega)^n \to \mathring{H}^1_{\rm div}(\Omega)^n$ and $\mathcal{P} : H^{-1}(\Omega)^n \to L^2(\Omega)/\mathbb{R}$ are linear continuous operators such that $\boldsymbol{\mathcal{L}}(\boldsymbol{\mathcal{U}}\mathbf{F}, \mathcal{P}\mathbf{F}) = \mathbf{F}$ for any $\mathbf{F} \in H^{-1}(\Omega)^n$. Applying the operator $\boldsymbol{\mathcal{L}}^{-1}$ to equation (7.29) we obtain the equivalent nonlinear system

$$\mathbf{u}_0 = \mathbf{U}\mathbf{u}_0, \tag{7.31}$$

$$\pi = P\mathbf{u}_0, \tag{7.32}$$

where $\mathbf{U} : \mathring{H}^1_{\rm div}(\Omega)^n \to \mathring{H}^1_{\rm div}(\Omega)^n$ and $P : \mathring{H}^1_{\rm div}(\Omega)^n \to L^2(\Omega)/\mathbb{R}$ are the nonlinear operators defined as

$$\mathbf{U}\mathbf{w} := \boldsymbol{\mathcal{U}}\,\mathbf{F}_{\boldsymbol{\varphi}}\mathbf{w}, \tag{7.33}$$

$$P\mathbf{w} := \mathcal{P}\,\mathbf{F}_{\boldsymbol{\varphi}}\mathbf{w}. \tag{7.34}$$

Since π is not involved in (7.31), we will first prove the existence of a solution $\mathbf{u}_0 \in \mathring{H}^1_{\rm div}(\Omega)^n$ to this equation and then use (7.32) as a representation formula. This provides the existence of a pressure field $\pi \in L^2(\Omega)/\mathbb{R}$.

In order to show the existence of a fixed point of the operator $\mathbf{U}$ and, thus, the existence of a weak solution of nonlinear problem (7.25), we employ Theorem 6.1.

Let us show first that $\mathbf{U}$ is continuous. Let $\mathbf{w}, \mathbf{w}' \in \mathring{H}^1_{\rm div}(\Omega)^n$. Then by (7.28), (C.6), (C.17) and (C.18) we obtain that

$$\begin{aligned}
\left\|\mathbf{F}_{\boldsymbol{\varphi}}\mathbf{w} - \mathbf{F}_{\boldsymbol{\varphi}}\mathbf{w}'\right\|_{H^{-1}(\Omega)^n} &\leq \left\|(\mathbf{w}\cdot\nabla)\mathbf{w} - (\mathbf{w}'\cdot\nabla)\mathbf{w}'\right\|_{H^{-1}(\Omega)^n} \\
&\quad + \left\|((\mathring{E}_{\Omega^+}\mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-}\mathbf{v}_{\boldsymbol{\varphi}}^-)\cdot\nabla)(\mathbf{w}-\mathbf{w}') + (\mathbf{w}-\mathbf{w}')\cdot(\mathring{E}_{\Omega^+}\nabla\mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-}\nabla\mathbf{v}_{\boldsymbol{\varphi}}^-)\right\|_{H^{-1}(\Omega)^n} \\
&\leq \left\|((\mathbf{w}-\mathbf{w}')\cdot\nabla)\mathbf{w} + (\mathbf{w}'\cdot\nabla)(\mathbf{w}-\mathbf{w}')\right\|_{H^{-1}(\Omega)^n} \\
&\quad + 2(c_1^*)^2\left\|\mathbf{w}-\mathbf{w}'\right\|_{H^1(\Omega)^n}(\|\mathbf{v}_{\boldsymbol{\varphi}}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{v}_{\boldsymbol{\varphi}}^-\|_{H^1(\Omega^-)^n}) \\
&\leq \left\|\mathbf{w}-\mathbf{w}'\right\|_{H^1(\Omega)^n}\big(c_1^2\|\mathbf{w}\|_{H^1(\Omega)^n} + c_1^2\|\mathbf{w}'\|_{H^1(\Omega)^n} \\
&\quad + 2(c_1^*)^2\|\mathbf{v}_{\boldsymbol{\varphi}}^+\|_{H^1(\Omega^+)^n} + 2(c_1^*)^2|\mathbf{v}_{\boldsymbol{\varphi}}^-\|_{H^1(\Omega^-)^n}\big).
\end{aligned}$$

This estimate shows that the operator $\mathbf{F}_{\boldsymbol{\varphi}} : \mathring{H}^1_{\rm div}(\Omega)^n \to H^{-1}(\Omega)^n$ is continuous. The operator $\mathbf{U} = \boldsymbol{\mathcal{U}}\,\mathbf{F}_{\boldsymbol{\varphi}} : \mathring{H}^1_{\rm div}(\Omega)^n \to \mathring{H}^1_{\rm div}(\Omega)^n$ is also continuous, as asserted.

Next we show that the operator $\mathbf{U}$ is compact. To this end, we assume that $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$ is a bounded sequence in the space $\mathring{H}^1_{\rm div}(\Omega)^n$ and prove that the sequence $\{\mathbf{F}_{\boldsymbol{\varphi}}\mathbf{w}_k\}_{k\in\mathbb{N}}$ contains a convergent subsequence in $H^{-1}(\Omega)^n$.

Let $M>0$ be such that $\|\mathbf{w}_k\|_{H^1(\Omega)^n}\le M$ for all $k\in\mathbb{N}$. By (7.28),

$$\begin{aligned}\left\|\mathbf{F}_{\boldsymbol{\varphi}}\mathbf{w}_k-\mathbf{F}_{\boldsymbol{\varphi}}\mathbf{w}_\ell\right\|_{H^{-1}(\Omega)^n}&\le\|((\mathbf{w}_k-\mathbf{w}_\ell)\cdot\nabla)\mathbf{w}_k+(\mathbf{w}_\ell\cdot\nabla)(\mathbf{w}_k-\mathbf{w}_\ell)\|_{H^{-1}(\Omega)^n}\\&+\left\|((\mathring{E}_{\Omega^+}\mathbf{v}^+_{\boldsymbol{\varphi}}+\mathring{E}_{\Omega^-}\mathbf{v}^-_{\boldsymbol{\varphi}})\cdot\nabla)(\mathbf{w}_k-\mathbf{w}_\ell)\right\|_{H^{-1}(\Omega)^n}\\&+\left\|(\mathbf{w}_k-\mathbf{w}_\ell)\cdot(\mathring{E}_{\Omega^+}\nabla\mathbf{v}^+_{\boldsymbol{\varphi}}+\mathring{E}_{\Omega^-}\nabla\mathbf{v}^-_{\boldsymbol{\varphi}})\right\|_{H^{-1}(\Omega)^n}.\end{aligned}$$

Employing estimates (C.6) and (C.15) to the first norm in the right hand side, (C.20) for the second norm, and estimate (C.18) for the third norm, we obtain

$$\begin{aligned}\left\|\mathbf{F}_{\boldsymbol{\varphi}}\mathbf{w}_k-\mathbf{F}_{\boldsymbol{\varphi}}\mathbf{w}_\ell\right\|_{H^{-1}(\Omega)^n}&\le\Big(c_1\|\mathbf{w}_k\|_{H^1(\Omega)^n}+c_1\|\mathbf{w}_\ell\|_{H^1(\Omega)^n}\\&+3c_1^*\|\mathbf{v}^+_{\boldsymbol{\varphi}}\|_{H^1(\Omega^+)^n}+3c_1^*\|\mathbf{v}^-_{\boldsymbol{\varphi}}\|_{H^1(\Omega^-)^n}\Big)\|\mathbf{w}_k-\mathbf{w}_\ell\|_{L^4(\Omega)^n}\\&+\Big((c_2^+)^2\|\gamma_{\Omega^+}\|^2\|\mathbf{v}^+_{\boldsymbol{\varphi}}\|_{H^1(\Omega^+)^n}+(c_2^-)^2\|\gamma_{\Omega^-}\|^2\|\mathbf{v}^-_{\boldsymbol{\varphi}}\|_{H^1(\Omega^-)^n}\Big)\|\gamma_\Sigma(\mathbf{w}_k-\mathbf{w}_\ell)\|_{L^3(\Sigma)^n}\\&\le\Big(2c_1M+3c_1^*\|\mathbf{v}^+_{\boldsymbol{\varphi}}\|_{H^1(\Omega^+)^n}+3c_1^*\|\mathbf{v}^-_{\boldsymbol{\varphi}}\|_{H^1(\Omega^-)^n}\Big)\|\mathbf{w}_k-\mathbf{w}_\ell\|_{L^4(\Omega)^n}\\&+\Big((c_2^+)^2\|\gamma_{\Omega^+}\|^2\|\mathbf{v}^+_{\boldsymbol{\varphi}}\|_{H^1(\Omega^+)^n}+(c_2^-)^2\|\gamma_{\Omega^-}\|^2\|\mathbf{v}^-_{\boldsymbol{\varphi}}\|_{H^1(\Omega^-)^n}\Big)\|\gamma_\Sigma(\mathbf{w}_k-\mathbf{w}_\ell)\|_{L^3(\Sigma)^n},\end{aligned}\tag{7.35}$$

where we denoted $\gamma_\Sigma(\mathbf{w}_k-\mathbf{w}_\ell):=r_\Sigma\gamma_{\Omega^+}(\mathbf{w}_k-\mathbf{w}_\ell)=r_\Sigma\gamma_{\Omega^-}(\mathbf{w}_k-\mathbf{w}_\ell)$ and took into account that $r_{\Gamma^+}\gamma_{\Omega^+}(\mathbf{w}_k-\mathbf{w}_\ell)=\mathbf{0}$ and $r_{\Gamma^-}\gamma_{\Omega^-}(\mathbf{w}_k-\mathbf{w}_\ell)=\mathbf{0}$.

Since $\|\mathbf{w}_k\|_{H^1(\Omega)^n}\le M$ and the embedding of the space $H^1(\Omega)^n$ into the space $L^4(\Omega)^n$ is compact (see, e.g., [1, Theorem 6.3]), there exists a subsequence of $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$, labelled as the sequence, which converges in $L^4(\Omega)^n$, and, hence, is a Cauchy sequence in $L^4(\Omega)^n$.

Since $\|\mathbf{w}_k\|_{H^1(\Omega)^n}\le M$, the sequence $\{\gamma_\Sigma\mathbf{w}_k\}_{k\in\mathbb{N}}$ is bounded in $H^{1/2}(\Sigma)^n$. Further, for $n=2,3$ the space $H^{1/2}(\Sigma)^n$ is compactly embedded in $L^3(\Sigma)^n$. For bounded Lipschitz domains in $\mathbb{R}^{n-1}$, this follows by the Rellich-Kondrachev compactness theorems, e.g., from the embedding results in [54, Section 2.2.4, Corollary 2(i)] for $\mathbb{R}^{n-1}$ and can be extended to $(n-1)$-dimensional bounded Lipschitz manifolds by standard arguments (cf. also a more general statement in [50, Proposition 3.8]). Then there exists a subsequence of $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$, labelled as the sequence, such that $\{\gamma_\Sigma\mathbf{w}_k\}_{k\in\mathbb{N}}$ converges in $L^3(\Sigma)^n$, and, hence, is a Cauchy sequence in $L^3(\Sigma)^n$.

Inequality (7.35) combined with this Cauchy property implies that the operator $\mathbf{F}_{\boldsymbol{\varphi}}:\mathring{H}^1_{\rm div}(\Omega)^n\to H^{-1}(\Omega)^n$ is compact. Hence, the operator $\mathbf{U}=\boldsymbol{\mathcal{U}}\mathbf{F}_{\boldsymbol{\varphi}}:\mathring{H}^1_{\rm div}(\Omega)^n\to\mathring{H}^1_{\rm div}(\Omega)^n$ is also compact, as asserted.

It remains to show that there is a constant $M_0>0$ such that if $\mathbf{w}\in\mathring{H}^1_{\rm div}(\Omega)^n$ satisfies the equation

$$\mathbf{w}=\lambda\mathbf{U}\mathbf{w}\tag{7.36}$$

for some $\lambda\in[0,1]$, then $\|\mathbf{w}\|_{H^1(\Omega)^n}\le M_0$. Let us introduce the function

$$q:=\lambda P\mathbf{w}.\tag{7.37}$$

By applying the operator $\boldsymbol{\mathcal{L}}$ to equations (7.36)–(7.37), and by using relations (7.33) and (7.34), we deduce that whenever the pair $(\mathbf{w},\lambda)\in\mathring{H}^1_{\rm div}(\Omega)^n\times\mathbb{R}$ satisfies equation (7.36),

the equation $\mathcal{L}(\mathbf{w}, q) = \lambda \mathbf{F}_{\boldsymbol{\varphi}} \mathbf{w}$ is also satisfied. (Recall the isomorphism property of operator (7.30).) Then the first Green identity (A.1) implies

$$a_{\mathbb{A};\Omega}(\mathbf{w}, \mathbf{v}) := \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{v}) \right\rangle_{\Omega} = -\langle \lambda \mathbf{F}_{\boldsymbol{\varphi}} \mathbf{w}, \mathbf{v} \rangle_{\Omega} \quad \forall\, \mathbf{v} \in \mathring{H}^1_{\mathrm{div}}(\Omega)^n, \tag{7.38}$$

which, in view of relation (7.28), takes the form

$$\begin{aligned} a_{\mathbb{A};\Omega}(\mathbf{w}, \mathbf{v}) = &- \lambda \langle \mathfrak{F}, \mathbf{v} \rangle_{\Omega} - \lambda \Big(\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{v}_{\boldsymbol{\varphi}}^+), E_{i\alpha}(\mathbf{v}) \right\rangle_{\Omega^+} + \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{v}_{\boldsymbol{\varphi}}^-), E_{i\alpha}(\mathbf{v}) \right\rangle_{\Omega^-} \Big) \\ &- \lambda \Big\langle \mathring{E}_{\Omega^+}\big[(\mathbf{v}_{\boldsymbol{\varphi}}^+ \cdot \nabla)\mathbf{v}_{\boldsymbol{\varphi}}^+\big] + \mathring{E}_{\Omega^-}\big[(\mathbf{v}_{\boldsymbol{\varphi}}^- \cdot \nabla)\mathbf{v}_{\boldsymbol{\varphi}}^-\big] \\ &+ ((\mathring{E}_{\Omega^+}\mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-}\mathbf{v}_{\boldsymbol{\varphi}}^-) \cdot \nabla)\mathbf{w} \\ &+ \mathbf{w} \cdot (\mathring{E}_{\Omega^+}\nabla\mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-}\nabla\mathbf{v}_{\boldsymbol{\varphi}}^-) + (\mathbf{w} \cdot \nabla)\mathbf{w}, \mathbf{v} \Big\rangle_{\Omega}, \ \forall\, \mathbf{v} \in \mathring{H}^1_{\mathrm{div}}(\Omega)^n. \end{aligned} \tag{7.39}$$

Moreover, formula (C.13) and the inclusion $\mathbf{w} \in \mathring{H}^1_{\mathrm{div}}(\Omega)^n$ imply the relation $\big\langle (\mathbf{w} \cdot \nabla)\mathbf{w}, \mathbf{w} \big\rangle_{\Omega} = 0$. Then by (7.39) we obtain the formula

$$\begin{aligned} a_{\mathbb{A};\Omega}(\mathbf{w}, \mathbf{w}) = &-\lambda \langle \mathfrak{F}, \mathbf{w} \rangle_{\Omega} - \lambda \Big(\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{v}_{\boldsymbol{\varphi}}^+), E_{i\alpha}(\mathbf{w}) \right\rangle_{\Omega^+} + \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{v}_{\boldsymbol{\varphi}}^-), E_{i\alpha}(\mathbf{w}) \right\rangle_{\Omega^-} \Big) \\ &- \lambda \Big\langle \mathring{E}_{\Omega^+}\big[(\mathbf{v}_{\boldsymbol{\varphi}}^+ \cdot \nabla)\mathbf{v}_{\boldsymbol{\varphi}}^+\big] + \mathring{E}_{\Omega^-}\big[(\mathbf{v}_{\boldsymbol{\varphi}}^- \cdot \nabla)\mathbf{v}_{\boldsymbol{\varphi}}^-\big], \mathbf{w} \Big\rangle_{\Omega} \\ &- \lambda \Big\langle \Big(\big(\mathring{E}_{\Omega^+}\mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-}\mathbf{v}_{\boldsymbol{\varphi}}^- \big) \cdot \nabla \Big) \mathbf{w}, \mathbf{w} \Big\rangle_{\Omega} \\ &- \lambda \Big\langle \mathbf{w} \cdot \Big(\mathring{E}_{\Omega^+}\nabla\mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-}\nabla\mathbf{v}_{\boldsymbol{\varphi}}^- \Big), \mathbf{w} \Big\rangle_{\Omega}, \end{aligned} \tag{7.40}$$

Arguments similar to those for estimate (6.20) combined with formula (7.40), the inclusion $\lambda \in [0, 1]$ and inequalities (C.2), (7.20), and (7.21) imply that

$$\begin{aligned} &\frac{1}{2} C_{\mathbb{A}}^{-1} \|\nabla \mathbf{w}\|^2_{L^2(\Omega)^{n\times n}} \le C_{\mathbb{A}}^{-1} \|\mathbb{E}(\mathbf{w})\|^2_{L^2(\Omega)^{n\times n}} \le a_{\mathbb{A};\Omega}(\mathbf{w}, \mathbf{w}) \\ &\le \|\mathfrak{F}\|_{H^{-1}(\Omega)^n} \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} + \|\mathbb{A}\| \Big(\|\nabla \mathbf{v}_{\boldsymbol{\varphi}}^+\|_{L^2(\Omega^+)^{n\times n}} + \|\nabla \mathbf{v}_{\boldsymbol{\varphi}}^-\|_{L^2(\Omega^-)^{n\times n}} \Big) \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} \\ &\quad + \Big(\|\mathbf{v}_{\boldsymbol{\varphi}}^+\|_{L^4(\Omega^+)^n} \|\nabla \mathbf{v}_{\boldsymbol{\varphi}}^+\|_{L^2(\Omega^+)^{n\times n}} + \|\mathbf{v}_{\boldsymbol{\varphi}}^-\|_{L^4(\Omega^-)^n} \|\nabla \mathbf{v}_{\boldsymbol{\varphi}}^-\|_{L^2(\Omega^-)^{n\times n}} \Big) \|\mathbf{w}\|_{L^4(\Omega)^n} \\ &\quad + \Big(\|\mathbf{v}_{\boldsymbol{\varphi}}^+\|_{L^4(\Omega^+)^n} + \|\mathbf{v}_{\boldsymbol{\varphi}}^-\|_{L^4(\Omega^+)^n} \Big) \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} \|\mathbf{w}\|_{L^4(\Omega)^n} \\ &\quad + \Big(\|\nabla \mathbf{v}_{\boldsymbol{\varphi}}^+\|_{L^2(\Omega^+)^{n\times n}} + \|\nabla \mathbf{v}_{\boldsymbol{\varphi}}^-\|_{L^2(\Omega^-)^{n\times n}} \Big) \|\mathbf{w}\|^2_{L^4(\Omega)^n} \\ &\le \|\mathfrak{F}\|_{H^{-1}(\Omega)^n} \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} + \|\mathbb{A}\| \Big(\|\mathbf{v}_{\boldsymbol{\varphi}}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{v}_{\boldsymbol{\varphi}}^-\|_{H^1(\Omega^-)^n} \Big) \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} \\ &\quad + c_0 c_1^* \Big(\|\mathbf{v}_{\boldsymbol{\varphi}}^+\|^2_{H^1(\Omega^+)^n} + \|\mathbf{v}_{\boldsymbol{\varphi}}^-\|^2_{H^1(\Omega^-)^n} \Big) \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} \\ &\quad + c_0 c_1^* \Big(\|\mathbf{v}_{\boldsymbol{\varphi}}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{v}_{\boldsymbol{\varphi}}^-\|_{H^1(\Omega^+)^n} \Big) \|\nabla \mathbf{w}\|^2_{L^2(\Omega)^{n\times n}} \\ &\quad + c_0^2 \Big(\|\mathbf{v}_{\boldsymbol{\varphi}}^+\|_{H^1(\Omega^+)^n} + \|\mathbf{v}_{\boldsymbol{\varphi}}^-\|_{H^1(\Omega^+)^n} \Big) \|\nabla \mathbf{w}\|^2_{L^2(\Omega)^{n\times n}} \\ &\le \|\mathfrak{F}\|_{H^{-1}(\Omega)^n} \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} + \|\mathbb{A}\| \, C_{\Sigma} \, \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet} \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} \\ &\quad 2 c_0 c_1^* C_{\Sigma}^2 \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|^2_{\mathcal{M}_\bullet} \|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}} \\ &\quad + 2(c_0 c_1^* + c_0^2) C_{\Sigma} \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet} \|\nabla \mathbf{w}\|^2_{L^2(\Omega)^{n\times n}}. \end{aligned} \tag{7.41}$$

Therefore, we obtain the estimate

$$\left(\frac{1}{2}C_{\mathbb{A}}^{-1} - 2c_0(c_1^* + c_0)C_\Sigma\|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}\right)\|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}}$$
$$\leq \|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + C_\Sigma\|\mathbb{A}\|\|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet} + 2c_0c_1^*C_\Sigma^2\|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}^2. \tag{7.42}$$

In view of assumption (7.22), estimate (7.42) can be written in the form

$$\|\nabla \mathbf{w}\|_{L^2(\Omega)^{n\times n}}$$
$$\leq \frac{\|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + C_\Sigma\|\mathbb{A}\|\|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet} + 2c_0c_1^*C_\Sigma^2\|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}^2}{\frac{1}{2}C_{\mathbb{A}}^{-1} - 2c_0(c_1^* + c_0)C_\Sigma\|(g^+, g^-, \boldsymbol{\varphi}_\Sigma, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}}, \tag{7.43}$$

that is, $\|\mathbf{w}\|_{H^1(\Omega)^n} \leq M_0$, where M_0 is given by the right hand side of (7.43) multiplied by the equivalence constant $\mathring{C}$ from (4.3).

Therefore, the operator $\mathbf{U} : \mathring{H}^1_{\rm div}(\Omega)^n \to \mathring{H}^1_{\rm div}(\Omega)^n$ given by (7.33) satisfies the hypothesis of Theorem 6.1 (for $\mathcal{X} = \mathring{H}^1_{\rm div}(\Omega)^n$), and hence it has a fixed point $\mathbf{u}_0 \in \mathring{H}^1_{\rm div}(\Omega)^n$, that is, $\mathbf{u}_0 = \mathbf{U}\mathbf{u}_0$. Then with $\pi \in L^2(\Omega)/\mathbb{R}$ as in (7.32), the couple $(\mathbf{u}_0, \pi) \in \mathring{H}^1_{\rm div}(\Omega)^n \times L^2(\Omega)/\mathbb{R}$ satisfies the nonlinear equation (7.29). Consequently, the couples $(\mathbf{u}_0^+ + \mathbf{v}_{\boldsymbol{\varphi}}^+, \pi^+, \mathbf{u}_0^- + \mathbf{v}_{\boldsymbol{\varphi}}^-, \pi^-) \in \mathfrak{X}_{\Omega^+,\Omega^-}$ provide a solution of the nonlinear Dirichlet-transmission (7.18) in the sense of relations (7.24). (Recall the equivalence between the nonlinear Dirichlet-transmission problem (7.25) and the nonlinear equation (7.29).)

(ii) Let us assume that condition (7.23) holds. Then it is immediate that condition (7.22) holds as well, and, thus, the nonlinear Dirichlet-transmission problem (7.18) has at least one solution in the space $\mathfrak{X}_{\Omega^+,\Omega^-}$.

Now, assume that the nonlinear problem (7.18) has two solutions, $(\mathbf{u}^{(1)+}, \pi^{(1)+}, \mathbf{u}^{(1)-}, \pi^{(1)-})$ and $(\mathbf{u}^{(2)+}, \pi^{(2)+}, \mathbf{u}^{(2)-}, \pi^{(2)-})$ in the space $\mathfrak{X}_{\Omega^+,\Omega^-}$. Let us represent the velocities in $\Omega^\pm$ in the form

$$\mathbf{u}^{(i)\pm} = \mathbf{v}_{\boldsymbol{\varphi}}^\pm + \mathbf{u}_0^{(i)\pm}, \quad i = 1, 2, \tag{7.44}$$

where $(\mathbf{v}_{\boldsymbol{\varphi}}^+, \mathbf{v}_{\boldsymbol{\varphi}}^-) \in H^1(\Omega^+)^n \times H^1(\Omega^-)^n$ satisfy relations (7.19) and (7.20), while

$$\mathbf{u}_0^{(i)}|_{\Omega^+} = \mathbf{u}_0^{(i)+}, \quad \mathbf{u}_0^{(i)}|_{\Omega^-} = \mathbf{u}_0^{(i)-}, \quad \pi^{(i)}|_{\Omega^+} = \pi^{(i)+}, \quad \pi^{(i)}|_{\Omega^-} = \pi^{(i)-}. \tag{7.45}$$

corresponds to the pairs $(\mathbf{u}_0^{(i)}, \pi^{(i)}) \in \mathring{H}^1(\Omega)^n \times L^2(\Omega)/\mathbb{R}$.

Let us also introduce the notations $\overline{\mathbf{u}}_0 := \mathbf{u}_0^{(1)} - \mathbf{u}_0^{(2)}$, $\overline{\pi} := \pi^{(1)} - \pi^{(2)}$,

$$\overline{\mathbf{u}}^\pm := \mathbf{u}^{(1)\pm} - \mathbf{u}^{(2)\pm} = \overline{\mathbf{u}}_0|_{\Omega^\pm}, \quad \overline{\pi}^\pm := \pi^{(1)\pm} - \pi^{(2)\pm} = \overline{\pi}|_{\Omega^\pm}.$$

Using (7.28) and (7.29), we obtain

$$\begin{aligned}\boldsymbol{\mathcal{L}}(\overline{\mathbf{u}}_0, \overline{\pi}) &= \mathring{E}_{\Omega^+}[(\mathbf{u}^{(1)+}\cdot\nabla)\mathbf{u}^{(1)+} - (\mathbf{u}^{(2)+}\cdot\nabla)\mathbf{u}^{(2)+}]\\ &\quad + \mathring{E}_{\Omega^-}[(\mathbf{u}^{(1)-}\cdot\nabla)\mathbf{u}^{(1)-} - (\mathbf{u}^{(2)-}\cdot\nabla)\mathbf{u}^{(2)-}]\\ &= \overline{\mathbf{u}}_0\cdot(\nabla\mathbf{u}_0^{(1)} + \mathring{E}_{\Omega^+}\nabla\mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-}\nabla\mathbf{v}_{\boldsymbol{\varphi}}^-)\\ &\quad + \big((\mathbf{u}_0^{(2)} + \mathring{E}_{\Omega^+}\mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-}\mathbf{v}_{\boldsymbol{\varphi}}^-)\cdot\nabla\big)\overline{\mathbf{u}}_0 \ \text{ in } \Omega\,.\end{aligned} \tag{7.46}$$

This implies that

$$\big\langle a_{ij}^{\alpha\beta} E_{j\beta}(\overline{\mathbf{u}}_0), E_{i\alpha}(\overline{\mathbf{u}}_0)\big\rangle_{\Omega} = -\big\langle (\overline{\mathbf{u}}_0 \cdot \nabla)\mathbf{u}_0^{(1)}, \overline{\mathbf{u}}_0\big\rangle_{\Omega} - \big\langle (\overline{\mathbf{u}}_0 \cdot (\mathring{E}_{\Omega^+} \nabla \mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-} \nabla \mathbf{v}_{\boldsymbol{\varphi}}^-), \overline{\mathbf{u}}_0\big\rangle_{\Omega}$$
$$-\big\langle ((\mathring{E}_{\Omega^+} \mathbf{v}_{\boldsymbol{\varphi}}^+ + \mathring{E}_{\Omega^-} \mathbf{v}_{\boldsymbol{\varphi}}^-) \cdot \nabla)\overline{\mathbf{u}}_0, \overline{\mathbf{u}}_0\big\rangle_{\Omega} - \big\langle (\mathbf{u}_0^{(2)} \cdot \nabla)\overline{\mathbf{u}}_0, \overline{\mathbf{u}}_0\big\rangle_{\Omega}. \tag{7.47}$$

Moreover, identity (C.13) and the inclusion $\overline{\mathbf{u}}_0 \in \mathring{H}^1_{\rm div}(\Omega)^n$ show that the last term in the right-hand side of (7.47) equals zero. Then inequality (4.10) and arguments similar to those for (7.14) imply that

$$\frac{1}{2} C_{\mathbb{A}}^{-1} \|\nabla \overline{\mathbf{u}}_0\|^2_{L^2(\Omega)^{n\times n}} \le \big\langle a_{ij}^{\alpha\beta} E_{j\beta}(\overline{\mathbf{u}}_0), E_{i\alpha}(\overline{\mathbf{u}}_0)\big\rangle_{\Omega}$$
$$\le c_0^2 \|\nabla \mathbf{u}_0^{(1)}\|_{L^2(\Omega)^{n\times n}} \|\nabla \overline{\mathbf{u}}_0\|^2_{L^2(\Omega)^{n\times n}}$$
$$+ 2c_0(c_1^* + c_0) C_{\Sigma} \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet} \|\nabla \overline{\mathbf{u}}_0\|^2_{L^2(\Omega)^{n\times n}}. \tag{7.48}$$

Using in (7.48) estimate (7.43) for $\mathbf{w} = \mathbf{u}_0^{(1)}$, after some simplifications, we obtain

$$\left(\frac{1}{2} C_{\mathbb{A}}^{-1} - 2c_0(c_1^* + c_0) C_{\Sigma} \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}\right)^2 \|\nabla \overline{\mathbf{u}}_0\|^2_{L^2(\Omega)^{n\times n}}$$
$$\le c_0^2 \Big(\|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + C_{\Sigma} \|\mathbb{A}\| \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}$$
$$+ 2c_0 c_1^* C_{\Sigma}^2 \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|^2_{\mathcal{M}_\bullet} \Big) \|\nabla \overline{\mathbf{u}}_0\|^2_{L^2(\Omega)^{n\times n}}, \tag{7.49}$$

and hence

$$\frac{1}{4} C_{\mathbb{A}}^{-2} \|\nabla \overline{\mathbf{u}}_0\|^2_{L^2(\Omega)^{n\times n}} \le \Big(c_0^2 \|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + c_0^2 C_{\Sigma} \|\mathbb{A}\| \, \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}$$
$$+ 2C_{\mathbb{A}}^{-1} c_0(c_1^* + c_0) C_{\Sigma} \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}$$
$$- c_0^2 \big(4(c_1^* + c_0)^2 - 2c_0 c_1^*\big) C_{\Sigma}^2 \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|^2_{\mathcal{M}_\bullet} \Big) \|\nabla \overline{\mathbf{u}}_0\|^2_{L^2(\Omega)^{n\times n}}$$
$$\le \Big(c_0^2 \|\mathfrak{F}\|_{H^{-1}(\Omega)^n} + c_0^2 C_{\Sigma} \|\mathbb{A}\| \, \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet}$$
$$+ 2C_{\mathbb{A}}^{-1} c_0(c_1^* + c_0) C_{\Sigma} \|(g^+, g^-, \boldsymbol{\varphi}_{\Sigma}, \boldsymbol{\varphi})\|_{\mathcal{M}_\bullet} \Big) \|\nabla \overline{\mathbf{u}}_0\|^2_{L^2(\Omega)^{n\times n}}.$$

In view of condition (7.23), this is possible only if

$$\|\nabla \overline{\mathbf{u}}_0\|^2_{L^2(\Omega)^{n\times n}} = \|\nabla(\mathbf{u}_0^{(1)} - \mathbf{u}_0^{(2)})\|_{L^2(\Omega)^{n\times n}} = 0.$$

Hence, $\mathbf{u}_0^{(1)} = \mathbf{u}_0^{(2)}$ in Ω and relations (7.44) imply that $\mathbf{u}^{(1)\pm} = \mathbf{u}^{(2)\pm}$ in $\Omega^{\pm}$. Finally, equation (7.46) leads to $\nabla \overline{\pi} = 0$, that is, $\pi^{(1)} = \pi^{(2)}$ in $L^2(\Omega)/\mathbb{R}$. □

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Declarations

Data availability statement Our paper has no associated data.

APPENDIX

A Generalized conormal derivative for the anisotropic Stokes system

Understanding the Stokes equation in (2.8) in the sense of distributions and taking into account that the Schwartz test function space $\mathcal{D}(\Omega)^n$ is dense in $\mathring{H}^1(\Omega)^n$, we obtain the following assertion that is essentially used in the main text of the paper.

Lemma A.1 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and let conditions (2.2), (2.3) be satisfied. Then the following first Green identity holds*

$$\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{w}) \right\rangle_{\Omega} - \langle \pi, \operatorname{div} \mathbf{w} \rangle_{\Omega} + \langle \mathbf{f}, \mathbf{w} \rangle_{\Omega} = 0\,, \ \forall\, \mathbf{w} \in \mathring{H}^1(\Omega)^n\,, \tag{A.1}$$

for all $(\mathbf{u}, \pi) \in H^1(\Omega)^n \times L^2(\Omega)$ and $\mathbf{f} \in H^{-1}(\Omega)^n$ such that $\mathcal{L}(\mathbf{u}, \pi) = \mathbf{f}$ in Ω.

As in [35, Definition 2.2] and [36, Definition 1], we introduce the generalized conormal derivatives as follows (see also [45, Lemma 4.3], [46, Definition 3.1, Theorem 3.2], [34, Definition 2.4], [51, Theorem 10.4.1]).

Definition A.2 Let conditions (2.2) and (2.3) be satisfied and

$$\boldsymbol{H}^1(\Omega, \mathcal{L}) := \left\{ (\mathbf{u}, \pi, \tilde{\mathbf{f}}) \in H^1(\Omega)^n \times L^2(\Omega) \times \widetilde{H}^{-1}(\Omega)^n : \mathcal{L}(\mathbf{u}, \pi) = \tilde{\mathbf{f}}|_{\Omega} \text{ in } \Omega \right\}.$$

If $(\mathbf{u}, \pi, \tilde{\mathbf{f}}) \in \boldsymbol{H}^1(\Omega, \mathcal{L})$, then the generalized conormal derivative $\mathbf{t}_{\Omega}(\mathbf{u}, \pi; \tilde{\mathbf{f}}) \in H^{-\frac{1}{2}}(\partial\Omega)^n$ is defined in the weak form as

$$\begin{aligned} \left\langle \mathbf{t}_{\Omega}(\mathbf{u}, \pi; \tilde{\mathbf{f}}), \boldsymbol{\Phi} \right\rangle_{\partial\Omega} &:= \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\gamma_{\Omega}^{-1} \boldsymbol{\Phi}) \right\rangle_{\Omega} - \left\langle \pi, \operatorname{div}(\gamma_{\Omega}^{-1} \boldsymbol{\Phi}) \right\rangle_{\Omega} \\ &\quad + \left\langle \tilde{\mathbf{f}}, \gamma_{\Omega}^{-1} \boldsymbol{\Phi} \right\rangle_{\Omega}\,, \ \forall\, \boldsymbol{\Phi} \in H^{\frac{1}{2}}(\partial\Omega)^n \end{aligned} \tag{A.2}$$

where $\gamma_{\Omega}^{-1} : H^{\frac{1}{2}}(\partial\Omega)^n \to H^1(\Omega)^n$ is a bounded right inverse of the trace operator $\gamma_{\Omega} : H^1(\Omega)^n \to H^{\frac{1}{2}}(\partial\Omega)^n$. We use the simplified notation $\mathbf{t}_{\Omega}(\mathbf{u}, \pi)$ for $\mathbf{t}_{\Omega}(\mathbf{u}, \pi; \mathbf{0})$.

B Extension results for Sobolev spaces on Lipschitz domains with Lipschitz interfaces

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain satisfying Assumption 5.1. Thus, $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$, where Σ is the $(n-1)$-dimensional Lipschitz interface between the disjoint Lipschitz sub-domains Ω^+ and Ω^-, and Σ meets transversally $\partial\Omega$. The boundary $\partial\Omega^{\pm}$ of $\Omega^{\pm}$ is partitioned into two relatively open subsets $\Gamma^{\pm}$ and Σ, while Γ^+ and Γ^- are not empty. Let $\gamma_{\Omega^{\pm}}$ be the trace operator from $H^1(\Omega^{\pm})$ to $H^{\frac{1}{2}}(\partial\Omega^{\pm})$.

The proof of the following extension property is based on similar arguments to those for Theorem 5.13 in [9] (see also Lemma C.1 in [36]). We omit the details for the shortness.

Lemma B.1 *The following assertions hold.*

(i) *Let $u^+ \in H^1(\Omega^+)$ and $u^- \in H^1(\Omega^-)$ be such that $\gamma_{\Omega^+}u^+ = \gamma_{\Omega^-}u^-$ on Σ. Then there exists a unique function $u \in H^1(\Omega)$ such that $u|_{\Omega^\pm} = u^\pm$. Moreover, there exists $C = C(n, \Omega^\pm) > 0$ such that $\|u\|_{H^1(\Omega)} \leq C\left(\|u^+\|_{H^1(\Omega^+)} + \|u^-\|_{H^1(\Omega^-)}\right)$.*

(ii) *If $u \in H^1(\Omega)$ then $\gamma_{\Omega^+}(u|_{\Omega^+}) = \gamma_{\Omega^-}(u|_{\Omega^-})$ on Σ.*

Lemma B.2 *Let Γ_1 and Γ_2 be two $(n-1)$-dimensional Lipschitz hyper-surfaces in $\mathbb{R}^n$, $n \geq 2$, that coincide on a relatively open $(n-1)$-dimensional subset Γ_0 (having a Lipschitz boundary if $n \geq 2$). Assume that either one of the following conditions holds:*

(1) *Γ_1 and Γ_2 are the graphs of two Lipschitz functions;*
(2) *Γ_1 and Γ_2 can be mapped by rigid rotations into two Lipschitz graphs;*
(3) *Γ_1 and Γ_2 are two bounded Lipschitz hyper-surfaces in $\mathbb{R}^n$.*

Let $0 \leq s \leq 1$ and functions $f_i \in L^2(\Gamma_i)$, $f_i = 0$ on $\Gamma_i \setminus \overline{\Gamma}_0$, $i = 1, 2$, and $f_2 = f_1$ on Γ_0. Then $f_1 \in H^s(\Gamma_1)$ if and only if $f_2 \in H^s(\Gamma_2)$.

Proof (1) Let Γ_1 and Γ_2 be graphs of two Lipschitz functions $x_n = \zeta_1(x')$ and $x_n = \zeta_2(x')$, $x' \in \mathbb{R}^{n-1}$, and Γ_0 be the image of a domain $S_0 \subset \mathbb{R}^{n-1}$, i.e., $x_n = \zeta_1(x') = \zeta_2(x')$ for $x' \in S_0 \subset \mathbb{R}^{n-1}$.

By the definition of the Sobolev spaces on Lipschitz graphs (see, e.g., [45, p. 98]), $f_1 \in H^s(\Gamma_1), 0 \leq s \leq 1$, means that $f_{\zeta_1} \in H^s(\mathbb{R}^{n-1})$, where $f_{\zeta_1}(x') = f_1(x', \zeta_1(x')), x' \in \mathbb{R}^{n-1}$. On the other hand, $f_{\zeta_2}(x') = f_2(x', \zeta_2(x')) = f_1(x', \zeta_1(x'))$ for $x' \in S_0$, and $f_{\zeta_2}(x') = f_2(x', \zeta_2(x')) = 0 = f_1(x', \zeta_1(x'))$ for $x' \in \mathbb{R}^{n-1} \setminus \overline{S}_0$. Hence $f_{\zeta_2}(x') = f_{\zeta_1}(x')$ for almost any $x' \in \mathbb{R}^{n-1}$ which implies $f_1 \in H^s(\Gamma_1)$ if and only if $f_2 \in H^s(\Gamma_2)$.

(2) Let further $i = 1, 2$. By the assumption of item (2), there exist constant invertible rotation matrices $\Phi_i \in \mathbb{R}^{n\times n}$ such that $\Gamma_i^* = \{x = \Phi_i y,\ y \in \Gamma_i\}$, $i = 1, 2$, are Lipschitz graphs, i.e., they are represented by two Lipschitz functions, $x_{1,n} = \zeta_1(x_1')$ and $x_{2,n} = \zeta_2(x_2')$, where $x_{1,n}, x_{2,n} \in \mathbb{R}$ and $x_1', x_2' \in \mathbb{R}^{n-1}$. By the definition of Sobolev spaces on Lipschitz hyper-surfaces, the inclusion $f_i \in H^s(\Gamma_i)$ implies that $f_{\zeta_i} \in H^s(\mathbb{R}^{n-1})$, where $f_{\zeta_i}(x') = f_i(\Phi_i^{-1}(x_i', \zeta_i(x_i')), x_i' \in \mathbb{R}^{n-1}$.

Let $\Gamma_{0i}^* \subset \Gamma_i^*$ be the images of Γ_0 through the above mapping, i.e., $\Gamma_{0i}^* \ni x_i = \Phi_i y$, $y \in \Gamma_0$, and, on the other hand, $\Gamma_{0i}^* \ni x_i = (x_i', \zeta_i(x_i'))$, $x_i' \in S_{0i}$, where S_{0i} are Lipschitz domains in $\mathbb{R}^{n-1}$. Then

$$\Phi_2^{-1}x_2 = \Phi_2^{-1}(x_2', \zeta_2(x_2')) = y = \Phi_1^{-1}(x_1', \zeta_1(x_1')) = \Phi_1^{-1}x_1, \quad \forall\, y \in \Gamma_0,\ x_1' \in S_{01},\ x_2' \in S_{02},$$

and

$$f_{\zeta_2}(x_2') = f_2(\Phi_2^{-1}(x_2', \zeta_2(x_2')) = f_2(y) = f_1(y) = f_1(\Phi_1^{-1}(x_1', \zeta_1(x_1')) = f_{\zeta_1}(x_1'), \quad \forall\, y \in \Gamma_0,\ x_1' \in S_{01},\ x_2' \in S_{02}. \tag{B.1}$$

By definition of the rigid rotation matrices and graph functions, we have,

$$(x_1', \zeta_1(x_1')) = \Phi_1 y = (\Phi_1' y, \Phi_{1,n} y), \quad (x_2', \zeta_2(x_2')) = \Phi_2 y = (\Phi_2' y, \Phi_{2,n} y), \quad y \in \Gamma_0,$$

where the matrices $\Phi_i' \in \mathbb{R}^{(n-1)\times n}$ and $\Phi_{i,n} \in \mathbb{R}^{1\times n}$ are parts of the corresponding matrices Φ_i. Then

$$x_1' = \Phi_1' y = \Phi_1' \Phi_2^{-1}\left(x_2', \zeta_2(x_2')\right),\ x_2' = \Phi_2' y = \Phi_2' \Phi_1^{-1}\left(x_1', \zeta_1(x_1')\right),\ x_1' \in \bar{S}_{01},\ x_2' \in \bar{S}_{02}. \tag{B.2}$$

Since ζ_1 and ζ_2 are Lipschitz functions, relations (B.2) imply that $x_1' = x_1'(x_2')$ and $x_2' = x_2'(x_1')$ are mutually inverse bi-Lipschitz mappings for $x_1' \in \overline{S}_{01}$, $x_2' \in \overline{S}_{02}$.

Assume now that $f_1 \in H^s(\Gamma_1)$ and $f_1 = 0$ on $\Gamma_1 \setminus \overline{\Gamma}_0$. Then $f_{\zeta_1} \in H^s(\mathbb{R}^{n-1})$ and $f_{\zeta_1} = 0$ on $\mathbb{R}^{n-1} \setminus \overline{S}_{01}$. Assume also that a function $f_2 \in L^2(\Gamma_2)$ is such that $f_2 = f_1$ on Γ_0 and $f_2 = 0$ on $\Gamma_2 \setminus \overline{\Gamma}_0$. From (B.1) we have for any $x_2' \in S_{02}$ that

$$f_{\zeta_2}(x_2') = f_{\zeta_1}(x_1'(x_2')), \tag{B.3}$$

where the Lipschitz map $x_1'(x_2')$ is defined for $x_2' \in S_{02}$ by (B.2). By the Kirszbraun theorem (cf., Lemma 1.29 and Theorem 1.31 in [57]) the map $x_1'(x_2')$ can be extended to all $x_2' \in \mathbb{R}^{n-1}$ with the same Lipschitz constant. Hence, taking into account that $f_{\zeta_1}(x_1') = 0$ for $x_1' \in \mathbb{R}^{n-1} \setminus \overline{S}_{01}$ and $f_{\zeta_2}(x_2') = 0$ for $x_2' \in \mathbb{R}^{n-1} \setminus \overline{S}_{02}$, we obtain that for such extension, relation (B.3) holds for almost any $x_2' \in \mathbb{R}^{n-1}$. Then, cf., e.g., Theorem 3.23 in [45], the inclusion $f_{\zeta_1} \in H^s(\mathbb{R}^{n-1})$ implies that $f_{\zeta_2} \in H^s(\mathbb{R}^{n-1})$ and thus $f_2 \in H^s(\Gamma_2)$.

(3) Assume now that Γ_1 and Γ_2 are bounded Lipschitz hyper-surfaces in $\mathbb{R}^n$ and arrange finite covers of both of them by open balls such that the intersections of each ball with the corresponding surface can be extended (possibly after some rigid rotations) to Lipschitz graphs. Moreover we choose the covers in such a way that the balls covering the closure of Γ_0 coincide for both surfaces. Arranging the subordinate partition of unity (see, e.g., [45, p. 98]) and employing item (2) for the balls intersecting the boundary of Γ_0 yield the asserted result. □

Let us show that the space $H^s_\bullet(\cdot)$ can be characterized as the weighted space $H^s_{00}(\cdot)$, whose counterpart on smooth domains in $\mathbb{R}^n$ was given in [42, Chapter 1, Theorem 11.7], see also Corollary 1.4.4.10 in [30] for Lipschitz domains.

Theorem B.3 *Let Γ be a $(n-1)$-dimensional Lipschitz graph or a bounded Lipschitz hyper-surface in $\mathbb{R}^n$, $n \geq 2$, and let Γ_0 be its relatively open $(n-1)$-dimensional subset with a $(n-2)$-dimensional Lipschitz boundary $\partial\Gamma_0$ if n>2; if n=2 $\partial\Gamma_0$ consists of two distinct points. Let $0 < s < 1$.*

Let $H^s_{00}(\Gamma_0)$ denote the space of all functions $\phi \in \mathring{H}^s(\Gamma_0)$, such that $\delta^{-s}\phi \in L^2(\Gamma_0)$, where $\delta(x)$ is the distance in $\mathbb{R}^n$ from x to the boundary $\partial\Gamma_0$.

Then the space $H^s_{00}(\Gamma_0)$ coincides with the space $H^s_\bullet(\Gamma_0)$, i.e., with the space of all functions from $H^s(\Gamma_0)$ such that their extensions by zero to Γ belong to $H^s(\Gamma)$.

Proof Let first Γ be graph of a Lipschitz function $x_n = \zeta(x') \in \mathbb{R}$, $x' \in \mathbb{R}^{n-1}$, and Γ_0 be the image of a domain $S_0 \subset \mathbb{R}^{n-1}$, i.e., $x_n = \zeta(x')$ for $x' \in S_0 \subset \mathbb{R}^{n-1}$. For all $x, \tilde{x} \in \overline{\Gamma}_0$, we have

$$|x' - \tilde{x}'| \leq |x - \tilde{x}| = \sqrt{|x' - \tilde{x}'|^2 + |\zeta(x') - \zeta(\tilde{x}')|^2} \leq \sqrt{1 + A^2}\, |x' - \tilde{x}'|. \tag{B.4}$$

where A is a finite Lipschitz constant of the function ζ on the domain S_0. The distance to the boundary is defined as

$$\delta(x) = \inf_{\tilde{x} \in \partial\Gamma_0} |x - \tilde{x}| = \inf_{\tilde{x}' \in \partial S_0} \sqrt{|x' - \tilde{x}'|^2 + |\zeta(x') - \zeta(\tilde{x}')|^2}, \tag{B.5}$$

Denoting $\delta'(x') = \inf_{\tilde{x}' \in \partial S_0} |x' - \tilde{x}'|$, we obtain from (B.5) and (B.4) that

$$\delta'(x') \leq \delta(x) \leq \sqrt{1 + A^2}\, \delta'(x'). \tag{B.6}$$

By the definition of the Sobolev spaces on Lipschitz graphs (see, e.g., [45, p. 98]), $\tilde{f} \in H^s(\Gamma)$, $0 < s < 1$, means that $\tilde{f}_\zeta \in H^s(\mathbb{R}^{n-1})$, where $f_\zeta(x') = f(x', \zeta(x'))$, $x' \in \mathbb{R}^{n-1}$ and $\phi \in \mathring{H}^s(\Gamma_0)$, $0 < s < 1$, means that $\phi_\zeta \in \mathring{H}^s(S_0)$.

Let $\phi \in H^s_{00}(\Gamma_0)$. Then $\phi \in \mathring{H}^s(\Gamma_0)$, $\delta^{-s}\phi \in L^2(\Gamma_0)$. The surface measure formula

$$d\sigma(x) = \sqrt{1 + |\text{grad}\, \zeta(x')|^2}dx' \tag{B.7}$$

(see, e.g. [45, Eq. (3.28)]) together with (B.6) implies that $\phi_\zeta \in \mathring{H}^s(S_0)$ and $(\delta')^{-s}\phi_\zeta \in L^2(S_0)$, where $\phi_\zeta(x') = \phi(x', \zeta(x'))$, $x' \in S_0$. Then by Corollary 1.4.4.10 in [30] we obtain that the extension of ϕ_ζ by zero from S_0 to $\mathbb{R}^{n-1}$ belongs to $H^s(\mathbb{R}^{n-1})$ and hence the extension of ϕ by zero from Γ_0 to Γ belongs to $H^s(\Gamma)$.

Conversely, let $\phi \in H^s(\Gamma_0)$ be such that its extension by zero from Γ_0 to Γ belongs to $H^s(\Gamma)$. Then $\phi_\zeta(x') = \phi(x', \zeta(x'))$ belongs to $H^s(S_0)$ and its extension by zero from to S_0 to $\mathbb{R}^{n-1}$ belongs to $H^s(\mathbb{R}^{n-1})$. Hence by Corollary 1.4.4.10 in [30] we obtain that $\phi_\zeta \in \mathring{H}^s(S_0)$ and $(\delta')^{-s}\phi_\zeta \in L^2(S_0)$, which by (B.6) and (B.7) implies that $\delta^{-s}\phi \in L^2(\Gamma_0)$ and thus $\phi \in H^s_{00}(\Gamma_0)$.

Let now Γ be a $(n-1)$-dimensional bounded Lipschitz hyper-surface. Let us arrange a finite cover of the hyper-surface by open balls such that, as usual, the intersections of each ball with the hyper-surface can be extended (maybe after corresponding rigid rotations) to a Lipschitz graph. Arranging the subordinate partition of unity (see, e.g., [45, p. 98]) and employing the above arguments to each of these graphs we obtain the asserted result. □

In addition, the following extension result holds (see also [55, p. 373]).

Lemma B.4 *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain satisfying* Assumption 5.1. *Then there exists a linear bounded extension operator $E_{\Omega^+\to\Omega}$ from the space $H^1_{\Gamma^+}(\Omega^+)^n$ to $\mathring{H}^1(\Omega)^n$.*

Proof Let $\mathbf{u}^+ \in H^1_{\Gamma^+}(\Omega^+)^n$. Let $\mathbf{u}_\Omega \in H^1_{\Gamma^+}(\Omega)^n$ be the function defined by $\mathbf{u}_\Omega := \mathcal{E}_{\Omega^+\to\Omega}\mathbf{u}^+$, where $\mathcal{E}_{\Omega^+\to\Omega} := r_\Omega \circ E_{\Omega^+\to\mathbb{R}^n}$, and $E_{\Omega^+\to\mathbb{R}^n}$ is the Rychkov extension operator from $H^1(\Omega^+)^n$ to $H^1(\mathbb{R}^n)^n$ (cf., e.g., [51, Theorem 2.4.1]). Thus, $r_{\Omega^+}(\mathbf{u}_\Omega) = \mathbf{u}^+$, where r_{Ω^+} denotes the restriction to Ω^+.

In addition, $(\gamma_\Omega \mathbf{u}_\Omega) \in \widetilde{H}^{\frac{1}{2}}(\Gamma^-; \partial\Omega)^n$ and $(\gamma_\Omega \mathbf{u}_\Omega)|_{\Gamma^-} \in H^{\frac{1}{2}}(\Gamma^-)^n$. Let $\mathring{E}_{\Gamma^-\to\Sigma}\left((\gamma_\Omega \mathbf{u}_\Omega)|_{\Gamma^-}\right)$ be the extension of $(\gamma_\Omega \mathbf{u}_\Omega)|_{\Gamma^-}$ by zero on Σ. Then Lemma B.2 (3) (applied to the functions $\mathbf{f}_1 = \gamma_\Omega \mathbf{u}_\Omega$ and $\mathbf{f}_2 = \mathring{E}_{\Gamma^-\to\Sigma}\left((\gamma_\Omega \mathbf{u}_\Omega)|_{\Gamma^-}\right)$, which are equal on Γ^- and vanish on Γ^+ and Σ, respectively) implies that $\mathring{E}_{\Gamma^-\to\Sigma}\left((\gamma_\Omega \mathbf{u})|_{\Gamma^-}\right) \in \widetilde{H}^{\frac{1}{2}}(\Gamma^-; \partial\Omega^-)^n$. Moreover, since the trace operator $\gamma_{\Omega^-} : H^1(\Omega^-) \to H^{\frac{1}{2}}(\partial\Omega^-)$ is onto, there exists $\mathbf{u}^- \in H^1_\Sigma(\Omega^-)^n$ such that

$$\mathbf{u}^- = \gamma^{-1}_{\Omega^-}\left(\mathring{E}_{\Gamma^-\to\Sigma}\left((\gamma_\Omega \mathbf{u}_\Omega)|_{\Gamma^-}\right)\right), \tag{B.8}$$

where $\gamma^{-1}_{\Omega^-} : H^{\frac{1}{2}}(\partial\Omega^-)^n \to H^1(\Omega^-)^n$ is a bounded linear right inverse of the trace operator $\gamma_{\Omega^-} : H^1(\Omega^-)^n \to H^{\frac{1}{2}}(\partial\Omega^-)^n$ (see [17], [46, Lemma 2.6], [51, Theorem 2.5.2]).

Thus, $\mathbf{u}^- \in H^1(\Omega^-)^n$, $(\gamma_{\Omega^-}\mathbf{u}^-)|_\Sigma = 0$, $\left(\gamma_{\Omega^-}\mathbf{u}^-\right)\big|_{\Gamma^-} = (\gamma_\Omega \mathbf{u}_\Omega)|_{\Gamma^-}$. Let $\mathbf{u}_0$ be the extension by zero of $\mathbf{u}^-$ in Ω^+, $\mathbf{u}_0 := \mathring{E}_{\Omega^-\to\Omega^+}\mathbf{u}^-$. Therefore, $\mathbf{u}_0 \in H^1(\Omega)^n$ and $\mathbf{u}_0 = 0$ in Ω^+. In addition, $(\gamma_\Omega \mathbf{u}_0)|_{\Gamma^-} = (\gamma_\Omega \mathbf{u}_\Omega)|_{\Gamma^-}$. Moreover,

$$\begin{aligned}\mathbf{u} := \mathbf{u}_\Omega - \mathbf{u}_0 &= \mathcal{E}_{\Omega^+\to\Omega}\mathbf{u}^+ - \mathring{E}_{\Omega^-\to\Omega^+}\mathbf{u}^- \\ &= \mathcal{E}_{\Omega^+\to\Omega}\mathbf{u}^+ - \mathring{E}_{\Omega^-\to\Omega^+}\left(\gamma^{-1}_{\Omega^-}\left(\mathring{E}_{\Gamma^-\to\Sigma}\left((\gamma_\Omega \mathbf{u}_\Omega)|_{\Gamma^-}\right)\right)\right).\end{aligned} \tag{B.9}$$

satisfies $\mathbf{u} \in H^1(\Omega)^n$ and, by construction, $(\gamma_\Omega \mathbf{u})|_{\Gamma^\pm} = 0$, i.e., $\gamma_\Omega \mathbf{u} = 0$ (a.e.) on Γ. Moreover, $\mathbf{u}|_{\Omega^+} = \mathbf{u}^+$ in Ω^+. Consequently, $\mathbf{u}$ is an extension of $\mathbf{u}^+$ from $H^1_{\Gamma^+}(\Omega^+)^n$ to $\mathring{H}^1(\Omega)^n$.

Finally, we define the extension operator $E_{\Omega^+\to\Omega} : H^1_{\Gamma^+}(\Omega^+)^n \to \mathring{H}^1(\Omega)^n$, such that $E_{\Omega^+\to\Omega}(\mathbf{u}^+) := \mathbf{u}$, where $\mathbf{u}$ has been constructed above. Consequently,

$$E_{\Omega^+\to\Omega}\mathbf{u}^+ := \mathcal{E}_{\Omega^+\to\Omega}\mathbf{u}^+ - \mathring{E}_{\Omega^-\to\Omega^+}\left(\gamma_{\Omega^-}^{-1}\left(\mathring{E}_{\Gamma^-\to\Sigma}\left(\gamma_\Omega\left(\mathcal{E}_{\Omega^+\to\Omega}\mathbf{u}^+\right)|_{\Gamma^-}\right)\right)\right). \tag{B.10}$$

An alternative construction of such an extension map can be consulted in [55, pp. 373, 374]. □

Let us introduce the space (cf., e.g., [45, p. 76]),

$$H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n = \{\boldsymbol{\Phi} \in H^{-1}(\mathbb{R}^n)^n : \operatorname{supp}\boldsymbol{\Phi} \subseteq \overline{\Gamma^+}\}. \tag{B.11}$$

Note that, $\boldsymbol{\Phi} \in H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n$ if and only if $\boldsymbol{\Phi} = \gamma^*_\Omega \boldsymbol{\phi}$, i.e.,

$$\langle \mathbf{v}, \boldsymbol{\Phi}\rangle_\Omega = \langle \gamma_\Omega \mathbf{v}, \boldsymbol{\phi}\rangle_{\partial\Omega} \quad \forall\, \mathbf{v} \in H^1(\Omega)^n \tag{B.12}$$

for some $\boldsymbol{\phi} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma^+)^n$, which is uniquely defined by $\boldsymbol{\Phi}$, cf. [46, Theorem 2.10(ii)].

Lemma B.5 *The dual* $\left(H^1_{\Gamma^+}(\Omega)^n\right)'$ *of the space* $H^1_{\Gamma^+}(\Omega)^n$ *can be identified with the space* $\widetilde{H}^{-1}(\Omega)^n / H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n$.

Proof First, we remark that due to (B.12), the space $H^1_{\Gamma^+}(\Omega^+)^n$ defined as in (5.2) can be also equivalently defined as

$$\begin{aligned} H^1_{\Gamma^+}(\Omega)^n &= \left\{\mathbf{v} \in H^1(\Omega)^n : \gamma_\Omega \mathbf{v} = 0 \text{ on } \Gamma^+\right\} \\ &= \left\{\mathbf{v} \in H^1(\Omega)^n : \langle \gamma_\Omega \mathbf{v}, \boldsymbol{\phi}\rangle_{\partial\Omega} = 0\,,\ \forall\, \boldsymbol{\phi} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma^+)^n\right\} \\ &= \left\{\mathbf{v} \in H^1(\Omega)^n : \langle \mathbf{v}, \boldsymbol{\Phi}\rangle_\Omega = 0\,,\ \forall\, \boldsymbol{\Phi} \in H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n\right\} =: H^1(\Omega)^n \perp H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n. \end{aligned}$$

Then a duality argument (see, e.g., [53, Sections 4.8, 4.9]) yields the required identification. □

Two more identifications are proved in the following assertion.

Lemma B.6 *The dual* $\left(H^1_{\Gamma^+}(\Omega^+)^n\right)'$ *of the space* $H^1_{\Gamma^+}(\Omega^+)^n$ *can be identified with the space* $\widetilde{H}^{-1}(\Omega^+)^n / H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n$ *and also with the space*

$$\left\{\boldsymbol{\varphi} \in H^{-1}(\Omega)^n : \boldsymbol{\varphi} = \mathbf{0} \text{ on } \Omega^-\right\}. \tag{B.13}$$

Proof The identification with $\widetilde{H}^{-1}(\Omega^+)^n / H^{-1}_{\overline{\Gamma^+}}(\mathbb{R}^n)^n$ follows from Lemma B.5 applied to Ω^+.

To prove the identification with the space in (B.13), assume first that $\boldsymbol{\varphi}$ belongs to the space defined in (B.13). Thus,

$$\boldsymbol{\varphi} \in H^{-1}(\Omega)^n = \left(\mathring{H}^1(\Omega)^n\right)' \text{ and } \langle \boldsymbol{\varphi}, \boldsymbol{\psi}^-\rangle_\Omega = \langle \boldsymbol{\varphi}, \boldsymbol{\psi}^-\rangle_{\Omega^-} = 0, \quad \forall\, \boldsymbol{\psi}^- \in \mathcal{D}(\Omega^-)^n\,, \tag{B.14}$$

i.e., $\boldsymbol{\varphi}$ has the support in $\overline{\Omega^+}$ ($\boldsymbol{\varphi}=\mathbf{0}$ on Ω^-). We have used the equivalent description of the space $\mathring{H}^1(\Omega)^n$ given in formula (3.7) and the identification of $\mathring{H}^1(\Omega)^n$ and $\widetilde{H}^1(\Omega)^n$.

Note that the functional $\boldsymbol{\varphi}:\mathring{H}^1(\Omega)^n\to\mathbb{R}$ is linear and bounded. Let $E_{\Omega^+\to\Omega}$ be a linear bounded extension operator from $H^1_{\Gamma^+}(\Omega^+)^n$ to $\mathring{H}^1(\Omega)^n$, which exists in view of Lemma B.4. Therefore, the functional

$$\boldsymbol{\phi}:=\boldsymbol{\varphi}\circ E_{\Omega^+\to\Omega}:H^1_{\Gamma^+}(\Omega^+)^n\to\mathbb{R},\quad \boldsymbol{\phi}(\boldsymbol{\psi}^+):=\boldsymbol{\varphi}\left(E_{\Omega^+\to\Omega}\boldsymbol{\psi}^+\right),\quad \forall\,\boldsymbol{\psi}^+\in H^1_{\Gamma^+}(\Omega^+)^n$$

is linear and bounded as well. Hence, we have that

$$\boldsymbol{\phi}(\boldsymbol{\psi}^+)=\boldsymbol{\varphi}(\boldsymbol{\psi}),\quad \forall\,\boldsymbol{\psi}^+\in H^1_{\Gamma^+}(\Omega^+)^n,\ \text{where } \boldsymbol{\psi}=E_{\Omega^+\to\Omega}(\boldsymbol{\psi}^+)\in\mathring{H}^1(\Omega)^n.$$

This definition agrees well with the condition that $\boldsymbol{\varphi}(\boldsymbol{\psi}^-)=\mathbf{0}$ for any $\boldsymbol{\psi}^-\in\mathcal{D}(\Omega^-)^n$, and shows that for a fixed $E_{\Omega^+\to\Omega}$, any functional $\boldsymbol{\varphi}$ from the space (B.13) can be identified with a functional $\boldsymbol{\phi}\in\left(H^1_{\Gamma^+}(\Omega^+)^n\right)'$.

To prove that the functional $\boldsymbol{\phi}$ does not depend on the extension operator $E_{\Omega^+\to\Omega}$, let us consider two such extension operators, $E'_{\Omega^+\to\Omega}$ and $E''_{\Omega^+\to\Omega}$ from $H^1_{\Gamma^+}(\Omega^+)^n$ to $\mathring{H}^1(\Omega)^n$, generating for a fixed $\boldsymbol{\varphi}$ two functionals, $\boldsymbol{\phi}':=\boldsymbol{\varphi}\circ E'_{\Omega^+\to\Omega}$ and $\boldsymbol{\phi}'':=\boldsymbol{\varphi}\circ E''_{\Omega^+\to\Omega}$. Then

$$\begin{aligned}\boldsymbol{\phi}'(\boldsymbol{\psi}^+)-\boldsymbol{\phi}''(\boldsymbol{\psi}^+)&=\boldsymbol{\varphi}\left(E'_{\Omega^+\to\Omega}\boldsymbol{\psi}^+\right)-\boldsymbol{\varphi}\left(E''_{\Omega^+\to\Omega}\boldsymbol{\psi}^+\right)\\&=\boldsymbol{\varphi}\left(E'_{\Omega^+\to\Omega}\boldsymbol{\psi}^+-E''_{\Omega^+\to\Omega}\boldsymbol{\psi}^+\right),\quad\forall\,\boldsymbol{\psi}^+\in H^1_{\Gamma^+}(\Omega^+)^n.\end{aligned}$$

Denoting $\boldsymbol{\psi}_0:=E'_{\Omega^+\to\Omega}\boldsymbol{\psi}^+-E''_{\Omega^+\to\Omega}\boldsymbol{\psi}^+$, we obtain that $\boldsymbol{\psi}_0\in\mathring{H}^1(\Omega)^n$ and $\boldsymbol{\psi}_0=\mathbf{0}$ in Ω^+, implying that $r_{\Omega^-}\boldsymbol{\psi}_0\in\mathring{H}^1(\Omega^-)^n$ and hence $\gamma_{\Omega^-}\boldsymbol{\psi}_0=\mathbf{0}$. Thus, there exist $\boldsymbol{\Psi}_0\in\widetilde{H}^1(\Omega^-)^n\subset\mathring{H}^1(\Omega)^n\subset H^1(\mathbb{R}^n)^n$ such that $\boldsymbol{\psi}_0=r_\Omega\boldsymbol{\Psi}_0$ and a sequence $\{\boldsymbol{\Psi}_i\}_{i\in\mathbb{N}}\subseteq\mathcal{D}(\Omega^-)^n$ converging to $\boldsymbol{\Psi}_0$ in $\widetilde{H}^1(\Omega^-)^n$. By (B.14) then $\boldsymbol{\varphi}(\boldsymbol{\psi}_0)=\boldsymbol{\varphi}(r_\Omega\boldsymbol{\Psi}_0)=\lim_{i\to\infty}\boldsymbol{\varphi}(r_\Omega\boldsymbol{\Psi}_i)=\lim_{i\to\infty}\boldsymbol{\varphi}(\boldsymbol{\Psi}_i)=0$, and, hence, the asserted independence property follows.

Conversely, assume that $\boldsymbol{\phi}\in\left(H^1_{\Gamma^+}(\Omega^+)^n\right)'$ and let $r_{\Omega\to\Omega^+}$ be the restriction operator from the space $\mathring{H}^1(\Omega)^n$ to $H^1_{\Gamma^+}(\Omega^+)^n$. Then the functional $\boldsymbol{\varphi}:=\boldsymbol{\phi}\circ r_{\Omega\to\Omega^+}:\mathring{H}^1(\Omega)^n\to\mathbb{R}$ is linear and bounded, i.e., $\boldsymbol{\varphi}\in H^{-1}(\Omega)^n$. In addition, for any $\boldsymbol{\psi}^-\in\mathcal{D}(\Omega^-)^n$, we have that $\boldsymbol{\psi}^-\in\mathring{H}^1(\Omega)^n$, and accordingly that $\boldsymbol{\varphi}\left(\boldsymbol{\psi}^-\right)=\boldsymbol{\phi}(\mathbf{0})=0$, where the last equality is provided by the linearity of the functional $\boldsymbol{\phi}:H^1_{\Gamma^+}(\Omega^+)^n\to\mathbb{R}$. Consequently, $\boldsymbol{\varphi}$ belongs to the space defined in (B.13).

For any $\boldsymbol{\psi}\in\mathring{H}^1(\Omega)^n$, $\boldsymbol{\varphi}$ from the space (B.13), and $\boldsymbol{\phi}=\boldsymbol{\varphi}\circ E_{\Omega^+\to\Omega}$, we have,

$$\begin{aligned}\boldsymbol{\varphi}(\boldsymbol{\psi})-\boldsymbol{\phi}\circ r_{\Omega\to\Omega^+}(\boldsymbol{\psi})&=\boldsymbol{\varphi}(\boldsymbol{\psi})-\varphi(E_{\Omega^+\to\Omega}\circ r_{\Omega\to\Omega^+}\boldsymbol{\psi})\\&=\boldsymbol{\varphi}(\boldsymbol{\psi}-E_{\Omega^+\to\Omega}\circ r_{\Omega\to\Omega^+}\boldsymbol{\psi})=0,\end{aligned}$$

because $\tilde{\boldsymbol{\psi}}:=\boldsymbol{\psi}-E_{\omega^+\to\Omega}\circ r_{\Omega\to\Omega^+}=0$ in Ω^+. Hence $r_{\Omega^-}\tilde{\boldsymbol{\psi}}\in\mathring{H}^1(\Omega^-)^n$ and thus this function can be approximated by functions from $\mathcal{D}(\Omega^-)^n$.

On the other hand, for any $\boldsymbol{\psi}^+\in H^1_{\Gamma^+}(\Omega^+)^n$, $\boldsymbol{\phi}\in(H^1_{\Gamma^+}(\Omega^+)^n)'$, and $\boldsymbol{\varphi}:=\boldsymbol{\phi}\circ r_{\Omega\to\Omega^+}$, we have,

$$\boldsymbol{\phi}(\boldsymbol{\psi}^+)-\boldsymbol{\varphi}(E_{\Omega^+\to\Omega}\boldsymbol{\psi}^+)=\boldsymbol{\phi}(\boldsymbol{\psi}^+)-\boldsymbol{\phi}\circ r_{\Omega^+\to\Omega^+}(E_{\Omega^+\to\Omega}\boldsymbol{\psi}^+)=\boldsymbol{\phi}(\boldsymbol{\psi}^+)-\boldsymbol{\phi}(\boldsymbol{\psi}^+)=0.$$

This implies that $\boldsymbol{\phi}$ can be identified with $\boldsymbol{\varphi}$ through the relations $\boldsymbol{\varphi}:=\boldsymbol{\phi}\circ r_{\Omega\to\Omega^+}:\mathring{H}^1(\Omega)^n\to\mathbb{R}$ and $\boldsymbol{\phi}=\boldsymbol{\varphi}\circ E_{\Omega^+\to\Omega}:H^1_{\Omega^+}(\Omega^+)^n\to\mathbb{R}$. □

C Useful norm estimates

In this appendix we provide several estimates, embeddings, and identities (some of them well known), used in the analysis of the Navier-Stokes problems. Let Ω denote a bounded Lipschitz domain in $\mathbb{R}^n$, $n \in \{2, 3\}$. Let $\mathring{E}_\Omega$ be the extension by zero operator from Ω to $\mathbb{R}^n$.

• By the Sobolev embedding theorem (see, e.g., [1, Theorem 6.3]), the space $H^1(\Omega)^n$ is compactly embedded in $L^4(\Omega)^n$ and there exists a constant $c_1 = c_1(\Omega, n) > 0$ such that

$$\|\mathbf{v}\|_{L^4(\Omega)^n} \leq c_1 \|\mathbf{v}\|_{H^1(\Omega)^n}\,, \quad \forall\, \mathbf{v} \in H^1(\Omega)^n\,. \tag{C.1}$$

Due to the equivalence in $\mathring{H}^1(\Omega)^n$ of the semi-norm $\|\nabla(\cdot)\|_{L^2(\Omega)^{n\times n}}$ with the norm $\|\cdot\|_{H^1(\Omega)^n}$ given by (3.3), estimate (C.1) also implies

$$\|\mathbf{v}\|_{L^4(\Omega)^n} \leq c_0 \|\nabla \mathbf{v}\|_{L^2(\Omega)^{n\times n}}\,, \quad \forall\, \mathbf{v} \in \mathring{H}^1(\Omega)^n, \tag{C.2}$$

with some constant $c_0 = c_0(\Omega, n) > 0$.

• By the Hölder inequality, we obtain for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in H^1(\Omega)^n$,

$$\begin{aligned} |\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_\Omega| &\leq \|\mathbf{v}_1\|_{L^4(\Omega)^n} \|\mathbf{v}_3\|_{L^4(\Omega)^n} \|\nabla \mathbf{v}_2\|_{L^2(\Omega)^{n\times n}} \\ &\leq c_1 \|\mathbf{v}_1\|_{L^4(\Omega)^n} \|\mathbf{v}_3\|_{H^1(\Omega)^n} \|\nabla \mathbf{v}_2\|_{L^2(\Omega)^{n\times n}}. \end{aligned} \tag{C.3}$$

This also means that for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in H^1(\Omega)^n$,

$$\begin{aligned} \left|\left\langle \mathring{E}_\Omega[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2], \mathbf{v}_3 \right\rangle_\Omega\right| &\leq \| \, |\mathring{E}_\Omega \mathbf{v}_1| \, |\mathbf{V}_3| \, \|_{L^2(\mathbb{R}^n)} \|\mathring{E}_\Omega \nabla \mathbf{v}_2\|_{L^2(\mathbb{R}^n)^{n\times n}} \\ &\leq \|\mathbf{v}_1\|_{L^4(\Omega)^n} \|\mathbf{v}_3\|_{L^4(\Omega)^n} \|\nabla \mathbf{v}_2\|_{L^2(\Omega)^{n\times n}} \\ &\leq c_1 \|\mathbf{v}_1\|_{L^4(\Omega)^n} \|\mathbf{v}_3\|_{H^1(\Omega)^n} \|\nabla \mathbf{v}_2\|_{L^2(\Omega)^{n\times n}}, \end{aligned} \tag{C.4}$$

where $\mathbf{V}_3 \in H^1(\mathbb{R}^n)^n$ is such that $r_\Omega \mathbf{V}_3 = \mathbf{v}_3$. This implies that $\mathring{E}_\Omega[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2]$ belongs to the space $\widetilde{H}^{-1}(\Omega)^n = \big(H^1(\Omega)^n\big)'$. Moreover, in view of (C.1), for all $\mathbf{v}_1, \mathbf{v}_2 \in H^1(\Omega)^n$,

$$\left\| \mathring{E}_\Omega[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2] \right\|_{\widetilde{H}^{-1}(\Omega)^n} \leq c_1^2 \|\mathbf{v}_1\|_{H^1(\Omega)^n} \|\mathbf{v}_2\|_{H^1(\Omega)^n}. \tag{C.5}$$

Taking $\mathbf{v}_3 \in \mathring{H}^1(\Omega)^n$ in (C.3), it follows that the term $(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2$ belongs to the dual of the space $\mathring{H}^1(\Omega)^n$, that is, to the space $H^{-1}(\Omega)^n$ and for all $\mathbf{v}_1, \mathbf{v}_2 \in H^1(\Omega)^n$,

$$\begin{aligned} \|(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2\|_{H^{-1}(\Omega)^n} &\leq c_1 \|\mathbf{v}_1\|_{L^4(\Omega)^n} \|\nabla \mathbf{v}_2\|_{L^2(\Omega)^{n\times n}} \\ &\leq c_1 \|\mathbf{v}_1\|_{L^4(\Omega)^n} \|\mathbf{v}_2\|_{H^1(\Omega)^n} \leq c_1^2 \|\mathbf{v}_1\|_{H^1(\Omega)^n} \|\mathbf{v}_2\|_{H^1(\Omega)^n}. \end{aligned} \tag{C.6}$$

• The dense embedding of the space $\mathcal{D}(\overline{\Omega})^n$ into $H^1(\Omega)^n$, the divergence theorem and estimate (C.6) imply the following identity for any $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in H^1(\Omega)^n$

$$\begin{aligned} \langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_\Omega &= \int_\Omega \nabla \cdot (\mathbf{v}_1 (\mathbf{v}_2 \cdot \mathbf{v}_3))\, d\mathbf{x} - \langle (\nabla \cdot \mathbf{v}_1)\mathbf{v}_3 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_\Omega \\ &= \langle \gamma_\Omega \mathbf{v}_1 \cdot \boldsymbol{\nu}, \gamma_\Omega \mathbf{v}_2 \cdot \gamma_\Omega \mathbf{v}_3\rangle_{\partial\Omega} - \langle (\nabla \cdot \mathbf{v}_1)\mathbf{v}_3 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_\Omega\,, \end{aligned} \tag{C.7}$$

where $\boldsymbol{\nu}$ is the normal vector on $\partial\Omega$ directed outward Ω.

To obtain an alternative versions of estimate (C.4), which does not involve $\|\nabla \mathbf{v}_2\|_{L^2(\Omega)^{n\times n}}$, let use (C.7) and take into account that $\gamma_\Omega \mathbf{v}_1, \gamma_\Omega \mathbf{v}_2, \gamma_\Omega \mathbf{v}_3 \in H^{1/2}(\partial\Omega)^n$. Further, we employ that for the Lipschitz domain $\Omega \in \mathbb{R}^n$, $n = 2, 3$, the space $H^{1/2}(\partial\Omega)^n$ is continuously embedded in $L^3(\partial\Omega)^n$ (e.g., by the embeddings in [54, Section 2.2.4, Corollary 2(i)] for $\mathbb{R}^{n-1}$ that can be extended to Lipschitz surfaces by standard arguments, cf. also a more

general statement in [50, Proposition 3.8]). Thus, there exists a constant $c_2 = c_2(\partial\Omega, n)$, such that

$$\|\boldsymbol{\phi}\|_{L^3(\partial\Omega)^n} \leq c_2 \|\boldsymbol{\phi}\|_{H^{1/2}(\partial\Omega)^n}, \quad \forall\, \boldsymbol{\phi} \in H^{1/2}(\partial\Omega)^n. \tag{C.8}$$

Hence by the Hölder inequality we have for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in H^1(\Omega)^n$,

$$\begin{aligned} \left|\langle \gamma_\Omega \mathbf{v}_1 \cdot \boldsymbol{\nu}, \gamma_\Omega \mathbf{v}_2 \cdot \gamma_\Omega \mathbf{v}_3 \rangle_{\partial\Omega}\right| &\leq \| \, |\gamma_\Omega \mathbf{v}_1| \, |\gamma_\Omega \mathbf{v}_2| \, \|_{L^{3/2}(\partial\Omega)} \|\gamma_\Omega \mathbf{v}_3\|_{L^3(\partial\Omega)^n} \\ &\leq c_2^2 \|\gamma_\Omega \mathbf{v}_1\|_{H^{1/2}(\partial\Omega)^n} \|\gamma_\Omega \mathbf{v}_2\|_{L^3(\partial\Omega)^n} \|\gamma_\Omega \mathbf{v}_3\|_{H^{1/2}(\partial\Omega)^n}. \end{aligned} \tag{C.9}$$

Then from (C.7) and (C.9) we obtain for any $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in H^1(\Omega)^n$

$$\begin{aligned} |\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3 \rangle_\Omega| &\leq c_2^2 \|\gamma_\Omega\|^2 \|\mathbf{v}_1\|_{H^1(\Omega)^n} \|\gamma_\Omega \mathbf{v}_2\|_{L^3(\partial\Omega)^n} \|\mathbf{v}_3\|_{H^1(\Omega)^n} \\ &\quad + 2c_1 \|\mathbf{v}_1\|_{H^1(\Omega)^n} \|\mathbf{v}_2\|_{L^4(\Omega)^n} \|\mathbf{v}_3\|_{H^1(\Omega)^n} \\ &= \|\mathbf{v}_1\|_{H^1(\Omega)^n} (c_2^2 \|\gamma_\Omega\|^2 \|\gamma_\Omega \mathbf{v}_2\|_{L^3(\partial\Omega)^n} + 2c_1 \|\mathbf{v}_2\|_{L^4(\Omega)^n}) \|\mathbf{v}_3\|_{H^1(\Omega)^n}, \end{aligned} \tag{C.10}$$

where $\|\gamma_\Omega\| := \|\gamma_\Omega\|_{H^1(\Omega)^n \to H^{1/2}(\partial\Omega)^n}$ is the norm of the trace operator.

• For $\mathbf{v}_3 \in \mathring{H}^1(\Omega)^n$, (C.7) simplifies to

$$\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3 \rangle_\Omega = -\langle (\nabla \cdot \mathbf{v}_1)\mathbf{v}_3 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_3, \mathbf{v}_2 \rangle_\Omega \,, \ \forall\, \mathbf{v}_1, \mathbf{v}_2 \in H^1(\Omega)^n, \ \mathbf{v}_3 \in \mathring{H}^1(\Omega)^n, \tag{C.11}$$

In view of (C.11) we also obtain the identity

$$\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3 \rangle_\Omega = -\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_3, \mathbf{v}_2 \rangle_\Omega \,, \ \forall\, \mathbf{v}_1 \in H^1_{\mathrm{div}}(\Omega)^n, \ \mathbf{v}_2 \in H^1(\Omega)^n, \ \mathbf{v}_3 \in \mathring{H}^1(\Omega)^n, \tag{C.12}$$

and hence the well known formula

$$\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_2 \rangle_\Omega = 0, \ \forall\, \mathbf{v}_1 \in H^1_{\mathrm{div}}(\Omega)^n, \ \mathbf{v}_2 \in \mathring{H}^1(\Omega)^n. \tag{C.13}$$

Arguments similar to those for (C.3) and identity (C.12) imply the estimate

$$|\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3 \rangle_\Omega| = |\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_3, \mathbf{v}_2 \rangle_\Omega| \leq \|\mathbf{v}_1\|_{L^4(\Omega)^n} \|\mathbf{v}_2\|_{L^4(\Omega)^n} \|\nabla \mathbf{v}_3\|_{L^2(\Omega)^{n\times n}} \tag{C.14}$$

for all $\mathbf{v}_1 \in H^1_{\mathrm{div}}(\Omega)^n$, $\mathbf{v}_2 \in H^1(\Omega)^n$, $\mathbf{v}_3 \in \mathring{H}^1(\Omega)^n$. Therefore,

$$\begin{aligned} \|(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2\|_{H^{-1}(\Omega)^n} &\leq \|(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2\|_{H^{-1}(\Omega)^n} \leq \|\mathbf{v}_1\|_{L^4(\Omega)^n} \|\mathbf{v}_2\|_{L^4(\Omega)^n} \\ &\leq c_1 \|\mathbf{v}_1\|_{H^1(\Omega)^n} \|\mathbf{v}_2\|_{L^4(\Omega)^n} \forall\, \mathbf{v}_1 \in H^1_{\mathrm{div}}(\Omega)^n, \ \mathbf{v}_2 \in H^1(\Omega)^n. \end{aligned} \tag{C.15}$$

• Let now Assumption 5.1 hold and Ω' be either Ω^+ or Ω^-. Similar to (C.4) we have for all $\mathbf{v}_1, \mathbf{v}_2 \in H^1(\Omega')^n$ and $\mathbf{v}_3 \in H^1(\Omega)^n$ that

$$\begin{aligned} \left|\left\langle \mathring{E}_{\Omega'}[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2], \mathbf{v}_3 \right\rangle_\Omega\right| &= |\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3 \rangle_{\Omega'}| \\ &\leq c_1' \|\mathbf{v}_1\|_{L^4(\Omega')^n} \|\mathbf{v}_3\|_{H^1(\Omega)^n} \|\nabla \mathbf{v}_2\|_{L^2(\Omega')^{n\times n}}, \end{aligned} \tag{C.16}$$

where $c_1' = c_1'(\Omega', n)$, cf. (C.1). Taking $\mathbf{v}_3 \in \mathring{H}^1(\Omega)^n$ in (C.16), we find that $r_\Omega \mathring{E}_{\Omega'}[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2]$ belongs to $H^{-1}(\Omega)^n$ and

$$\begin{aligned} \left\| \mathring{E}_{\Omega'}[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2] \right\|_{H^{-1}(\Omega)^n} &\leq c_1' \|\mathbf{v}_1\|_{L^4(\Omega')^n} \|\mathbf{v}_2\|_{H^1(\Omega')^n} \\ &\leq (c_1')^2 \|\mathbf{v}_1\|_{H^1(\Omega')^n} \|\mathbf{v}_2\|_{H^1(\Omega')^n}, \ \forall\, \mathbf{v}_1, \mathbf{v}_2 \in H^1(\Omega')^n. \end{aligned} \tag{C.17}$$

If, moreover, $\mathbf{v}_1 \in H^1(\Omega)^n$, then (C.17) implies

$$\left\| \mathring{E}_{\Omega'}[(r_{\Omega'}\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2] \right\|_{H^{-1}(\Omega)^n} \leq c_1' \|\mathbf{v}_1\|_{L^4(\Omega')^n} \|\mathbf{v}_2\|_{H^1(\Omega')^n}$$
$$\leq (c_1')^2 \|\mathbf{v}_1\|_{H^1(\Omega)^n} \|\mathbf{v}_2\|_{H^1(\Omega')^n}, \ \forall\, \mathbf{v}_1 \in H^1(\Omega)^n, \ \mathbf{v}_2 \in H^1(\Omega')^n. \tag{C.18}$$

• Let again Assumption 5.1 hold. In order to obtain some alternative versions of estimates (C.16) and (C.17), which do not involve $\|\nabla \mathbf{v}_2\|_{L^2(\Omega')^{n\times n}}$, we implement (C.10) for Ω' and find that for all $\mathbf{v}_1, \mathbf{v}_2 \in H^1(\Omega')^n$ and $\mathbf{v}_3 \in H^1(\Omega)^n$

$$\left| \left\langle \mathring{E}_{\Omega'}[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2], \mathbf{v}_3 \right\rangle_{\Omega} \right| = |\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3 \rangle_{\Omega'}|$$
$$\leq \|\mathbf{v}_1\|_{H^1(\Omega')^n} (c_2'^2 \|\gamma_{\Omega'}\|^2 \|\gamma_{\Omega'}\mathbf{v}_2\|_{L^3(\partial\Omega')^n} + 2c_1' \|\mathbf{v}_2\|_{L^4(\Omega')^n}) \|\mathbf{v}_3\|_{H^1(\Omega)^n}, \tag{C.19}$$

where $c_2' = c_2(\partial\Omega', n)$. If we take $\mathbf{v}_3 \in \mathring{H}^1(\Omega)^n$, then (C.19) implies

$$\left\| \mathring{E}_{\Omega'}[(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2] \right\|_{H^{-1}(\Omega)^n} \leq \|\mathbf{v}_1\|_{H^1(\Omega')^n} \Big(c_2'^2 \|\gamma_{\Omega'}\|^2 \|\gamma_{\Omega'}\mathbf{v}_2\|_{L^3(\partial\Omega')^n}$$
$$+ 2c_1' \|\mathbf{v}_2\|_{L^4(\Omega')^n} \Big), \quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in H^1(\Omega')^n. \tag{C.20}$$

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