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# A comparative analysis of several nonlocal fracture criteria

**L. P. Isupov, S. E. Mikhailov**



**Summary** Comparative analysis has been carried out for three nonlocal fracture criteria (NLFC) in application to plane problems: the average stress fracture criterion (ASFC), the minimum stress fracture criterion (MSFC) and the fictitious crack fracture criterion (FCFC). Each of them may be considered as an equality for a particular form of the general nonlocal strength functional. The criteria contain two material parameters: a characteristic length and the tensile strength (ASFC and MSFC) or the critical stress intensity factor (FCFC).

The criteria have been used for a strength description of a plate containing a smooth stress concentrator (circular hole) or a singular stress concentrator (central straight crack). It has been ascertained that ASFC and FCFC lead to identical results for the symmetrically loaded central straight crack. ASFC and MSFC may be successfully used for the description of strength of bodies with smooth as well as singular concentrators, while FCFC gives incorrect predictions for large smooth concentrators and for some other cases. A comparison of the predicted and experimental data has shown that ASFC is preferable in most cases; nevertheless, there exists a systematic deviation of experimental points from the criterion predictions.

**Key words** Fracture mechanics, strength, nonlocal criteria, stress concentrator

## 1
## Introduction

In the traditional local approach, the strength of a body in an analyzed point $y$ is characterized by the value of some function of stress tensor components at the same point, without consideration of the stress state in other points. A local fracture criterion can be represented, e.g. in the form

$$f(\sigma_{ij}(y)) = \sigma_c \quad ,$$

where $f$ is a material function and $\sigma_c$ is a material constant. Such criteria give a good description of experimental data when the stress distribution is close to a uniform stress state.

There are several problems of strength and fracture mechanics that can not be solved or are tedious to solve, by use of traditional strength conditions. Such problems relate to small-scale effects, singular stress concentrators like corner points, intersection of interfaces generating

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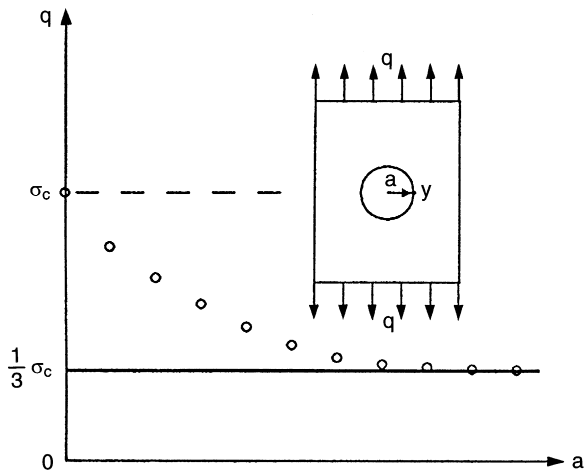


**Fig. 1.** Dependence of the plate strength $q$ on the hole radius $a$

singularities with different exponents, unification of strength conditions for bodies with smooth and singular concentrators. Some examples of these problems are given in Fig. 1–3.

An example of the small-scale effect on strength is presented in Fig. 1. An infinite elastic plate with a circular hole is considered, which is loaded at the infinity by a uniform traction $q$. It is known from the elasticity theory that the maximum stress is independent of the hole radius $a$, is equal to $3q$, and is realized in the boundary point $y$. The plate strength evaluated by use of the fracture criterion $\sigma_{\theta\theta} = \sigma_c$ is then independent of the hole radius and is equal to one third of the strength $\sigma_c$ of the plate without a hole (the solid line). However, appropriate fracture test data for plates with small holes, e.g. [1–3], pointed out schematically in the figure show that the plate strength depends actually on the hole radius.

Another example of the strength small-scale effect is delivered by a plate with a crack having a length $2a$, Fig. 2. The linear elasticity yields the value for the stress intensity factor $K_1(y) = q\sqrt{\pi a}$. From the linear fracture mechanics one has the fracture criterion $K_1(y) = K_{1c}$,

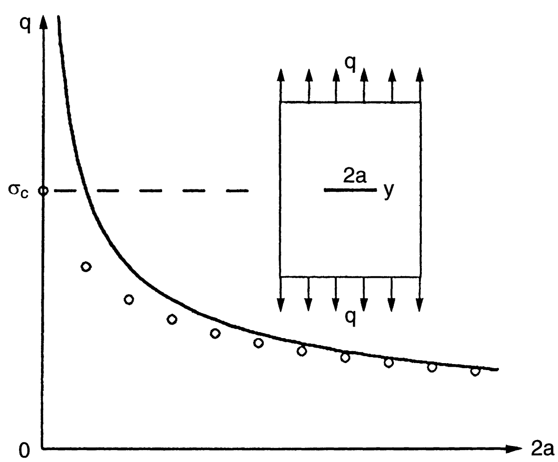


**Fig. 2.** Dependence of the plate strength $q$ on the crack length $a$

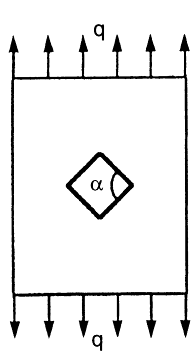


**Fig. 3.** Plate with a diamond-shaped hole

where $K_{1c}$ is a material constant-fracture toughness. Using these two expressions, we get the theoretical dependence of the plate strength from the crack length (the solid line), according to which the plate strength tends to infinity as the crack length tends to zero. But the experiments for short cracks, e.g. [2, 4, 5] show that strength tends to a finite value.

The same plate, but now with a diamond-shaped hole, Fig. 3, delivers an example of a singular concentrator with a not square-root singularity. According to the linear elasticity, the stress behavior near the corner points is $\sigma_{ij}(\rho,\theta) \sim K_1(y;\alpha)\rho^{-\gamma(\alpha)}f_{ij}(\theta;\alpha)$, where the exponent $\gamma$ depends only on the angle $\alpha$. It is not possible to estimate the strength of a body with such stress behavior either by the traditional local strength condition or by the local linear fracture mechanics condition. In principle, one can try to use the strength condition $K_1(y;\gamma) = K_{1c}(\gamma)$, which is analogous to the linear fracture mechanics condition. However, one must then determine the critical strength intensity factor $K_{1c}$ experimentally for each $\gamma$, that is, for each angle $\alpha$, what is rather tedious and expensive. Moreover, the same difficulties occur with the small-scale effects as for short cracks or small circular holes. 

These three examples show the necessity of a more general strength theory. Such a theory should describe the small-scale effects and be applicable to bodies without and with cracks as well as with other singular concentrators. These conditions are met by non-local strength theories.

A functional approach to nonlocal strength conditions and fracture criteria was suggested in [6, 7, 8], where a general form of nonlocal strength condition based on a nonlinear space strength functional was proposed. The strength functional or the functional safety factor is associated with the supremum of a positive factor, by which a given stress field may be multiplied to obtain a nonfracturing stress field.

There is a number of nonlocal fracture criteria proposed earlier, which can be considered as particular cases of the general strength functional form. The objective of this work is a comparative analysis of some of these nonlocal fracture criteria on the basis of experimental data for the bodies containing smooth and singular concentrators.

## 2
## Three nonlocal fracture criteria for plane problems

Three most popular nonlocal fracture criteria for the two-dimensional case are represented in some generalized forms in [6]. Let $(\rho,\theta)$ be a local polar coordinate system with the center at an analyzed point $y$ of a body; let $\eta(\theta)$ be a unit vector making an angle $\theta$ with the coordinate axis and let $\sigma_{\rho\rho}, \sigma_{\theta\theta}, \sigma_{\rho\theta}$ be the stress components in this coordinate system.

### 2.1
### Fracture criterion based on average stress over a characteristic length (ASFC)

This approach was considered by Neuber [9], Novozhilov [10], and other author e.g. [2]. It can be written in the following generalized form:

$$\frac{1}{d_1}\max_{-\pi<\theta\le\pi}\int_0^{d_1}\sigma_{\theta\theta}(y+\rho\eta(\theta))\mathrm{d}\rho = \sigma_c \ . \tag{1}$$

Here, $\sigma_c$ is the strength of a body without concentrators under uniform traction, $d_1$ is a material constant with length measure.

If the direction $\theta_0$, where the maximum in (1) is realized, is known, the ASFC has the simplest form

$$\frac{1}{d_1}\int_0^{d_1}\sigma_{\theta\theta}(y+\rho\eta(\theta_0))\mathrm{d}\rho = \sigma_c \ , \tag{2}$$

where the integration is performed along this direction.

### 2.2
### Fracture criterion based on a minimum stress over a characteristic length (MSFC)

This approach was used in [2] and by other authors. In a generalized form, this criterion may be written as

$$\max_{-\pi<\theta\leq\pi}\left[\min_{0\leq\rho\leq d_2}\sigma_{\theta\theta}(y+\rho\eta(\theta))\right]=\sigma_c \quad . \tag{3}$$

Here, $\sigma_c$ and $d_2$ are material constants. In the case of a known direction $\theta_0$, where the maximum in (3) is realized, the MSFC may be written in a more simple form

$$\min_{0\leq\rho\leq d_2}\sigma_{\theta\theta}(y+\rho\eta(\theta_0))=\sigma_c \quad . \tag{4}$$



## 2.3
## Fracture criterion based on a model of fictitious crack with a characteristic length (FCFC)

Criterion of such type was used in [1], [11], [12] and by other authors. It is supposed that there exists a fictitious crack with a characteristic length $d_3$ originating from the considered point $y$ of the body. After some modification, [6], this criterion may be represented in the following form:

$$\max_{-\pi<\theta\leq\pi}\min_i K_{1i}(y,\theta,d_3)=K_{1c} \quad . \tag{5}$$

Here, $K_{1c}$ and $d_3$ are material constants, and $K_{11}=K_1(y)$ and $K_{12}=K_1(y+d_3\eta(\theta))$ are the stress intensity factors at the ends of the fictitious crack oriented at the $\eta(\theta)$ direction.

If the direction $\theta_0$ of fracture is known, Eq. (5) yields

$$\min(K_1(y),K_1(y+d_3\eta(\theta_0))=K_{1c} \quad . \tag{6}$$

For an edge crack beginning from the body boundary, there exists only one stress intensity factor $K_1(y+d_3\eta(\theta_0))$ and the minimum disappears in (6).

Each of the criteria written above includes two material parameters: a characteristic length $d_i$ and the strength parameter $\sigma_c$ or $K_{1c}$. These two parameters can be determined from two independent macro-experiments. The following fracture tests may be, for example, chosen: a tensile loading of a smooth specimen without concentrators and tensile loading of a specimen with a transverse crack or with a circular hole.

## 3
## Strength of a plate containing a central straight crack

A body containing a straight central crack provides a convenient object for the analysis of the NLFC validity. Consider a straight crack of a length $2a$ in a plate of infinite extent. The origin of the $(x_1,x_2)$-coordinate system coincides with the center of the crack. If a uniform tensile traction $\sigma$ is applied parallel to the $x_2$-axis at infinity, then $\sigma_{22}(x_1,0)$ near the crack tip is approximated asymptotically, e.g. [13], by the expression

$$\sigma_{22}(x_1,0)=\frac{K_1}{\sqrt{2\pi(x_1-a)}} \quad , \tag{7}$$

where $K_1$ is the mode 1 stress intensity factor given by the expression

$$K_1=\sigma\sqrt{\pi a} \quad . \tag{8}$$

Equation (7) presents only the main part of the asymptotic decomposition of the stress field near the crack tip, and it is usually considered to be quite correct for $(x-a)/a\leq 0.1$. For small cracks, however, the fracture criteria may require an accurate knowledge of the stress distribution near the crack tip. An exact expression for the normal stress ahead of the crack can be obtained, e.g. as a limiting case of the solution for an elliptical hole in an infinite plate [14]. It should be noted that the result is the same for isotropic and anisotropic plates

$$\sigma_{22}(x_1,0)=\frac{K_1x_1}{\sqrt{\pi a(x_1^2-a^2)}} \quad . \tag{9}$$

The line of maximal tensile stress coincides with the continuation of the crack direction along the $x$ axis and all NLFC given above may be used in the simplest forms (2), (4) and (6).

### 3.1
### Average stress fracture criterion (ASFC)

Applying the criterion (2) to the end point of the crack in conjunction with Eq. (9) yields

$$K_{1c} = \sigma_c \sqrt{\frac{\pi d_1 \eta_1}{1 + \eta_1}} \ , \qquad (10)$$

where $\eta_1 = a/(a + d_1)$ is the normalized crack length. Let us denote $q = \sigma$ the tensile strength of the plate with crack. Then, from Eqs. (8) and (10), one can obtain the relation between the normalized plate strength and crack length



$$\frac{q}{\sigma_c} = \sqrt{\frac{1 - \eta_1}{1 + \eta_1}} \ . \qquad (11)$$

It follows from (11), that $q/\sigma_c \to 1$ when $\eta_1 \to 0$ $(a \to 0)$ for the short crack; and $q/\sigma_c \to 0$ when $\eta_1 \to 1$ $(a \to \infty)$ for the long crack.

It is interesting to note that the critical stress intensity factor $K_{1c}$ used generally in the linear fracture mechanics is not a material constant, when a nonlocal fracture criterion is applied. It is the function of the crack length given by (10).

As $a$ increases, the value $K_{1c}$ asymptotically approaches the limiting constant value $K_{1c}^{\infty}$ for large cracks

$$K_{1c}^{\infty} = \sigma_c \sqrt{\pi d_1 / 2} \ . \qquad (12)$$

This value of $K_{1c}$ may be obtained at once if we use criterion (2) and stress asymptotics (7) approximating (9) for sufficiently large cracks. Just this value of $K_{1c}^{\infty}$, determined from the long crack tests, should be regarded as a material constant from the view point of non-local fracture criteria.

If $\sigma_c$ and $K_{1c}^{\infty}$ are defined from independent experiments, we can express $d_1$ from (12):

$$d_1 = \frac{2}{\pi} \left( \frac{K_{1c}^{\infty}}{\sigma_c} \right)^2 \ .$$

If $d_1$ is obtained from a plate strength $q'$ experimentally measured for a crack length $2a'$ and from $\sigma_c$ also known from a fracture test, then

$$d_1 = 2a' \frac{\left( \frac{q'}{\sigma_c} \right)^2}{1 - \left( \frac{q'}{\sigma_c} \right)^2} \ . \qquad (13)$$

### 3.2
### Minimum stress fracture criterion (MSFC)

According to criterion (4) and Eq. (9), the following relation is valid for the dependence of the critical stress intensity factor on the crack length:

$$K_{1c} = \sigma_c \sqrt{\pi d_2 \eta_2 (1 + \eta_2)} \ , \qquad (14)$$

where $\eta_2 = a/(a + d_2)$. Using (8) for $\sigma = q$ together with (14), one can obtain

$$\frac{q}{\sigma_c} = \sqrt{1 - \eta_2^2} \ . \qquad (15)$$

As it was done in the previous section, the following results can be easily derived:

$$K_{1c}^{\infty} = \sigma_c \sqrt{2\pi d_2}; \quad d_2 = \frac{1}{2\pi} \left( \frac{K_{1c}^{\infty}}{\sigma_c} \right)^2 \ ,$$

or

$$d_2 = a' \frac{1 - \sqrt{1 - \left(\frac{q'}{\sigma_c}\right)^2}}{\sqrt{1 - \left(\frac{q'}{\sigma_c}\right)^2}} \quad , \tag{16}$$

when $d_2$ is evaluated from a test for a plate with a finite crack of length $2a'$ and from $\sigma_c$.
Comparison of (13) and (16) given

$$d_2 = \alpha_2 d_1, \quad \alpha_2 = \frac{1}{2\left[1 + 1/\sqrt{1 - \left(\frac{q'}{\sigma_c}\right)^2}\right]} = \frac{1}{2\left[1 + \sqrt{\frac{1+\eta_1'}{2\eta_1'}}\right]} \quad , \tag{17}$$



where $\eta_1' := a'/(a' + d_1')$. Since $q'/\sigma_c \to 0$ when $a \to \infty$, we have from (17) the relation $d_1 = 4d_2$ for the case when $d_1$ and $d_2$ are obtained from the critical stress intensity factor $K_{1c}^{\infty}$ for a long crack and from the longitudinal strength $\sigma_c$ of the plate without concentrators.

### 3.3
### Fictitious crack fracture criterion (FCFC)

In accordance with this approach, one has to add a fictitious crack of the length $d_3$ to the considered end of the main crack of the length $2a$. The stress intensity factor for this composed crack may be written in the form

$$K_1 = \sigma \sqrt{\pi \left(a + \frac{d_3}{2}\right)} \quad ,$$

and criterion (6) gives

$$K_{1c}^{\infty} = q \sqrt{\pi \left(a + \frac{d_3}{2}\right)} \quad .$$

For a plate without a crack, we have $a = 0$ and $K_{1c}^{\infty} = \sigma_c \sqrt{\pi d_3/2}$. Thus,

$$d_3 = \frac{2}{\pi} \left(\frac{K_{1c}^{\infty}}{\sigma_c}\right)^2 \quad , \tag{18}$$

and

$$\frac{q}{\sigma_c} = \sqrt{\frac{1 - \eta_3}{1 + \eta_3}} \quad , \tag{19}$$

where

$$\eta_3 = \frac{a}{a + d_3} \quad . \tag{20}$$

If $d_3$ is evaluated from a test with a plate with a crack of a finite length $2a'$ and from $\sigma_c$, then

$$d_3 = 2a' \frac{\left(\frac{q'}{\sigma_c}\right)^2}{1 - \left(\frac{q'}{\sigma_c}\right)^2} \quad . \tag{21}$$

It follows from (13), (21), that ASFC and FCFC contain the same characteristic length parameters $d_1 = d_3$, when they are obtained from experiments with a plate without and with a crack, and lead to the same dependencies (11), (19) of the strength $q$ on the crack length $a$.

The plate strength $q = K_{1c}^{\infty}/\sqrt{\pi a}$ predicted by the linear fracture mechanics can be also presented in terms of the same nondimensional variables if we express $a$ in terms of $\eta_3$ and $K_{1c}^{\infty}$ from relations (20), (18)

$$\frac{q}{\sigma_c} = \sqrt{\frac{1-\eta_3}{2\eta_3}} \ . \tag{22}$$

One usually evaluates the critical stress intensity factor $K'_{1c}$, instead of $K_{1c}^{\infty}$, from the formula $q' = K'_{1c}/\sqrt{\pi a'}$, applying the test with a crack of a finite length $2a'$. Using (21), one can rewrite the linear fracture mechanics strength prediction $q/\sigma_c = (q'/\sigma_c)\sqrt{a'/a}$ in the normalized form and get



$$\frac{q}{\sigma_c} = \sqrt{\frac{1-\eta_3}{2\eta_3}}\sqrt{1-\frac{q'^2}{\sigma_c^2}} \ , \tag{23}$$

instead of (22).

Note that in deriving Eq. (22) it was supposed that the ratio of the strength predictions of the linear fracture mechanics and of FCFC tends to unity as $a \to \infty$. In contrast to that, in deriving (23) it was supposed that this ratio is equal to unity at a finite $a = a'$. This explains the difference between (22) and (23).

## 3.4
## Predicted results and comparison with experimental data

To represent all results in the same coordinates, let us introduce a common nondimensional crack length parameter $\eta = a/(a + d_0)$, then

$$\eta_i = \frac{\eta}{\eta + \alpha_i(1-\eta)} \ , \tag{24}$$

where $\alpha_i = d_i/d_0$. In what follows, it is assumed that $d_0 = d_1 = d_3$ and all plots are presented in the nondimensional coordinates $\eta = a/(a + d_0)$ and $q/\sigma_c$. Then, $\eta_1 = \eta_3 = \eta$, but $\eta_2$ is given by (24), where $\alpha_2$ is determined by (17).

Note, that all the nonlocal fracture criteria give $q/\sigma_c \to 1$ when $\eta \to 0$ for small cracks, as would be expected, while the linear fracture mechanics criterion gives the unrealistic prediction $q/\sigma_c \to \infty$, as can be seen from (22).

From Eqs. (11), (15), (17), and (24), one can express the relative difference $\delta_{21} := (q_2 - q_1)/q_1$ between the predictions of the strength of a plate with a crack according to the ASFC (coinciding with FCFC) and the MSFC. It is always positive, its maximum with respect to $\eta$ equals to $\delta_{21}^m = (1-\alpha_2)/\sqrt{1-2\alpha_2} - 1$ and is reached at $\eta = \alpha_2^2/(\alpha_2^2 - 3\alpha_2 + 1)$, where $0 \le \alpha_2 \le 1/4$ according to (17). Thus, the maximum of $\delta_{21}$ with respect to $\alpha_2$ and $\eta$ is reached at $\alpha_2 = 1/4$, i.e. in the case when $d_1, d_2, d_3$ are obtained from the following two plate fracture tests: without any crack and with an infinitely large crack at $\eta = 0.2$ (i.e., at $a = d_2 = d_1/4 = d_3/4$); it is equal to $\delta_{21}^* = (3\sqrt{2})/4 - 1 \approx 0.06$. It means that if $d_1, d_2, d_3$ are obtained from a fracture test without any crack and from a test with a crack of any length, the maximal relative difference between the ASFC/FCFC and the MSFC predictions for the cracked plate do not exceed 6%.

Data obtained from experiments for laminated composite plates containing short cracks [2, 4, 5] have been chosen to apply each of the criteria described above. The test result for the crack of maximal length has been used to calculate $d_i$ for each material by (13), (16), (21). Plots for the three considered criteria, whose parameters are obtained by experiments on various composites, are shown in Fig. 4–8 together with the corresponding experimental points. The dashed line corresponds to the prediction of the linear fracture mechanics (23). All of the experimental data used have been taken with an allowance for the finite widths of the specimens.

The use of the standard nondimensional variables $\eta$ and $\lambda = q/\sigma_c$ for the presentation of the obtained results permits the marking of data of several series of experiments for different materials on the same plot, for a chosen fracture criterion. Such common pictures of experi-

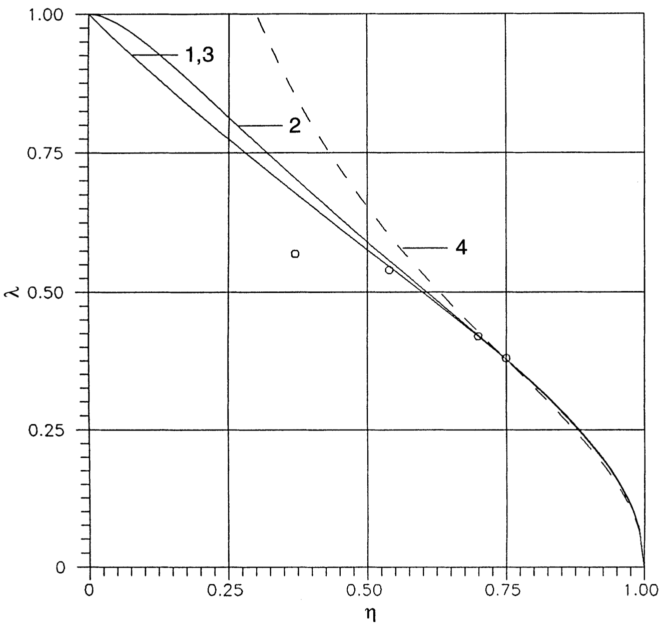


**Fig. 4.** Comparison of predicted and experimental results: 1,3-ASFC and FCFC; 2-MSFC; 4-LFMC; ○-experimental data for a $[0/+45/-45]_{2s}$-graphite-epoxy laminate [2]

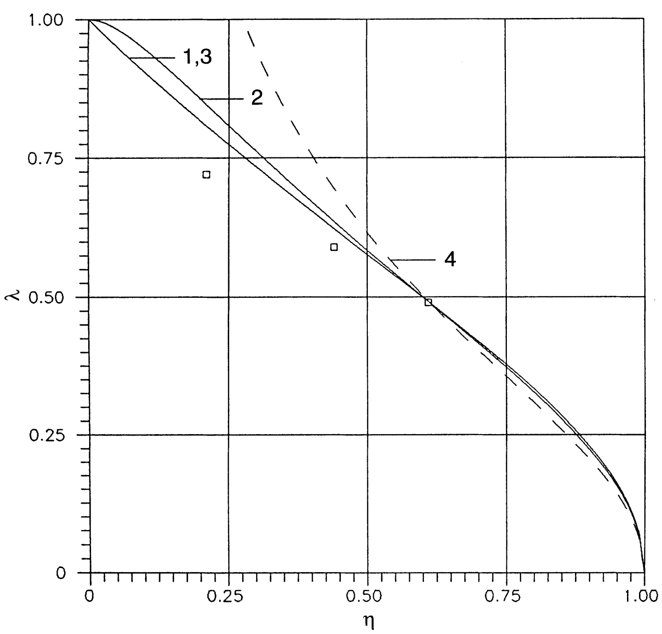


**Fig. 5.** Comparison of predicted and experimental results: 1,3-ASFC and FCFC; 2-MSFC; 4-LFMC; □-experimental data for a $[+45/-45/0/90]_s$-graphite-epoxy laminate [4]

mental data in comparison with theoretical predictions of the ASFC (coinciding with the FCFC) and the MSFC are given in Fig. 9 and 10, respectively. Obviously, the $\eta$-normalizations of the experimental points differ for the different materials because of the different values of $d_i$ obtained. In addition, the limiting value 1/4 for the factor $\alpha_2$ was used for the $\eta$-normalizations of the experimental points in Fig. 10.

As can be seen from these figures, all the fracture criteria overestimate somewhat the plate strength for most short cracks, when $d_i$ are determined from the tests with the maximum length cracks.

## 4
## Strength of a plate containing a circular hole

Consider a circular hole of a radius $a$ in an isotropic plate of infinite extent. The origin of an $(x_1, x_2)$-coordinate system coincides with the center of the hole. The solution of linear theory of elasticity for a plate with a hole is well-known, see, e.g. [13, 14]. If a uniform tensile stress $\sigma$ is applied parallel to the $x_2$ axis at infinity, then the distribution of the normal stress $\sigma_{22}(x_1, 0)$ along the $x_1$ axis $(x_1 \geq a)$ is given by the expression

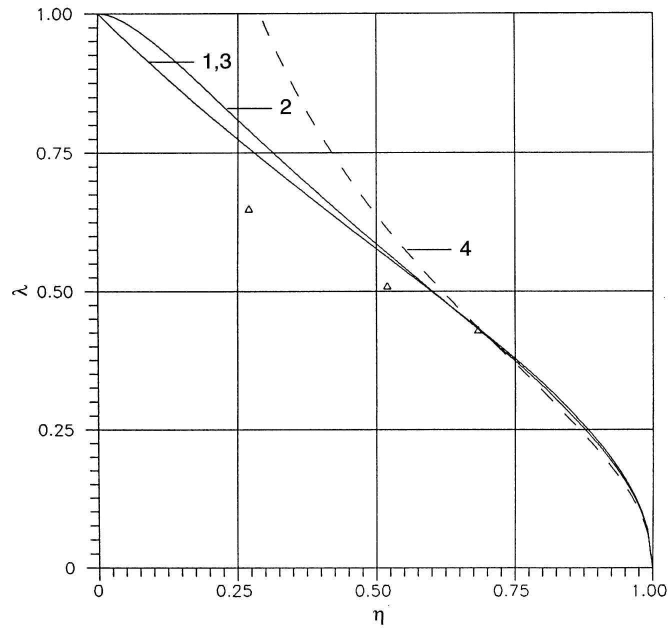


**Fig. 6.** Comparison of predicted and experimental results: 1,3-ASFC and FCFC; 2-MSFC; 4-LFMC; △-experimental data for a $[90/0+45/-45]_s$-graphite-epoxy laminate [4]

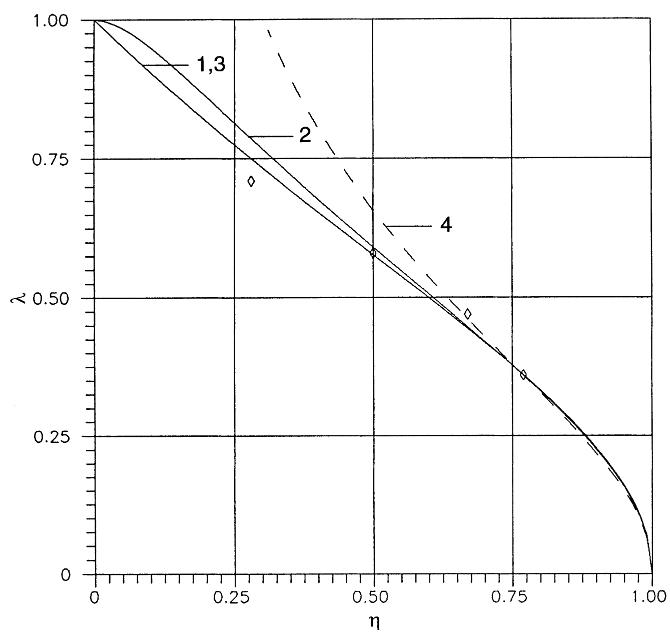


**Fig. 7.** Comparison of predicted and experimental results: 1,3-ASFC and FCFC; 2-MSFC; 4-LFMC; ◇-experimental data for a $[0/+45/-45/90]_s$-glass-epoxy laminate [4]

$$\frac{\sigma_{22}(x_1,0)}{\sigma} = 1 + \frac{1}{2}\left(\frac{a}{x_1}\right)^2 + \frac{3}{2}\left(\frac{a}{x_1}\right)^4 . \tag{25}$$

The stress distribution in the nondimensional coordinates, as it follows from (25), is independent of the hole size, and the stress concentration factor at the edge of a hole is independent of the hole radius too, $\sigma_{22}(a,0) = 3\sigma$. However, the size of the stress concentration region depends on the hole radius.

Let us consider the strength predictions for this plate with the smooth concentrator by the three NLFC given above. Let us introduce, as in Sec. 3, the normalized hole radius parameters $\eta_i = a/(a+d_i)$, where $d_i$ are the characteristic lengths of the corresponding criteria.

## 4.1
## Average stress fracture cirterion (ASFC)

Substituting Eq. (25) into (2), and performing the integration yield the fracture criterion for the most stressed point $(a,0)$ at the plate with the hole

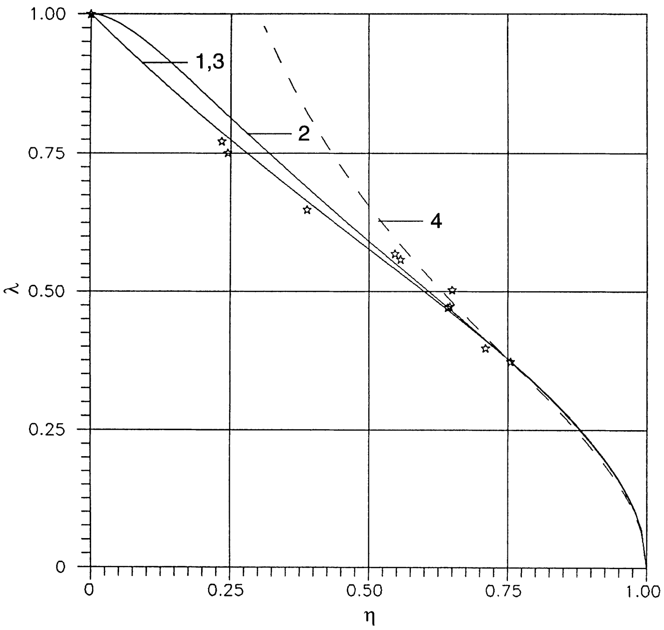


**Fig. 8.** Comparison of predicted and experimental results: 1,3-ASFC and FCFC; 2-MSFC; 4-LFMC; ⋆-experimental data for a fibrous composite Al/B [5]

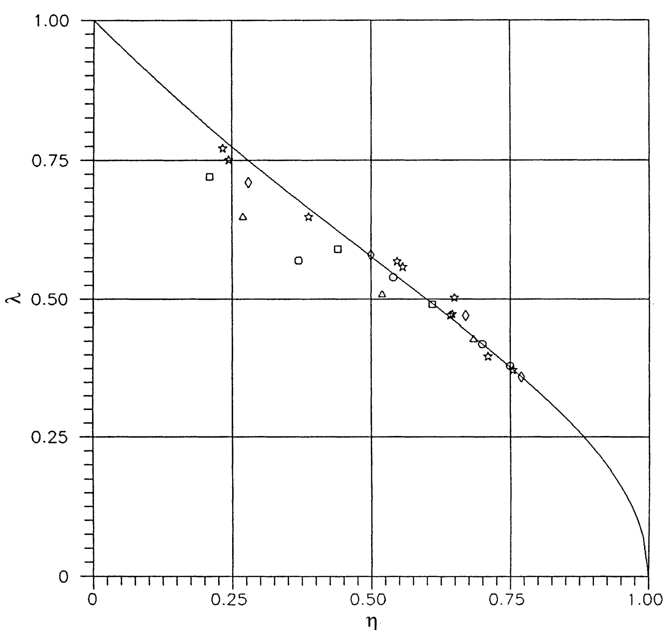


**Fig. 9.** Description of the common set of experimental data on the basis of ASFC/FCFC. Marks of experimental points are the same as in Figs. 4–8

$$\frac{q}{\sigma_c} = \frac{2}{(1+\eta_1)(2+\eta_1^2)} \quad . \tag{26}$$

One can easily see that for large values of the radius a, i.e. when $\eta_1 \to 1$, the plate's strength reduction caused by the hole is determined by the factor $q/\sigma_c \to 1/3$, while for small values of $a$, i.e. when $\eta \to 0$ no strength reduction is predicted, $q/\sigma_c \to 1$.

## 4.2
## Minimum stress fracture criterion (MSFC)

According to criterion (4) and (25), one can obtain the following fracture criterion:

$$\frac{q}{\sigma_c} = \frac{2}{2+\eta_2^2+3\eta_2^4} \quad . \tag{27}$$

For the limiting cases of large holes ($\eta_2 \to 1$) and small ones ($\eta_2 \to 0$) it follows from (27) that $q/\sigma_c \to 1/3$ and $q/\sigma_c \to 1$, respectively

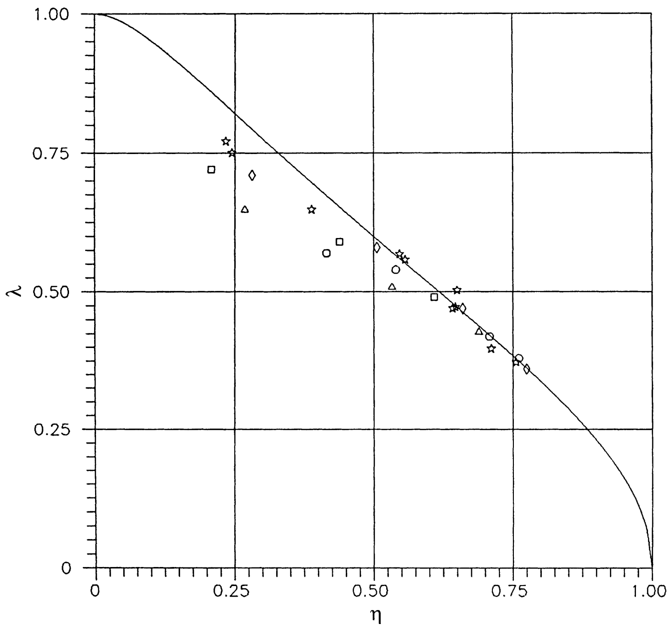


**Fig. 10.** Description of the common set of experimental data on the basis of MSFC. Marks of experimental points are the same as in Figs. 4–8

### 4.3
### Fictitious crack fracture criterion (FCFC)

Placing a fictitious crack of a length $d_3$ at the most stressed point $(a, 0)$ of the plate along the $x_1$-axis direction, one obtains a concentrator of nonsymmetric form: a circular hole with a single crack originating at its boundary. The symmetric form with two cracks used earlier [1, 4] corresponds to the simultaneous fracture at two points. It seems more natural not to demand the fracture symmetry.

A linear elastic solution for the concentrator of such a form was obtained in [15, 16]. This solution gives the following expression for the stress intensity factor:

$$K_1 = \sigma\sqrt{\pi d_3} f(\eta_3) \ , \tag{28}$$

where the function $f(\eta_3)$ (more exactly, $\tilde{f}(\varepsilon)$ where $\varepsilon = d_3/a$) is given in [15] in a table form.

In the limiting case, when the hole radius $a$ tends to zero, $\eta_3 \to 0$ and $f(\eta_3) \to 1/\sqrt{2}$. So, the following fracture criterion takes place for the plate without a hole:

$$K_{1c} = \sigma_c\sqrt{\pi d_3/2} \ . \tag{29}$$

The fracture criterion $K_1 = K_{1c}$ together with Eqs. (28) and (29) gives

$$\frac{q}{\sigma_c} = \frac{1}{\sqrt{2} f(\eta_3)} \ . \tag{30}$$

Note that for the hole with the radius $a \to \infty$, $\eta_3 \to 1$ and the limiting geometry corresponds to the half-plane with the edge crack under the tensile load $3\sigma$. Using the known solution of this problem, see, e.g. [17], one can calculate the limiting value $\lim_{\eta_3 \to 1} f(\eta_3) \simeq 3.3645$.

### 4.4
### Predicted results and comparison with experimental data

To compare the predictions of all the criteria considered, we will use, as in Sec. 3, the common normalization $\lambda = q/\sigma_c$ and $\eta = a/(a + d_0)$; then $\eta_i$ is connected with $\eta$ by (24), where $\alpha_i = d_i/d_0$. It is taken further that $d_0 = d_1$.

It should be noted in advance that the fictitious crack fracture criterion is not correct for holes of large radius, since the reduction of strength described by (30) equals approximately 0.21 in the limiting case $a \to \infty$ $(\eta \to 1)$, what is more intensive than 1/3 given by linear elasticity stress concentration factor. However, the value 1/3 is true for large holes, as experiments show.

Let us suppose that, for the three fracture criteria considered, the parameters $\sigma_c$ (or $K_{1c}$) and $d_i$ are obtained from the following two fracture tests: for a plate without a hole and for a plate with a hole of a radius $a'$. One can then get the normalized hole diameter $\eta'_i$ from the plate strength $q' = q(a')$ and from formulas (26), (27), (30) for each criterion; the characteristic lengths, thus, are $d_i = (a'/\eta'_i) - a'$.

Let us now compare the maximum relative difference between the strength predictions given by the ASFC and the MSFC. From (24), (26), (27) we have,

$$\delta_{21} := \frac{q_2 - q_1}{q_1} = \frac{(1+\eta)(2+\eta^2)}{2 + \eta^2(\eta + \alpha_2(1-\eta))^{-2} + 3\eta^4(\eta + \alpha_2(1-\eta))^{-4}} - 1 \ . \tag{31}$$

Using (26) and (27), we can express $\eta'_2$ in terms of $\eta'_1 = \eta'$ and then, using (24), express $\alpha_2$ in terms of $\eta'$

$$\alpha_2 = \frac{\eta'}{1-\eta'}\left(\sqrt{\frac{6}{-1 + \sqrt{12(1+\eta')(2+\eta'^2) - 23}}} - 1\right) \ .$$

Analysis of this expression shows that $0 < \alpha_2 < 1/2$ for $0 < \eta' < 1$, $\alpha_2 \to 0$ when $\eta' \to 0$, and $\alpha_2 \to 1/2$ when $\eta' \to 1$. Analysing the maximum of expression (31) with respect to $\eta$ and $\alpha_2$ such that $0 < \eta' < 1$, $0 < \alpha_2 < 1/2$, we note that the maximum is reached at $\alpha_2 \to 1/2$ (i.e. at $a' \to \infty$) and at $\eta \approx 0.29888$ (i.e. at $a \approx 0.42629 d_1 \approx 0.85258 d_2$), and is equal to $\delta^*_{21} \approx 0.15659$. Analogously, the minimum of (31) is reached at $\alpha_2 \to 0$ (i.e. at $a' \to 0$) and $\eta \to 0$ (i.e. at $a \to 0$), and is equal to $\delta^{**}_{21} = -2/3$. It means that if $d_1$, $d_2$ are obtained from the two fracture test without and with a circular hole, the MSFC strength predictions can amount to between 33.3% and 115.7% of the ASFC predictions for the plate with a circular hole.

Comparison of the predicted and experimental [1–3] data for strength of multi-layer composite materials are given in the Fig. 11–17. The characteristic lengths $d_i (i = 1, 2, 3)$ were determined for all criteria on the basis of the experiments for the concentrator of the maximal size, except for the experiment on the graphite-epoxy laminate with holes of large radius [1] in Fig. 15–17, where the minimal size concentrator was used. The dashed line corresponds to the strength evaluation on the basis of elastic stress concentration factor.

As in the corresponding pictures of Sec. 3, the $\eta$-normalizations of the experimental points on Fig. 15–17, including the common set of experimental data, differ for the different materials because of the different values of $d_i$. In addition, the values $\alpha_2 = 1/4$ and $\alpha_3 = 1$ have been used for the $\eta$-normalizations of the experimental points in Fig. 16–17.

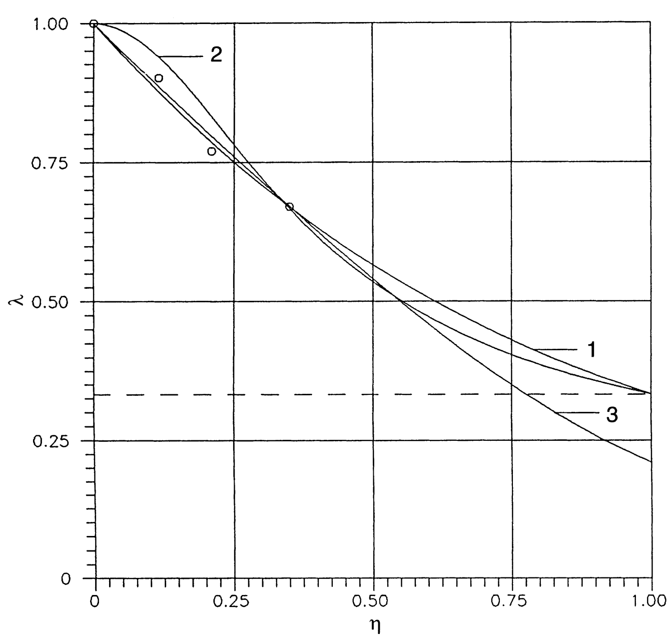


**Fig. 11.** Comparison of predicted and experimental results: 1-ASFC; 2-MSFC; 3-FCFC; ○-experimental data for a $[0/+45/-45]_s$-graphite-epoxy laminate [1]

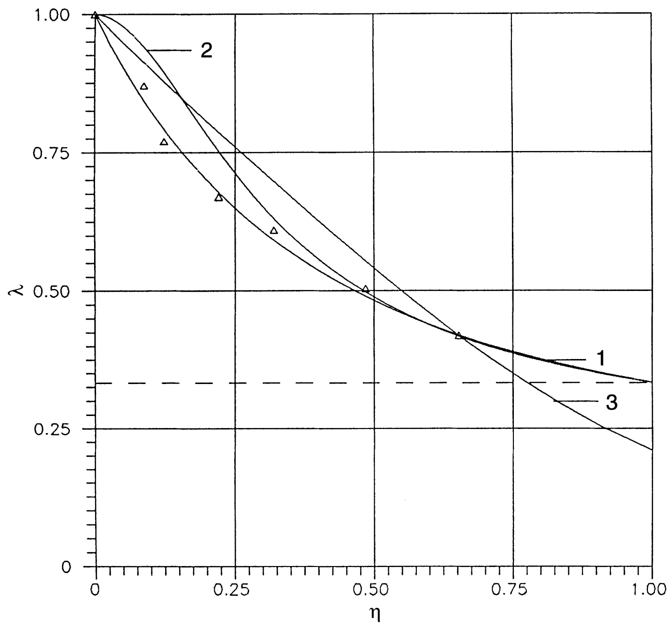


**Fig. 12.** Comparison of predicted and experimental results: 1-ASFC; 2-MSFC; 3-FCFC; △-experimental data for a quasi-isotropic glass-epoxy composite [2]

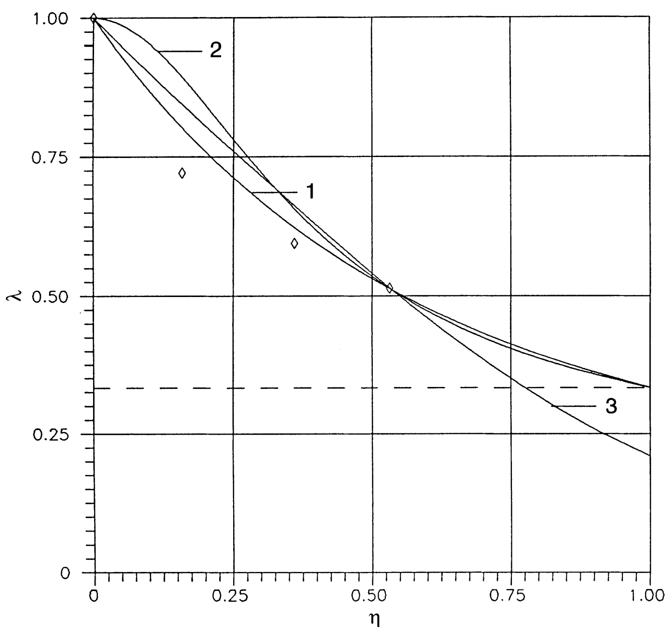


**Fig. 13.** Comparison of predicted and experimental results: 1-ASFC; 2-MSFC; 3-FCFC; ◇-experimental data for a $[+45/-45/0/90]_s$-graphite-epoxy laminate [3]

## 5
## Some remarks about the fictitious crack fracture criteria

In addition to the remark given in Sec. 4.4, some more criticism of the FCFC can be given.

Let us consider a rectilinear elastic plate under the action of a uniform tensile tractions $q$ applied to two of its sides, Fig. 18. Placing the fictitious crack parallel to the loaded sides of the plate at a distance $h$ from one of these sides, one can see (e.g. from [17]) that the stress intensity factor $K_1$ at the crack ends tends to infinity, when $h$ tends to zero. It means, that for any arbitrarily small $q$ there exists a sufficiently small distance $h$ such that FCFC will predict the fracture, what leads to a contradiction with experimental data and experience.

In defence of this criterion, one can say that the type of the boundary conditions such as positive tractions prescribed, is only a mathematical or mechanical model, and does not exist in reality. But the rejection of this model seems to be too heavy a sacrifice for saving the criterion.

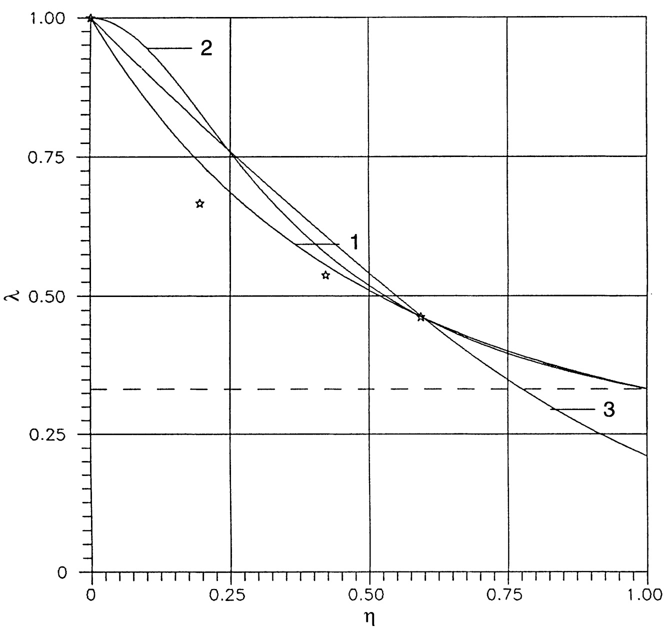


**Fig. 14.** Comparison of predicted and experimental results: 1-ASFC; 2-MSFC; 3-FCFC; $\star$-experimental data for a $[90/0/+45/-45]_s$-graphite-epoxy laminate [3]

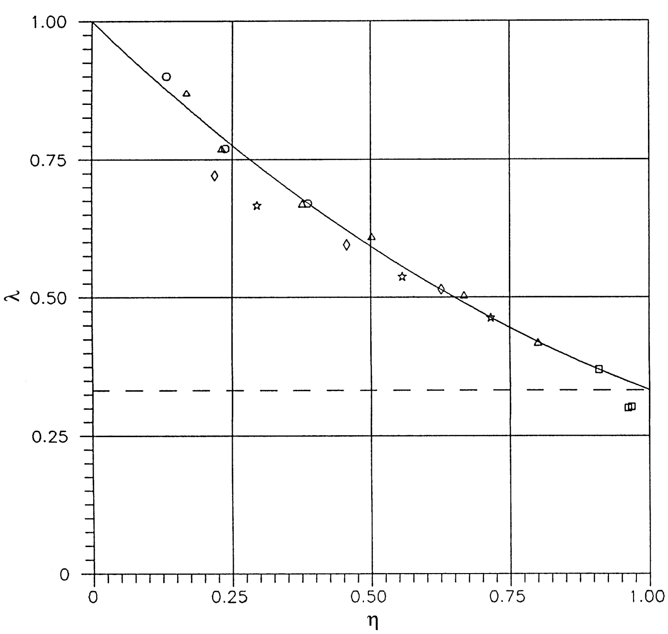


**Fig. 15.** Description of the common set of experimental data on the basis of ASFC. Marks of experimental points are the same as in Figs. 11–14; □-experimental data for the graphite-epoxy laminate with holes of large radius [1]

## Conclusion

The following conclusions may be drawn on the basis of the performed analysis:

1. All the nonlocal fracture criteria considered allow to describe the strength of bodies containing smooth concentrators as well as singular ones. It has been shown that the FCFC coincides completely with the ASFC for a plate with a central crack under symmetric loading.
2. Both the ASFC and the MSFC describe the dependence between body strength and the size of concentrator qualitatively correct. The FCFC gives incorrect results for large smooth concentrators and does not enable the limiting transition to the local strength condition for the half-plane. In addition, it gives extremely incorrect results even for the uniform stress state under boundary tractions.
3. The best description of experimental data was obtained on the basis of the ASFC.
4. The considered criteria containing two material parameters do not enable the known experimental data on the strength of bodies with small concentrators to be described exactly. There are systematic deviations of the predicted results from the experimental data. Thus, the nonlocal fracture criteria considered need to be improved.

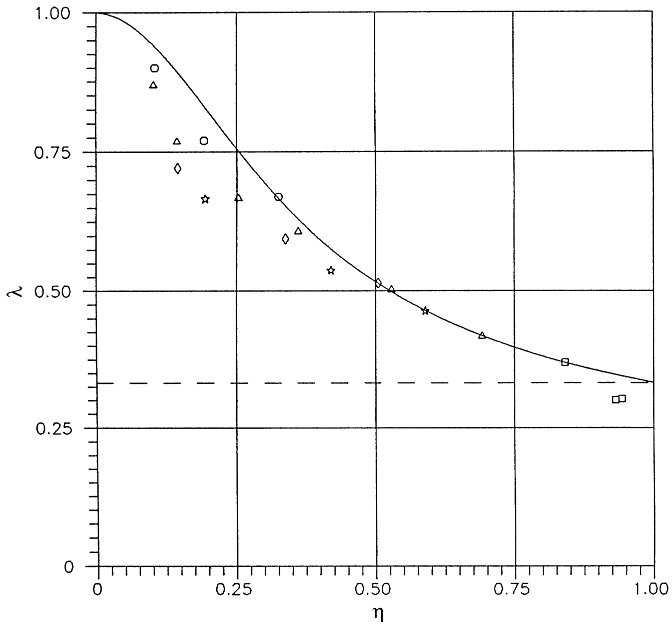


**Fig. 16.** Description of the common set of experimental data on the basis of MSFC. Marks of experimental points are the same as in Figs. 11–14; □-experimental data for the graphite-epoxy laminate with holes of large radius [1]

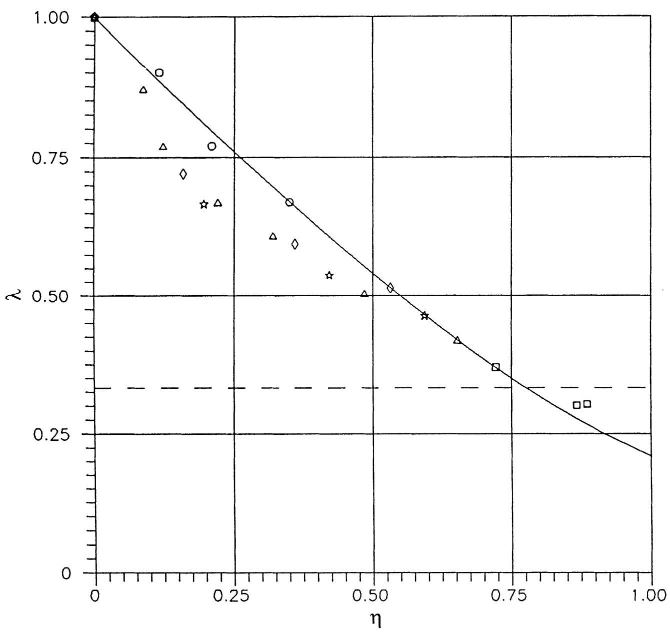


**Fig. 17.** Description of the common set of experimental data on the basis of FCFC. Marks of experimental points are the same as in Fig. 11–14; □-experimental data for the graphite-epoxy laminate with holes of large radius [1]

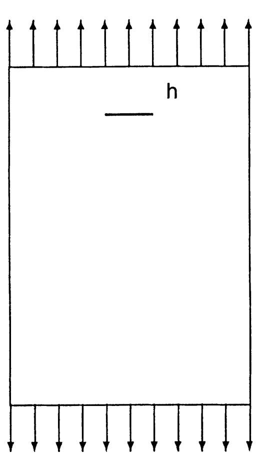


**Fig. 18.** A crack near the plate side loaded by tractions

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