ZAMM, 2000, Vol. 80, S483-S486

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Non-local Strength Conditions Based on Generalized δ_c Cohesive Models

Generalizations of the Leonov-Panasyuk-Dugdale δ_c cohesive model applicable not only for crack tips but also for any internal or boundary point are described. These models are interpreted also as non-local strength conditions and can be used for description of strength small scale effects, for stress fields with stress singularities other than the usual crack tip square root singularity (e.g. for corner points), for bodies with smooth and singular stress concentrators.

1. Introduction and Motivation

There exist at least three problems of strength and fracture mechanics that can not be solved by use of traditional (local) fracture criteria: (i) the strength small–scale effects; (ii) description of the fracture of bodies with singular stress concentrators generating stress singularities other than $1/\sqrt{\rho}$; (iii) the problem of unification of the fracture criteria for bodies with smooth and singular concentrators. The problems can be solved by use of non-local strength conditions, according to which the strength at a point y depends on the stress tensor components not only at this point but on the stress field in a neighbourhood of y. The conditions analysis and examples are given in [1–3]. Another one such condition is presented here.

Let us consider the classical δ_c cohesive model (Leonov-Panasyuk [4], Dugdale [5]). The model (Fig. 1) includes three main elements:
(A) Rectilinear crack in a two-dimensional elastic body symmetric to the crack line is considered.
(B) Plastic (cohesive) zones Γ arise on the crack prolongation with the zone boundary conditions $\sigma_{\theta\theta}^{(\Gamma)}|_\Gamma = \sigma_c$, an additional condition $\sigma_{\rho\theta}^{(\Gamma)}|_\Gamma = 0$ follows from the problem symmetry. The cohesive zones lengths are such that the stress field have no any power-wise singularities at the zone tips (the stress intensity factors are zero).
(C) The strength condition (condition that the crack does not propagate): the crack opening displacements $\delta = [u^{(\Gamma)}]$ at the crack tips (which are the beginning points of the cohesive zones) are less than a critical opening δ_c (material parameter), $\delta < \delta_c$.

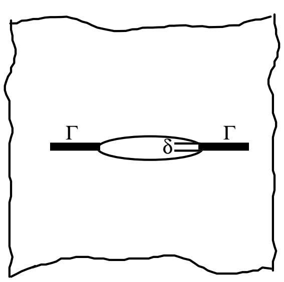

Fig. 1

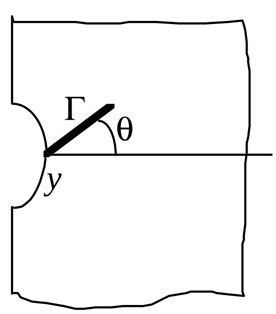

Fig. 2

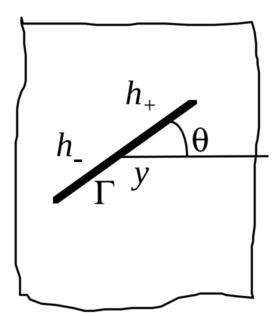

Fig. 3

It is possible to note that the classical δ_c model can be successfully used for description of the small-scale effect for small cracks: it predicts a finite strength as the crack length tends to zero, in contrast to the infinite strength predicted by the linear fracture mechanics for a body with a crack. Thus, the point (i) of the above list of problems is fulfilled. However, the δ_c model in its original form is applicable only to crack tips. A variant of the δ_c model generalization for boundary points was considered in [6]. The goal of this paper is an extension of this model to arbitrary boundary and internal body points to fulfil also points (ii) and (iii) and to show that it can be considered as a non-local strength condition after such generalization.

2. Generalized δ_c model for boundary points

One of possible generalisations of the δ_c model for boundary points includes three main elements:
(A) Strength at any boundary point y in a two-dimensional elastic body with respect to eventual fracture in a direction $\eta(\theta)$ is analysed (Fig. 2).

(B) A rectilinear cohesive zone $\Gamma(y,\theta;\sigma)$ (with the zone boundary conditions $\sigma^{(\Gamma)}_{\theta\theta}|_\Gamma=\sigma_c,\ \sigma^{(\Gamma)}_{\rho\theta}|_\Gamma=0$) originates at the point y in the direction $\eta(\theta)$. The zone length $h(y,\theta;\sigma)=|\Gamma(y,\theta;\sigma)|$ is such that the mode I stress intensity factor K_{1+} is zero at the end point $y+h\eta(\theta)$ of the cohesive zone and is positive for any cohesive zone $\Gamma^\vee$ originating from y in the same direction $\eta(\theta)$ and being smaller than Γ:

$$K^{(\Gamma)}_{1+}(\Gamma)=0;\qquad K^{(\Gamma)}_{1+}(\Gamma^\vee)>0\quad\forall\ \bar{\Gamma}^\vee(y,\theta;\sigma)\subset\Gamma(y,\theta;\sigma). \tag{1}$$

(C) The strength condition: the cohesive zone opening at all points x of $\Gamma(y,\theta;\sigma)$ is less than a critical opening δ_c (material parameter):

$$[u^{(\Gamma)}_\theta(x)]<\delta_c\quad\forall\ x\in\Gamma(y,\theta;\sigma). \tag{2}$$

Let $\sigma_{ij}(x)$ be the stress field in the original elastic problem without cohesive zones. Let us denote by $\sigma_{\theta\theta}(y,\theta)$ a definite (finite or $+\infty$ or $-\infty$) limit of the stress $\sigma_{\theta\theta}(y+\rho\eta(\theta))$ as $\rho\to0$ if the limit does exist. Note that $\sigma_{\theta\theta}(y+\rho\eta(0))\sim K_1/(\sqrt{2\pi\rho})\to\sigma_{\theta\theta}(y,0)=+\infty$ as $\rho\to0$ in the classical case, when y is a tip of an opened crack ($K_1>0$). One can remark that if $\sigma_{\theta\theta}(y,\theta)>\sigma_c$, then second condition of (1) is satisfied for all sufficiently small $\Gamma^\vee$; otherwise, if $\sigma_{\theta\theta}(y,\theta)<\sigma_c$, then opposite condition $K^{(\Gamma)}_{1+}(\Gamma^\vee)<0$ is satisfied for all sufficiently small $\Gamma^\vee$. Hence, the second condition (1) ensures that the cohesive zone can not arise from the point y in the direction θ where $\sigma_{\theta\theta}(y,\theta)<\sigma_c$.

Let $l(y,\theta)$ be the distance from y to another nearest boundary point in the direction $\eta(\theta)$. It may happen that there exists no any solution $h(y,\theta)$ of (1) such that $h(y,\theta)<l(y,\theta)$ since $K^{(\Gamma)}_{1+}(\Gamma^\vee)>0$ for all $|\Gamma^\vee|<l(y,\theta)$. One can mechanically interpret this as the cohesive zone $\Gamma(y,\theta;\sigma)$ extending up to the boundary point $y+l(y,\theta)\eta(\theta)$, that is, one should take $h(y,\theta;\sigma)=l(y,\theta)$. For such case (when the cohesive zone come out to a boundary point where tractions are prescribed) the cohesive zone tip $y+l(y,\theta)\eta(\theta)$ is open and condition (1) is satisfied. Strength condition (2) is applicable also for this situation.

The fracture criterion for a point y is

$$\sup_\theta\{\sup_{x\in\Gamma(y,\theta;\sigma)}[u^{(\Gamma)}_\theta(x)]\}=\delta_c. \tag{3}$$

The angle θ, at which the supremum is realised gives the fracture direction. The fracture appearance can be attributed either to the boundary point y or to the point x, at which the supremum in (3) is realised, or to the whole of the zone Γ. Note that another option to choose the zone Γ and the eventual fracture direction $\eta(\theta)$ is the demand that not only K_{1+} but also the mode II stress intensity factor K_{2+} is equal to zero at the cohesive zone end point; then the supremum with respect to θ disappears in (3) and we arrive at another δ_c model generalization.

3. Generalized δ_c model for internal points

A possible generalization of the δ_c model for internal points is similar to that described above for boundary points and includes three main elements:
(A) Strength at any internal point y in a two-dimensional elastic body with respect to eventual fracture in a direction $\eta(\theta)$ is analysed (Fig. 3).
(B) A rectilinear cohesive zone $\Gamma(y,\theta;\sigma)$, (with the zone boundary conditions $\sigma^{(\Gamma)}_{\theta\theta}|_\Gamma=\sigma_c,\ \sigma^{(\Gamma)}_{\rho\theta}|_\Gamma=0$) goes trough the point y in the direction $\eta(\theta)$. The zone length is $|\Gamma|=h_-+h_+>0$ where the lengths $h_\mp$ of the zone parts before and after y are such that the mode I stress intensity factors are zero at the tips $y\mp h_\mp\eta(\theta)$ of Γ, and Γ is continuously deformable to zero such that the mode I stress intensity factors are positive at the tips of each intermediate (deformed) zone $\Gamma^\vee$:

$$K^{(\Gamma)}_{1\mp}(\Gamma)=0;\qquad K^{(\Gamma)}_{1\mp}(\Gamma^\vee)>0,\quad\bar{\Gamma}^\vee\subset\Gamma(y,\theta;\sigma) \tag{4}$$

(C) The strength condition is given by (2): the crack opening at all points x of $\Gamma(y,\theta;\sigma)$ is less than a critical opening δ_c (material parameter).

Let $l_\mp(y,\theta)$ be the distances from y to the nearest boundary points in the directions $\mp\eta(\theta)$. It may happen that there exists no any cohesive zone Γ satisfying point (B) above such that $h_\mp(y,\theta)<l_\mp(y,\theta)$ since $K^{(\Gamma)}_{1\mp}(\Gamma^\vee)>0$ for all $\Gamma^\vee\subset(y-l_-(y,\theta),y+l_+(y,\theta))$. One can mechanically interpret this as the cohesive zone $\Gamma(y,\theta;\sigma)$ extending between the boundary points $y-l_-(y,\theta)\eta(\theta)$ and $y+l_+(y,\theta)\eta(\theta)$, that is, one should take $h_\mp(y,\theta;\sigma)=l_\mp(y,\theta)$. For such case (when the cohesive zone spread between two boundary points where tractions are prescribed) the cohesive zone tips $y \mp l_{\mp}(y,\theta)\eta(\theta)$ are open and conditions $K_{1\mp}^{(\Gamma)}(\Gamma)=0$, are satisfied. Cases when only one zone tip come out to a boundary point can be also analysed. Strength condition (2) is applicable also to all those situations.

The fracture criterion for the point y is given by the same condition (3) and the fracture direction $\eta(\theta)$ is given by the angle θ that realises the supremum in (3). The fracture appearance can be attributed either to the point x, at which the supremum in (3) is realised, or to the whole of the zone Γ. Similar to the previous section, second condition of (4) ensures that no any cohesive zone can occur from a point y in a direction θ where $\sigma_{\theta\theta}(y,\theta)<\sigma_c$.

Note that strength condition (4), (2) (and (1), (2) for a boundary point y) is not generally applicable if $K_{1\mp}^{(\Gamma)}(\Gamma^\vee)$ is not continuous in $\Gamma^\vee$. A generalized δ_c strength condition applicable also to those situations can be presented in the form $\sup_{\Gamma\subset\mathcal{G}(y,\theta;\sigma)}\{\sup_{x\in\Gamma}[u_\theta^{(\Gamma)}(x)]\}<\delta_c$, where $\mathcal{G}(y,\theta;\sigma)$ is a set of all cohesive zones $\Gamma(y,\theta;\sigma)$ that are continuously deformable to zero such that $K_{1\mp}^{(\Gamma)}(\Gamma^\vee)\geq 0$ for each intermediate (deformed) zone $\Gamma^\vee\subset\Gamma(y,\theta;\sigma)$. This form of the δ_c strength condition can be also used for a three-dimensional body with a plane cohesive zone.

4. Generalized δ_c model in terms of the original stress field

The strength conditions may be represented in terms of the original stress field. We will give the representation for the δ_c model for boundary points, the reasoning for internal points is similar.

We can write a decomposition $u_i^{(\Gamma)}(x)=u_i(x)+\tilde{u}_i^{(\Gamma)}(x)$, $\sigma_{ij}^{(\Gamma)}(x)=\sigma_{ij}(x)+\tilde{\sigma}_{ij}(x)$. Here σ_{ij} and u_i are original stress and displacement fields in the body D without cohesive zones. Then the auxiliary fields $\tilde{u}_i^{(\Gamma)}(x)$, $\tilde{\sigma}_{ij}^{(\Gamma)}(x)$ satisfy the homogeneous elasticity equations in D, the corresponding homogeneous boundary conditions on the boundary ∂D, and the boundary conditions

$$\tilde{\sigma}_{\theta\theta}^{(\Gamma)}(x)\Big|_\Gamma=\sigma_c-\sigma_{\theta\theta}(x)\Big|_\Gamma,\quad \tilde{\sigma}_{\rho\theta}^{(\Gamma)}(x)\Big|_\Gamma=-\sigma_{\rho\theta}(x)\Big|_\Gamma \tag{5}$$

on Γ. Then $K_{1+}^{(\Gamma)}(\Gamma)=K_1(y+h\eta(\theta))+\tilde{K}_{1+}^{(\Gamma)}(\Gamma)$, $\quad [u_\theta^{(\Gamma)}(x)]=[u_\theta(x)]+[\tilde{u}_\theta^{(\Gamma)}(x)]$.

The stress intensity factor $K_1(y+h\eta(\theta))$ equals to zero since the original stress field $\sigma_{ij}(x)$ is bounded at the point $y+h\eta(\theta)$. The jump $[u_\theta(x)]$ equals to zero for $x\in\Gamma$ since the original displacement field $u_i(x)$ is continuous on Γ. Taking into account the linear dependence of $\tilde{K}_{1+}^{(\Gamma)}$ and $\tilde{u}_i^{(\Gamma)}(x)$ on the right hand sides of the boundary conditions (5), we can write strength condition (1), (2) in the form

$$\tilde{K}_{1+}^{(\Gamma)}(\Gamma;-\sigma_{ij}n_j(\theta))=\tilde{K}_{1+}^{(\Gamma)}(\Gamma;-\sigma_c n_i(\theta)),\quad \tilde{K}_{1+}^{(\Gamma)}(\Gamma^\vee;-\sigma_{ij}n_j(\theta))>\tilde{K}_{1+}^{(\Gamma)}(\Gamma^\vee;-\sigma_c n_i(\theta))\quad \forall\,\bar{\Gamma}^\vee\subset\Gamma; \tag{6}$$

$$[\tilde{u}_\theta^{(\Gamma)}(x;\Gamma;-\sigma_{ij}n_j(\theta))]<\delta_c+[\tilde{u}_\theta^{(\Gamma)}(x;\Gamma;-\sigma_c n_i(\theta))]\ \forall x\in\Gamma(y,\theta;\sigma) \tag{7}$$

where $n_i^{\pm}(\theta)$ are the external normals on the $\Gamma(y,\theta;\sigma)$ shores.

Only the original stress field σ_{ij} is involved in (6), (7) (in addition to the functions depending on the material parameter σ_c and the body geometry). Thus, inequality (7) may be considered as a non–local strength condition where the dimension $h=|\Gamma(y,\theta;\sigma)|$ is determined by (6) if $\sigma_{\theta\theta}(y,\theta)>\sigma_c$, and $[\tilde{u}_\theta^{(\Gamma)}(x)]=0$ is taken otherwise.

Let $N_{i1+}(\xi;\Gamma)$ be the Green–type functions (Bueckner functions) for the body D with the crack Γ, which give the values of the stress intensity factor K_1 induced at the crack tip $y+h\eta(\theta)$ by two unit oppositely directed concentrated forces, applied to the opposite crack shores at a point $\xi\in\Gamma(y,\theta)$. Let $[U_{i\theta}](\xi;x;\Gamma)$ be the Green–type functions, which give the displacement jump $[u_i(x)]$ induced at the crack by the same unit forces. Then

$$\tilde{K}_{1+}^{(\Gamma)}(\Gamma;-\sigma_{ij}n_j(\theta)) = \int_\Gamma[N_{\rho1+}(\xi;\Gamma)\sigma_{\rho\theta}(\xi)+N_{\theta1+}(\xi;\Gamma)\sigma_{\theta\theta}(\xi)]d\xi, \tag{8}$$

$$\tilde{K}_{1+}^{(\Gamma)}(\Gamma;-\sigma_c n_i(\theta)) = \sigma_c\int_\Gamma N_{\theta1+}(\xi;\Gamma)d\xi, \tag{9}$$

$$[\tilde{u}_\theta^{(\Gamma)}(x;\Gamma;-\sigma_{ij}n_j(\theta))] = \int_\Gamma\{[U_{\theta\rho}](\xi;x;\Gamma)\sigma_{\rho\theta}(\xi)+[U_{\theta\theta}](\xi;x;\Gamma)\sigma_{\theta\theta}(\xi)\}d\xi, \tag{10}$$

$$[\tilde{u}_\theta^{(\Gamma)}(x;\Gamma;-\sigma_c n_i(\theta))] = \sigma_c\int_\Gamma[U_{\theta\theta}](\xi;x;\Gamma)d\xi. \tag{11}$$

Substitution of (8)–(11) represents strength condition (6)–(7) in a more explicit form.

5. Functional form of the δ_c non–local strength condition

For a given stress field $\sigma_{ij}(x)$, the functional safety factor $\underline{\lambda}(\sigma; y, \theta))$ (see [1]) is a solution λ^* of the equation $\sup_{x\in\Gamma(y,\theta;\lambda^*\sigma)}[u_\theta^{(\Gamma)}(x;\lambda^*\sigma)] = \delta_c$ if its left hand side is a continuous function of λ^* monotonously growing from zero when λ^* grows from zero. Similar to (7), this equation can be presented in the form

$$\sup_{x\in\Gamma^*(y,\theta)} \left\{[\tilde{u}_\theta^{(\Gamma)}(x;\Gamma^*;-\lambda^*\sigma_{ij}n_j(\theta))]/\{\delta_c + [\tilde{u}_\theta^{(\Gamma)}(x;\Gamma^*;-\sigma_c n_i(\theta))]\}\right\} = 1 \tag{12}$$

Let y be a boundary point. According to (6), the cohesive zone $\Gamma^*(y,\theta) := \Gamma(y,\theta;\lambda^*\sigma)$ (i.e., its length h^*) is such that

$$\tilde{K}_{1+}^{(\Gamma)}(\Gamma^*;-\lambda^*\sigma_{ij}n_j(\theta)) = \tilde{K}_{1+}^{(\Gamma)}(\Gamma^*;-\sigma_c n_i(\theta)), \quad \tilde{K}_{1+}^{(\Gamma)}(\Gamma^\vee;-\lambda^*\sigma_{ij}n_j(\theta)) > \tilde{K}_{1+}^{(\Gamma)}(\Gamma^\vee;-\sigma_c n_i(\theta)) \quad \forall\, \bar{\Gamma}^\vee \subset \Gamma^*. \tag{13}$$

Using (12) and linearity of $[\tilde{u}_\theta^{(\Gamma)}]$ and $\tilde{K}_{1+}^{(\Gamma)}$ in λ^*, conditions (13) can be rewritten in a form, independent of λ^*:

$$\begin{aligned}
\inf_{x\in\Gamma^*(y,\theta)} \left\{\frac{\delta_c + [\tilde{u}_\theta^{(\Gamma)}(x;\Gamma^*;-\sigma_c n_i(\theta))]}{[\tilde{u}_\theta^{(\Gamma)}(x;\Gamma^*;-\sigma_{ij}n_j(\theta))]}\right\} \tilde{K}_{1+}^{(\Gamma)}(\Gamma^*;-\sigma_{ij}n_j(\theta)) &= \tilde{K}_{1+}^{(\Gamma)}(\Gamma^*;-\sigma_c n_i(\theta)), \\
\inf_{x\in\Gamma^*(y,\theta)} \left\{\frac{\delta_c + [\tilde{u}_\theta^{(\Gamma)}(x;\Gamma^*;-\sigma_c n_i(\theta))]}{[\tilde{u}_\theta^{(\Gamma)}(x;\Gamma^*;-\sigma_{ij}n_j(\theta))]}\right\} \tilde{K}_{1+}^{(\Gamma)}(\Gamma^\vee;-\sigma_{ij}n_j(\theta)) &> \tilde{K}_{1+}^{(\Gamma)}(\Gamma^\vee;-\sigma_c n_i(\theta)) \quad \forall\, \bar{\Gamma}^\vee \subset \Gamma^*.
\end{aligned} \tag{14}$$

Then

$$\underline{\lambda}(\sigma;y,\theta) = \frac{1}{\underline{\Lambda}(\sigma;y,\theta)} = \inf_{x\in\Gamma^*(y,\theta)} \left\{\frac{\delta_c + [\tilde{u}_\theta^{(\Gamma)}(x;\Gamma^*;-\sigma_c n_i(\theta))]}{[\tilde{u}_\theta^{(\Gamma)}(x;\Gamma^*;-\sigma_{ij}n_j(\theta))]}\right\} = \frac{\tilde{K}_{1+}^{(\Gamma)}(\Gamma^*;-\sigma_c n_i(\theta))}{\tilde{K}_{1+}^{(\Gamma)}(\Gamma^*;-\sigma_{ij}n_j(\theta))}$$

and the strength condition has the form

$$\underline{\Lambda}(\sigma;y,\theta) < 1,$$

where Γ^* is a solution of (14).

Acknowledgements

This work was completed under the research grant GR/M24592 of the Engineering and Physical Sciences Research Council, UK.

6. References

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