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Concept of Normalised Equivalent Stress Functionals for Cyclic Fatigue

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Abstract

Generalised fatigue durability diagrams and their properties are considered for a material under multi-axial loading given by a (non-regularly) oscillating function of time. Material strength and durability under such loading is described in terms of durability and normalised equivalent stress. Relations between these functionals are analysed. Phenomenological strength conditions are presented in terms of the normalised equivalent stress. It is shown that the damage based fatigue durability analysis is reduced to a particular case of such strength conditions. Examples of the reduction are presented for some known durability models. Some complex strength conditions applicable to the durability description at fatigue, creep, dynamic loading and their combination are presented. The normalised equivalent stress functional interpolation along classical periodic S–N diagrams is introduced.

Keywords: Durability; Strength conditions; Endurance limit; Cyclic loading

Nomenclature

$\tau,\ t$ = the natural continuous time;

$\sigma(\tau)$ = a uniaxial or multiaxial loading process in time;

$\sigma_{ij}(\tau)$ = a multiaxial loading process in time;

$m,\ n,\ j$ = a (quasi–)cycle number, discrete time;

$\sigma^{ct}(m) = \{\sigma_{ij}(\tau); \tau_{m-1} \le \tau \le \tau_m\}$ = a (quasi–)cycle with time parametrisation;

$\sigma^{cs}(m)$ = a (quasi–) cycle as an oriented loop in the stress space without time parametrisation;

$\sigma^{c}(m) = \sigma^{ct}(m)$ or $\sigma^{cs}(m)$

$\{\sigma^c(m)\}_{m=1,2,\ldots} = \{\sigma(m)\}_{m=1,2,\ldots} = \{\sigma^c\} = \{\sigma\}$ = a (quasi–)cyclic loading process;

$\sigma_{max}(j)$ = maximum stress in j-th cycle;

$\sigma_{min}(j)$ = minimum stress in j-th cycle;

$\sigma_a(j) = [\sigma_{max}(j) - \sigma_{min}(j)]/2$ = stress amplitude in j-th cycle;

$\sigma_m(j) = [\sigma_{max}(j) + \sigma_{min}(j)]/2$ = mean stress in j-th cycle;

$R = \frac{\sigma_m - \sigma_a}{\sigma_m + \sigma_a}$ = cycle asymmetry ratio;

σ_a^{max} = maximum stress amplitude in a loading process;

σ_{ij}^0 = constant stress tensor;

A_R; b_R; $\tilde{A}_R$ = material parameters in the Wöhler S–N diagram;

$\sigma_i = \frac{1}{\sqrt{2}}\sqrt{(\hat{\sigma}_1-\hat{\sigma}_2)^2+(\hat{\sigma}_2-\hat{\sigma}_3)^2+(\hat{\sigma}_3-\hat{\sigma}_1)^2}$, von Mises intensity of a stress tensor σ_{kj} possessing principal stresses $\hat{\sigma}_1 > \hat{\sigma}_2 > \hat{\sigma}_3$;

$\sigma_{i,a}$ = von Mises intensity of a stress amplitude tensor in multiaxial loading process;

$\sigma_{i,m}$ = von Mises intensity of a mean stress tensor in multiaxial loading process;

σ_r = strength under monotone uniaxial loading;

$\sigma_R^*(n)$ = stress amplitude to reach rupture on n-th cycle under uniaxial periodic loading with asymmetry factor R;

σ_{eq} = equivalent stress amplitude e.g. von Mises or Tresca;

$n^*\{\sigma\}$ = cyclic durability under a process $\{\sigma(m)\}_{m=1,2..}$;

$t^*(\sigma)$ = durability under a process $\{\sigma(\tau)\}$;

$n_R^*(\sigma_a)$ = number of cycles up to rupture under periodic loading with stress amplitude σ_a and cyclic asymmetry factor R;

λ = a non-negative number;

$\underline{\lambda}^N(\{\sigma\};n)$ = (a value of) the cyclic safety factor functional;

$\underline{\Lambda}^N(\{\sigma\};n)$ = (a value of) the cyclic normalized equivalent stress functional, NESF;

$\sigma_{-1\infty}^*$ = fatigue limit amplitude under uniaxial reversing ($R = 0$) tension-compression loading;

$\sigma_{0\infty}^*$ = fatigue limit (amplitude) under repeated (with zero minimum stress) tension loading;

$\sigma_{H,m}$ = mean cyclic hydrostatic pressure;

$\sigma_{H,max}$ = maximum cyclic hydrostatic pressure;

S_{ij} = stress deviator;

P = period in a periodic process;

α, β = material constants in critical surface criteria;

$\sigma_{\eta\eta,a}(\vec{\eta},\tau)$ = normal stress amplitude on a plane with a normal $\vec{\eta}$;

$\tau_a(\vec{\eta},\tau)$ = shear stress amplitude acting in on a plane with a normal $\vec{\eta}$;

$\sigma_{\eta\xi,a}$ = shear stress acting in a direction $\vec{\xi}$ on a plane with a normal $\vec{\eta}$;

$|\sigma|$ = a matrix norm of a tensor σ_{ij}, e.g. $|\sigma| = \sqrt{\sum_{i,j=1}^3 \sigma_{ij}^2}$;

$\|\sigma^c(m)\|$ = a norm of a tensor function $\sigma^c(m)$ on the $m-$th cycle, e.g. $\|\sigma^c(m)\| = \sup_{\sigma\in\sigma^c(m)}|\sigma|$;

$\tilde{\sigma}^c(m) = \sigma^c(m)/\|\sigma^c(m)\|$ = normalised shape of a tensor function $\sigma^c(m)$ on the $m-$th cycle;

$|\vec{S}|$ = Euclidian norm of a vector (the vector length).

1 Introduction

In the traditional approach to cyclic strength of a material, the fatigue durability is evaluated from the inequality

$$\omega(\{\sigma\}, n) < 1 \tag{1}$$

where the damage measure $\omega(\{\sigma\}, n)$ is a functional of the loading history $\{\sigma(m)\}_{m=1,2,...}$ and m, n are the cycle numbers. When the number of cycles is sufficiently large such that inequality (1) is violated (transfers into the equality), a material rupture occurs. A particular form of $\omega(\{\sigma\}, n)$ for a non-periodic cycling (i.e. with varying cycles $\sigma(m)$) is usually related to a particular damage accumulation law. For example,

$$\omega(\{\sigma\}, n) = \sum_{m=1}^{n} \frac{1}{n^{*p}[\sigma(m)]} \tag{2}$$

for a material obeying the Palmgren-Miner hypothesis of linear damage accumulation, where $n^{*p}(\sigma)$ is the number of cycles to rupture in the *periodic* process, given by an appropriate S–N Wöhler diagram with all the cycles equal to $\sigma(m, y)$. By this way, a durability under non-periodic cyclic process is related to experimental data for corresponding periodic cyclic processes. However the linear accumulation rule does not work for some cyclic processes and many attempts to improve it lead to some abstract damage measures that one can not obtain (or verify) from direct durability experiments. Even more methodical problems appear when the process under consideration is presented by *random (with unclosed cycles)* oscillations where the known procedures for a reduction of such processes to cyclic ones (e.g. the rain flow method) give not accurate prediction or not applicable at all (say, for out-of-phase random multiaxial oscillations).

To overcome those difficulties with processes sufficiently general in time, appearing not only for oscillating but also for dynamic and creep loading, and to create a more robust and experimentally verifiable tool for durability analysis, some theoretical backgrounds of a functional approach to durability description under a loading program (process) $\sigma_{ij}(\tau)$, which is a (tensor) function of time τ, were considered by Mikhailov (1999, 2000). However, the time dependence is not essential for the durability description of the materials, whose rupture depends only on the loading sequence but not on time itself or on the loading process rate (for considered loading conditions). The pure fatigue rupture (without creep and ageing) is an example of such behavior. Although such processes may still be considered with respect to time as a natural parameter and the approach described by Mikhailov (1999, 2000) holds true, some other parameterizations seem to be more relevant for fatigue.

The (quasi–) cycle number m can be considered then as the so-called *discrete time* (see e.g. Bolotin 1989) and the main issues developed by Mikhailov (1999, 2000) for the natural continuous time will be adopted here for the discrete time. We arrive after this at the notions of durability (number of cycles to rupture) $n^*\{\sigma\}$, generalised Wöhler S–N diagram $\lambda \mapsto n^*\{\lambda\sigma\}$, functional safety factor $\underline{\lambda}^N(\{\sigma\}, n)$ and normalised equivalent stress functional $\underline{\Lambda}^N(\{\sigma\}, n)$ defined on a loading process $\{\sigma(m)\}_{m=1,2,...}$. This leads to a fatigue strength and durability phenomenological description in terms of these mechanically meaningful and experimentally identifiable functionals without necessary involvement of a geometrical, stiffness-related or abstract damage measure. On the other hand, this approach allows rather simple deviation from a standard linear Palmgren-Miner damage accumulation rule to describe better the experimental observations.

Fatigue under stress fields independent of the space coordinates is mainly analysed in this paper with obvious reasoning about application to moderately inhomogeneous stress fields . Extension to highly inhomogeneous stress field incorporating a non-local approach by Mikhailov (1995) will be considered elsewhere (Mikhailov & Namestnikova 2002).

2 Cyclic and quasi-cyclic processes and their parametrisations

We will call a multiaxial process $\sigma_{ij}(\tau)$, $\tau \geq 0$ *cyclic* if it can be represented as a *temporal* sequence of connected loops (cycles) $\sigma^{ct}(m) = \{\sigma_{ij}(\tau); \tau_{m-1} \leq \tau \leq \tau_m\}$ closed in the stress space, $\sigma_{ij}(\tau_m) = \sigma_{ij}(\tau_{m-1})$, where $m = 1, 2, ...$ is the loop number. Here the notation $\sigma^{ct}(m)$ includes information on the loop position $[\tau_{m-1}, \tau_m]$ in time, the loop parametrisation with respect to time and consequently the loop shape in the stress space. If $\tau_m - \tau_{m-1} = P = const$ for $m = m_1, ...m_2$ and $\sigma_{ij}(\tau + P) = \sigma_{ij}(\tau)$, $\tau_{m_1-1} \leq \tau,\ \tau + P \leq \tau_{m_2}$, we call the process $\{\sigma^{ct}(m)\}_{m=1,2,...}$ *periodic* on $\tau \in [\tau_{m_1-1}, \tau_{m_2}]$ with the period P.

If a *uniaxial* process $\sigma(\tau)$ is randomly (not cyclically) oscillating, like in Fig. 1, we can denote the local minima or local maxima or middle points between them as τ_m and call the parts of the process $\sigma(\tau)$ on the segments $[\tau_{m-1}, \tau_m]$ *quasi–cycles* $\sigma^{ct}(m)$. If

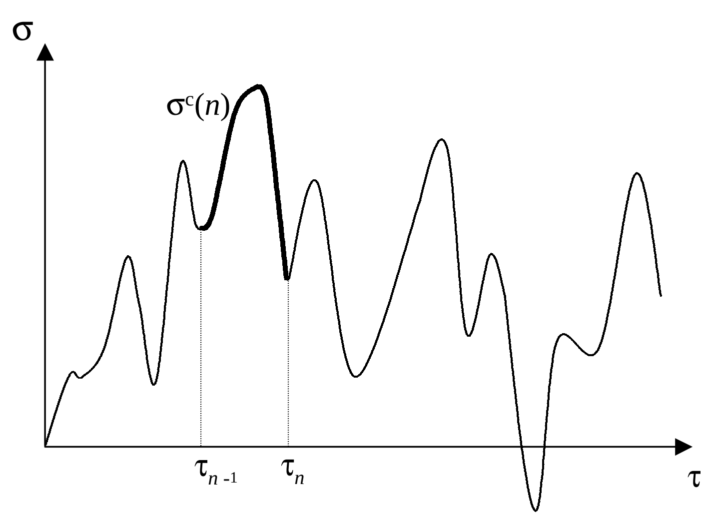

Figure 1: A parametrisation of a randomly oscillating process

$\sigma_{ij}(\tau)$ is a *multiaxial in-phase (proportional)* process, i.e., $\sigma_{ij}(\tau) = \sigma^0_{ij} f(\tau)$, where σ^0_{ij} is a constant tensor and $f(\tau)$ is a scalar function, we can associate the quasi–cycle boundaries τ_m with the local minima or local maxima or local middle points of the function $f(\tau)$. If $\sigma_{ij}(\tau)$ is a *general multiaxial* process, we can distribute points τ_m by a more or less reasonable way, such that $\tau_0 = 0$, $\tau_m > \tau_{m-1}$, and call the parts of the process $\sigma_{ij}(\tau)$ on the time segments $[\tau_{m-1}, \tau_m]$ *quasi–cycles* $\sigma^{ct}(m)$. After such a parameterization we can consider any process as quasi–cyclic. Note that since a (quasi–) cycle $\sigma^{ct}(m)$ includes all the information about the process $\sigma_{ij}(\tau)$ behavior on the segment $\tau_{m-1} \leq \tau \leq \tau_n$, the temporal sequence $\{\sigma^{ct}(m)\}_{m=1,2,...}$ is equivalent to the original form $\sigma_{ij}(\tau)$, $0 \le \tau$, of the process presentation.

However, the particular values of the instants τ_m as well as the natural time parametrisation $\sigma_{ij}(\tau)$ of a loop $\sigma^{ct}(m)$ is not essential for pure fatigue, for which only an order of the loops, their shapes and orientations in the stress space is essential. To deal with such less informative presentation of loading processes, we will consider also the sequence $\{\sigma^{cs}(m)\}_{m=1,2,...}$ of connected oriented (quasi–) loops $\sigma^{cs}(m)$ in the *stress space*, for which a particular time parametrisation is not prescribed. If the loops coincide in the stress space, $\sigma^{cs}(m) = \sigma^{cs}$, for $m = m_1, ...m_2$, we will say the loading is *periodic* in the block of (quasi–) cycles $[m_1, m_2]$. Evidently, $\{\sigma^{cs}(m)\}_{m=1,2,...}$ is uniquely defined by $\{\sigma^{ct}(m)\}_{m=1,2,...}$ but inverse is not true. For example, two different temporal cyclic processes:

$$\begin{aligned}
\{\sigma^{ct(1)}(m)\}_{m=1,2,...} &= \{\sin(\tau);\ 2\pi(m-1) \le \tau \le 2\pi m\}_{m=1,2,...} \\
\{\sigma^{ct(2)}(m)\}_{m=1,2,...} &= \{\sin(\tau^2);\ \sqrt{2\pi(m-1)} \le \tau \le \sqrt{2\pi m}\}_{m=1,2,...}
\end{aligned}$$

define one and the same stress space loop sequences $\{\sigma^{cs(1)}(m)\}_{m=1,2,...} = \{\sigma^{cs(2)}(m)\}_{m=1,2,...}$, where each loop lies on the segment $-1 \le \sigma \le 1$ in the one-dimensional stress space.

Let $\{\sigma\}$ be a uniaxial *cyclic regular* process, i.e. $\sigma(\tau)$ is defined in each instant τ of the cycle and has not more than one internal local infimum and one internal local supremum on each cycle. Let us denote by $\sigma_{\max}(m)$ and $\sigma_{\min}(m)$ the global supremum and infimum of $\sigma(\tau)$ during a cycle m including its start and end points, by $R(m) = \sigma_{\min}(m)/\sigma_{\max}(m)$ the cycle asymmetry ratio, by $\sigma_a(m) = (\sigma_{\max}(m) - \sigma_{\min}(m))/2$ the stress cycle amplitude and by $\sigma_m(m) = (\sigma_{\max}(m) + \sigma_{\min}(m))/2$ the stress cycle mean value. Then the sequence $\{\sigma^{cs}(m)\}_{m=1,2,...}$ is equivalently presented by the sequences $\{\sigma_{\max}(m), \sigma_{\min}(m)\}_{m=1,2,...}$, $\{\sigma_a(m), R(m)\}_{m=1,2,...}$, $\{\sigma_a(m), \sigma_m(m)\}_{m=1,2,...}$ or $\{\sigma_m(m), R(m)\}_{m=1,2,...}$.

Let $\sigma_{ij}(\tau)$ be a *multiaxial in-phase (proportional) cyclic regular* process, i.e., $\sigma_{ij}(\tau) = \sigma^0_{ij} f(\tau)$, where σ^0_{ij} is a constant tensor and $f(\tau)$ is a scalar function defined in each instant τ of the cycle, which has not more than one internal local infimum and one internal local supremum on each cycle. Let us denote by $f_{\max}(m)$ and $f_{\min}(m)$ the global supremum and infimum of $f(\tau)$ during the cycle including its start and end points, by $R(m) = f_{\min}(m)/f_{\max}(m)$ the cycle asymmetry ratio, and by

$$\begin{aligned}
\sigma_{ij,\max}(m) = \sigma^0_{ij} f_{\max}(m), &\qquad \sigma_{ij,\min}(m) = \sigma^0_{ij} f_{\min}(m), \\
\sigma_{ij,a}(m) = \sigma^0_{ij}(f_{\max}(m) - f_{\min}(m))/2, &\qquad \sigma_{ij,m}(m) = \sigma^0_{ij}(f_{\max}(m) + f_{\min}(m))/2
\end{aligned} \tag{3}$$

the stress cycle maximum, minimum, amplitude and mean value tensors, respectively. Then similar to the previous paragraph, the sequence $\{\sigma^{cs}(m)\}_{m=1,2,...}$ is equivalently presented by the sequences $\{\sigma_{ij,\max}(m), \sigma_{ij,\min}(m)\}_{m=1,2,...}$, $\{\sigma_{ij,a}(m), R(m)\}_{m=1,2,...}$, $\{\sigma_{ij,a}(m), \sigma_{ij,m}(m)\}_{m=1,2,...}$ or $\{\sigma_{ij,m}(m), R(m)\}_{m=1,2,...}$.

We will write $\{\sigma^c(m)\}_{m=1,2,...}$ when considering the both presentations $\{\sigma^{ct}(m)\}_{m=1,2,...}$ and $\{\sigma^{cs}(m)\}_{m=1,2,...}$ at the same time. We will further omit sometimes the superscript c and write $\sigma^t(m)$, $\sigma^s(m)$ or $\sigma(m)$ for a (quasi–) cycle if this lead to no confusion. Sometimes we denote a process $\{\sigma^c(m)\}_{m=1,2,...}$ as $\{\sigma^c\}$ or $\{\sigma\}$.

3 (Quasi–) Cyclic Durability and Generalised S–N Diagram

Let a material undergo a multiaxial quasi–cyclic loading process $\{\sigma(m)\}_{m=1,2,...}$. We will discuss here rupture without specifying the rupture type and only assume that (i) one can unambiguously detect at the end of any (quasi–) cycle whether the material is ruptured or not, and (ii) if the material is ruptured during a (quasi–) cycle n_1, it will be ruptured also at any $n_2 > n_2$ (no repairing mechanism). The (quasi–) cycle number $n^*\{\sigma\}$, during which a rupture for the material appears under a loading process $\{\sigma\}$ is called *(quasi–) cyclic durability* or *(quasi–) cyclic life time*. For different loading processes $\sigma^1_{ij}(m)$, $\sigma^2_{ij}(m)$, $m = 1, 2, ...$, the durability has generally different values $n^*\{\sigma^1\}$, $n^*\{\sigma^2\}$.

3.1 Classical S–N diagram under periodic loading

Let $\{\sigma\}$ be a uniaxial *periodic regular* process, i.e. a regular cyclic process, where all the cycles $\sigma^c(m)$ are independent of m. Under such a loading, it is usual to determine experimentally *the Wöhler S–N diagram* in the axes $\sigma_a \mapsto n^*_R(\sigma_a)$, where $n^*_R(\sigma_a) = n^*\{\sigma\}$ is the number of cycles up to rupture.

An example of a simple S–N diagram given by a power law (a straight line in the double logarithmic coordinates) is usually written as

$$n^*\{\sigma\} = n^*_R(\sigma_a) = \left[\frac{\sigma^*_{R1}}{\sigma_a}\right]^{-b_R}. \qquad (4)$$

where σ^*_{R1} and b_R are positive material parameters considered as depending generally on the asymmetry ratio R but not on the amplitude σ_a.

Taking into account that $n^*\{\sigma\}$ can take only integer values, this should be rewritten more precisely as

$$n^*\{\sigma\} = n^*_R(\sigma_a) = \mathrm{Int}_+\left(\left[\frac{\sigma^*_{R1}}{\sigma_a}\right]^{-b_R}\right) \qquad (5)$$

where $\mathrm{Int}_+(x)$ is the minimal integer greater or equal to real x. However, both forms (4) and (5) for the durability $n^*\{\sigma\}$ lead to equivalent results when it is used in the strength condition like $n < n^*\{\sigma\}$ for an integer cycle number n, and we will not distinguish the forms sometimes.

If we introduce a notation $\|\sigma^c\| = \max(|\sigma_{max}|, |\sigma_{min}|)$, then the above relation can be rewritten also in the form

$$n^*\{\sigma\} = \mathrm{Int}_+\left(\left[\frac{\check{\sigma}^*_{R1}}{\|\sigma^c\|}\right]^{-b_R}\right] \qquad (6)$$

where $\check{\sigma}^*_{R1} = \sigma^*_{R1max} = \begin{cases} 2\sigma^*_{R1}(1-R\quad), & |R| \le 1 \\ 2\sigma^*_{R1}(1-R^{-1}), & |R| > 1 \end{cases}$ and $b = b_R$.

Let now $\{\sigma\}$ be an *arbitrary periodic* multi-axial cyclic loading process, i.e. the shape of the periodicity loop σ^c is arbitrary. For multiaxial cases, let $|\sigma|$ denote a matrix norm of tensor σ_{ij}, for example, $|\sigma| = \sqrt{\sum_{i,j=1}^3 \sigma^2_{ij}}$. Let $\|\sigma^c(m)\|$ denote a norm of the tensor function σ^c, for example, $\|\sigma^c(m)\| = \sup_{\sigma\in\sigma^c(m)} |\sigma|$. Then power dependence (6) can be used also for general periodic processes if we suppose that $\check{\sigma}^*_{R1} = \check{\sigma}^*_{R1}(\sigma^c/\|\sigma^c\|)$ and

$b = b(\sigma^c/\|\sigma^c\|)$ are positive material parameters depending generally on the cycle σ^c shape in the stress space but not on the cycle norm $\|\sigma^c\|$.

3.2 Generalised S–N diagram under arbitrary loading

To present a *generalised S–N diagram* for an arbitrary multi-axial *quasi–cyclic* process $\{\sigma^c(m)\}_{m=1,\dots}$, let us consider a family of proportional processes $\{\lambda\sigma^c(n)\}_{n=1,\dots}$, obtained from the original process $\{\sigma^c\}$ after its multiplication by a non-negative constant number λ. Then the durability $n^*\{\lambda\sigma\}$ becomes an integer-valued (piece-wise constant) function

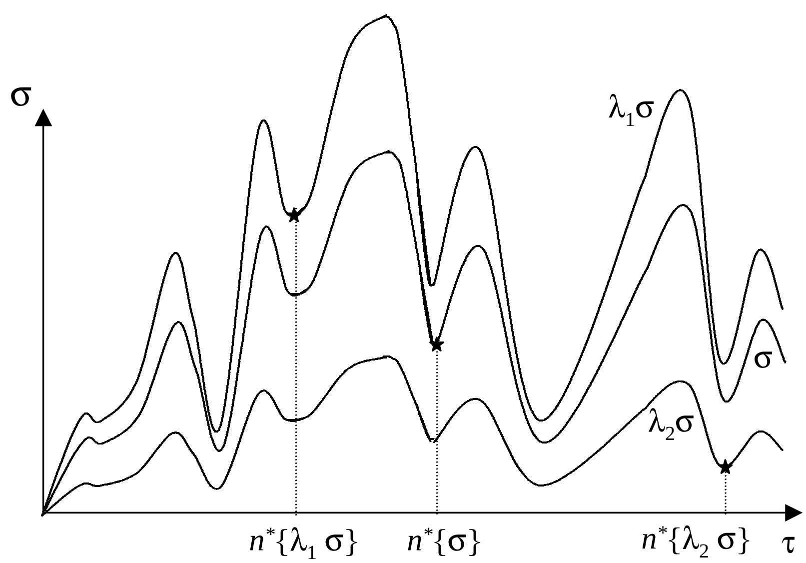

Figure 2: Proportional loading processes and durabilities, $0 < \lambda_2 < 1 < \lambda_1$.

of the parameter λ (see on Fig. 2 an example for a quasi–cyclic uniaxial process, where the quasi–cycle boundaries are associated with the local minima of σ).

The generalised S–N diagram for a process $\{\sigma^c\}$ is the dependence of the durability $n^*\{\lambda\sigma^c\}$ on parameter $\lambda \geq 0$.

The concept propounded in the paper concerns mainly materials, which strength depends on the oscillating loading history but should work also in the particular case of materials independent of history and for rather simple loading processes split into quasi–cycles. We will widely use the latter for illustrations.

Let us consider, for example, a material independent of history (that is, its strength is determined only by the instants stress value) and obeying the strength condition $|\sigma| < \sigma_r$ under uniaxial loading, where σ_r is a material constant. Suppose the material is periodically loaded by a uniaxial process $\{\sigma^c\}$, where some minimal and maximal values σ_{min} and σ_{max} are reached during the cycle σ^c. Then the S–N diagram is given by the constant sequence $\lambda_m = \sigma_r/\|\sigma^c\|,\ m = 1, 2, ...,$ where $\|\sigma^c\| = \max(|\sigma_{max}|, |\sigma_{min}|)$ that is, $n^*\{\lambda\sigma\} = \left\{ \begin{array}{ll} \infty, & \lambda < \sigma_r/\|\sigma^c\| \\ 1, & \lambda \geq \sigma_r/\|\sigma^c\| \end{array} \right\}.$

Suppose the same material is loaded by a uniaxial loading process $\{\sigma^c\}$ such that some minimum $\sigma_{min}(m)$ and maximum $\sigma_{max}(m)$ are reached during the cycle $\sigma^c(m)$ and their values grow linearly with the cycle number, hence $\|\sigma^c(m)\| = \max(|\sigma_{max}(m)|, |\sigma_{min}(m)|) = am$, where a is a constant. Then the S–N diagram is given by a discrete hyperbola $n^*\{\lambda\sigma\} = \text{Int}_+[\sigma_r/(a\lambda)]$.

Let us consider general properties of the *generalised* Wöhler S–N diagram $n^*\{\lambda\sigma\}$ for a material under an arbitrary process $\{\sigma^c\}$. The function $\lambda \mapsto n^*\{\lambda\sigma\}$ is defined on the half axis $\lambda \in [0, \infty)$ and is non-negative and integer-valued. When λ varies, different situations can arise. On Fig. 3a, we plot schematically an S–N diagram $n^*\{\lambda\sigma\}$ (step-like graphs) over corresponding durability diagram $t^*\{\lambda\sigma\}$ (smooth curves, see Mikhailov 2000, Fig. 3a) for the same process. The diagram $t^*\{\lambda\sigma\}$ can be always obtained for a $\{\sigma^{ct}\}$ presentation of the process. Although we consider $n^*\{\lambda\sigma\}$ as a function of λ, the choice of the axes directions on the plot is traditional for the fatigue analysis. The curves a, b, c at large λ and curves d, e, f at small λ present different possible cases of the diagram behaviour, that is, one of the curves a, b or c continues by one of the curves d, e or f for a particular material under a particular loading $\{\sigma\}$.

To be consistent, we keep below the item numbering by Mikhailov (2000).

(A-B): The rupture can occur at $t = t^*(\lambda^0\sigma) = 0$ for a finite but sufficiently large λ^0, curve a on Fig. 3a, or $t^*(\lambda\sigma_{ij})$ can be non-zero at any finite λ but tends to zero as λ tends to infinity, curve b on the Fig. 3a, i.e. $\lambda^0 = \infty$. Since the rupture/strength state of the material is checked only at the (quasi–) cycle ends, when using the discrete time description, the rupture is attributed to the first (quasi–) cycle in the both cases, that is, $n^*\{\sigma\} = 1$ when λ is sufficiently close to λ^0 for the curve a, or sufficiently large for the curve b.

(C): There exist loadings for some materials (or material models), that do not cause rupture however large these loadings are. An example is the uniform three-axes contraction cycling. Suppose a loading process $\sigma_{ij}(\tau)$ is represented by such a loading on a beginning time interval $0 \le \tau \le t_+$ followed by a loading able to cause rupture at some time. Then there is no rupture on $0 \le \tau \le t_+$ for any non-negative λ, curve c on the Fig. 3a, and we can put $\lambda = \lambda^0 = \infty$ on this segment. For the discrete parametrisation, the rupture for some λ can appear not earlier than during a (quasi–) cycle n_+ for which t_+ is either start or internal point, i.e., $\tau_{n_+-1} \le t_+ < \tau_{n_+}$.

Let us consider the fatigue durability behaviour at large n^*, that is, at small λ.

Let $\lambda = 0$. The durability $n^*\{0\}$, when no loading is applied, is either finite or infinite.

(0): The first case means that rupture at $t = t^*(0) < \infty$, that attributed to a (quasi–) cycle $n^*\{0\}$ including that point is caused not by a mechanical load $\{\sigma(m)\}_{m=1,2,...}$ but for another reason, for example, by a previous loading history at $t < 0$. Other possible reasons for such behaviour can be radiation, corrosion or other chemical reactions, dissolution etc., which we can refer to as natural or artificial ageing leading to the complete degradation of the material at the time $t^*(0)$ ((quasi–) cycle $n^*\{0\}$). We call the material *self-degrading* if $t^*(0) < \infty$.

Generally, ageing does not necessarily lead to complete degradation but means that a shift of a loading process in time does not cause the same shift in durability for an ageing material (see Mikhailov 2000). Note that pure fatigue processes can not be ageing or, by other words, pure fatigue models will be not sufficient for materials with strength ageing.

(D): Curve d on Fig. 3a presents the case when the durability $t^*(\lambda\sigma)$ tends to a finite value $t^{*0}(\sigma) \le t^*(0)$ and $n^*\{\lambda\sigma\}$ tends to a finite value $n^{*0}\{\sigma\} \le n^*\{0\}$ as λ tends to 0. For example, this can be the case for a singular stress $\sigma_{ij}(\tau)$ infinitely growing during the

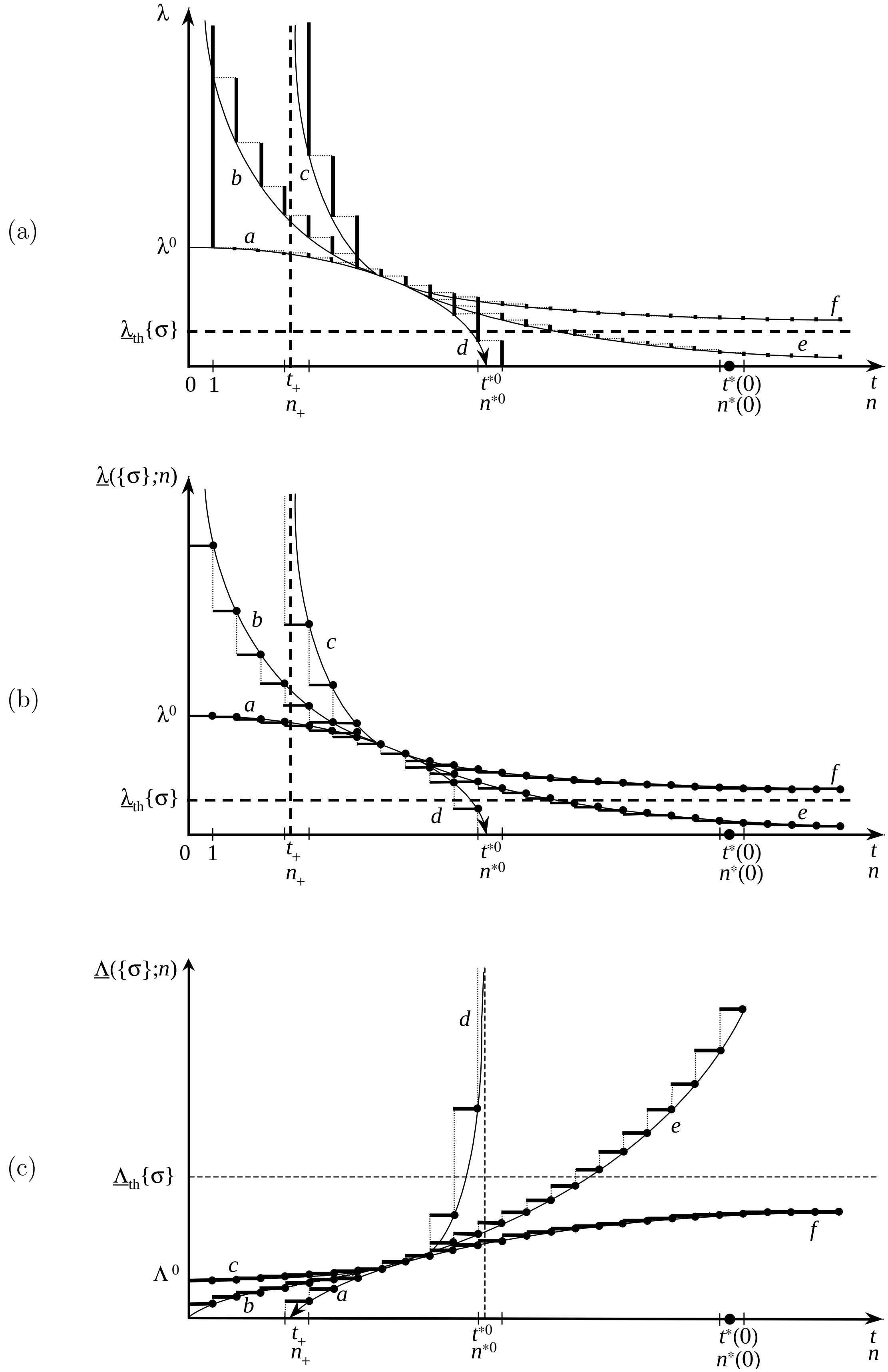

Figure 3: (a) S–N diagram for a process $\{\sigma\}$. (b) Safety factor vs. n and t for the process. (c) Normalised equivalent stress vs. n and t for the process.

quasi–cycle $n^{*0}\{\sigma\}$ as τ tends to $t^{*0} \in (\tau_{n^{*0}-1}, \tau_{n^{*0}}]$, e.g., for $\sigma_{ij}(\tau) = \sigma^0_{ij}/(t^{*0} - \tau)$.

(E): $n^*\{\lambda\sigma\} \to n^{*0}\{\sigma\} = \infty$ as $\lambda \to 0$ and there exists no non-zero threshold, that is, the durability $n^*\{\lambda\sigma\}$ monotonously grows up to infinity with diminishing λ but is always finite at $\lambda > 0$, curve e on Fig. 3a.

(F): $n^*\{\lambda\sigma_{ij}\} \to n^{*0}\{\sigma\} = \infty$ as $\lambda \to 0$ and there exists a threshold value $\lambda_{th}(\sigma) > 0$ such that $n^*\{\lambda\sigma\} = \infty$ for all λ such as $0 \leq \lambda \leq \lambda_{th}\{\sigma\}$ and $n^*\{\lambda\sigma\} < \infty$ for all $\lambda > \lambda_{th}\{\sigma\}$, curve f on Fig. 3a.

(G): $n^*\{\lambda\sigma\}$ has no definite limit $n^{*0}(\sigma)$, this means it is not monotonous as $\lambda \to 0$. This can happen for materials and processes that are not monotonously damaging (see below and Mikhailov 2000).

Cases E and F seem to be most usual in the fatigue durability analysis.

Analysing the S–N diagram for intermediate λ, we should remark that the dependence $n^*\{\lambda\sigma\}$ on λ can be either monotonously non-increasing or not.

If $n^*\{\lambda_1\sigma\} \geq n^*\{\lambda_2\sigma\}$ for any numbers $\lambda_2 > \lambda_1 \geq 0$, the (quasi–) cyclic process will be called monotonously damaging (MD). A material is monotonously damaging if all processes are monotonously damaging for it (see Mikhailov 2000).

Note that there exist materials that are not monotonously damaging. For example, strength and durability of solidifying or cemented materials can be essentially increased, if the contracting (cyclic) loading is increased during the solidification or cementation phase, see Fig. 4.

Further, in addition to the finite jumps along λ axis caused by the discrete numbering of the (quasi–) cycles, the S–N diagrams can have finite jumps along the $n^*\{\lambda\sigma\}$ axis as well. Fig. 5, 6, and 7 give some examples of the loading processes with such features for a material independent of time and history, in which rupture appears at $\sigma = \sigma_r$.

3.3 Quasi– cyclic strength stability in proportional load perturbations

Quasi–cyclic strength is said to be stable with respect to proportional load perturbations ($\lambda-$stable) during a (quasi–) cycle $n < \infty$ under a (quasi–) cyclic process $\{\sigma(m)\}_{m=1,2,...}$ if there is no rupture during and before the (quasi–) cycle n under $\{\sigma\}$ and under slightly higher and lower loadings. More precisely, there exists $\epsilon > 0$ such that there is no rupture at and before the (quasi–) e n under the process $\{\lambda\sigma\}$ for any $\lambda \in (1-\epsilon, 1+\epsilon)$.

This implies that if the strength in a material is $\lambda-$unstable on a (quasi–) cycle n_1, it can not become $\lambda-$stable on any (quasi–) cycle $n_2 > n_1$.

We will denote by $n^{**}\{\sigma\}$ the critical (quasi–) cycle number, that is such that (quasi–) cyclic strength is $\lambda-$stable at all (quasi–) cycles $n < n^{**}\{\sigma\}$ but either rupture or strength $\lambda-$instability exists on the (quasi–) cycle $n = n^{**}\{\sigma\}$.

It is evident, that the critical (quasi–) cycle number $n^{**}\{\sigma\}$ is not greater than the durability $n^*\{\sigma\}$. If $n^{**}\{\sigma\} < n^*\{\sigma\}$, then strength is $\lambda-$unstable on (quasi–) cycles $n \in [n^{**}\{\sigma\}, n^*\{\sigma\} - 1]$. This means, the S–N diagram has at $\lambda = 1$ a horizontal jump from $n^{**}\{\sigma\}$ to $n^*\{\sigma\}$ (see Fig. 7b at $\sigma^\wedge = \sigma_r$).

For $n = \infty$, the above reasoning can be modified by the following way.

Endurance is said to be stable with respect to proportional load perturbations ($\lambda-$stable) under a (quasi–) cyclic process $\{\sigma(m)\}_{m=1,2,...}$, if there is no rupture under $\{\sigma\}$ and under slightly higher and lower loadings at any (quasi–) cycle. More precisely, there exists $\epsilon > 0$

(a)

(b)

(c)

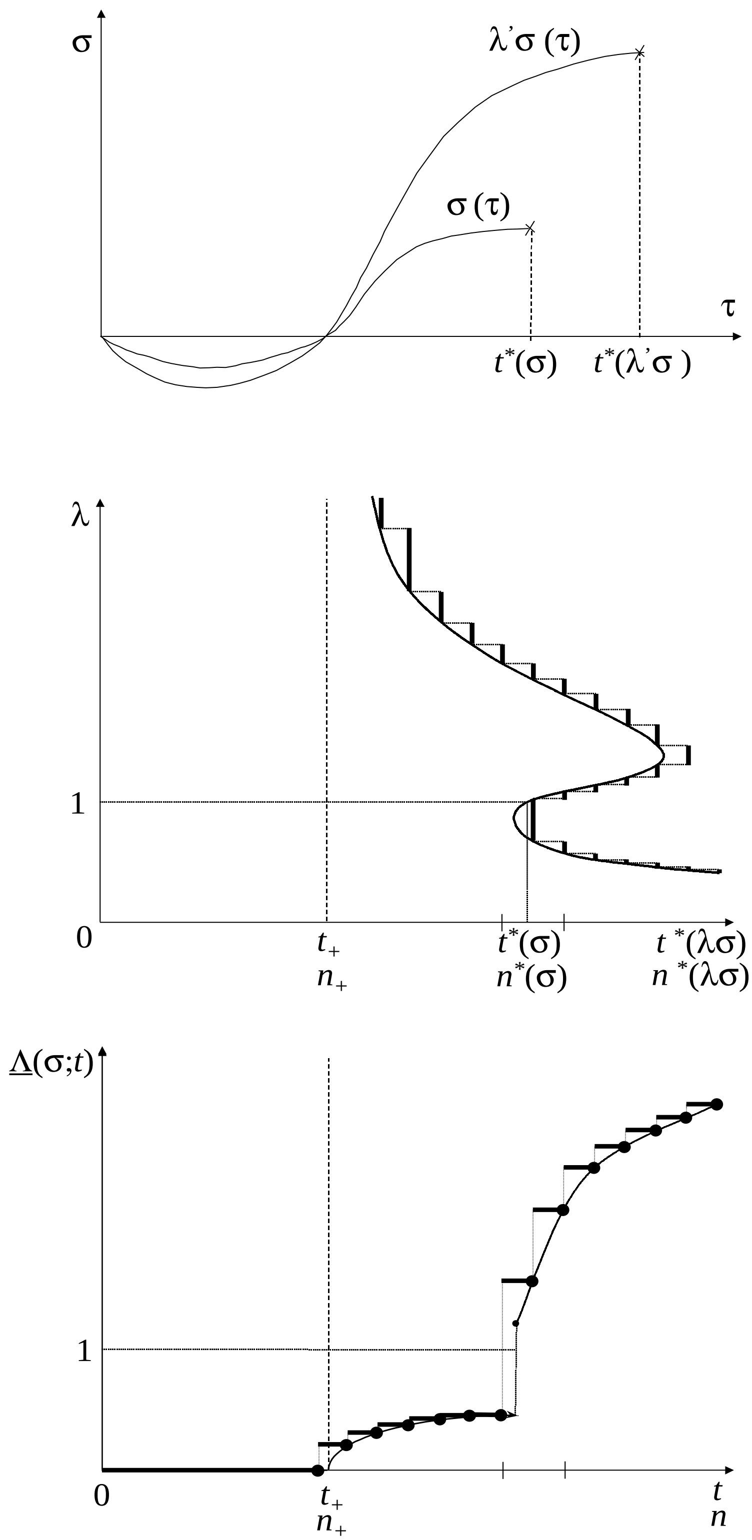

Figure 4: (a) Proportional non-monotonously damaging loading processes for $\lambda = 1$ and $\lambda > 1$. (b) S–N diagram for the process. (c) The normalised equivalent stress for the process.

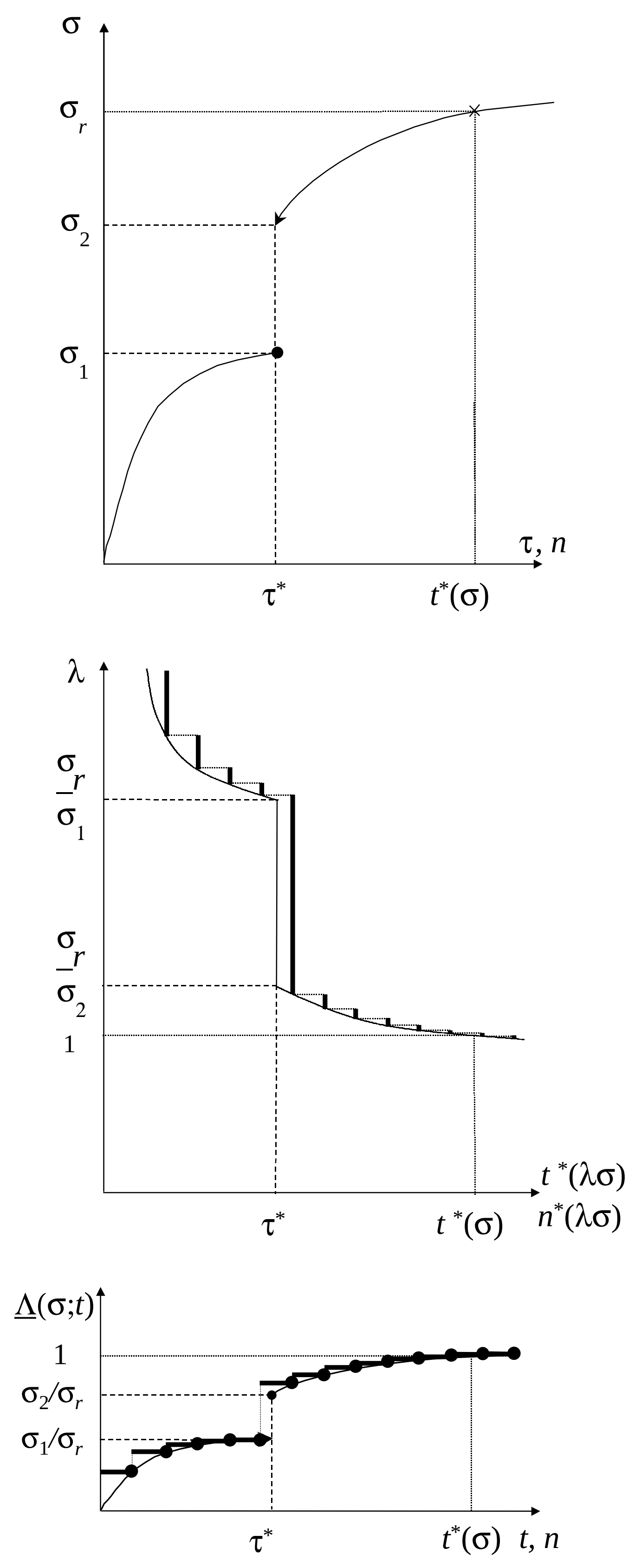

Figure 5: (a) Monotonous piecewise continuous loading process. (b) S–N diagram generated by the process. (c) The normalised equivalent stress for the process.

(a)

(b)

(c)

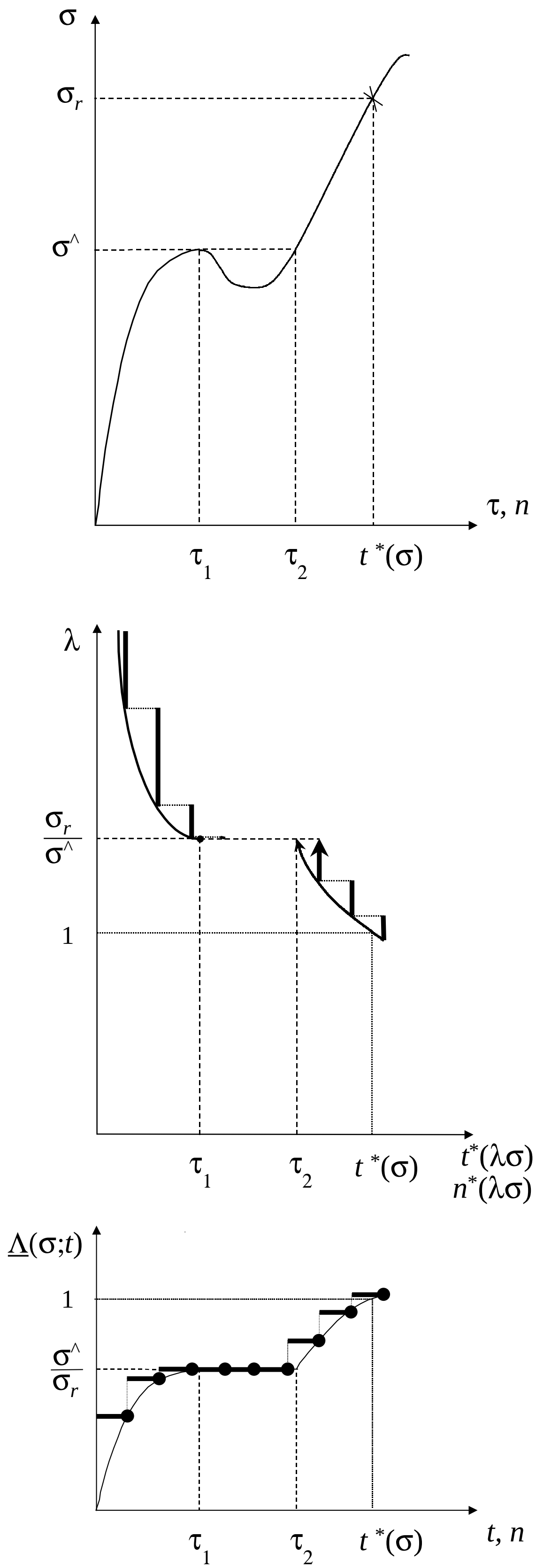

Figure 6: (a) Non-monotonous continuous loading process. (b) S–N diagram generated by the process. (c) The normalised equivalent stress for the process.

(a)

(b)

(c)

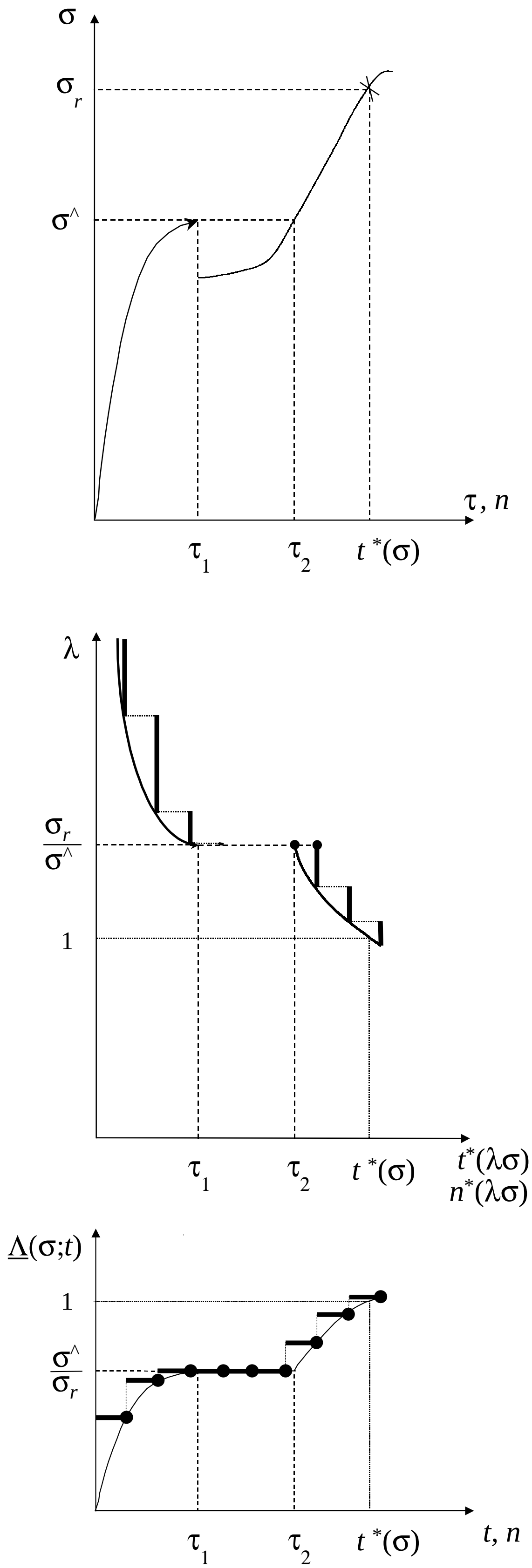

Figure 7: (a) Non-monotonous right-continuous loading process. (b) S–N diagram generated by the process. (c) The normalised equivalent stress for the process.

such that there is no rupture at all (quasi–) cycles $n < \infty$ under the process $\{\lambda\sigma\}$ for any $\lambda \in (1-\epsilon, 1+\epsilon)$.

4 Quasi–cyclic safety factor and normalised equivalent stress

For a given process $\{\sigma(m)\}_{m=1,2,...}$, we can determine (experimentally) a unique finite or infinite durability $n^*\{\lambda\sigma\}$ for any number $\lambda \geq 0$. Consider the inverse task: for any (quasi–) cycle number $n \geq 0$, to determine a number $\lambda^*(\{\sigma\};n)$ such that $n^*\{\lambda^*(\{\sigma\};n)\sigma\} = n$. This is equivalent to interpreting the S–N diagram $\lambda \mapsto n^*\{\lambda\sigma\}$ as the dependence $n \mapsto \lambda^*(\{\sigma\};n)$. Examples of the diagrams on Fig. 3a, 4b, 5b, 6b, 7b show this is not uniquely possible and for some cases is not possible at all since the dependence is not defined for some (quasi–) cycles n. To overcome the difficulty, we introduce a notion of *(quasi–) cyclic safety factor functional* and *(quasi–) cyclic normalised equivalent stress functional* (NESF).

Definition 1 *Let $\{\sigma(m)\}_{m=1,2,...}$ be a quasi–cyclic process. The (quasi–) cyclic safety factor $\underline{\lambda}^N(\{\sigma\};n)$ is supremum of $\lambda \geq 0$ such that $n^*\{\lambda''\sigma\} > n$ for any $\lambda'' \in [0,\lambda]$; if there is no such λ, we take $\underline{\lambda}(\{\sigma\};n) = 0$.*

The (quasi–) cyclic normalized equivalent stress $\underline{\Lambda}^N(\{\sigma\};n)$ is defined as $1/\underline{\lambda}^N(\{\sigma\};n)$ if $\underline{\lambda}^N(\{\sigma\};n) \neq 0$ and $\underline{\Lambda}^N(\{\sigma\};n) := \infty$ otherwise.

The mappings $(\{\sigma\};n) \mapsto \underline{\lambda}^N(\{\sigma\};n)$ and $(\{\sigma\};n) \mapsto \underline{\Lambda}^N(\{\sigma\};n)$ defined on a set of processes $\{\sigma\}$ and (quasi–) cycle numbers n are called the (quasi–) cyclic safety factor functional $\underline{\lambda}^N$ and the (quasi–) cyclic normalized equivalent stress functional $\underline{\Lambda}$, respectively.

For monotonously damaging processes the definition can be simplified as follows.

Definition 1MD *The (quasi–) cyclic safety factor $\underline{\lambda}(\{\sigma\};n)$ for a quasi–cyclic MD process $\{\sigma(m)\}_{m=1,2,...}$ is supremum of $\lambda \geq 0$ such that $n^*\{\lambda\sigma\} > n$; if there is no such λ, we take $\underline{\lambda}^N(\{\sigma\};n) = 0$. The (quasi–) cyclic normalized equivalent stress $\underline{\Lambda}^N(\{\sigma\};n)$ is defined as $1/\underline{\lambda}^N(\{\sigma\};n)$ if $\underline{\lambda}^N(\{\sigma\};n) \neq 0$ and $\underline{\Lambda}^N(\{\sigma\};n) := \infty$ otherwise.*

We call both $\underline{\lambda}^N$ and $\underline{\Lambda}^N$ also *the (quasi–) cyclic strength functionals.* They are material characteristics reflecting the influence of the endured (quasi–) cyclic loading process on the material strength. Using these definitions, the functionals values can be obtained from experiments for any process $\{\sigma\}$ and any (quasi–) cycle n.

Particularly, if the durability functional $n^*\{\lambda\sigma\}$ is known for all $\lambda \geq 0$, the value of the normalized equivalent stress $\underline{\Lambda}^N(\{\sigma\};n)$ for any n is a supremum of solutions to the scalar equation

$$n^*\{\sigma/\Lambda\} = n$$

for each (quasi–) cycle n and loading process $\{\sigma\}$ for which solutions do exist; otherwise $\underline{\Lambda}^N(\{\sigma\};n)$ can be determined directly from Definition 1. Note that values of $\underline{\Lambda}^N(\{\sigma\};n)$ are uniquely defined in the both cases.

But what information about $\underline{\Lambda}^N$ can one extract from durability measurement $n^*\{\sigma\}$ under only one process $\{\sigma\}$? From Definition 1, one can get for an MD material (see

Appendix A) only the inequality

$$\underline{\Lambda}^N(\{\sigma\}; n^*\{\sigma\} - 1) \le 1 \le \underline{\Lambda}^N(\{\sigma\}; n^*\{\sigma\}) \tag{7}$$

This uncertainty is quite natural and is connected with the fact that the loadings slightly higher and slightly lower than $\{\sigma\}$ can cause rupture during the same (quasi–) cycle $n^*\{\sigma\}$. In fact, it is a payment for identifying rupture only at the (quasi–) cycle end points but not at the (quasi–) cycle internal points.

Remark 1 *One can observe from Definition 1 that one can replace the durability* $n^*\{\lambda\sigma\}$ *by the critical (quasi–) cycle number* $n^{**}\{\lambda\sigma\}$ *in the definition to arrive at exactly the same functionals,* $\underline{\lambda}^{**}(\{\sigma\}; n) = \underline{\lambda}(\{\sigma\}; n)$*,* $\underline{\Lambda}^{**}(\{\sigma\}; n) = \underline{\Lambda}(\{\sigma\}; n)$ *(see proof in Appendix B).*

The (quasi–) cyclic safety factor $\underline{\lambda}^N(\{\sigma\}; n)$ and (quasi–) cyclic normalized equivalent stress $\underline{\Lambda}^N(\{\sigma\}; n)$ are counterparts of the time-dependent safety factor $\underline{\lambda}^T(\sigma; t)$ and normalized equivalent stress $\underline{\Lambda}^T(\sigma; t)$ (Mikhailov 1999, 2000) and of the non-local safety factor functional $\underline{\lambda}(\sigma; x)$ and non-local normalized equivalent stress (load factor) functional $\underline{\Lambda}(\sigma; y)$ defined by Mikhailov (1995-I).

For brevity, we will drop the superscript N further in the paper if this will not lead to a confusion.

To justify the title *normalized equivalent stress* for $\underline{\Lambda}$, we consider a regular periodic multiaxial in-phase process $\{\sigma\}$ where $\sigma_{ij}(\tau)$ varies on each cycle from 0 to a tensor σ_{aij} and back to 0. Let, for example, the material cyclic strength under such loading be associated with the von Mises equivalent stress $\sigma_{eq}(\sigma) = \sqrt{[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]/2}$, that is the strength condition has the form $\sigma_{eq}(\sigma_a) < \sigma_0^*(n)$, where the function $\sigma_0^*(n)$ is a material characteristic (classical S–N diagram under the uniaxial periodic cycling with $R = 0$) and $\sigma_1,\ \sigma_2,\ \sigma_3$ are the principal stresses. Then $\underline{\Lambda}(\{\sigma\}; n)$ is defined from the equation $\sigma_{eq}(\sigma_a/\Lambda) = \sigma_0^*(n)$, that is

$$\underline{\Lambda}(\{\sigma\}; n) = \sigma_{eq}(\sigma_a)/\sigma_0^*(n). \tag{8}$$

Formula (8) holds true not only for the von Mises equivalent stress but also for the Tresca and other equivalent stress representations $\sigma_{eq}(\sigma_a)$ that are functions positively homogeneous of the order +1.

One can see from the above definitions (proof is similar to Mikhailov, 2000) that the safety factor is a non-increasing and the normalised equivalent stress is a non-decreasing function of the (quasi–) cycle number, that is,

$$\underline{\lambda}(\{\sigma\}; n_2) \le \underline{\lambda}(\{\sigma\}; n_1),\ \ \underline{\Lambda}(\{\sigma\}; n_2) \ge \underline{\Lambda}(\{\sigma\}; n_1) \text{ if } n_2 > n_1. \tag{9}$$

It follows from the definitions (see Mikhailov 2000, Appendix C) that *for any* n*, the safety factor functional and the normalised equivalent stress functional are non-negative positively-homogeneous functionals of the orders -1 and +1 respectively, that is*

$$\underline{\lambda}(\{k\sigma\}; n) = \frac{1}{k}\underline{\lambda}(\{\sigma\}; n) \ge 0, \quad \underline{\Lambda}(\{k\sigma\}; n) = k\underline{\Lambda}(\{\sigma\}; n) \ge 0, \quad \text{for any } k > 0. \tag{10}$$

For infinite n we get the following corresponding definitions of the (quasi–) cyclic endurance safety factor and normalised equivalent stress.

Definition 2 *The (quasi–) cyclic endurance (threshold) safety factor* $\underline{\lambda}^N_{th}\{\sigma\}$ *is supremum of* $\lambda \geq 0$ *such that there is no body rupture under the process* $\{\lambda''\sigma\}$ *for any* $\lambda'' \in [0, \lambda]$ *for all* $n < \infty$*; if there is no such* λ*, we take* $\underline{\lambda}^N_{th}\{\sigma\} = 0$.

The (quasi–) cyclic endurance (threshold) normalised equivalent stress is defined as $\underline{\Lambda}^N_{th}\{\sigma\} = 1/\underline{\lambda}^N_{th}\{\sigma\}$ *if* $\underline{\lambda}^N_{th}\{\sigma\} \neq 0$ *and* $\underline{\Lambda}^N_{th}\{\sigma\} := \infty$ *otherwise.*

The mappings $\sigma \mapsto \underline{\lambda}^N_{th}(\{\sigma\})$, $\sigma \mapsto \underline{\Lambda}^N_{th}\{\sigma\}$ *defined on a set of processes* $\{\sigma\}$ *are called the (quasi–) cyclic endurance (threshold) safety factor functional* $\underline{\lambda}^N_{th}$ *and the endurance (threshold) normalised equivalent stress functional* $\underline{\Lambda}^N_{th}$ *respectively.*

Owing to monotonicity (9), we can define the endurance functionals also as

$$\underline{\lambda}_{th}\{\sigma\} = \underline{\lambda}(\{\sigma\};\infty) \;:=\; \lim_{n\to\infty} \underline{\lambda}(\{\sigma\};n) = \inf_{n<\infty} \underline{\lambda}(\{\sigma\};n), \tag{11}$$

$$\underline{\Lambda}_{th}\{\sigma\} = \underline{\Lambda}(\{\sigma\},\infty) \;:=\; \lim_{n\to\infty} \underline{\Lambda}(\{\sigma\};n) = \sup_{n<\infty} \underline{\Lambda}(\{\sigma\};n). \tag{12}$$

We can point out the cases, described in the previous section, for which $\underline{\lambda}_{th}\{\sigma\} = 0$: case (0) when material is self-degrading, i.e. $n^*\{\sigma\} < \infty$; case (D), i.e. $n^*(\{\lambda\sigma\}) \to n^{*0}\{\sigma\} \neq \infty$ as $\lambda \to 0$; case (E); case (G) since the absence of a limit of the function $n^*(\{\lambda\sigma\})$ as $\lambda \to 0$ implies that there exists $n < \infty$ such that $\underline{\lambda}(\{\sigma\};n) = 0$.

Evidently, the endurance safety factor and the endurance normalised equivalent stress make sense as material characteristics only for non-self-degrading materials. As follows from the self-degradation definition above, a material is self-degrading, if and only if there exists a (quasi–) cycle $n^*\{0\}$ such that $\underline{\Lambda}(\{0\};n) = 0$ for $n < n^*\{0\}$ and $\underline{\Lambda}(\{0\};n) = \infty$ for $n \geq n^*\{0\}$. This statement gives an equivalent definition of self-degradation in terms of the safety factor $\underline{\lambda}$ behaviour.

The safety factor $\underline{\lambda}(\{\sigma\};n)$ as function of n at a given process $\{\sigma\}$ can also be considered as a generalised S–N diagram $n \mapsto \underline{\lambda}(\{\sigma\};n)$. As a function of discrete integer argument n, it takes only discrete values and hence is presented not by a curve but by a discrete set of points on the $(n, \underline{\lambda})$ plane. At each n, the point is placed on the bottom of the vertical segment (lowest if it is not unique), corresponding to the n, on the $\lambda \mapsto n^*\{\lambda\sigma\}$ diagram, see Fig. 4b. The points are also placed for integer n from the jump segment $[n^*\{(\lambda+0)\sigma\}, n^*\{(\lambda-0)\sigma\}]$ for some λ where no values of the $\lambda \mapsto n^*\{\lambda\sigma\}$ diagram do exist, see Fig. 6b, 7b. As shown above in this section, the $n \mapsto \underline{\lambda}(\{\sigma\};n)$ diagram is monotonously non-increasing in n. The collection of such diagrams for all possible processes in fact defines the functional $\underline{\lambda}^N$.

One can also associate with each $n-$th (quasi–) cycle not a point n but a segment $[n-1, n]$ on the $n-$axis. Then, remaining in the discrete time description related with the strength/rupture status detection only at the (quasi–) cycle end point, one should extend the status to the whole (quasi–) cycle except its start points being also end point of the previous (quasi–) cycles. Using such approach, one can extend the point-wise S–N diagram $n \mapsto \underline{\lambda}(\{\sigma\};n)$ to the piece-wise constant left-continuous function coinciding with the monotonous parts of the corresponding diagram $\lambda \mapsto n^*\{\lambda\sigma\}$ at the (quasi–) cycle ends and remaining constant at other points of the (quasi–) cycles. It cuts off the non-monotonous (multi-valued) parts of the diagram $\lambda^*(\{\sigma\};n)$ (connecting by the above way to the branch with the lowest λ^* and making a corresponding finite jump in $\underline{\lambda}(\{\sigma\};n)$ in the branch beginning, see Fig. 4). It continues also the diagram onto the jump segment $[n^*\{(\lambda+0)\sigma\}, n^*\{(\lambda-0)\sigma\}]$ where $\lambda^*(\{\sigma\};n)$ does not exist, see Fig. 7b.

From the generalised S–N diagram $n \mapsto \underline{\lambda}(\{\sigma\};n)$ for a given process $\{\sigma\}$, presented e.g. on Fig. 3b, we can obtain the corresponding diagram $n \mapsto \underline{\Lambda}(\{\sigma\};n) = 1/\underline{\lambda}(\{\sigma\};n)$ for the normalised equivalent stress $\underline{\Lambda}(\{\sigma\};n)$, Fig. 3c. Different curves correspond to different possible cases of its behaviour described in points (A)-(F) of Section 3. Generally, $n \mapsto \underline{\Lambda}(\{\sigma\};n)$ is a non-decreasing function of the (quasi–) cycle number n (see above). Some examples are given on Fig. 4c, 5c, 6c, 7c.

The diagram can be used in two ways. First, it gives a number $\underline{\Lambda}(\{\sigma\};n)$ such that there is no rupture up to (quasi–) cycle n for any process $\{\sigma/\Lambda'\}$ with $\Lambda' > \underline{\Lambda}(\{\sigma\};n)$. For example, if the diagram includes curve f (see Fig. 3), then the process $\{\sigma/\Lambda'\}$ with $\Lambda' > \underline{\Lambda}_{th}\{\sigma\}$ causes no rupture for any n. Another way is to use the diagram together with the stable strength condition (14) below for given $\{\sigma\}$ and n. For example, if the diagram includes the curve f, then the process $\{\sigma\}$ causes no rupture for any n if $\underline{\Lambda}_{th}\{\sigma\} < 1$.

Consider *existence* and *uniqueness* of the NESF $\underline{\Lambda}$. Suppose the material (quasi–) cyclic strength under a process $\{\sigma\}$ on a (quasi–) cycle n is described by a strength condition

$$\underline{F}(\{\sigma\};n) < 1 \tag{13}$$

where $\underline{F}$ is a non-linear functional non-decreasing in n, known from experimental data approximation or from a (quasi–) cyclic durability theory on the processes $\{\lambda\sigma\}$ for all $\lambda \geq 0$ for the considered n. Then the NESF $\underline{\Lambda}$ is *uniquely determined* from (13) by Definition 1 for the process $\{\sigma\}$ and the (quasi–) cycle n, although analytical expressing $\underline{\Lambda}$ in terms of $\underline{F}$ is not always possible.

However, if $\underline{F}(\{\sigma\};n)$ is a non-negative positively homogeneous of order +1 functional of $\{\sigma\}$, then simply $\underline{\Lambda}(\{\sigma\};n) = \underline{F}(\{\sigma\};n)$ (proof is similar to Mikhailov 2000). This relation will be used in Section 6 to obtain the NESF from known strength conditions of some (quasi–) cyclic durability theories.

5 (Quasi–) cyclic strength and endurance conditions

Let $\{\sigma\}$ be a process and n be a (quasi–) cycle number. From Definition 1 for the strength functionals, we have the following conclusions:
(i) The inequality

$$\underline{\Lambda}(\{\sigma\};n) < 1 \tag{14}$$

implies $\lambda-$stable strength under the process $\{\sigma\}$ on or before the (quasi–) cycle n.
(ii) The equality

$$\underline{\Lambda}(\{\sigma\};n) = 1 \tag{15}$$

implies either rupture or $\lambda-$unstable strength under the process $\{\sigma\}$ on or before the (quasi–) cycle n.
(iii) The inequality

$$\underline{\Lambda}(\{\sigma\};n) > 1 \tag{16}$$

implies rupture under the process $\{\sigma\}$ on or before the (quasi–) cycle n if $\{\sigma\}$ is an MD process.

Inversely, if strength is $\lambda-$stable for an MD process $\{\sigma\}$ at and before a (quasi–) cycle n then (14) is satisfied (see proof by Mikhailov, 2000). Consequently, we have the following

Statement 1 *Inequality (14) gives a sufficient (and necessary, if $\{\sigma\}$ is an MD process) condition of $\lambda-$stable (quasi–) cyclic strength on and before a (quasi–) cycle n under a process $\{\sigma\}$.*

By the same way, we have from Definition 2 the following conclusions for the endurance functionals:

(i) The inequality

$$\underline{\Lambda}_{th}\{\sigma\} < 1 \tag{17}$$

implies $\lambda-$stable endurance for the process $\{\sigma\}$.

(ii) The equality

$$\underline{\Lambda}_{th}\{\sigma\} = 1 \tag{18}$$

implies either rupture on a (quasi–) cycle $n < \infty$ that is, $n^*\{\sigma\} < \infty$, or $\lambda-$unstable endurance, that is, there is no rupture under the process $\{\sigma\}$ on any (quasi–) cycle but for any $\lambda > 1$ there exists $\lambda'' \in (1, \lambda]$ such that $n^*\{\lambda''\sigma\} < \infty$.

(iii) The inequality

$$\underline{\Lambda}_{th}\{\sigma\} > 1 \tag{19}$$

implies rupture on a (quasi–) cycle $n < \infty$, that is, $n^*\{\sigma\} < \infty$ if $\{\sigma\}$ is an MD process.

Similarly to Statement 1, we have the following

Statement 2 *Inequality (17) gives a sufficient (and necessary if $\{\sigma\}$ is an MD process) condition of $\lambda-$stable (quasi–) cyclic endurance under a process $\{\sigma\}$.*

Conditions (14)-(19) together with the homogeneity of $\underline{\Lambda}^N$, $\underline{\Lambda}^N_{th}$ also show that the functionals do really play the role of normalised equivalent stresses.

It follows from the $\underline{\Lambda}$ definition that if the durability $n^*\{\lambda\sigma\}$ is known for a process $\{\sigma\}$ for all $\lambda \geq 0$, then the normalised equivalent stress $\underline{\Lambda}(\{\sigma\}; n)$ can be obtained for $\{\sigma\}$ for any $n = 1, 2, ...$ Consider now if it is possible to obtain values of the (quasi–) cyclic durability functional $n^*\{\lambda\sigma\}$ for a process $\{\sigma\}$ for any $\lambda \geq 0$ if the values of the NESF $\underline{\Lambda}(\{\sigma\}; n)$ are known for the process $\{\sigma\}$ for any $n = 1, 2, ...$

It is evident that this is not possible if $\{\sigma\}$ is not an MD process, since the information about the non-monotonous behaviour of $n^*\{\lambda\sigma\}$ as function of λ is lost in $\underline{\Lambda}(\{\sigma\}; n)$. However, as one can prove similar to Mikhailov 2000, the following inequality holds for any process,

$$\underline{\Lambda}(\{\sigma\}; n^*\{\lambda\sigma\}) \geq 1/\lambda \quad \text{for all } \lambda > 0. \tag{20}$$

The discussion above shows that in addition to the non-sensitivity to non-monotonous behaviour of the S–N diagram, the NESF $\underline{\Lambda}(\{\sigma\}; n)$ does not also distinguish rupture from not $\lambda-$stable strength. For this reason it is the critical (quasi–) cycle $n^{**}\{\sigma\} \leq n^*\{\sigma\}$, who can be obtained from NESF $\underline{\Lambda}(\{\sigma\}; n)$, but generally not the durability $n^*\{\sigma\}$. The following statement is proved in Appendix C.

Statement 3 *Let $\{\sigma\}$ be an MD process. Let $n^{**}_-\{\sigma\}$ be supremum of n such that*

$$\underline{\Lambda}(\{\sigma\}; n) < 1. \tag{21}$$

Then the critical (quasi–) cycle is $n^{**}\{\sigma\} = n^{**}_-\{\sigma\} + 1$ if $n^{**}_-\{\sigma\} < \infty$, otherwise the (quasi–) cyclic endurance is $\lambda-$stable under the process $\{\sigma\}$.

Taking into account that $\underline{\Lambda}(\{\sigma\}; n)$ is monotonously non-decreasing in n and that if $n^{**}_{-}\{\sigma\} = n^{**}\{\sigma\} - 1$ is finite, it does satisfy inequality (21) but $n^{**}\{\sigma\}$ does not, we have the following corollary from Statement 3.

Corollary 1 *A (quasi–) cycle n^{**} is critical for an MD process $\{\sigma\}$, i.e. $n^{**} = n^{**}\{\sigma\}$, if and only if it satisfies the inequality*

$$\underline{\Lambda}(\{\sigma\}; n^{**} - 1) < 1 \leq \underline{\Lambda}(\{\sigma\}; n^{**}). \tag{22}$$

If such n^{**} does not exist, the (quasi–) cyclic endurance is $\lambda-$stable under the process $\{\sigma\}$.

Thus, inequality (22) is a criterion of rupture or strength instability on a (quasi–) cycle n^{**} under a (quasi–) cyclic MD process.

Applying Statement 3 to a process $\{\lambda\sigma\}$ and using the positive homogeneity of $\underline{\Lambda}(\{\sigma\}; n)$, we get some generalisation of the Statement:

Corollary 2 *Let $\{\sigma\}$ be an MD process. Let $n^{**}_{-}\{\lambda\sigma\}$ be supremum of n such that*

$$\underline{\Lambda}(\{\sigma\}; n) < 1/\lambda. \tag{23}$$

Then for any $\lambda > 0$, the critical (quasi–) cycle is $n^{**}\{\lambda\sigma\} = n^{**}_{-}\{\lambda\sigma\} + 1$ if $n^{**}_{-}\{\lambda\sigma\} < \infty$, otherwise the (quasi–) cyclic endurance is $\lambda-$stable under the process $\{\lambda\sigma\}$.

Remark 2 *As noted in Remark 1, one can replace the durability $n^{*}\{\lambda\sigma\}$ by the critical (quasi–) cycle number $n^{**}\{\lambda\sigma\}$ in Definition 1 to arrive at exactly the same functional, $\underline{\Lambda}^{**N} = \underline{\Lambda}^{N}$. Thus, if the critical (quasi–) cycle number $n^{**}\{\lambda\sigma\}$ is known for a process $\{\sigma\}$ at all $\lambda \geq 0$, then values of the NESF $\underline{\Lambda}(\{\sigma\}; n)$ are uniquely determined for the process $\{\sigma\}$ at any n. Conversely, if values of the NESF $\underline{\Lambda}(\{\sigma\}; n)$ are known for an MD process $\{\sigma\}$ at all n, then numbers of the critical (quasi–) cycles $n^{**}\{\lambda\sigma\}$ are uniquely determined for the process $\{\sigma\}$ at any $\lambda \geq 0$ and particularly at $\lambda = 1$.*

Note that namely the critical (quasi–) cycle number $n^{**}\{\sigma\}$ is necessary for practical design since, for the cases when $n^{**}\{\sigma\} \neq n^{*}\{\sigma\}$, the material strength is $\lambda-$unstable for $n \in [n^{**}\{\sigma\}, n^{*}\{\sigma\})$ and the material can be ruptured by an arbitrarily small increase of loading $\{\sigma\}$.

5.1 General remarks

The (quasi–) cyclic NESF $\underline{\Lambda}^{N}$ as well as durability n^{*} and critical (quasi–) cycle n^{**} functionals are supposed to be material characteristics in the sense that they may have different values on the same processes $\{\sigma\}$ for different materials but must give the same values on the same processes for the same material independent particularly of the shape of the body consisting of the material.

The durability $n^{*}\{\sigma\}$ does not depend on a particular time parametrisation of (quasi–) cycles $\sigma^{c}(m)$ for materials with *pure fatigue* responses. Thus the pure fatigue NESF $\underline{\Lambda}^{N}$ must also be time-independent and non-sensitive to the loop time parametrisation but may be sensitive to the loop shape and direction in the stress space as well as to the (quasi–) cycle order in the sequence $\{\sigma^c(m)\}_m = 1, 2, ...$ This means the functional $\underline{\Lambda}^N$ should be defined on sequences $\{\sigma^{cs}\}$ for pure fatigue.

On the other hand, there are materials that manifest a rupture dependence on time (e.g. under creep or dynamic loading) along with (quasi–) cyclic fatigue effects. For such materials, the time sensitivity should be reflected also in the functional $\underline{\Lambda}^N$ that will be than defined on sequences $\{\sigma^{ct}\}$ to take into account the time-history-fatigue interaction.

All the written above in Sections 3-5 can be referred to each of the both loading process representations $\{\sigma^c(m)\}_{m=1,2,...}$.

As material characteristics, the functionals n^{**} and $\underline{\Lambda}^N$ for a particular material can be (approximately) identified from experimental data on durability $n^*\{\sigma\}$. Evidently only a finite number of the durability tests $n^*\{\sigma\}$ for different processes $\{\sigma\}$ can be done and identification of $n^*\{\sigma\}$ or $\underline{\Lambda}^N(\{\sigma\}; n)$ for other processes and (quasi–) cycle numbers should be done along those test results using an interpolation/approximation procedure.

In spite of the fact that it is just the durability $n^*\{\sigma\}$, which values are obtained directly from experiments for some test processes $\{\sigma\}$, it is usually more straightforward to approximate along those date first the NESF $\underline{\Lambda}^N$ rather than n^*. This is because, although the both functionals are nonlinear, the functional $\underline{\Lambda}^N(\{\sigma\}; n)$ is homogeneous and can be considered as bounded with respect to $\{\sigma\}$ in appropriate function spaces.

For the (quasi–) cyclic NESFs, any one durability test for a process $\{\sigma\}$ will allocate, according to uncertainty inequality (7), the value 1 between the values of NESF for the neighbouring (quasi–) cycles, $\underline{\Lambda}^N(\{\sigma\}; n^*\{\sigma\})$ and $\underline{\Lambda}^N(\{\sigma\}; n^*\{\sigma\}) - 1$. Typical numbers of (quasi–) cycles under fatigue loading vary between 10^3 and 10^7, and consequently one can suppose rather small changes of $\underline{\Lambda}^N(\{\sigma\}; n)$ between the (quasi–) cycles and attribute $\underline{\Lambda}^N(\{\sigma\}; n^*\{\sigma\}) \approx 1$. Then, taking into account the functional homogeneity, we obtain the NESF for the one-dimensional linear set, $\underline{\Lambda}^N(\{k\sigma\}; n^*\{\sigma\}) = k \ \forall k \geq 0$.

In what is written above in the paper, we analysed strength and durability under a (quasi–) cyclic loading process $\{\sigma^c\}$ where the stress field $\sigma = \sigma_{ij}(\tau)$ is independent of the space coordinates. If the stress field depends not only on time (or (quasi–) cycle number) but also on the space coordinates $x = (x_1, x_2, x_3)$, i.e. $\sigma = \sigma_{ij}(x, \tau)$, it is usually supposed that rupture is local, that is, rupture at a point y depends only on the stress (quasi–cyclic) history at the point y, that is on $\sigma_{ij}(y, \tau)$ (or on $\{\sigma^c(y, m)\}_{m=1,2,...}$ for a (quasi–) cyclic process) and does not depend on the stress history at other points of the body. This means, one can use the durability n^* and the NESF $\underline{\Lambda}^N$, obtained from tests under homogeneous in space loading processes, to predict rupture under inhomogeneous in space loadings. Particularly, the counterpart of condition (14) for prediction of stable (quasi–) cyclic strength in a point y during a (quasi–) cycle n will then have form

$$\underline{\Lambda}(\{\sigma(y)\}; n) < 1.$$

This approach works well for moderately inhomogeneous stress fields but fails, when a stress field vary rather sharp, e.g. near a crack tip or other stress concentrator. Some non-local approaches able to deal with such stress fields for time and history independent materials were considered by Mikhailov (1995–I&II). Their extension to history dependent materials under (quasi–) cyclic loadings is developed by Mikhailov & Namestnikova (2002).

6 Examples of (quasi–) cyclic normalised equivalent stress functionals for uniaxial loading

6.1 Uniaxial periodic loading processes for material independent of history

Let us consider a material independent of history (that is, its strength is determined only by the instants stress value) and obeying the strength condition

$$|\sigma| < \sigma_r$$

where σ_r is a material constant (the material strength under monotone uniaxial tension). Using the example in section 3.2, we have from the definition of NESF,

$$\underline{\Lambda}^N(\{\sigma\}, n) = \max_{1\le m\le n} \frac{\|\sigma^c(m)\|}{\sigma_r},$$

where as above $\|\sigma^c(m)\| = \max(|\sigma_{max}(m)|, |\sigma_{min}(m)|)$.

6.2 Uniaxial regular periodic loading processes

The fatigue strength conditions at an n-th cycle of a uniaxial regular periodic loading process $\{\sigma\}$ can be written in the following form

$$\sigma_a < \sigma^*_R(n), \tag{24}$$

or inversely,

$$n < n^*_R(\sigma_a)$$

Here σ_a is a cycle amplitude and $n^*_R(\sigma_a)$ is the classical S-N diagram, depending on the asymmetry ratio R. Sometimes the maximum stress value σ_{max} is used instead of the stress amplitude σ_a in a cycle if $\sigma_{max} > 0$. Then the strength condition at the n-th cycle can be rewritten as

$$\sigma_{max} < \sigma^*_{R,max}(n), \tag{25}$$

where $\sigma^*_{R,max}(n)$ is the S–N diagram for σ_{max}. Inversely,

$$n < n^*_{R,max}(\sigma_{max})$$

Note that at $R = -1$ the stress amplitude coincides with the maximum stress value in a cycle. According to the Definition 1 of the NESF, we have for arbitrary stress asymmetry ratio R

$$\underline{\Lambda}^N(\{\sigma\}; n) = \frac{\sigma_a}{\sigma^*_R(n)} \tag{26}$$

or, respectively,

$$\underline{\Lambda}^N(\{\sigma\}; n) = \frac{\sigma_{max}}{\sigma^*_{R,max}(n)} \tag{27}$$

Let the Wöhler S–N diagram $\sigma^*_R(n)$ be approximated by a power law

$$\sigma^*_R(n) = \sigma^*_{R1} n^{-1/b_R}, \tag{28}$$

where σ^*_{R1} and b_R are positive material parameters generally dependent on R. Then the NESF is

$$\underline{\Lambda}^N(\{\sigma\}; n) = \frac{\sigma_a}{\sigma^*_{R1}} n^{1/b_R}$$

Let the Wöhler diagram $\sigma^*_R(n)$ for arbitrary R be approximated in terms of the Wöhler diagram for symmetric cycling, $\sigma^*_{-1}(n)$, as

$$\sigma^*_R(n) = \sigma^*_{-1}(n) \left[1 + \frac{1+R}{1-R} \quad \frac{\sigma^*_{-1}(n)}{\sigma_r}\right]^{-1} \tag{29}$$

then we arrive at the strength condition at an n-th cycle, associated with the Haigh diagram

$$\frac{\sigma_a}{\sigma^*_{-1}(n)} + \frac{\sigma_m}{\sigma_r} < 1 \tag{30}$$

where $\sigma_m = \frac{\sigma_{max}+\sigma_{min}}{2}$ is mean stress in the cycle. Note that the Haigh diagram can be considered as the fatigue strength diagram at a fixed number of cycles n (constant-lifetime diagram) and also as the endurance diagram (then $n = \infty$).

The NESF corresponding to (29), (30) is

$$\underline{\Lambda}^N(\{\sigma\}; n) = \frac{\sigma_a}{\sigma^*_{-1}(n)} + \frac{\sigma_m}{\sigma_r}$$

6.3 Uniaxial non-periodic regular cyclic loading processes

The approaches analysed below in this section were originally developed for block-periodic loadings. The block-periodicity usually means a large number of periodic cycles in a small number of blocks. For this reason, the number of transitions between the blocks is also small and can be neglected in the durability analysis. However, all the block-periodic theories considered below can be equally applied also to arbitrary uniaxial loading with closed cycles, that is, periodicity in blocks is not necessary and there can be many transitions between the blocks if the transitions are closed cycles.

6.3.1 Palmgren-Miner linear damage accumulation rule

For a uniaxial block-periodic loading process $\{\sigma^c(m)\}$, the cyclic strength condition according Palmgren(1924,1945)-Miner(1945) linear damage accumulation rule can be written as the inequality

$$\sum_{j=1}^{n} \frac{1}{n^*_{R(j)}(\sigma_a(j))} < 1. \tag{31}$$

The value of $n^*_{R(j)}(\sigma_a(j))$ is found from an appropriate classical S–N curve $\sigma_a \mapsto n^*_R(\sigma_a)$ for the same material under corresponding *periodic* cycling. For a process $\{\lambda\sigma^c(m)\}$, where $\lambda \geq 0$, we have thus the strength condition

$$\sum_{j=1}^{n} \frac{1}{n^*_{R(j)}(\lambda\sigma_a(j))} < 1. \tag{32}$$

Definition 1 together with the strength condition (32) allow to obtain the NESF for any particular S–N diagram $n^*_R(\sigma_a)$ although not always explicitly. If any *periodic* loading is monotonously damaging for the considered material, that is, $n^*_R(\sigma_a)$ is a non-increasing function of σ_a (what is usually the case for structural materials), then the left hand side of (32) is non-decreasing in λ and consequently any block-periodic loading is monotonously damaging for the material obeying the linear damage accumulation rule. In this case one can apply a simplified Definition 1MD and obtain that $\underline{\Lambda}^N(\{\sigma\};n) = 1/\underline{\lambda}^N(\{\sigma\};n)$, where $\underline{\lambda}^N(\{\sigma\};n)$ is the supremum of numbers $\lambda \geq 0$ satisfying inequality (32).

Although as follows from the cyclic durability definition, $n^*_R(\sigma_a)$ is a piece-wise constant integer-valued function of σ_a, some continuous or piece-wise continuous interpolation of $n^*_R(\sigma_a)$ is usually used for simplicity in damage accumulation rules like (31) and we will often follow this tradition below. This leads to an error less then one cycle for $n^*_R(\sigma_a)$, which is negligible in comparison with the durabilities $10^3 - 10^7$ typical for fatigue.

If $n^*_R(\sigma_a)$ is a continuous monotonously decreasing function of σ_a, then the left hand side of (32) is continuous and monotonously increasing in λ, and instead of taking supremum according to Definition 1MD, one can determine the NESF as $\underline{\Lambda}^N(\{\sigma\};n) = 1/\lambda$, where λ is a solution of the equation

$$\sum_{j=1}^{n} \frac{1}{n^*_{R(j)}(\lambda\sigma_a(j))} = 1. \tag{33}$$

In the case when $n^*_R(\sigma_a)$ is a piece-wise continuous monotonously decreasing function of σ_a, one can also try to find the NESF in form $\underline{\Lambda}^N(\{\sigma\};n) = 1/\lambda$, where λ is a solution of equation (33) if the solution does exist, or use more general Definition 1MD otherwise.

Suppose, particularly, that $n^*_R(\sigma_a)$ is given by the power law inverse to (28), that is,

$$n^*_R(\sigma_a) = \Big(\frac{\sigma^*_{R1}}{\sigma_a}\Big)^{b_R} \tag{34}$$

where $\sigma^*_{R1}, b_R > 0$. The so–defined $n^*_R(\sigma_a)$ is a continuous monotonously decreasing function of σ_a. Then $\underline{\Lambda}^N(\{\sigma\};n) = 1/\lambda$, where λ is a solution of the equation

$$\sum_{j=1}^{n} \left[\frac{\lambda\sigma_a(j)}{\sigma^*_{R(j)1}}\right]^{b_{R(j)}} = 1 \tag{35}$$

If $b_R = b$ does not depend on R, or $R(j) = R$ does not depend on the cycle number j, then the equation can be solved explicitly and the NESF is

$$\underline{\Lambda}^N(\{\sigma\};n) = \left[\sum_{j=1}^{n} \left(\frac{\sigma_a(j)}{\sigma^*_{R(j)1}}\right)^b\right]^{1/b} \tag{36}$$

in the first case and

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{1}{\sigma^*_{R1}} \left(\sum_{j=1}^{n} \sigma_a^{b_R}(j)\right)^{1/b_R} \tag{37}$$

in the second case.

Note, perhaps the most significant shortcoming of the Palmgren-Miner hypothesis is that it does not account for sequence effects; that is, that damage caused by a stress cycle is independent of where it occurs in the load history.

6.3.2 Marin damage accumulation rule

Marin (1962) proposed the damage accumulation rule for a block-periodic process with the stress asymmetry ratio $R=-1$. Using Marin's hypothesis the fatigue strength condition can be written in form

$$\frac{1}{n^*_{-1}(\sigma_a^{max}(n))}\sum_{j=1}^{n}\left[\frac{\sigma_a(j)}{\sigma_a^{max}(n)}\right]^d < 1 \tag{38}$$

Here $\sigma_a(j)$ is the stress amplitude in the j-th cycle, $\sigma_a^{max}(n)=\max\limits_{j=1,\dots n}\sigma_a(j)$ is the maximum stress amplitude in the process $\{\sigma(j)\}_{j=1}^n$, and $n^*_{-1}(\sigma_a)$ is the number of cycles up to rupture in a periodic process with the stress amplitude σ_a and the asymmetry ratio $R=-1$, d is a material constant.

As above, under assumption (34) we can rewrite (38) in the following form

$$\left[\frac{\sigma_a^{max}(n)}{\sigma^*_{-1,1}}\right]^{b_{-1}}\sum_{j=1}^{n}\left[\frac{\sigma_a(j)}{\sigma_a^{max}(n)}\right]^d < 1 \tag{39}$$

We should note that strength condition (39) can be used only for $d\le b_{-1}$. Otherwise, if we consider for example, a cyclic process with a constant amplitude σ_a, then, according to (39), the addition to the process of only one cycle with the amplitude $2\sigma_a$ *increases* the number of cycles to rupture by almost $2^{d-b_{-1}}$ times, what does not seem to be natural.

From (39), the durability $n^*(\lambda\sigma)$ under the process $\{\lambda\sigma(j)\}_{j=1,2\dots}$ is determined from the equation

$$\left[\frac{\lambda\sigma_a^{max}(n^*)}{\sigma^*_{-1,1}}\right]^{b_{-1}}\sum_{j=1}^{n^*}\left[\frac{\sigma_a(j)}{\sigma_a^{max}(n^*)}\right]^d = 1$$

Finally we find

$$\lambda=\frac{\sigma^*_{-1,1}}{\sigma_a^{max}(n^*)}\left(\sum_{j=1}^{n^*}\left[\frac{\sigma_a(j)}{\sigma_a^{max}(n^*)}\right]^d\right)^{-1/b_{-1}}$$

and have the following representation for the NESF,

$$\underline{\Lambda}^N(\{\sigma\};n)=\frac{\sigma_a^{max}(n)}{\sigma^*_{-1,1}}\left(\sum_{j=1}^{n}\left[\frac{\sigma_a(j)}{\sigma_a^{max}(n)}\right]^d\right)^{1/b_{-1}}=\frac{\left(\sigma_a^{max}(n)\right)^{1-\frac{d}{b_{-1}}}}{\sigma^*_{-1,1}}\left[\sum_{j=1}^{n}[\sigma_a(j)]^d\right]^{1/b_{-1}} \tag{40}$$

for $d\le b_{-1}$.

Note, that the same expression (38) for the durability was obtained by Corten & Dolan (1956) under other assumptions than by Marin. Consequently, the functional $\underline{\Lambda}^N$ for the Corten & Dolan fatigue model is also determined by (40) if the Wöhler diagram is used in form (34). When $n^*_{-1}(\sigma_a)$ is given by (34) and $d=b_{-1}$, the both damage accumulation rules are reduced to strength condition (31) for the Palmgren-Miner linear damage accumulation rule.

6.3.3 Pavlov non-linear accumulation rule

A non-linear strength condition, taking into account instantaneous rupture, was proposed by Pavlov (1988)

$$\sum_{j=1}^{n}\tilde{f}[\sigma_{max}(j),R(j)]+\frac{\sigma_{max}(n)}{\sigma_r} < 1 \tag{41}$$

Where $\sigma_{max}(j)$ is the maximum stress in the j-th cycle. Under block-periodic loading, the function $\tilde{f}[\sigma_{max}(j), R(j)]$ was taken in the form

$$\tilde{f}[\sigma_{max}(j), R(j)] = \left(1 - \frac{\sigma_{max}(j)}{\sigma_r}\right) \frac{1}{n^*_{R(j)max}[\sigma_{max}(j)]} \tag{42}$$

Substituting (42) into (41) we have

$$\sum_{j=1}^{n} \left(1 - \frac{\sigma_{max}(j)}{\sigma_r}\right) \frac{1}{n^*_{R(j)max}[\sigma_{max}(j)]} + \frac{\sigma_{max}(n)}{\sigma_r} < 1 \tag{43}$$

Assuming the Wöhler diagram in form

$$n^*_{R,max}(\sigma_{max}) = \Big(\frac{\sigma^*_{R1,max}}{\sigma_{max}}\Big)^{b_R}, \tag{44}$$

where b_R, σ^*_{R1} are material characteristics generally depending on R, we obtain after some algebraic manipulations the following equation for $\underline{\Lambda}^N$,

$$\Lambda^{b_R} \frac{\sigma_{max}(n)}{\sigma_r} + \Lambda \sum_{j=1}^{n} \Big(\frac{\sigma_{max}(j)}{\sigma^*_{R1,max}}\Big)^{b_R} - \Lambda^{b_R+1} = \sum_{j=1}^{n} \Big(\frac{\sigma_{max}(j)}{\sigma^*_{R1,max}}\Big)^{b_R} \frac{\sigma_{max}(j)}{\sigma_r}, \tag{45}$$

which can be solved numerically. Here $b_R = b_{R(j)}$, $\sigma^*_{R1} = \sigma^*_{R(j)1}$.

Note that the strength condition (43) can be rewritten in the following form

$$\frac{1}{\sigma_r - \sigma_{max}(n)} \sum_{j=1}^{n} \frac{\sigma_r - \sigma_{max}(j)}{n^*_{R(j),max}[\sigma_{max}(j)]} < 1$$

This last inequality was generalized by Pavlov (1988) to the non-linear damage accumulation strength condition

$$l[\sigma_{max}(n); R(n)] \sum_{j=1}^{n} \frac{1}{l[\sigma_{max}(j); R(j)] n^*[\sigma_{max}(j), R(j)]} < 1. \tag{46}$$

Here $l[\sigma_{max}(j); R(j)]$ is a parameter depending not only on the maximum stress in a cycle but also on the cycle asymmetry ratio.

Assuming the Wöhler diagram in form (44), the durability $n^*(\lambda\sigma)$ under the process $\{\lambda\sigma(j)\}_{j=1,2...}$ is determined from the equation

$$l[\lambda\sigma_{max}(n^*); R(n^*))] \sum_{j=1}^{n^*} \frac{1}{l[\lambda\sigma_{max}(j); R(j)]} \left[\frac{\lambda\sigma_{max}(j)}{\sigma^*_{R(j)1,max}}\right]^{b_{R(j)}} = 1 \tag{47}$$

Then, from (47), the NESF $\underline{\Lambda}^N(\{\sigma\}; n)$ is a solution of the equation

$$l[\Lambda^{-1}\sigma_{max}(n^*); R(n^*)] \sum_{j=1}^{n} \frac{\Lambda^{-b_{R(j)}}}{l[\Lambda^{-1}\sigma_{max}(j); R(j)]} \left[\frac{\sigma_{max}(j)}{\sigma^*_{R(j)1}}\right]^{b_{R(j)}} = 1 \tag{48}$$

which can be solved numerically. Note that if $l(\sigma_{max}; R) = const$ then (46) degenerates into (31) and (48) into (37).

6.3.4 Serensen-Kogaev model

To get a better agreement to the test data under a non-regular in time loading $\{\sigma_a(j)\}_{j=1,2,...}$ with the stress asymmetry ratio $R=-1$, it was proposed by Serensen et al (1975) and Kogaev et al (1985) (see also in English Lagoda (2001)) to use an improved Palmgren-Miner hypothesis, for which we can write the strength condition in the form

$$\sum_{\substack{j=1 \\ \sigma_a(j)>\sigma^*_{-1\infty}}}^{n} \frac{1}{n^*_{-1}(\sigma_a(j))} < a_p(\{\sigma\};n) \tag{49}$$

The value $a_p(\{\sigma\};n)$ is defined as

$$a_p(\{\sigma\};n) = \max\left[\frac{\tilde{\sigma}(n)-0.5\sigma^*_{-1\infty}}{\sigma_a^{max}(n)-0.5\sigma^*_{-1\infty}}, \quad 0.1\right] \tag{50}$$

Here

$$\tilde{\sigma}(n) = \frac{1}{\tilde{n}} \sum_{\substack{j=1 \\ \sigma_a(j)>0.5\sigma^*_{-1\infty}}}^{n} \sigma_a(j) \tag{51}$$

Here $\tilde{n}$ is a number of cycles out of n with the stress amplitudes $\sigma_a(j) > 0.5\sigma^*_{-1\infty}$, $\quad j = 1,...,n$; $\sigma^*_{-1\infty} = \sigma^*_{-1}(\infty)$ is the fatigue limit for the symmetric loading ($R=-1$).

Note that as follows from (51), $0.5\sigma^*_{-1\infty} \leq \tilde{\sigma} \leq \sigma_a^{max}(n)$ if not all $\sigma_a(j) < 0.5\sigma^*_{-1\infty}$. Then owing to (50), $0.1 \leq a_p \leq 1$. Hence the strength criterion (49) can describe only an decrease but not an increase of the durability in comparison with the prediction of the classical Palmgren-Miner linear damage accumulation hypothesis (31).

The Wöhler curve $n^*_{-1}(\sigma_a)$ is presented by two straight line parts in the double logarithmic coordinates. One of them, $\sigma_a = \sigma^*_{-1\infty}$ for the large number of cycles, is parallel to the abscissa axis. The non-constant part of the Wöhler diagram is presented by the power law (34). Then the coordinates for the point of the lines intersection are $(n^*_G, \sigma^*_{-1\infty})$, where $n^*_G = (\sigma^*_{-1,1}/\sigma^*_{-1\infty})^{b_{-1}}$. Substituting (34) into (49), we have

$$\frac{1}{[\sigma^*_{-1,1}]^{b_{-1}}} \sum_{\substack{j=1 \\ \sigma_a(j)>\sigma^*_{-1\infty}}}^{n} [\sigma_a(j)]^{b_{-1}} < a_p(\{\sigma\};n)$$

Then the durability $n^*(\lambda\sigma)$ for the process $\{\lambda\sigma\}$ is determined from the equation

$$\frac{1}{[\sigma^*_{-1,1}]^{b_{-1}}} \sum_{\substack{j=1 \\ \lambda\sigma_a(j)>\sigma^*_{-1\infty}}}^{n^*} \left[\lambda\sigma_a(j)\right]^{b_{-1}} = a_p(\{\lambda\sigma\};n^*) \tag{52}$$

The NESF can be determined as $\underline{\Lambda}^N(\{\sigma\};n) = 1/\lambda$, where λ is a solution of equation (52) if the solution does exist, or from Definition 1MD otherwise.

If the fatigue limit $\sigma^*_{-1\infty}$ is equal to zero, strength condition (49) reduces to the following inequality,

$$\sum_{j=1}^{n} \frac{1}{n^*_{-1}(\sigma_a(j))} < a_p(\{\sigma\}; n) = \max\left[\frac{1}{n\sigma_a^{max}(n)} \sum_{j=1}^{n} \sigma_a(j),\ 0.1\right] \tag{53}$$

Using (34), we obtain the NESF for the case in the form

$$\underline{\Lambda}^N(\{\sigma\}; n) = \frac{1}{\sigma^*_{-1,1}} \left[\frac{1}{a_p(\{\sigma\}; n)} \sum_{j=1}^{n} [\sigma_a(j)]^{b_{-1}}\right]^{1/b_{-1}} \tag{54}$$

6.4 Remarks on applications to random uniaxial loading processes

Under a random loading $\{\sigma^c(m)\}_{m=1,2,...}$, the quasi-cycles $\sigma^c(m)$ are not closed and consequently direct applying a linear or non-linear summation rule to reduce a durability description to a material characteristic determined under corresponding periodic processes is impossible. Usual in this case is an intermediate step: to define an auxiliary reduced process, or better to say, a finite set of closed cycles $\{\tilde{\sigma}^c(\tilde{m}; n)\}_{m=1}^{\tilde{n}}$ deemed to be equivalent to a finite subsequence $\{\sigma^c(m)\}_{m=1}^{n}$ of the original process, using one of the cycle counting methods (for example the rainflow method), see e.g. Dowling N.E. (1972), Downing S.D. & Socie D.F. (1982), Collins (1993), Rychlik (1987), the British Standards-5400 (1980). For a given process $\{\sigma^c(m)\}_{m=1,2,...}$, both the reduced cycles $\tilde{\sigma}^c(\tilde{m}; n)$ and their number $\tilde{n}$ in the set depend on n. After this step, normally the linear summation rule is applied.

The cycle counting methods have several features important for our analysis:

(i) The sequence effect is lost in the reduced cycle set $\{\tilde{\sigma}^c(\tilde{m}; n)\}_{m=1}^{\tilde{n}}$.

(ii) If $\{\tilde{\sigma}^c(\tilde{m}; n)\}_{m=1}^{\tilde{n}}$ is a reduced set to a subsequence $\{\sigma^c(m)\}_{m=1}^{n}$, then $\{\lambda\tilde{\sigma}^c(\tilde{m}; n)\}_{m=1}^{\tilde{n}}$ is a reduced set to the subsequence $\{\lambda\sigma^c(m)\}_{m=1}^{n}$ for any $\lambda \geq 0$.

(iii) Suppose $\{\sigma^c(m)\}_{m=1}^{n_1}$ and $\{\sigma^c(m)\}_{m=1}^{n_2}$ are two finite subsequences of an original sequences $\{\sigma^c(m)\}_{m=1,2,...}$ and $n_2 > n_1$, then $\{\sigma^c(m)\}_{m=1}^{n_1}$ constitutes a part of $\{\sigma^c(m)\}_{m=1}^{n_2}$. For the reduced sets, $\tilde{n}_2 > \tilde{n}_1$ but the reduced set $\{\tilde{\sigma}^c(m\tilde{;} n_1)\}_{m=1}^{\tilde{n_1}}$ does not generally belong to the reduced set $\{\tilde{\sigma}^c(m\tilde{;} n_2)\}_{m=1}^{\tilde{n_2}}$.

Using a cycle counting method and a summation rule, one can estimate durability under a process $\{\sigma^c(m)\}_{m=1,2,...}$ by the durabilities under the reduced sets $\{\tilde{\sigma}^c(\tilde{m}; n)\}_{m=1}^{\tilde{n}}$. Then one can estimate the NESF $\underline{\Lambda}(\{\sigma^c\}; n)$ by the NESF under the reduced load set,

$$\underline{\Lambda}(\{\sigma^c\}; n) = \underline{\Lambda}(\{\tilde{\sigma}^c(\tilde{m}; n)\}_{m=1}^{\tilde{n}(n)}; \tilde{n}(n)).$$

As follows from item (ii) above, the NEFS $\underline{\Lambda}(\{\sigma^c\}; n)$ obtained in this way satisfy the homogeneity property (10) with respect to $\{\sigma^c\}$. However, the monotone non-decreasing with respect to n, see (9), is not evident as follows from item (iii) above, and a detailed analysis of particular cycle counting method is necessary to investigate this.

7 Examples of cyclic normalised equivalent stress functionals for multiaxial loading

The fatigue strength criteria for multiaxial case can be classified roughly into three main categories: criteria based on equivalent stress concept (approaches based on the stress invariants), critical surface criteria and strain energy criteria. The approach based on an equivalent stress σ_{eq}, which is applicable to a cyclic loading with synchronous (proportional, coaxial, in-phase) cycling for all stress components, is described e. g. by Pavlov (1988), Lebedev (1990). The equivalent stress is defined as a function of the stress amplitudes and the mean stresses in a cycle and then is substituted into the corresponding durability expressions known for symmetric uniaxial periodic loading. In this way, the durability analysis under asymmetric multiaxial cyclic loading is reduced to one S-N diagram under symmetric uniaxial cyclic loading.

The so-called critical surface criteria for in-phase loading, based on the von Mises equivalent stress and taking into account an influence of the hydrostatic stress on the fatigue endurance were proposed by Crossland (1956) and Sines (1959). The both criteria have similar analytical formulas. The difference is that Sines used the mean on cycle hydrostatic stress $\sigma_{H,m}$ while Crossland used in his criterion the maximal on cycle hydrostatic stress $\sigma_{H,max}$. Another criterion, based on the Tresca maximum shear stress was proposed by Dang Van (1973). Modifying the Sines criterion, Kakuno (1979) proposed to take into account an effect of the hydrostatic pressure amplitude. Endurance limits (or S–N diagrams) under several types of periodic loading are necessary to obtain the criteria parameters (or their dependence on the cycle number). All these criteria were extended for the case of existing residual stress by Flavenot & Skalli (1989). Papadopoulos et al (1997) made an attempt to extend the critical surface approaches to non-proportional periodic processes.

7.1 Multiaxial loading processes for materials independent of history

Let us consider a material independent of history (that is, its strength is determined only by the instants stress value) and obeying the strength condition

$$\sigma_{eq}(\sigma) < \sigma_r$$

where $\sigma_{eq}(\sigma)$ is the von Mises, Tresca or other equivalent stress (positively homogeneous function of order +1), σ_r is a material constant (the material strength under monotone uniaxial tension). Then we have from the definition of NESF,

$$\underline{\Lambda}^N(\{\sigma\}, n) = \max_{1\le m\le n} \max_{\sigma\in\sigma^c(m)} \frac{\sigma_{eq}(\sigma)}{\sigma_r} = \max_{1\le m\le n} \frac{\|\sigma^c(m)\|}{\sigma^*(\tilde{\sigma}^c(m))}. \tag{55}$$

Here $\|\sigma^c(m)\|$ denotes a norm of a tensor function $\sigma^c(m)$ describing the stress tensor behaviour on the $m-$th cycle, for example, $\|\sigma^c(m)\| = \sup_{\sigma\in\sigma^c(m)} |\sigma|$, where the supremum is taken along the $m-$th cycle and $|\sigma|$ denotes a matrix norm of a tensor σ_{ij}, for example, $|\sigma| = \sqrt{\sum_{i,j=1}^3 \sigma_{ij}^2}$;

$$\sigma^*(\tilde{\sigma}^c(m)) = \min_{\sigma\in\tilde{\sigma}^c(m)} [\sigma_r/\sigma_{eq}(\sigma)], \qquad \tilde{\sigma}^c(m) = \sigma^c(m)/\|\sigma^c(m)\|. \tag{56}$$

7.2 Multiaxial regular proportional periodic loading

7.2.1 Equivalent stress concept for regular periodic loading

Let $\sigma_{eq}(\sigma^c)$ be an equivalent stress expressed in terms of stress amplitude tensor $\sigma_{ij,a}$ and asymmetric ratio R or of maximum stresses tensor $\sigma_{ij,max}$ and R, in a multiaxial proportional cycle σ^c. The main assumption is that the number of cycles up to rupture $n^*\{\sigma^c\}$ under multiaxial cyclic loading $\{\sigma^c_{ik}\}$ with an equivalent stress $\sigma_{eq}(\sigma^c)$ can be found from fatigue curves $n^*_{-1}(\sigma_a)$ for uniaxial symmetric cyclic loading with the stress amplitude $\sigma_a = \sigma_{eq}(\sigma^c)$ and the asymmetry ratio $R = -1$,

$$n^*\{\sigma^c\} = n^*_{-1}(\sigma_{eq}(\sigma^c)) \tag{57}$$

The function σ_{eq} is chosen such that $\sigma_{eq}(\sigma^c) = \sigma_a$ for any uniaxial process σ^c with an amplitude σ_a and the asymmetry ratio $R = -1$.

For example, if the uniaxial durability is described by (34), then for a multiaxial periodic loading we obtain from (57),

$$n^*(\sigma^c) := n^*_{-1}(\sigma_{eq}(\sigma^c)) = \left(\frac{\sigma^*_{-1,1}}{\sigma_{eq}(\sigma)}\right)^{b_{-1}}, \tag{58}$$

where $\sigma^*_{-1,1}$ and b_{-1} are positive material constants.

There are many possibilities to introduce the value $\sigma_{eq}(\sigma)$. let us consider two expressions presented by Pavlov (1988). First one is

$$\sigma_{eq1}(\sigma^c) = \frac{\sigma_{i,a}}{1 - \frac{\sigma_{i,m}}{\sigma_r}}, \tag{59}$$

where σ_r is the material strength under monotone uniaxial loading and $\sigma_{i,a}$, $\sigma_{i,m}$ are the intensities (the von Mises equivalent stress) of stress amplitude and mean stress tensors introduced in (3), respectively. In terms of the tensor principal values $\sigma_{k,a}$, $\sigma_{k,m}$, $k = 1, 2, 3$,

$$\sigma_{i,a} = \frac{1}{\sqrt{2}}\left[(\hat\sigma_{1,a} - \hat\sigma_{2,a})^2 + (\hat\sigma_{2,a} - \hat\sigma_{3,a})^2 + (\hat\sigma_{3,a} - \hat\sigma_{1,a})^2\right]^{1/2},$$

$$\sigma_{i,m} = \frac{1}{\sqrt{2}}\left[(\hat\sigma_{1,m} - \hat\sigma_{2,m})^2 + (\hat\sigma_{2,m} - \hat\sigma_{3,m})^2 + (\hat\sigma_{3,m} - \hat\sigma_{1,m})^2\right]^{1/2}.$$

Expression (59) can be considered as a generalization of Haigh diagram for the multiaxial case.

The second example is

$$\sigma_{eq2}(\sigma^c) = \frac{\xi_a}{1 - \frac{\xi_m}{\sigma_r}}, \tag{60}$$

$$\xi_a = \zeta\sigma_{i,a} + (1-\zeta)\hat\sigma_{1,a}, \qquad \xi_m = \zeta\sigma_{i,m} + (1-\zeta)\hat\sigma_{1,m}, \quad \zeta = \frac{1}{\sqrt{3}-1}\left(\frac{\sigma^*_{-1\infty}}{\tau^*_{-1\infty}} - 1\right)$$

Here $\sigma^*_{-1\infty} := \sigma^*_{-1}(\infty)$ and $\tau^*_{-1\infty} := \tau^*_{-1}(\infty)$ are respectively the fatigue limits under periodic symmetric uniaxial and pure shearing cycling; $\hat\sigma_{1,a}$ is the amplitude of the maximum principal stress, $\hat\sigma_{1,m}$ is the mean value of the maximum principal stress. The values η_a and η_m are the modified stress amplitude and the modified mean stress respectively.

Another expression for σ_{eq} under asymmetric cycling was given by Lebedev (1990)

$$\sigma_{eq3}(\sigma^c) = \sigma_{i,a}\left[\left(1-\frac{2}{3}\frac{\sigma_{i,m}}{\sigma_r}\right)^2 - \frac{1}{9}\left(\frac{\sigma_{i,m}}{\sigma_r}\right)^2\right]^{-1/2} \tag{61}$$

For brittle materials under symmetric cycling $R = -1$ the maximal normal stress amplitude, that is, maximal principal stress amplitude $\hat{\sigma}_{1,a}$ can be used as an equivalent stress (e.g. Lebedev, 1990),

$$\sigma_{eq4}(\sigma^c) = \hat{\sigma}_{1,a} \tag{62}$$

Under the multiaxial regular periodic loading, the fatigue strength conditions for (59), (60), (61) and (62) at the n-cycles can be written in the following form respectively

$$\frac{\sigma_{i,a}}{1-\frac{\sigma_{i,m}}{\sigma_r}} < \sigma^*_{-1}(n), \tag{63}$$

$$\frac{\xi_a}{1-\frac{\xi_m}{\sigma_r}} < \sigma^*_{-1}(n), \tag{64}$$

$$\sigma_{i,a}\left[\left(1-\frac{2}{3}\frac{\sigma_{i,m}}{\sigma_r}\right)^2 - \frac{1}{9}\left(\frac{\sigma_{i,m}}{\sigma_r}\right)^2\right]^{-1/2} < \sigma^*_{-1}(n). \tag{65}$$

$$\hat{\sigma}_{1,a} < \sigma^*_{-1}(n), \tag{66}$$

Hence, according to the definition of the NESF, we have for (63), (64), (65) and (66), respectively,

$$\begin{aligned}
\underline{\Lambda}^N(\{\sigma\};n) &= \frac{\sigma_{i,a}}{\sigma^*_{-1}(n)} + \frac{\sigma_{i,m}}{\sigma_r}\\
\underline{\Lambda}^N(\{\sigma\};n) &= \frac{\xi_a}{\sigma^*_{-1}(n)} + \frac{\xi_m}{\sigma_r}\\
\underline{\Lambda}^N(\{\sigma\};n) &= \frac{2}{3}\frac{\sigma_{i,m}}{\sigma_r} + \sqrt{\left(\frac{\sigma_{i,a}}{\sigma^*_{-1}(n)}\right)^2 + \frac{1}{9}\left(\frac{\sigma_{i,m}}{\sigma_r}\right)^2}\\
\underline{\Lambda}^N(\{\sigma\};n) &= \frac{\hat{\sigma}_{1,a}}{\sigma^*_{-1}(n)}
\end{aligned}$$

7.2.2 Invariant based critical surface approaches for regular periodic loading

Popular fatigue endurance conditions under in-phase multiaxial periodic loading, based on the von Mises equivalent stress and taking into account an influence of the hydrostatic stress were proposed by Sines (1959),

$$\sigma_{i,a} + \alpha\sigma_{H,m} < \beta, \tag{67}$$

and Crossland (1956),

$$\sigma_{i,a} + \alpha\sigma_{H,max} < \beta. \tag{68}$$

Here $\sigma_{H,m} = \sigma_{jj,m}/3$ is the mean and $\sigma_{H,max} = -\sigma_{jj,\max}/3$ is the maximum on cycle hydrostatic stresses. The material parameters α and β can be found, for example, from tension-compression tests with $R = 0$ and $R = -1$. Then we have,

$$\alpha = 3\left(\frac{\sigma^*_{-1\infty}}{\sigma^*_{0\infty}} - 1\right), \quad \beta = \sigma^*_{-1\infty} \tag{69}$$

for the Sines criterion and

$$\alpha = 3\left(\frac{\sigma^*_{-1\infty} - \sigma^*_{0\infty}}{2\sigma^*_{0\infty} - \sigma^*_{-1\infty}}\right), \quad \beta = \frac{\sigma^*_{-1\infty}\sigma^*_{0\infty}}{2\sigma^*_{0\infty} - \sigma^*_{-1\infty}} \tag{70}$$

for the Crossland criterion. The value $\sigma^*_{0\infty} = \sigma_0(\infty)$ denotes the fatigue limit at the reversed (with zero minimum stress, i.e. at $R = 0$) tension test. Note that the use of the maximum on cycle hydrostatic stress $\sigma_{H,max}$ in the Crossland endurance condition allows to describe, in particular, a difference between torsional and tensile or bending symmetric ($R = -1$) fatigue tests.

Another fatigue endurance condition, based on the Tresca equivalent stress i.e. on the maximum shear stress and also taking into account the hydrostatic stress, was proposed by Dang Van (1973), Dang Van et al (1989),

$$\hat{\tau}_a + \alpha\sigma_{H,max} < \beta. \tag{71}$$

Here $\hat{\tau}_a$ is the shear stress amplitude acting on the plane of maximum shear. The unknown constants α and β can be also determined from the tension tests with $R = -1$ and $R = 0$,

$$\alpha = \frac{3}{2}\left(\frac{\sigma^*_{-1\infty} - \sigma^*_{0\infty}}{2\sigma^*_{0\infty} - \sigma^*_{-1\infty}}\right), \quad \beta = \frac{\sigma^*_{-1\infty}\sigma^*_{0\infty}}{2(2\sigma^*_{0\infty} - \sigma^*_{-1\infty})} \tag{72}$$

Note that for the Crossland and the Dang Van endurance conditions to be applicable to compression fatigue tests (with $\sigma_{H,max} \le 0$), the constant β in (68) and (71) should be positive. Then relations (70), (72) imply $2\sigma^*_{0\infty} = \sigma^*_{0\infty,max} > \sigma^*_{-1\infty,max} = \sigma^*_{-1\infty}$, that is, the endurance limit under uniaxial tension-compression is higher under the cycling with zero minimum stress than with the negative one, in terms of the maximal stress. This demand seems to be rather natural for structural materials.

Kakuno & Kawada (1979) proposed to take into account an effect of both the amplitude and the mean value of hydrostatic stress, modifying the Sines criterion in the following way

$$\sigma_{i,a} + \alpha_1\sigma_{H,a} + \alpha_2\sigma_{H,m} < \beta \tag{73}$$

The material constants are defined from three simple tests, for example fully reversed torsion test, fully reversed tension-compression test and repeating (zero minimum stress) tension test,

$$\alpha_1 = 3\sqrt{3}\frac{\tau^*_{-1\infty}}{\sigma^*_{-1\infty}} - 3, \quad \alpha_2 = 3\sqrt{3}\tau^*_{-1\infty}\left(\frac{1}{\sigma^*_{0\infty}} - \frac{1}{\sigma^*_{-1\infty}}\right), \quad \beta = \sqrt{3}\tau^*_{-1\infty}. \tag{74}$$

Here $\tau^*_{-1\infty}$ is the fatigue limit in fully reversed torsion, $\sigma^*_{-1\infty}$ is the fatigue limit in fully reversed tension-compression. The value $\sigma^*_{0\infty}$ denotes here the fatigue limit at repeated (zero minimum stress) tension. Note that the parameters $\sigma^*_{-1\infty}$, $\sigma^*_{0\infty}$ can be also referred to the bending tests.

Flavenot & Skalli (1989) extended all these criteria for the case of existing residual stress. The total hydrostatic stress was assumed to be the sum of the mean stress induced by the external load and of the residual stresses.

Inequalities (67), (68), (71) and (73) can be used not only as the fatigue endurance (i.e. at $n = \infty$) conditions but also as the fatigue strength conditions at an arbitrary number of cycles n if one replaces the constants α, β, α_1 and α_2 by corresponding functions. To find the functions $\alpha(n)$, $\beta(n)$, $\alpha_1(n)$ and $\alpha_2(n)$, one could use e.g. corresponding relations (69), (70), (72), where $\sigma^*_{-1\infty}$, $\sigma^*_{0\infty}$ and $\tau^*_{-1\infty}$ should be replaced by $\sigma^*_{-1}(n)$, $\sigma^*_0(n)$ and $\tau^*_{-1}(n)$ respectively.

Using the fatigue strength conditions (67), (68), (71) and (73) we find the NESFs:

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{1}{\beta(n)}\left(\sigma_{i,a} + \alpha(n)\sigma_{H,m}\right) \tag{75}$$

for the Sines criterion;

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{1}{\beta(n)}\left(\sigma_{i,a} + \alpha(n)\sigma_{H,max}\right) \tag{76}$$

for the Crossland criterion;

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{1}{\beta(n)}\left(\hat{\tau}_a + \alpha(n)\sigma_{H,max}\right) \tag{77}$$

for the Dang Van criterion and

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{1}{\beta(n)}\left(\sigma_{i,a} + \alpha_1(n)\sigma_{H,a} + \alpha_2(n)\sigma_{H,m}\right) \tag{78}$$

for the modified Sines (Kakuno & Kawada) criterion.

In particular cases, when the parameters $\alpha(n)$, $\beta(n)$, $\alpha_1(n)$ and $\alpha_2(n)$ are expressed in terms of $\sigma^*_{-1}(n)$, $\sigma^*_0(n)$ and $\tau^*_{-1}(n)$, we can rewrite the NESF in terms of the uniaxial S–N diagrams under tension-compression and torsion periodic loadings:

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{\sigma_{i,a}(n)}{\sigma^*_{-1}(n)} + 3\left(\frac{1}{\sigma^*_0(n)} - \frac{1}{\sigma^*_{-1}(n)}\right)\sigma_{H,m}(n) \tag{79}$$

for the Sines criterion;

$$\underline{\Lambda}^N(\{\sigma\};n) = \left(\frac{2}{\sigma^*_{-1}(n)} - \frac{1}{\sigma^*_0(n)}\right)\sigma_{i,a}(n) + 3\left(\frac{1}{\sigma^*_0(n)} - \frac{1}{\sigma^*_{-1}(n)}\right)\sigma_{H,max}(n) \tag{80}$$

for the Crossland criterion;

$$\underline{\Lambda}^N(\{\sigma\};n) = 2\left(\frac{2}{\sigma^*_{-1}(n)} - \frac{1}{\sigma^*_0(n)}\right)\hat{\tau}_a(n) + 3\left(\frac{1}{\sigma^*_0(n)} - \frac{1}{\sigma^*_{-1}(n)}\right)\sigma_{H,max}(n) \tag{81}$$

for the Dang Van criterion and

$$\begin{aligned}\underline{\Lambda}^N(\{\sigma\};n) &= \frac{1}{\sqrt{3}\tau^*_{-1}(n)}\sigma_{i,a}(n) + \left[\frac{3}{\sigma^*_{-1}(n)} - \frac{\sqrt{3}}{\tau^*_{-1}(n)}\right]\sigma_{H,a}(n) \\ &+ 3\left[\frac{1}{\sigma^*_0(n)} - \frac{1}{\sigma^*_{-1}(n)}\right]\sigma_{H,m}(n)\end{aligned} \tag{82}$$

for the modified Sines criterion.

7.2.3 Critical surface approaches for critical planes under regular periodic loading

The criteria based on the stress invariants (stress intensity amplitude and hydrostatic stress) are not able to determine a fracture (critical) plane. Including in a criterion some non-invariant values (e.g. normal or shear stresses acting on a considered plane) allowes to predict not only a fracture point but also a fracture plane. One of such criteria can be obtained by a modification of the Matake criterion.

Matake criterion. Let $\vec{\eta}$ denote a normal vector to a material plane. Matake (1977) assumed that the critical plane $\vec{\eta}^*$ is a plane on which the shear stress amplitude reaches its maximum and one can write the endurance condition in the form

$$\tau_a(\vec{\eta}) + \alpha\sigma_{\eta\eta,max} < \beta, \tag{83}$$

for $\vec{\eta} = \vec{\eta}^*$, where α and β are some material constants; $\sigma_{\eta\eta,max}$ is the maximum along the cycle of the normal stress $\sigma_{\eta\eta}$ acting on the plane $\vec{\eta}$. Using the definition of the endurance functional $\underline{\Lambda}^N_{th}(\{\sigma\}) = \underline{\Lambda}^N(\{\sigma\},\infty)$ we obtain from (83)

$$\underline{\Lambda}^N_{th}(\{\sigma\}) = \frac{1}{\beta}\left[\tau_a(\vec{\eta}^*) + \alpha\sigma_{\eta^*\eta^*,max}\right] \tag{84}$$

In particular, the constants can be identified from fully reversed torsion test and fully reversed tension-compression test

$$\alpha = \frac{2\tau^*_{-1\infty}}{\sigma^*_{-1\infty}} - 1, \quad \beta = \tau^*_{-1\infty} \tag{85}$$

Then the endurance functional $\underline{\Lambda}^N_{th}$ is

$$\underline{\Lambda}^N_{th}(\{\sigma\}) = \frac{\tau_a(\vec{\eta}^*)}{\tau^*_{-1\infty}} + \left[\frac{2}{\sigma^*_{-1\infty}} - \frac{1}{\tau^*_{-1\infty}}\right]\sigma_{\eta^*\eta^*,max} \tag{86}$$

Inequality (83) can be used not only as the fatigue endurance condition (i.e. at $n=\infty$) but also as the fatigue strength condition at an arbitrary number of cycles n if one replaces the constants α and β by corresponding functions $\alpha(n)$ and $\beta(n)$, $\alpha_1(n)$. Then we find the NESF for the Matake criterion

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{1}{\beta(n)}\left[\tau_a(\vec{\eta}^*) + \alpha(n)\sigma_{\eta^*\eta^*,max}\right] \tag{87}$$

Modified Matake criterion. If we consider the left hand side of (83) not for the maximal shear stress plane but for an arbitrary plane $\vec{\eta}$, we can realise that the maximum of this expression with respect to $\vec{\eta}$ will be not necessarily on the maximal shear stress plane. Thus it seems to be natural to modify the Matake criterion in the following way. Let

$$\tau_a(\vec{\eta}) + \alpha\sigma_{\eta\eta,max}(\vec{\eta}) < \beta, \tag{88}$$

be the endurance condition for a plane $\vec{\eta}$. The the endurance condition for a material point, that is for all planes passing through the point, is

$$\max_{\vec{\eta}}[\tau_a(\vec{\eta}) + \alpha\sigma_{\eta\eta,max}(\vec{\eta})] < \beta, \tag{89}$$

We assume now critical plane $\vec{\eta}^*$ is that, where the maximum in the left hand side of (89) is achieved, that is,

$$\tau_a(\vec{\eta}^*) + \alpha\sigma_{\eta^*\eta^*,max} = \max_{\vec{\eta}}[\tau_a(\vec{\eta}) + \alpha\sigma_{\eta\eta,max}]. \tag{90}$$

If the S–N diagrams $\tau^*_{-1}(n)$, $\sigma^*_{-1}(n)$ or at least their limits $\tau^*_{-1\infty} = \tau^*_{-1}(\infty)$, $\sigma^*_{-1\infty} = \sigma^*_{-1}(\infty)$ are available, we can interpret (89) as a Coulomb-Mohr strength condition and after some trigonometric manipulations obtain a counterpart of (85) for the modified Matake criterion

$$\alpha(n) = \frac{\dfrac{2\tau^*_{-1}(n)}{\sigma^*_{-1}(n)} - 1}{\sqrt{\dfrac{4\tau^*_{-1}(n)}{\sigma^*_{-1}(n)} - \left(\dfrac{2\tau^*_{-1}(n)}{\sigma^*_{-1}(n)}\right)^2}}\,, \qquad \beta(n) = \frac{\tau^*_{-1}(n)}{\sqrt{\dfrac{4\tau^*_{-1}(n)}{\sigma^*_{-1}(n)} - \left(\dfrac{2\tau^*_{-1}(n)}{\sigma^*_{-1}(n)}\right)^2}} \tag{91}$$

The normalised equivalent endurance functional for the modified Matake criterion is

$$\underline{\Lambda}^N_{th}(\{\sigma\}) = \max_{\vec{\eta}} \underline{\Lambda}^N_{th}(\{\sigma\};\vec{\eta}) = \max_{\vec{\eta}} \left\{\frac{1}{\beta}\left[\tau_a(\vec{\eta}) + \alpha\sigma_{\eta\eta,max})\right]\right\} \tag{92}$$

and the NESF for a finite n is

$$\underline{\Lambda}^N(\{\sigma\};n) = \max_{\vec{\eta}} \underline{\Lambda}^N(\{\sigma\};\vec{\eta};n) = \max_{\vec{\eta}} \left\{\frac{1}{\beta(n)}\left[\tau_a(\vec{\eta}) + \alpha(n)\sigma_{\eta\eta,max})\right]\right\} \tag{93}$$

McDiarmid criterion. If one chooses the parameters α and β in (83) as

$$\alpha = \frac{\tau^*_{-1\infty}}{2\sigma_r}, \quad \beta = \tau^*_{-1\infty},$$

one arrives at a criterion proposed by McDiarmid (1991, 1994)

$$\tau_a(\vec{\eta}) + \frac{\tau^*_{-1\infty}}{2\sigma_r}\sigma_{\eta\eta} < \tau^*_{-1\infty} \tag{94}$$

where $\vec{\eta} = \vec{\eta}^*$ is the critical plane supposed to be the maximum shear stress amplitude plane. Then for the endurance functional $\underline{\Lambda}_{th}$ we obtain

$$\underline{\Lambda}_{th}(\{\sigma\}) = \frac{\tau_a(\vec{\eta}^*)}{\tau^*_{-1\infty}} + \frac{\sigma_{\eta^*\eta^*}(\vec{\eta}^*)}{2\sigma_r}. \tag{95}$$

The NESF for the McDiarmid criterion is

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{\tau_a(\vec{\eta}^*)}{\tau^*_{-1}(n)} + \frac{\sigma_{\eta^*\eta^*}(\vec{\eta}^*)}{2\sigma_r} \tag{96}$$

One can also consider a modified version of criterion (94) interpreting it as an endurance condition of a plane $\vec{\eta}$. The critical plane is taken to be that subjected to the greatest value of the left hand side of (94) (McDiarmid, 1994). Then the normalised equivalent endurance functional becomes

$$\underline{\Lambda}^N_{th}(\{\sigma\}) = \max_{\vec{\eta}} \underline{\Lambda}^N_{th}(\{\sigma\};\vec{\eta}) = \max_{\vec{\eta}} \left\{\frac{\tau_a(\vec{\eta})}{\tau^*_{-1\infty}} + \frac{\sigma_{\eta\eta}(\vec{\eta})}{2\sigma_r}\right\} \tag{97}$$

and the NESF for a finite n becomes

$$\underline{\Lambda}^N(\{\sigma\};n) = \max_{\vec{\eta}} \underline{\Lambda}^N(\{\sigma\};\vec{\eta};n) = \max_{\vec{\eta}} \left\{\frac{\tau_a(\vec{\eta})}{\tau^*_{-1}(n)} + \frac{\sigma_{\eta\eta}(\vec{\eta})}{2\sigma_r}\right\} \tag{98}$$

A more general form of a critical surface-type fatigue life criterion was proposed by Papadopoulos (1998) for a plane, where the shear stress amplitude reaches its maximum value (considered by the author as a critical plane),

$$F(\tau_a, |\tau_m|, \sigma_{\eta\eta,a}, \sigma_{\eta\eta,m}, n) \leq 1 \tag{99}$$

Here τ_m is the mean shear stress acting on the plane of maximum shear, $\sigma_{\eta\eta,a}$ is the normal stress amplitude acting on the plane of the maximum shear stress and $\sigma_{\eta\eta,m}$ is the mean value of the normal stress. From (99) the durability $n^*(\lambda\sigma)$ under a process $\{\lambda\sigma\}$, $\lambda \geq 0$, is determined from the equation

$$F(\lambda\tau_a, \lambda|\tau_m|, \lambda\sigma_{\eta\eta,a}, \lambda\sigma_{\eta\eta,m}, n) \leq 1 \tag{100}$$

If the left hand side of (100) is a function non-decreasing in n and positive homogenous of order +1 with respect to λ, then, as pointed out at the end of Section 4, we have the following representation for the NESF,

$$\underline{\Lambda}^N(\{\sigma\}; n) = F[\tau_a, |\tau_m|, \sigma_{\eta\eta,a}, \sigma_{\eta\eta,m}, n].$$

Note that as for the Matake criterion, it seems to be more justified to apply condition (99) to any plane and consider as critical the plane where F reaches maximum; such a plane may generally not be the maximum shear stress amplitude plane.

7.3 Palmgren-Miner hypothesis for multiaxial regular proportional cyclic loading

As for the uniaxial case, the damage accumulation theories considered in this section are equally applicable to multiaxial proportional loadings with a large number of periodic cycles in a small number of blocks and to arbitrary proportional loadings with closed cycles, that is, periodicity in blocks is not necessary and there can be many transitions between the blocks if the transitions are closed cycles.

The strength condition for the Palmgren-Miner linear damage accumulation rule (31) can be easily generalised to a multiaxial cyclic loading,

$$\sum_{j=1}^{n} \frac{1}{n^{*p}(\sigma^c(j))} < 1 \tag{101}$$

Here $\sigma^c(j)$ is the stress trajectory on the $j-$th cycle; $n^{*p}(\sigma^c(j)) = n^*\{\sigma^c\}|_{\sigma^c=\sigma^c(j)}$ is an appropriate S–N Wöhler diagram, that is, the number of cycles up to rupture under a multiaxial *periodic* loading $\{\sigma^c_{ik}(m)\}_{m=1,2,\ldots}$, where $\sigma^c_{ik}(m) = \sigma^c_{ik}(j)$.

The other uniaxial damage accumulation from Section 6.3 can be generalised to the multiaxial closed-cyclic loading by in a similar way.

7.3.1 Equivalent stress concept for regular proportional cyclic loading

Using the equivalent stress concept given for *periodic* loading by (57), one can express the multiaxial S–N diagram $n^{*p}(\sigma^c(m))$ in terms of the uniaxial one. Then the strength condition (101) can be written in the form

$$\sum_{j=1}^{n} \frac{1}{n^*_{-1}(\sigma_{eq}(\sigma^c(j)))} < 1. \tag{102}$$

Using the equivalent stress $\sigma_{eq1}(\sigma)$ given by (59) and the S–N diagram in form (58), we have the equations for finding the durability $n^*(\{\lambda\sigma)\}$ for a process $\{\lambda\sigma_{ik}\}$ for any $\lambda \geq 0$ from linear damage accumulation rule (102),

$$\sum_{j=1}^{n}\left[\frac{\lambda\sigma_{i,a}(j)}{\sigma^*_{-1,1}\left(1-\frac{\lambda\sigma_{i,m}(j)}{\sigma_r}\right)}\right]^{b_{-1}} = 1. \tag{103}$$

The functional safety factor $\underline{\lambda}^N$ is the solution of the equations (103) and the NESF is $\underline{\Lambda}^N = 1/\underline{\lambda}^N$. Equation (103) can be analytically solved e.g. if $\sigma_{i,m}(j) = \sigma_{i,m} = const.$,

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{\sigma_{i,m}}{\sigma_r} + \frac{1}{\sigma^*_{-1,1}}\left[\sum_{j=1}^{n}\sigma_{i,a}^{b_{-1}}(j)\right]^{1/b_{-1}}. \tag{104}$$

Repeating the same manipulations for the equivalent stress $\sigma_{eq2}(\sigma)$ given by (60) and taking the S–N diagram in form (58), we obtain an equation similar to (103) for determining $\underline{\lambda}$ and $\underline{\Lambda}$, which can be analytically solved for the case of constant mean stresses $\sigma_{ik,m}(j) = \sigma_{ik,m} = const$

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{\xi_m}{\sigma_r} + \frac{1}{\sigma^*_{-1,1}}\left[\sum_{j=1}^{n}\xi_a^{b_{-1}}(j)\right]^{1/b_{-1}}. \tag{105}$$

If we consider the equivalent stress $\sigma_{eq3}(j)$ given by (61) and the S–N diagram in form (58), then the corresponding equation, obtained from (102) for the determination of the NESF $\underline{\Lambda}^N = 1/\lambda$ (31), can be also analytically solved if $\sigma_{i,m}(j) = \sigma_{i,m} = const$

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{2}{3}\frac{\sigma_{i,m}}{\sigma_r} + \sqrt{\frac{1}{(\sigma^*_{-1,1})^2}\left[\sum_{j=1}^{n}\sigma_{i,a}^{b_{-1}}(j)\right]^{2/b_{-1}} + \frac{1}{9}\left(\frac{\sigma_{i,m}}{\sigma_r}\right)^2} \tag{106}$$

We can also use damage accumulation rule (102) and the equivalent stress $\sigma_{eq4}(j)$ given by (62). If we take the S–N diagram in form (58), then we find,

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{1}{\sigma^*_{-1,1}}\left[\sum_{j=1}^{n}\hat{\sigma}_{1,a}^{b_{-1}}(j)\right]^{1/b_{-1}}. \tag{107}$$

7.3.2 Critical surface approaches for regular proportional cyclic loading

The critical surface approach can be also applied to find the multiaxial S–N $n^*\{\sigma^c\}$ diagram under a periodic loading for (101). To do this, one should have the functions $\alpha(n)$, $\beta(n)$, $\alpha_1(n)$ and $\alpha_2(n)$ and solve with respect to n an equation, obtained after replacement the inequality by the equality sign in (67), (68), (71) or (73). Equivalently, one can solve the equation $\underline{\Lambda}^N(\{\sigma\};n) = 1$ with respect to n, where $\underline{\Lambda}^N(\{\sigma\};n)$ is given by (76)–(78) or, the same, by (79)–(82).

Let us use presentations (79)–(82) and suppose the S–N diagrams for $\sigma^*_{-1}(n)$, $\sigma^*_0(n)$ and if necessary for $\tau^*_{-1}(n)$ can be approximated by a power law with the same parameter b,

$$\sigma^*_{-1}(n) = \sigma^*_{-1,1}n^{-1/b}, \quad \sigma^*_0(n) = \sigma^*_{0,1}n^{-1/b}, \quad \tau^*_{-1}(n) = \tau^*_{-1,1}n^{-1/b}, \tag{108}$$

where $\sigma^*_{-1,1}$, $\sigma^*_{0,1}$, $\tau^*_{-1,1}$ and b are material parameters. Then the find the S–N diagrams under a multiaxial periodic loading,

$$n^*(\{\sigma\}) = \left[\frac{\sigma_{i,a}}{\sigma^*_{-1,1}} + 3\left(\frac{1}{\sigma^*_{0,1}} - \frac{1}{\sigma^*_{-1,1}}\right)\sigma_{H,m}\right]^{-b} \tag{109}$$

for the Sines criterion,

$$n^*(\{\sigma\}) = \left[\left(\frac{2}{\sigma^*_{-1,1}} - \frac{1}{\sigma^*_{0,1}}\right)\sigma_{i,a} + 3\left(\frac{1}{\sigma^*_{0,1}} - \frac{1}{\sigma^*_{-1,1}}\right)\sigma_{H,max}\right]^{-b} \tag{110}$$

for the Crossland criterion,

$$n^*(\{\sigma\}) = \left[2\left(\frac{2}{\sigma^*_{-1,1}} - \frac{1}{\sigma^*_{0,1}}\right)\hat{\tau}_a + 3\left(\frac{1}{\sigma^*_{0,1}} - \frac{1}{\sigma^*_{-1,1}}\right)\sigma_{H,max}\right]^{-b} \tag{111}$$

for the Dang Van criterion and

$$n^*(\{\sigma\}) = \left[\frac{1}{\sqrt{3}\tau^*_{-1,1}}\sigma_{i,a} + \left(\frac{3}{\sigma^*_{-1,1}} - \frac{\sqrt{3}}{\tau^*_{-1,1}}\right)\sigma_{H,a} + 3\left(\frac{1}{\sigma^*_{0,1}} - \frac{1}{\sigma^*_{-1,1}}\right)\sigma_{H,m}\right]^{-b} \tag{112}$$

for the modified Sines criterion.

After substituting (109) into the damage accumulation laws (31) or (43) (see also (49) or (54)) we have the expressions for finding the durability $n^*(\lambda\sigma)$ for the process $\lambda\sigma_{ij}$. For example, using (31) and solving the corresponding equation with respect to λ, we find the NESF $\underline{\Lambda}^N = 1/\lambda$

$$\underline{\Lambda}^N(\{\sigma\};n) = \left[\sum_{j=1}^{n}\left[\frac{\sigma_{i,a}(j)}{\sigma^*_{-1,1}} + 3\left(\frac{1}{\sigma^*_{0,1}(j)} - \frac{1}{\sigma^*_{-1,1}}\right)\sigma_{H,m}(j)\right]^b\right]^{1/b} \tag{113}$$

Repeating the same manipulations for the Crossland criterion we obtain

$$\underline{\Lambda}^N(\{\sigma\};n) = \left[\sum_{j=1}^{n}\left[\left(\frac{2}{\sigma^*_{-1,1}} - \frac{1}{\sigma^*_{0,1}}\right)\sigma_{i,a}(j) + 3\left(\frac{1}{\sigma^*_{0,1}} - \frac{1}{\sigma^*_{-1,1}}\right)\sigma_{H,max}(j)\right]^b\right]^{1/b} \tag{114}$$

For the Dang Van criterion we have

$$\underline{\Lambda}^N(\{\sigma\};n) = \left[\sum_{j=1}^{n}\left[2\left(\frac{2}{\sigma^*_{-1,1}} - \frac{1}{\sigma^*_{0,1}}\right)\tau_a(j) + 3\left(\frac{1}{\sigma^*_{0,1}} - \frac{1}{\sigma^*_{-1,1}}\right)\sigma_{H,max}(j)\right]^b\right]^{1/b} \tag{115}$$

Finally, the NESF for the modified Sines criterion is

$$\begin{aligned}\underline{\Lambda}^N(\{\sigma\};n) \;&=\; \left[\sum_{j=1}^{n}\left[\frac{1}{\sqrt{3}\tau^*_{-1,1}}\sigma_{i,a}(j) + \left(\frac{3}{\sigma^*_{-1,1}} - \frac{\sqrt{3}}{\tau^*_{-1,1}}\right)\sigma_{H,a}(j)\right.\right.\\ &+\; 3\left.\left.\left(\frac{1}{\sigma^*_{0,1}} - \frac{1}{\sigma^*_{-1,1}}\right)\sigma_{H,m}(j)\right]^b\right]^{1/b} \end{aligned}\tag{116}$$

7.4 Critical surface approaches for multiaxial non-proportional periodic process

General multiaxial periodic processes consist of the loops in the stress space that do not necessarily lie on a line. Consequently, the notions of the amplitude and the mean stress become ambiguous and the approaches used in the case of proportional cycling need some generalisations.

To extend the critical surface approaches on the case of non-proportional periodic process, some generalised definitions of the mean value and amplitude of the hydrostatic stress σ_H and shear stress for multiaxial non-proportional periodic process were considered by Papadopoulos et al (1997) and Papadopoulos (1998). The maximum, the minimum, the mean and the amplitude of the hydrostatic stress per cycle with a period P was defined as

$$\begin{aligned} \sigma_{H,max} &= \max_{t\in[0,P]}[\sigma_H(t)], \quad \sigma_{H,min} = \min_{t\in[0,P]}[\sigma_H(t)] \\ \sigma_{H,m} &= \frac{\sigma_{H,max}+\sigma_{H,min}}{2}, \quad \sigma_{H,a} = \frac{\sigma_{H,max}-\sigma_{H,min}}{2} \end{aligned} \tag{117}$$

Let us consider a material plane (with a normal vector) $\vec{\eta}$. Let $\vec{\sigma}(\vec{\eta},t)$ be the stress vector (traction) on the plane $\vec{\eta}$ and $\sigma_{\eta\eta}(\vec{\eta},t)$ be the normal stress tensor component on the plane $\vec{\eta}$. Then the maximum, the minimum, the mean and the amplitude of $\sigma_{\eta\eta}(\vec{\eta},t)$ per cycle can be defined as

$$\begin{aligned} \sigma_{\eta\eta,max} &= \max_{t\in[0,P]}[\sigma_{\eta\eta}(t)], \quad \sigma_{\eta\eta,min} = \min_{t\in[0,P]}[\sigma_{\eta\eta}(t)] \\ \sigma_{\eta\eta,m} &= \frac{\sigma_{\eta\eta,max}+\sigma_{\eta\eta,min}}{2}, \quad \sigma_{\eta\eta,a} = \frac{\sigma_{\eta\eta,max}-\sigma_{\eta\eta,min}}{2} \end{aligned} \tag{118}$$

During a cycle, the tip of the shear stress vector $\vec{\tau}(\vec{\eta},t) = \vec{\sigma}(\vec{\eta},t) - \sigma_{\eta\eta}(\vec{\eta},t)\vec{\eta}$ on a plane $\vec{\eta}$ is moving along a closed curve Ψ on the plane. To define the mean value of shear stress $\tau_m(\vec{\eta},t)$ and the shear stress amplitude $\tau_a(\vec{\eta},t)$ the minimum circumscribed circle to the curve Ψ is constructed. The circle radius is taken as the shear stress amplitude on the plane $\vec{\eta}$,

$$\tau_a(\vec{\eta}) := \min_{\vec{\tau}'}\Big[\max_{t\in[0,P]}|\vec{\tau}(\vec{\eta},t)-\vec{\tau}'|\Big]. \tag{119}$$

The length of the vector, which points to the centre of this circle, is taken as mean shear stress $\tau_m(\vec{\eta})$ on the plane $\vec{\eta}$, that is, $\tau_m(\vec{\eta}) := |\vec{\tau}_m(\vec{\eta})|$, where $\vec{\tau}_m(\vec{\eta})$ is such that

$$\max_{t\in[0,P]}|\vec{\tau}(\vec{\eta},t)-\vec{\tau}_m(\vec{\eta})| = \tau_a(\vec{\eta}). \tag{120}$$

The stress deviator $S_{kj}(t) = \sigma_{kj}(t) - \frac{1}{3}\sigma_{ll}(t)\delta_{kj}$ can be mapped onto a vector $\vec{S}(t)$ in a five-dimensional Euclidian space. The length of the vector $\vec{S}$ is equal to the stress intensity σ_i multiplied by $\sqrt{2/3}$. During a periodic loading the tip of the vector $\vec{S}(t)$ describes a closed curve Φ. To obtain the unique value of $\sigma_{i,a}$, Papadopoulos et al (1997) and Papadopoulos (1998) constructed the unique minimum five-dimensional hypersphere circumscribed to the curve Φ. The radius of this sphere is taken as $\sqrt{2/3}\ \sigma_{i,a}$, that is,

$$\sigma_{i,a} = \sqrt{3/2}\ \min_{S'}\Big[\max_{t\in[0,P]}|\vec{S}(t)-\vec{S}'|\Big]. \tag{121}$$

The length of the vector $\vec{S}_m$ which points to the centre of this hypersphere is taken as $\sqrt{2/3}\ \sigma_{i,m}$, that is, $\sigma_{i,m} = \sqrt{3/2}\ |\vec{S}_m|$, where $\vec{S}_m$ is such that

$$\max_{t\in[0,P]} |\vec{S}(t) - \vec{S}_m| = \sqrt{2/3}\ \sigma_{i,a}.$$

After the above generalisations for the cycle parameters, the Sines criterion, the Crossland criterion, the Kakuno-Kawada criterion, the Matake criterion and its modification, and the McDiarmid criterion and its modification can be extended for non-proportional periodic loading if one uses in (67), (68), (73), (83) and (94) the values given by (121) for $\sigma_{i,a}$, by (117) for $\sigma_{H,max}$, $\sigma_{H,m}$ and $\sigma_{H,a}$ and by (118) for $\sigma_{\eta\eta,max}$. The same generalised cycle parameters should be used then in the corresponding NESF expressions (113), (114), (116), (87), (93), (96) and (98).

Consider some generalisation of the Dang Van criterion (71). Let $\vec{\tau}(t)$ be the shear stress vector in a plane $\vec{\eta}$. Let us consider its projection $\sigma_{\eta\xi}(t) = \vec{\tau}(\vec{\eta}, t)\vec{\xi}$ on a direction $\vec{\xi}$ in this plane and introduce the shear stress amplitude

$$\sigma_{\eta\xi,a} = \frac{\max_{t\in[0,P]} \sigma_{\eta\xi}(t) - \min_{t\in[0,P]} \sigma_{\eta\xi}(t)}{2}.$$

Following Papadopoulos et al (1997), the generalised Dang Van criterion can be written in the form

$$\sqrt{\langle \tau_a^2 \rangle} + \alpha\sigma_{H,max} < \beta \tag{122}$$

Here $\langle \tau_a^2 \rangle$ is the average value of the shear stress amplitude projection square,

$$\langle \tau_a^2 \rangle = \frac{5}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} \sigma_{\eta\xi,a}^2 d\chi_\xi \sin(\vartheta_\eta) d\vartheta_\eta d\phi_\eta. \tag{123}$$

The angles ϑ_η and ϕ_η spherical coordinates of the vector $\vec{\eta}$, and χ_ξ is a polar angle of the vector $\vec{\xi}$ in the plane $\vec{\eta}$.

In particular, the constants can be identified from fully reversed torsion test and fully reversed tension-compression test

$$\alpha = 3\frac{\tau^*_{-1\infty}}{\sigma^*_{-1\infty}} - \sqrt{3}, \quad \beta = \tau^*_{-1\infty} \tag{124}$$

Then the NESF for the generalised Dang Van criterion a non-proportional periodic process $\{\sigma\}$ will have the form

$$\underline{\Lambda}^N(\{\sigma\}; n) = \frac{1}{\beta(n)} \left(\sqrt{\langle \tau_a^2 \rangle} + \alpha(n)\sigma_{H,max} \right). \tag{125}$$

It can be shown that $\langle \tau_a^2 \rangle$ is equal to the square of the stress intensity amplitude multiplied by a constant number, for a proportional periodic process. Hence this criterion is reduced to the Crossland criterion in that case.

Another modification of the Dang Van endurance criterion (71) applicable to non-proportional periodic loading was presented by Ballard P. et al (1995),

$$\max_{t\in[0,P]} \left[\hat{\tau}(t) + \alpha\sigma_H(t) \right] < \beta, \tag{126}$$

where $\hat{\tau}$ is the maximum (along all planes) shear stress. Then according to the definition, the endurance functional $\underline{\Lambda}_{th}$ is

$$\underline{\Lambda}_{th}\{\sigma\} = \frac{1}{\beta} \max_{t\in[0,P]} \left[\hat{\tau}(t) + \alpha\sigma_H\right] \tag{127}$$

The corresponding NESF for an arbitrary n can be written as

$$\underline{\Lambda}(\{\sigma\}; n) = \frac{1}{\beta(n)} \max_{t\in[0,P]} \left[\hat{\tau}(t) + \alpha(n)\sigma_H\right] \tag{128}$$

7.5 Palmgren-Miner hypothesis for multiaxial non-proportional cyclic loading

The Palmgren-Miner linear damage accumulation rule for this case will have the same form (101), where $n^{*p}(\sigma^c(m)) = n^*\{\sigma^c\}|_{\sigma^c=\sigma^c(m)}$ is is an appropriate S–N Wöhler diagram for a multiaxial *periodic* non-proportional loading $\{\sigma^c_{ij}(k)\}_{k=1,2,...}$, where $\sigma^c_{ij}(k) = \sigma^c_{ij}(m)$.

Suppose the Wöhler diagram $n^{*p}(\sigma^c_{ij})$ is given by the power law (a straight line in the double logarithmic coordinates):

$$n^{*p}(\sigma^c) = \left[\frac{\|\!|\sigma^c|\!\|}{\sigma^*_{\bowtie 1}(\tilde{\sigma}^c)}\right]^{-b(\tilde{\sigma}^c)}. \tag{129}$$

Here $\tilde{\sigma}^c = \sigma^c/\|\!|\sigma^c|\!\|$ is the normalised shape of the cycle playing for the non-proportional cycling the same role as the asymmetry ratio R in regular uniaxial processes; $\sigma^*_{\bowtie 1}(\tilde{\sigma}^c)$ and $b(\tilde{\sigma}^c)$ should be taken as positive material characteristics in the power S–N diagram for the corresponding multiaxial *periodic* process, depending generally on the cycle σ^c shape in the stress space but not on the cycle norm $\|\!|\sigma^c|\!\|$.

Then we can determine NESF $\underline{\Lambda}(\{\sigma\}; n) = 1\lambda$, where λ is a solution to the equation

$$\sum_{m=1}^{n^*} \left[\frac{\lambda\|\!|\sigma^c(m)|\!\|}{\sigma^*_{\bowtie 1}(\tilde{\sigma}^c(m))}\right]^{b(\tilde{\sigma}^c(m))} = 1. \tag{130}$$

We consider below two cases when it is possible to solve the equation analytically.

7.5.1 Self-similar multi-axial cyclic process

Suppose $\sigma_{ij}(t)$ is a *self-similar* multi-axial cyclic process $\sigma^c_{ij}(m) = k(m)\sigma^c_{ij}(1)$, $k(m) = \|\!|\sigma^c(m)|\!\|/\|\!|\sigma^c(1)|\!\| \geq 0$. Then the cycle shape $\tilde{\sigma}^c(m)$ is independent on m and consequently $b(\tilde{\sigma}^c(m)) = b = const$ and $\sigma^*_{\bowtie 1}(\tilde{\sigma}^c(m)) = \sigma^*_{\bowtie 1} = const$, and we have from (130),

$$\lambda = \sigma^*_{\bowtie 1} \left[\sum_{m=1}^{n^*} \|\!|\sigma^c(m)|\!\|^b\right]^{-1/b}. \tag{131}$$

The right hand side of (131) is a monotonously non-increasing function of n^*. From the definition of $\underline{\Lambda}$, we then have,

$$\underline{\Lambda}^N(\{\sigma\}; n) = \frac{1}{\sigma^*_{\bowtie 1}} \left[\sum_{m=1}^{n} \|\!|\sigma^c(m)|\!\|^b\right]^{1/b}. \tag{132}$$

7.5.2 Power S–N diagram with a constant exponent

Let $\{\sigma\}$ be a (not necessarily self-similar) multi-axial cyclic process but the exponent in (129) does depend on the cycle shape, $b(\tilde{\sigma}^c(m)) = b = const$. Then we can solve (130) and obtain the formula for the NESF,

$$\underline{\Lambda}^N(\{\sigma\}; n) = \left[\sum_{m=1}^{n} \left(\frac{\|\sigma^c(m)\|}{\sigma^*_{\bowtie 1}(\tilde{\sigma}^c(m))}\right)^b\right]^{1/b}. \tag{133}$$

Specific functions $\sigma^*_{\bowtie 1}(\tilde{\sigma}^c(m))$ for an S–N diagram under a non-proportional cyclic loading can be obtained e.g. from some equivalent stress concepts or critical surface approaches described in Section 7.2 if one substitute there expressions for $\sigma_{i,a}$, $\sigma_{i,m}$, $\sigma_{H,m}$, $\sigma_{H,max}$ and $\sigma_{H,a}$ from Section 7.4. Then one arrives for the NESF at the same expressions (104) and (106) for the equivalent stresses σ_{eq1} and σ_{eq3} if $\sigma_{i,m}$ is constant, and expressions (113), (114), (116) for the Sines, the Crossland, and the modified Sines (Kakuno & Kawada) criteria, where the mentioned substitutions should be done.

The generalised Dang Van criterion (122) for non-proportional cyclic loading can be also used in the similar way. Substituting the constants (124), we obtain another the (115) expression of the NESF

$$\underline{\Lambda}^N(\{\sigma\}; n) = \left[\sum_{j=1}^{n} \left[\frac{1}{\tau^*_{-1,1}}\sqrt{\langle\tau_a^2\rangle(j)} + \left(3\frac{1}{\sigma^*_{-1,1}} - \frac{\sqrt{3}}{\tau^*_{-1,1}}\right)\sigma_{H,max}(j)\right]^b\right]^{1/b} \tag{134}$$

7.6 Fatigue endurance under multiaxial random loading

Macha introduced several fatigue endurance criteria (Macha (1976), Macha (1984)) for different cases of loading generalised later (Macha 1989) for random loading. The corresponding endurance condition can be written in the form

$$\max_{0<t<\infty} [\alpha_1 \sigma_{\eta\xi}(t) + \alpha_2 \sigma_{\eta\eta}(t)] < \beta, \tag{135}$$

where $\sigma_{\eta\eta}$ is the normal stress amplitude on a plane (with a normal vector) $\vec{\eta}$, $\sigma_{\eta\xi,a}$ is the shear stress acting in a direction $\vec{\xi}$ on the plane with the normal vector $\vec{\eta}$. The constants α_1, α_2 and β are the material constants determined from three endurance tests under periodic loadings. Condition (135) was applied on a critical plane $\vec{\eta} = \vec{\eta}^*$ for a critical direction $\vec{\xi} = \vec{\xi}^*$, which were determined by mean of the stress tensor $\sigma_{kj}(t)$ principal directions and by the mean direction of the maximum shear stress vector.

Using the definition of the endurance functional, we then have

$$\underline{\Lambda}_{th}\{\sigma\} = \max_{0\le t<\infty} \left[\frac{\alpha_1}{\beta}\sigma_{\eta^*\xi^*}(t) + \frac{\alpha_2}{\beta}\sigma_{\eta^*\eta^*}(t)\right] \tag{136}$$

As for some fatigue strength and endurance conditions considered previously, it seems to be more natural to modify the Macha approach and consider (135) as an endurance condition not for a critical plane and direction pre-determined by other reasoning but for all planes $\vec{\eta}$ and for all directions $\vec{\xi}$ on those planes. Then the critical (fracture) plane $\vec{\eta}^*$

and the direction $\vec{\xi}^*$ can be defined are those where the left hand side of (135) reaches maximum. Thus we arrive at the strength condition

$$\max_{0<t<\infty}[\alpha_1\sigma_{\eta^*\xi^*}(t)+\alpha_2\sigma_{\eta^*\eta^*}(t)] = \max_{\vec{\eta}}\max_{(\vec{\xi}\vec{\eta})=0}\max_{0<t<\infty}[\alpha_1\sigma_{\eta\xi}(t)+\alpha_2\sigma_{\eta\eta}(t)] < \beta \tag{137}$$

The corresponding endurance functional then becomes

$$\underline{\Lambda}_{th}\{\sigma\} = \max_{\vec{\eta}}\max_{(\vec{\xi}\vec{\eta})=0}\max_{0\le t<\infty}\left[\frac{\alpha_1}{\beta}\sigma_{\eta\xi}(t)+\frac{\alpha_2}{\beta}\sigma_{\eta\eta}(t)\right] \tag{138}$$

In the case of bending and torsion, the empirical Gough-Pollard "ellipse quadrant" criterion (Gough-Pollard, 1951) was also generalized on the case of a random loading in the following way by Macha (Macha (1976), Macha (1984))

$$\max_{0\le t<\infty}\left[\left(\frac{\sigma_{\eta\eta}(t)}{\sigma^*_{-1\infty}}\right)^2+\left(\frac{\tau(t)}{\tau^*_{-1\infty}}\right)^2\right]=1 \tag{139}$$

The endurance functional $\underline{\Lambda}_{th}(\vec{\eta})$ in this case (139) is

$$\underline{\Lambda}_{th}(\vec{\eta}) = \max_{0\le t<\infty}\sqrt{\left[\left(\frac{\sigma_{\eta\eta}(t)}{\sigma^*_{-1\infty}}\right)^2+\left(\frac{\tau(t)}{\tau^*_{-1\infty}}\right)^2\right]} \tag{140}$$

Note, the NESF $\underline{\Lambda}(\tilde{t})$ for some $\tilde{t}<\infty$ is obtained from (136), (138), (140) if the maximum in t is considered on the interval $0\le t\le\tilde{t}$.

8 Complex NESF for combined fatigue, creep and instant loading

The NESF $\underline{\Lambda}^N$ is a material characteristic which is not necessary connected with a geometrical, stiffness-related or abstract damage measure and can be identified from some cyclic durability tests under homogeneous stress process fields. As was shown in the previous sections, any cyclic strength condition written in terms of a damage measure can be expressed in terms of a corresponding NESF (although not always analytically). Let us show some simple ways constructing NESFs to include e.g. instant overloading, creep or dynamic effects. Most of the damage accumulation rules mentioned above (except the Pavlov rule) did not take into account sequence effects, that is damage caused by a stress cycle is independent of where it occurs in the load history. We will see that this shortcoming can be overcome in a simple way choosing a proper structure of NESFs.

Suppose one has a NESF $\underline{\Lambda}^N(\{\sigma\};n)$ obtained e.g. from a damage measure approach, which do not take into account an influence of instantaneous overloads of material, especially a finite strength σ_r under instantaneous loading. Particularly, NESFs (36), (37), (40), (113)–(116) based on the power-type S–N diagrams give such examples. If one would like to avoid this shortcoming, one can introduce a new NESF and arrive at a local strength condition e.g. in the form

$$\underline{\Lambda}^{IN}(\{\sigma\};n(t)) = \sup_{0\le t'\le t}\left\{\frac{\sigma_{eq}(\sigma(t'))}{\sigma_r}+\underline{\Lambda}^N(\{\sigma\};n(t'))\right\}<1. \tag{141}$$

Here $\sigma_{eq}(\sigma)$ is e.g. von Mises, Tresca or other instantaneous equivalent stress; $\underline{\Lambda}^N$ is normalised equivalent stress functional defined, for example, from (132) or (113)-(116); σ_r is a material strength under uniaxial monotone tensile loading; the cycle number $n(t)$ is a function of time t.

If one would like to take into account an influence of both the instantaneous overloads of material and creep durability, one can add a term $\underline{\Lambda}^T(\sigma;t')$ connected with a durability under creep (see Mikhailov 2000), and arrive at another one complex NESF and local strength condition,

$$\underline{\Lambda}^{ITN}(\{\sigma\};n(t)) = \sup_{0\le t'\le t}\left\{\frac{\sigma_{eq}(\sigma(t'))}{\sigma_r} + \underline{\Lambda}^T(\sigma;t') + \underline{\Lambda}^N(\{\sigma\};n(t'))\right\} < 1 \tag{142}$$

For example, if we take $\underline{\Lambda}^N(\{\sigma\};n)$ in form (133) and associate $\underline{\Lambda}^T(\sigma;t')$ with the Robinson rule of time-dependent damage accumulation and power time-durability diagram (see Mikhailov 2000),

$$\underline{\Lambda}^T(\sigma;t') = \left[\int_0^{t'} \frac{|\sigma(t''|^{b_T}}{A_T(\tilde{\sigma}(t''))}dt''\right]^{1/b_T}, \tag{143}$$

the NESF (142) will take form

$$\begin{aligned}\underline{\Lambda}^{ITN}(\{\sigma\};n(t)) \quad &= \quad \sup_{0\le t'\le t}\left\{\frac{\sigma_{eq}(\sigma(t'))}{\sigma_r} + \right.\\ &\left.\left[\int_0^{t'} \frac{|\sigma(t'')|^{b_T}}{A_T(\tilde{\sigma}(t''))}dt''\right]^{1/b_T} + \left[\sum_{m=1}^{n(t')}\left(\frac{\|\sigma^c(m)\|}{\sigma^*_{\bowtie 1}(\tilde{\sigma}^c(m))}\right)^{b_N}\right]^{1/b_N}\right\}.\end{aligned} \tag{144}$$

where b_T, b_N are material parameters $A_T(\tilde{\sigma}(t''))$, is a material functions of the normalised stress tensor $\tilde{\sigma}_{ij}(t'') = \sigma_{ij}(t'')/|\sigma(t'')|$ at the instant t'', $\sigma^*_{\bowtie 1}(\tilde{\sigma}^c(m))$ is a material functions of the normalised shape of the stress cycle.

If there exist also dynamic effects on the material strength, one can replace the instant strength term by a corresponding dynamic NESF $\underline{\Lambda}^D(\sigma(t')$ (see e.g. Mikhailov 2000) and arrive e.g. at the complex NESF and strength condition,

$$\underline{\Lambda}^{DTN}(\{\sigma\};n(t)) = \sup_{0\le t'\le t}\left\{\underline{\Lambda}^D(\sigma;t') + \underline{\Lambda}^T(\sigma;t') + \underline{\Lambda}^N(\{\sigma\};n(t'))\right\} < 1 \tag{145}$$

For example, we can take $\underline{\Lambda}^N(\{\sigma\};n)$ in form (133), $\underline{\Lambda}^T(\sigma;t')$ in form (143), and $\underline{\Lambda}^D(\sigma;t')$ associated with the Morozov, Petrov and Utkin criterion (see Morozov and Petrov, 2000) generalised on the multiaxial case (see Mikhailov 2000), in the form

$$\underline{\Lambda}^D(\sigma;t') = \frac{1}{\sigma_r}\sigma_{eq}\left(\bar{\sigma}(t';t_r)\right), \quad \bar{\sigma}_{kj}(t';t_r) = \frac{1}{t_r}\int_{t'-t_r}^{t'} \sigma_{kj}(t'')dt'', \tag{146}$$

where σ_r is, as before, a longitudinal material strength and $\sigma_{eq}(\sigma)$ is e.g. von Mises, Tresca or other instantaneous equivalent stress and t_r is a material constant. Then the

NESF (145) will take form

$$\underline{\Lambda}^{DTN}(\{\sigma\};n(t)) = \sup_{0\le t'\le t}\left\{\frac{1}{\sigma_r}\sigma_{eq}\left(\bar\sigma(t';t_r)\right) + \left[\int_0^{t'}\frac{|\sigma(t'')|^{b_T}}{A_T(\tilde\sigma(t''))}dt''\right]^{1/b_T} + \left[\sum_{m=1}^{n(t')}\left(\frac{\|\sigma^c(m)\|}{\sigma^*_{\bowtie 1}(\tilde\sigma^c(m))}\right)^{b_N}\right]^{1/b_N}\right\}. \quad (147)$$

Note that all the three strength conditions (141), (142) and (145) lead to a non-linear summation rule.

Note also that presentations (141), (142) and (145) are not uniquely possible and one can use not only the sum but also other homogeneous combinations of the four terms $\sigma_{eq}(\sigma)/\sigma_r$, $\underline{\Lambda}^D$, $\underline{\Lambda}^T$ and $\underline{\Lambda}^N$ to get other possible simple forms of the NESF describing interaction of instant, dynamic, time-dependent and cycle-dependent effects on the durability. For example, one can take,

$$\underline{\Lambda}^{DTN}(\{\sigma\};n(t)) = \sup_{0\le t'\le t}\left\{[\underline{\Lambda}^D(\sigma;t')]^q + [\underline{\Lambda}^T(\sigma;t')]^q + [\underline{\Lambda}^N(\{\sigma\};n(t'))]^q\right\}^{\frac{1}{q}}, \quad (148)$$

where $q > 0$ can be considered as a material parameter. If $q = 1$, (148) is reduced to (145). The limiting case $q \to \infty$ corresponds to the NESF

$$\underline{\Lambda}^{DTN}(\{\sigma\};n(t)) = \sup_{0\le t'\le t}\max\left\{\underline{\Lambda}^D(\sigma;t'), \underline{\Lambda}^T(\sigma;t'), \underline{\Lambda}^N(\{\sigma\};n(t'))\right\}. \quad (149)$$

Evidently, which form fits better to a particular material behaviour, can be determined from comparison with experimental data.

9 Direct interpolation of NESF for cyclic loadings

As one can see in section 6.2, the cyclic strength conditions (24) for the stress amplitude or (25) for the maximum stress under a uniaxial periodic regular process $\{\sigma^c\}$ are expressed in terms of the corresponding S–N diagrams $\sigma^*_R(n)$ or $\sigma^*_{R,max}(n)$, respectively. Then the NESF on such processes is also expressed by very simple formulas (26) and (27).

For a multiaxial periodic process, similarly, the cyclic strength conditions can be written in the form

$$\|\sigma^c\| < \sigma^*(\tilde\sigma^c;n) \quad (150)$$

where a scalar function $\sigma^*(\tilde\sigma^c;n)$ is the S–N diagram under the multiaxial periodic process, depending on the normalised shape $\tilde\sigma^c$ of the tensor loading process on the cycle. Then we get from the NESF definition, that

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{\|\sigma^c\|}{\sigma^*(\tilde\sigma^c;n)} \quad (151)$$

under such processes. Sections 7.2 and 7.4 specify the NESF given by (151) for the cases when the S–N diagram $\sigma^*(\tilde\sigma^c;n)$ under an arbitrary periodic multiaxial process can be expressed in terms of the S–N diagrams under some periodic uniaxial regular processes, using one or another cyclic strength criterion.

For non-periodic process, it is usual to apply some damage accumulation theories to reduce a strength and durability analysis to some S–N diagrams under periodic processes. We discussed in sections 6.3 and 7.3 some of existing approaches. For any of them one can obtain a corresponding NESF, e.g., numerically but analytical formulas are not always available and we given examples only for power S–N diagrams.

However, the damage accumulation theories, being quite suitable for the analysis in terms of the damage (abstract or partial life time), is in fact not necessary for the NESF concept. Instead, we will try to interpolate an NESF directly along the S–N diagrams for periodic processes. Together with some additional properties implied either by the NESF definition or by some properties of the S–N diagrams for periodic processes, the interpolating NESF is to meet the following conditions

(i) $\underline{\Lambda}^N(\{\sigma\};n)$ must satisfy interpolation condition (151) on periodic processes $\{\sigma\}$;

(ii) $\underline{\Lambda}^N(\{\sigma\};n)$ must be homogeneous in $\{\sigma\}$;

(iii) $\underline{\Lambda}^N(\{\sigma\};n)$ must be non-decreasing in n;

(iv) the interpolation formula should be applicable to history independent materials, for which $\sigma^*(\tilde{\sigma}^c;n)$ is given by (56).

9.1 Direct interpolation of NESF for uniaxial regular cyclic loading

9.1.1 Direct interpolation of NESF by a linear rule

We will try to write down a "linear" accumulation rule not in the partial life times but in the normalised stress partial increments and start from the uniaxial case.

Let $\{\sigma^c\}$ be first a uniaxial regular cyclic process with $R = -1$. If the process is periodic, then we have from (26),

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{\sigma_a}{\sigma^*_{-1}(n)} = \frac{\sigma_a(1)}{\sigma^*_{-1}(n)} \tag{152}$$

for the NESF value at an $n-$th cycle. If the cycle amplitude increases after the first cycle and the process becomes periodic from the second cycle, we can take into account the amplitude increment as follows,

$$\underline{\Lambda}^N(\{\sigma\};n) = \frac{\sigma_a(1)}{\sigma^*_{-1}(n)} + \frac{\sigma_a(2)-\sigma_a(1)}{\sigma^*_{-1}(n-1)}. \tag{153}$$

If the cycle amplitude may increase on each cycle, we can continue the formula as

$$\begin{aligned}\underline{\Lambda}^N(\{\sigma\};n) &= \sum_{m=1}^{n} \frac{\sigma_a(m)-\sigma_a(m-1)}{\sigma^*_{-1}(n-m+1)} \\ &= \frac{\sigma_a(n)}{\sigma^*_{-1}(1)} + \sum_{m=1}^{n-1} \sigma_a(m)\left[\frac{1}{\sigma^*_{-1}(n-m+1)} - \frac{1}{\sigma^*_{-1}(n-m)}\right],\end{aligned} \tag{154}$$

where the condition $\sigma_a(0) = 0$ was taken into account in the first part of (154).

To extend the NESF representation to an arbitrary change of the cycle amplitude from cycle to cycle and comply with condition (iii) above, we write

$$\begin{aligned}\underline{\Lambda}^N(\{\sigma\};n) &= \max_{1\le n'\le n}\left\{\sum_{m=1}^{n'}\frac{\sigma_a(m)-\sigma_a(m-1)}{\sigma^*_{-1}(n'-m+1)}\right\}\\ &= \max_{1\le n'\le n}\left\{\frac{\sigma_a(n')}{\sigma^*_{-1}(1)}+\sum_{m=1}^{n'-1}\sigma_a(m)\left[\frac{1}{\sigma^*_{-1}(n'-m+1)}-\frac{1}{\sigma^*_{-1}(n'-m)}\right]\right\}\end{aligned} \quad (155)$$

for an arbitrary uniaxial regular cyclic process with $R=-1$.

To consider uniaxial regular cyclic process with variable asymmetry ratio $R(m)$, we use the second part of representation (155) in the modified form,

$$\underline{\Lambda}^N(\{\sigma\};n) = \max_{1\le n'\le n}\left\{\frac{\sigma_a(n')}{\sigma^*_{R(n')}(1)}+\sum_{m=1}^{n'-1}\sigma_a(m)\left[\frac{1}{\sigma^*_{R(m)}(n'-m+1)}-\frac{1}{\sigma^*_{R(m)}(n'-m)}\right]\right\} \quad (156)$$

Let us check now the above conditions (i)-(iv) for NESF (156). For a periodic process, $\sigma_a(m)=\sigma_a(1)$ and after some manipulations we have that $\underline{\Lambda}^N(\{\sigma\};n)$ is reduced to (152), that is interpolation condition (i) is satisfied. Homogeneity condition (ii) and monotonicity condition (iii) are obviously satisfied too.

Let us check condition (iv). Under definition, $\sigma^*_{R(m)}(m')$ does not depend on m' for a history independent material and is an amplitude such that $\max_{\sigma\in\sigma^c(m)}\sigma_{eq}(\sigma)=\sigma_r$. Then $\sigma^*_{R(m)}(m')=\sigma_r/\max_{\sigma\in\sigma^c(m)}\sigma_{eq}(\sigma)$. Consequently, NESF (156) for such material becomes

$$\underline{\Lambda}^N(\{\sigma\};n) = \max_{1\le n'\le n}\frac{\sigma_a(n')}{\sigma^*_{R(n')}(1)} = \max_{1\le n'\le n}\max_{\sigma\in\sigma^c(n')}\frac{\sigma_r}{\sigma_{eq}(\sigma)}$$

what coincides with (56). Thus condition (iv) is satisfied.

As an example, let us consider a particular case, when the S–N diagram $\sigma^*_R(n)$ is given by power function (29). Then NESF (156) becomes

$$\underline{\Lambda}^N(\{\sigma\};n) = \max_{1\le n'\le n}\left\{\frac{\sigma_a(n')}{\sigma^*_{R(n'),1}}+\sum_{m=1}^{n'-1}\frac{\sigma_a(m)}{\sigma_{R(m),1}}\left[(n'-m+1)^{1/b_{R(m)}}-(n'-m)^{1/b_{R(m)}}\right]\right\} \quad (157)$$

As one can see, unlike to the Palmgren-Miner rule described in Section 6.3.1, this approach gives an analytical presentation of NESF for an arbitrary process and a general dependence b_R, however it is does not generally reduces to (36) or (37) for a constant b_R or constant R.

9.1.2 Direct interpolation of NESF by a nonlinear rule

The linear interpolation of NESFs is not unique option. Let us consider a nonlinear interpolation in the following form,

$$\underline{\Lambda}^N(\{\sigma\};n) =$$

$$\max_{1\le n'\le n}\left\{\left(\frac{\sigma_a(n')}{\sigma^*_{R(n')}(1)}\right)^b+\sum_{m=1}^{n'-1}\left[\left(\frac{\sigma_a(m)}{\sigma^*_{R(m)}(n'-m+1)}\right)^b-\left(\frac{\sigma_a(m)}{\sigma^*_{R(m)}(n'-m)}\right)^b\right]\right\}^{\frac{1}{b}} \quad (158)$$

Here $b > 0$ is considered as a material constant and the NESF (158) degenerates into (156) if $b = 1$. In the same way as for the linear rule, one can check the conditions (i)–(iv) are satisfied.

Let us consider the particular case, when the S–N diagram $\sigma^*_R(n)$ is given by power function (29). Then NESF (158) becomes

$$\underline{\Lambda}^N(\{\sigma\}; n) =$$

$$\max_{1\le n'\le n}\left\{\left(\frac{\sigma_a(n')}{\sigma^*_{R(n'),1}}\right)^b + \sum_{m=1}^{n'-1}\left(\frac{\sigma_a(m)}{\sigma_{R(m),1}}\right)^b\left[(n'-m+1)^{b/b_{R(m)}} - (n'-m)^{b/b_{R(m)}}\right]\right\}^{\frac{1}{b}} \quad (159)$$

One can see, that if $b_R(m) = b$ then (159) coincides with representation (36) obtained from the Palmgren-Miner rule for the case of constant $b_R(m)$.

9.2 Direct interpolation of NESF for multiaxial cyclic loading

A multiaxial counterpart of uniaxial NESF (156) can be obtained if one takes into account that the cycle amplitude σ_a can be replaced by the cycle shape norm $\|\sigma^c\|$, the role of the asymmetry ratio R is plaid by the normalised cycle shape $\tilde{\sigma^c}$ and the uniaxial S–N diagram $\sigma^*_R(n)$ for the amplitude σ_a should be replaced by the multiaxial S–N diagram $\sigma^*(\tilde{\sigma^c}; n)$ for the norm $\|\sigma^c\|$. Then the normalised stress linear accumulation rule gives the NESF

$$\begin{aligned}\underline{\Lambda}^N(\{\sigma\}; n) \quad = \quad & \max_{1\le n'\le n}\left\{\frac{\|\sigma^c(n')\|}{\sigma^*(\tilde{\sigma^c}(n'); 1)} + \right. \\ & \left.\sum_{m=1}^{n'-1}\|\sigma^c(m)\|\left[\frac{1}{\sigma^*(\tilde{\sigma^c}(m); n'-m+1)} - \frac{1}{\sigma^*(\tilde{\sigma^c}(m); n'-m)}\right]\right\} \quad (160)\end{aligned}$$

For a normalised stress nonlinear (power) accumulation rule, we similarly have a multiaxial counterpart of (158),

$$\begin{aligned}\underline{\Lambda}^N(\{\sigma\}; n) \quad = \quad & \max_{1\le n'\le n}\left\{\left(\frac{\|\sigma^c(n')\|}{\sigma^*(\tilde{\sigma^c}(n'); 1)}\right)^b + \right. \\ & \left.\sum_{m=1}^{n'-1}\left[\left(\frac{\|\sigma^c(m)\|}{\sigma^*(\tilde{\sigma^c}(m); n'-m+1)}\right)^b - \left(\frac{\|\sigma^c(m)\|}{\sigma^*(\tilde{\sigma^c}(m); n'-m)}\right)^b\right]\right\}^{\frac{1}{b}} \quad (161)\end{aligned}$$

Here b is considered as a material constant and the NESF (161) degenerates into (160) if $b = 1$.

If the periodic S–N diagram $\sigma^*(\tilde{\sigma^c}; n)$, used in (160) and (161), is not available for a particular cycle shape σ^c, it can be expressed in terms of the uniaxial S–N diagrams for the same material, using the approaches described in Sections 7.2 and 7.4.

Similar to the uniaxial case, one can check conditions (i)-(iv) are satisfied for NESFs (160) and (161).

As we have seen, NESF (161) interpolate the periodic S–N diagrams for any $b > 0$, including the NESF linear counterparts appearing for $b = 1$. However, if experiments on non-periodic cyclic processes show some deviations of the formula predictions, one can percept the NESF (161) as a first approximations to a NESF specific for a considered material and use an interpolation procedure to refine the approximation. Particularly the approaches presented by Mikhailov (1997, 1998) can be adopted to this case.

10 Conclusion

The notions of generalised S–N diagram and normalised equivalent stress functional introduced by Mikhailov (2000) for arbitrary loading processes were adopted in this paper to oscillating loading by consideration some quasi-cyclic (discrete) parameterization more relevant for cyclic fatigue.

The NESF concept reduces different durability and strength fatigue models to a unique form what facilitates their comparison. Numerous examples of the reduction are presented in the paper for some known fatigue durability and strength models. Some complex NESFs describing interaction of instant , creep, fatigue and dynamic loading were introduced.

The NESF is a mechanically meaningful material characteristic, which can be approximated from a finite number of durability tests for different loading processes without any other information such as micro-cracks or stiffness change. The NESFs interpolating the classical S–N diagrams for periodic processes were presented in the paper. Methods for a refined NESF identification (interpolation) from a finite number of experimental data is to be developed further. Adaptation of the identification approaches by Mikhailov (1997, 1998) to NESFs looks promising.

Fatigue under stress fields independent of the space coordinates was mainly analysed in this paper with obvious reasoning about application to moderately inhomogeneous stress fields . Extension to highly inhomogeneous stress field incorporating a non-local approach by Mikhailov (1995) will be considered elsewhere (Mikhailov & Namestnikova 2002).

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Appendix

A Proof of inequality (7) for MD material

From Definition 1MD,

$$\begin{aligned}\underline{\lambda}^N(\{\sigma\}; n^*\{\sigma\}) &= \sup\{\lambda : \; n^*\{\lambda\sigma\} > n^*\{\sigma\}\} = \\ &\quad \sup\{\lambda : \; \lambda \le 1, \; n^*\{\lambda\sigma\} > n^*\{\sigma\}\} \le 1\end{aligned}$$

since $n^*\{\lambda\sigma\} \le n^*\{\sigma\}$ for all $\lambda > 1$ for MD materials. Similarly,

$$\begin{aligned}\underline{\lambda}^N(\{\sigma\}; n^*\{\sigma\} - 1) &= \sup\{\lambda : \; n^*\{\lambda\sigma\} > n^*\{\sigma\} - 1\} = \\ &\quad \sup\{\lambda : \; \lambda \ge 1, \; n^*\{\lambda\sigma\} > n^*\{\sigma\} - 1\} \ge 1\end{aligned}$$

since $n^*\{\sigma\} > n^*\{\sigma\} - 1$ and consequently $n^*\{\lambda\sigma\} > n^*\{\sigma\} - 1$ for all $\lambda \leq 1$ for MD materials.

B Proof of Remark 1

We define $\underline{\lambda}^{**}(\{\sigma\};n) := \sup\{\lambda : \ n^{**}\{\lambda''\sigma\} > n \quad \forall\ \lambda'' \in [0,\lambda]\}$. Since $n^{**}\{\lambda''\sigma\} \leq n^*\{\lambda''\sigma\}$ then $\underline{\lambda}^{**}(\{\sigma\};n) \leq \underline{\lambda}(\{\sigma\};n)$.

Suppose $\underline{\lambda}^{**}(\{\sigma\};n) < \underline{\lambda}(\{\sigma\};n)$. Then for any λ_0, λ_{00} such that $\underline{\lambda}^{**}(\{\sigma\};n) < \lambda_0 < \lambda_{00} < \underline{\lambda}(\{\sigma\};n)$, we have,

$$n^{**}\{\lambda_0\sigma\} \leq n < n^*\{\lambda_0\sigma\}, \qquad n^{**}\{\lambda_{00}\sigma\} \leq n < n^*\{\lambda_{00}\sigma\} \tag{162}$$

Consequently, strength is $\lambda-$unstable on the (quasi–) cycle n under the process $\{\lambda_0\sigma\}$, that is, for any $\epsilon > 0$ there exists $\lambda_{00} \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ such that $n^*\{\lambda_{00}\sigma\} \leq n$. Choosing $\epsilon \leq \min[\underline{\lambda}(\{\sigma\};n) - \lambda_0, \lambda_0 - \underline{\lambda}^{**}(\{\sigma\};n)]$, we have, $\underline{\lambda}^{**}(\{\sigma\};n) < \lambda_{00} < \underline{\lambda}(\{\sigma\};n)$, and arrive at a contradiction with the last inequality in (162). Thus $\underline{\lambda}^{**}(\{\sigma\};n) = \underline{\lambda}(\{\sigma\};n)$.

C Proof of Statement 3

Let us complete here the definition of $n^{**}\{\sigma\}$ by ∞ if the endurance under $\{\sigma\}$ is $\lambda-$stable. Let $n^{**}_-\{\sigma\} < \infty$, then supremum can be replaced by maximum in the definition of $n^{**}_-\{\sigma\}$. Suppose first $n^{**}_-\{\sigma\} + 1 < n^{**}\{\sigma\} \leq \infty$. For any $n > n^{**}_-\{\sigma\}$ and particularly for $n = n^{**}_-\{\sigma\} + 1$, condition (21) is violated, that is $\underline{\lambda}(\{\sigma\};n) \leq 1$. Consequently, $n^*\{\lambda'\sigma\} \leq n^{**}_-\{\sigma\} + 1 < n^{**}\{\sigma\}$ for any $\lambda' > 1$ due to the definition of $\underline{\lambda}(\sigma;t)$ for MD processes. However the last inequality contradicts to the definition of the critical (quasi–) cycle $n^{**}\{\sigma\}$ since the strength appears to be $\lambda-$unstable on the (quasi–) cycle $n^{**}_-\{\sigma\} + 1 < n^{**}\{\sigma\}$ under the process $\{\sigma\}$. Consequently $n^{**}_-\{\sigma\} + 1$ can not be less than $n^{**}\{\sigma\}$. If $n^{**}_-\{\sigma\} = \infty$ then evidently $n^{**}_-\{\sigma\} + 1$ also can not be less than $n^{**}\{\sigma\}$.

Suppose now first $n^{**}\{\sigma\} < n^{**}_-\{\sigma\} + 1 < \infty$. Then we obtain from the definition of $n^{**}_-\{\sigma\}$ that condition (21) holds for any $n \leq n^{**}_-\{\sigma\}$ and particularly for $n = n^{**}\{\sigma\}$. Suppose then $n^{**}\{\sigma\} < n^{**}_-\{\sigma\} + 1 = \infty$. This means condition (21) holds for any $n < \infty$ and particularly for $n = n^{**}\{\sigma\}$. That is, in the both cases there exists $\lambda' > 1$ such that $n^*\{\lambda''\sigma\} > n^{**}\{\sigma\}$ for all $\lambda'' \in [0, \lambda']$. This implies $\lambda-$stable strength on the (quasi–) cycle $n^{**}(\sigma)$ under the MD process $\{\sigma\}$, which contradicts to the definition of the critical (quasi–) cycle $n^{**}\{\sigma\}$. The contradiction proves that $n^{**}\{\sigma\}$ can not be less than the finite or infinite $n^{**}_-\{\sigma\} + 1$.

Hence $n^{**}_-\{\sigma\} + 1 = n^{**}\{\sigma\}$ in all cases. For $n^{**}_-\{\sigma\} = \infty$ this means $n^{**}\{\sigma\} = \infty$, that is, endurance is $\lambda-$stable under the process $\{\sigma\}$.

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