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Analysis of Boundary-Domain Integral Equations for Variable-Coefficient Drichlet BVP in 2D

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15.1 Preliminaries

In this paper, the Dirichlet boundary value problem for the second order "stationary heat transfer" elliptic partial differential equation with variable coefficient is considered in 2D. Using an appropriate parametrix (Levi function), this problem is reduced to some direct segregated systems of Boundary-Domain Integral Equations (BDIEs). Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. Consequently, we need to set conditions on the domain for the invertibility of corresponding parametrix-based integral layer potentials and hence the unique sovability of BDIEs. The properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BDIEs are analysed and the invertibility of the BDIE operators is proved.

Let Ω be a domain in $\mathbb{R}^2$ bounded by simple closed infinitely differentiable curve $\partial\Omega$, the set of all infinitely differentiable function on Ω with compact support is denoted by $\mathcal{D}(\Omega)$. The function space $\mathcal{D}'(\Omega)$ consists of all continuous linear functionals over $\mathcal{D}(\Omega)$. For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^2)$ the Bessel potential space. Note that the space $H^1(\mathbb{R}^2)$ coincides with the Sobolev space $W_2^1(\mathbb{R}^2)$ with equivalent norm and $H^{-s}(\mathbb{R}^2)$ is the dual space to $H^s(\mathbb{R}^2)$. For any non-empty open set $\Omega \in \mathbb{R}^n$ we define $H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) : u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n)\}$. The space $\widetilde{H}^s(\Omega)$ is defined to be the closure of $\mathcal{D}(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^n)$. For $s \in (-\frac{1}{2}, \frac{1}{2})$, $H^s(\Omega)$ can be identified with $\widetilde{H}^s(\Omega)$, see e.g. [McL00, HW08].

We shall consider the scalar elliptic differential equation

$$Au(x) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x) \qquad \text{in } \Omega \tag{15.1}$$

with $a(x) \in C^\infty(\mathbb{R}^2)$, $a(x) > 0$.

For given functions $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$, we will consider the Dirichlet boundary value problem for function $u \in H^1(\Omega)$,

$$Au = f \qquad \text{in } \Omega, \tag{15.2}$$

$$\gamma^+ u = \varphi_0 \qquad \text{on } \partial\Omega. \tag{15.3}$$

Here equation (15.2) is understood in the distributional sense and (15.3) in the trace sense.

In applications, the BVP (15.2)-(15.3) may describe a stationary heat transfer boundary value problem in isotropic inhomogeneous two-dimensional body Ω, where $u(x)$ is an unknown temperature, $a(x)$ is a known variable thermo-conductivity coefficient, $f(x)$ is a known distributed heat source, $\varphi_0(x)$ is known temperature on the boundary.

We define as in [Gri85, Cos88, Mik11], the subspace

$$H^{1,0}(\Omega; A) := \{g \in H^1(\Omega) : Ag \in L_2(\Omega)\}$$

endowed with the norm $\|g\|^2_{H^{1,0}(\Omega;A)} := \|g\|^2_{H^1(\Omega)} + \|Ag\|^2_{L_2(\Omega)}$. For $u \in H^{1.0}(\Omega; A)$ we can define the (canonical) co-normal derivative $T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ in the weak form (see, e.g. [Cos88, Mik11] and the references therein),

$$\langle T^+u, w\rangle := \int_\Omega [(\gamma^+_{-1}w)Au + E(u, \gamma^+_{-1}w)]dx \qquad \forall w \in H^{\frac{1}{2}}(\partial\Omega), \tag{15.4}$$

where $\gamma^+_{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is a continuous right inverse of the continuous interior trace operator $\gamma^+ : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, while $E(u, v) := a(x)\nabla u(x) \cdot \nabla v(x)$ is the symmetric bilinear form.

For $u \in H^s(\Omega)$, $s > 3/2$, the canonical conormal derivative defined by (15.4) coincides with the classical one, defined in the trace sense, i.e.,

$$T^+u = an \cdot \gamma^+\nabla u, \tag{15.5}$$

where $n(x)$ is the exterior unit normal vector.

Remark 1. The first Green identity holds for any $u \in H^{1,0}(\Omega; A)$ and $v \in H^1(\Omega)$ ([Cos88, Mik11]), i.e,

$$\int_\Omega E(u, v)dx = -\langle Au, v\rangle_\Omega + \langle T^+u, \gamma^+v\rangle_{\partial\Omega}$$

and the second Green identity holds for any $u, v \in H^{1,0}(\Omega; A)$,

$$\int_\Omega (vAu - uAv)dx = \langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T^+v, \gamma^+u\rangle_{\partial\Omega}.$$

15.2 Parametrix-Based Potential Operators

A function $P(x,y)$ is a parametrix (Levi function) for the operator A if

$$A_x P(x,y) = \delta(x-y) + R(x,y),$$

where δ is the Dirac-delta distribution, while $R(x,y)$ is a remainder possessing at most a weak singularity at $x=y$.

In particular, see e.g. [Mik02], the function

$$P(x,y) = \frac{1}{2\pi a(y)} \log|x-y|, \quad x,y \in \mathbb{R}^2$$

is a parametrix for the operator A and the corresponding remainder is

$$R(x,y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x-y|^2} \frac{\partial a(x)}{\partial x_i}, \quad x,y \in \mathbb{R}^2.$$

If $a(x) = 1$, then A becomes the Laplace operator, Δ, and the parametrix $P(x,y)$ becomes its fundamental solution, $P_\Delta(x,y)$.

If $u \in H^{1,0}(\Omega; A)$, then from the second Green's identity, we have the following parametrix-based third Green identity for $y \in \Omega$, [Mik02],

$$\begin{aligned} u(y) = & \int_{\partial\Omega} [\gamma^+ u(x) T_x^+ P(x,y) - P(x,y) T^+ u(x)] dx \\ & - \int_{\Omega} R(x,y) u(x) dx + \int_{\Omega} P(x,y) f(x) dx, \quad y \in \Omega. \end{aligned} \tag{15.6}$$

Note that the direct substitution of $v(x)$ by $P(x,y)$ in the second Green identity is not possible as it has singularity at $x=y$. This difficulty is avoided by replacing Ω by $\Omega \setminus B(y,\varepsilon)$, where $B(y,\varepsilon)$ is a disc of radius ε centered at y; taking the limit $\varepsilon \to 0$ we then arrive at (15.6), cf. e.g. [Mir70].

The parametrix-based logarithmic and remainder potential operators are defined, similar to [CMN09a] in the 3D case, as

$$\mathcal{P}g(y) := \int_{\Omega} P(x,y) g(x) dx, \qquad \mathcal{R}g(y) := \int_{\Omega} R(x,y) g(x) dx.$$

The single and double layer potential operators, corresponding to the parametrix $P(x,y)$, are defined for $y \notin \partial\Omega$ as

$$Vg(y) := -\int_{\partial\Omega} P(x,y) g(x) ds_x, \qquad Wg(y) := -\int_{\partial\Omega} T_x^+ P(x,y) g(x) ds_x.$$

The following boundary integral (pseudo-differential) operators are also defined for $y \in \partial\Omega$,

$$\mathcal{V}g(y) := -\int\limits_{\partial\Omega} P(x,y)g(x)ds_x, \qquad \mathcal{W}g(y) := -\int\limits_{\partial\Omega} T_x^+ P(x,y)g(x)ds_x,$$

$$\mathcal{W}'g(y) := -\int\limits_{\partial\Omega} T_y^+ P(x,y)g(x)ds_x, \qquad \mathcal{L}^+ g(y) := T_y^+ W g(y).$$

Let $\mathcal{P}_\Delta, V_\Delta, W_\Delta, \mathcal{V}_\Delta, \mathcal{W}_\Delta, \mathcal{L}_\Delta^+$ denote the potentials corresponding to the operator $A = \Delta$. Then the following relations hold (cf. [CMN09a] for 3D case),

$$\mathcal{P}g = \frac{1}{a}\mathcal{P}_\Delta g, \qquad \mathcal{R}g = \frac{-1}{a(y)}\sum_{i=1}^{2} \partial_i \mathcal{P}_\Delta[g(\partial_i a)], \tag{15.7}$$

$$Vg = \frac{1}{a}V_\Delta g, \qquad Wg = \frac{1}{a}W_\Delta(ag) \tag{15.8}$$

$$\mathcal{V}g = \frac{1}{a}\mathcal{V}_\Delta g, \qquad \mathcal{W}g = \frac{1}{a}\mathcal{W}_\Delta(ag), \tag{15.9}$$

$$\mathcal{W}'g = \mathcal{W}'_\Delta g + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]\mathcal{V}_\Delta g, \tag{15.10}$$

$$\mathcal{L}^+ g = \mathcal{L}_\Delta^+(ag) + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right] W_\Delta^+(ag). \tag{15.11}$$

Theorem 1. *For $s \in \mathbb{R}$, the following operators are continuous,*

$$V : H^s(\partial\Omega) \to H^{s+\frac{3}{2}}(\Omega),$$

$$W : H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega),$$

$$\mathcal{V} : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega),$$

$$\mathcal{W}, \mathcal{W}' : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega),$$

$$\mathcal{L}^+ : H^s(\partial\Omega) \to H^{s-1}(\partial\Omega).$$

Proof. We have the corresponding mappings for the corresponding constant-coefficient operators. Then (15.8)-(15.11) imply the theorem claim. □

Theorem 2. *Let $u \in H^{-\frac{1}{2}}(\partial\Omega)$ and $v \in H^{\frac{1}{2}}(\partial\Omega)$. Then the following jump relation hold on $\partial\Omega$*

$$\gamma^+ V u(y) = \mathcal{V}u(y), \tag{15.12}$$

$$\gamma^+ W v(y) = -\frac{1}{2}v(y) + \mathcal{W}v(y), \tag{15.13}$$

$$T^+ V u(y) = \frac{1}{2}u(y) + \mathcal{W}'u(y). \tag{15.14}$$

Proof. For the constant coefficient case, this theorem is well known. Then taking into account the relations (15.8)-(15.10), we can prove the theorem for the variable positive coefficient $a \in C^\infty(\mathbb{R}^2)$ as well.

Theorem 3. *Let Ω be a bounded open domain in $\mathbb{R}^2$ with closed, infinitely smooth boundary $\partial\Omega$. The following operators are continuous.*

$$\mathcal{P} : \widetilde{H}^s(\Omega) \to H^{s+2}(\Omega), \qquad s \in \mathbb{R}; \tag{15.15}$$

$$: H^s(\Omega) \to H^{s+2}(\Omega), \qquad s > -\frac{1}{2}; \tag{15.16}$$

$$\mathcal{R} : \widetilde{H}^s(\Omega) \to H^{s+1}(\Omega), \qquad s \in \mathbb{R}; \tag{15.17}$$

$$: H^s(\Omega) \to H^{s+1}(\Omega), \qquad s > -\frac{1}{2}; \tag{15.18}$$

$$\gamma^+\mathcal{P} : \widetilde{H}^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \qquad s > -\frac{3}{2}; \tag{15.19}$$

$$: H^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \qquad s > -\frac{1}{2}; \tag{15.20}$$

$$\gamma^+\mathcal{R} : \widetilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > -\frac{1}{2}; \tag{15.21}$$

$$: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > -\frac{1}{2}; \tag{15.22}$$

$$T^+\mathcal{P} : \widetilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > -\frac{1}{2}; \tag{15.23}$$

$$: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > -\frac{1}{2}; \tag{15.24}$$

$$T^+\mathcal{R} : \widetilde{H}^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \qquad s > \frac{1}{2}; \tag{15.25}$$

$$: H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \qquad s > \frac{1}{2}. \tag{15.26}$$

Proof. The operator $\mathcal{P}_\Delta$ is a homogeneous pseudo-differential operator of order -2 on $\mathbb{R}^2$, mapping $\mathcal{P}_\Delta : H^s_{\text{comp}}(\mathbb{R}^2) \to H^{s+2}_{\text{loc}}(\mathbb{R}^2)$ continuously for any $s \in \mathbb{R}$. Hence the application of trace theorem along with the relations (15.7), the operators (15.15), (15.17),(15.19),(15.21), (15.23) and (15.25) are continuous. For $s \in (-\frac{1}{2}, \frac{1}{2})$, $\widetilde{H}^s(\Omega)$ is identified with $H^s(\Omega)$, and (15.16) directly follows from (15.15). To prove the case $s \in (\frac{1}{2}, \frac{3}{2})$, we implement the Gauss divergence theorem and the fact that $\dfrac{\partial}{\partial x_j} \log|x-y| = -\dfrac{\partial}{\partial y_j} \log|x-y|$ and obtain

$$\begin{aligned}
\frac{\partial}{\partial y_j}(\mathcal{P}_\Delta g)(y) &= -\frac{1}{2\pi}\int_\Omega g(x)\frac{\partial}{\partial x_j}\log|x-y|dx \\
&= \frac{1}{2\pi}\int_\Omega \log|x-y|\frac{\partial}{\partial x_j}g(x)dx - \frac{1}{2\pi}\int_{\partial\Omega}\log|x-y|n_j\gamma^+g(x)ds_x \\
&= \mathcal{P}_\Delta(\partial_j g)(y) + V_\Delta(n_j\gamma^+g)(y).
\end{aligned} \tag{15.27}$$

Now for $s \in (\frac{1}{2}, \frac{3}{2})$, since $\partial_j : H^s(\Omega) \to H^{s-1}(\Omega)$ is continuous, we have $\mathcal{P}_\Delta\partial_j : H^s(\Omega) \to H^{s+1}(\Omega)$ is continuous, and from trace theorem $\gamma^+g \in H^{s-\frac{1}{2}}(\partial\Omega)$ and the properties of the single layer potential, we conclude that

$\nabla\mathcal{P}_\Delta : H^s(\Omega) \to H^{s+1}(\Omega)$ is continuous. This implies that $\mathcal{P}_\Delta : H^s(\Omega) \to H^{s+2}(\Omega)$ is continuous, which along with the relation $\mathcal{P}g = \frac{1}{a}\mathcal{P}_\Delta$ leads to the continuity of operator (15.16) for $s \in (\frac{1}{2}, \frac{3}{2})$.

Further, with the help of these results and the relation (15.27), we can verify by induction that the operator (15.16) is continuous for $s \in (k - \frac{1}{2}, k + \frac{1}{2})$, where k is an arbitrary nonnegative integer. For the values $s = k + \frac{1}{2}$ the continuity of the operator (15.16) then follows due to the complex interpolation properties of Bessel potential spaces.

The trace theorem will give the continuity proof for the operators (15.19) and (15.20). We can follow the same procedure to prove the claim of the theorem concerning the operator $\mathcal{R}$. The continuity of the operators (15.23)-(15.26) follows if we remark that for the chosen s the conormal derivative can be understood in the classical sense (15.5). □

By the Rellich compact embedding theorem (see e.g [McL00, Theorem 3.27]), Theorems 1 and 3 imply the following two assertions.

Corollary 1. *Let $s \in \mathbb{R}$. The following operators are compact,*

$$\mathcal{V} : H^s(\partial\Omega) \to H^s(\partial\Omega) \tag{15.28}$$
$$\mathcal{W} : H^s(\partial\Omega) \to H^s(\partial\Omega) \tag{15.29}$$
$$\mathcal{W}' : H^s(\partial\Omega) \to H^s(\partial\Omega) \tag{15.30}$$

Corollary 2. *The following operators are compact for any $s > \frac{1}{2}$,*

$$\mathcal{R} : H^s(\Omega) \to H^s(\Omega),$$
$$\gamma^+\mathcal{R} : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega),$$
$$T^+\mathcal{R} : H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega).$$

15.3 Invertibility of the single layer potential operator.

This is well-known (see e.g. [CC00, Remark 1.42(ii)], [Ste08, proof of Theorem 6.22]) that for some 2D domains the kernel of the operator $\mathcal{V}_\Delta$ is non-zero, which by (15.9) also implies that $\ker\mathcal{V} \neq \{0\}$ for the same domains.

In order to have invertibility for the single layer potential operator in 2D, we define the following subspace of the space $H^{-\frac{1}{2}}(\partial\Omega)$, see e.g. [Ste08, Eq. (6.30)],

$$H_*^{-\frac{1}{2}}(\partial\Omega) := \{\phi \in H^{-\frac{1}{2}}(\partial\Omega) : \langle\phi, 1\rangle_{\partial\Omega} = 0\},$$

where the norm in $H_*^{-\frac{1}{2}}(\partial\Omega)$ is the induced by the norm in $H^{-\frac{1}{2}}(\partial\Omega)$.

Theorem 4. *If $\psi \in H_*^{-\frac{1}{2}}(\partial\Omega)$ satisfies $\mathcal{V}\psi = 0$ on $\partial\Omega$, then $\psi = 0$.*

Proof. The theorem holds for the operator $\mathcal{V}_\Delta$ (see e.g. [McL00, Corollay 8.11(ii)]), which by (15.9) implies it for the operator $\mathcal{V}$ as well. □

Theorem 5. *Let $\Omega \subset \mathbb{R}^2$ have the diameter* $\mathrm{diam}(\Omega) < 1$*. Then the single layer potential $\mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is invertible.*

Proof. By [Ste08, Theorem 6.23], for $diam(\Omega) < 1$ the operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is $H^{-\frac{1}{2}}(\partial\Omega)-$elliptic and since it is also bounded, c.f. Theorem 1 for $s = -1/2$, the Lax-Milgram theorem implies its invertibility. Then by the first relation in (15.10) the invertibility of the operator $\mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ also follows. □

15.4 The Third Green Identity

For $u \in H^{1,0}(A;\Omega)$, let us write the third Green's identity (15.6) using the surface and volume potential operator notations,

$$u + \mathcal{R}u - VT^+u + W\gamma^+u = \mathcal{P}Au \qquad \text{in } \Omega. \tag{15.31}$$

Applying the *trace operator* to equation (15.31) and using the jump relations from Theorem 2, we have

$$\frac{1}{2}\gamma^+u + \gamma^+\mathcal{R}u - \mathcal{V}T^+u + \mathcal{W}\gamma^+u = \gamma^+\mathcal{P}Au \qquad \text{on } \partial\Omega. \tag{15.32}$$

Similarly, applying the *co-normal derivative operator* to equation (15.31), and using again the jump relation, we obtain

$$\frac{1}{2}T^+u + T^+\mathcal{R}u - \mathcal{W}'T^+u + \mathcal{L}^+\gamma^+u = T^+\mathcal{P}Au \qquad \text{on } \partial\Omega. \tag{15.33}$$

For some functions f, Ψ and Φ let us consider a more general indirect integral relation associated with equation (15.31).

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f \qquad \text{in } \Omega. \tag{15.34}$$

Lemma 1. *Let $u \in H^1(\Omega)$, $f \in L_2(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, and $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy equation (15.34). Then u belongs to $H^{1,0}(\Omega;A)$ and is a solution of PDE $Au = f$ in Ω and*

$$V(\Psi - T^+u)(y) - W(\Phi - \gamma^+u)(y) = 0, \qquad y \in \Omega$$

Proof. The proof follows word for word the corresponding proof in 3D case in [CMN09a, Theorem 4.1]. □

Lemma 2. *(i) Let either $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and* $\mathrm{diam}(\Omega) < 1$*, or $\Psi^* \in H_*^{-\frac{1}{2}}(\partial\Omega)$. If $V\Psi^* = 0$ in Ω, then $\Psi^* = 0$ on $\partial\Omega$.*

(ii) Let $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$. If $W\Phi^* = 0$ in Ω, then $\Phi^* = 0$ on $\partial\Omega$.

Proof. (i) Taking the trace of equation in Lemma 2(i) on $\partial\Omega$, by the jump relation (15.13) we have $\mathcal{V}\Psi^*(y) = 0$ on $\partial\Omega$. If $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathrm{diam}(\Omega) < 1$, then the result follows from the invertibility of the single layer potential given by Theorem 5. On the other hand, if $\Psi^* \in H_*^{-\frac{1}{2}}(\partial\Omega)$, then the result is implied by Theorem 4.

(ii) Let us take the trace of equation in Lemma 2(ii) on $\partial\Omega$, and use the jump relation (15.14) to obtain,

$$-\frac{1}{2}\Phi^* + \mathcal{W}\Phi^* = 0 \ \text{ on } \partial\Omega.$$

Multiplying this equation by $a(y)$, denoting $\hat{\Phi}^* = a\Phi^*$ and using the second relation in (15.9), we obtain equation

$$-\frac{1}{2}\hat{\Phi}^* + \mathcal{W}_\Delta\hat{\Phi}^* = 0 \ \text{ on } \partial\Omega.$$

It is well known that this equation has only the trivial solution. It is particularly due to the contraction property of the operator $\frac{1}{2}I + \mathcal{W}_\Delta$, see [SW01, Theorem 3.1]. Since $a(y) \neq 0$, the result follows. □

15.5 Boundary-Domain Integral Equations (BDIEs)

To reduce the variable-coefficient Dirichlet BVP (15.2)-(15.3) to a *segregated* boundary-domain integral equation system, let us denote the unknown conormal derivative as $\psi := T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ and will further consider ψ as formally independent on u.

Assuming that the function u satisfies PDE $Au = f$, by substituting the Dirichlet condition into the third Green identity (15.31) and either into its trace (15.32) or into its co-normal derivative (15.33) on $\partial\Omega$, we can reduce the BVP (15.2)-(15.3) to two different systems of Boundary Domain-Integral Equations for the unknown function $u \in H^{1,0}(\Omega; A)$ and $\psi := T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$.

BDIE system (D1) obtained from equations (15.31) and (15.32) is

$$\begin{aligned} u + \mathcal{R}u - V\psi &= F_0 \qquad \text{in } \ \Omega, \\ \gamma^+\mathcal{R}u - \mathcal{V}\psi &= \gamma^+F_0 - \varphi_0 \qquad \text{on } \ \partial\Omega, \end{aligned}$$

where

$$F_0 := \mathcal{P}f - W\varphi_0 \ \text{ in } \Omega. \tag{15.35}$$

The system can be written in matrix form as $\mathcal{A}^1\mathcal{U} = \mathcal{F}^1$, where $\mathcal{U} := [u, \psi]^\top \in H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega)$ and

$$\mathcal{A}^1 := \begin{bmatrix} I + \mathcal{R} & -V \\ \gamma^+\mathcal{R} & -\mathcal{V} \end{bmatrix}, \qquad \mathcal{F}^1 = \begin{bmatrix} F_0 \\ \gamma^+F_0 - \varphi_0 \end{bmatrix}.$$

From the mapping properties of W in Theorem 1 and $\mathcal{P}$ in Theorem 3, we get the inclusion $F_0 \in H^{1,0}(\Omega; A)$, and the trace theorem implies $\gamma^+ F_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Therefore, $\mathcal{F}^1 \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved in $\mathcal{A}^1$, the operator $\mathcal{A}^1 : H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded.

BDIE system (D2) obtained from equations (15.31) and (15.33) is

$$\begin{aligned} u + \mathcal{R}u - V\psi = F_0 \qquad &\text{in } \ \Omega, \\ \frac{1}{2}\psi + T^+\mathcal{R}u - \mathcal{W}'\psi = T^+F_0 \qquad &\text{on } \ \partial\Omega, \end{aligned}$$

where F_0 is given by (15.35). In matrix form it can be written as $\mathcal{A}^2\mathcal{U} = \mathcal{F}^2$, where

$$\mathcal{A}^2 = \begin{bmatrix} I + \mathcal{R} & -V \\ T^+\mathcal{R} & \frac{1}{2}I - \mathcal{W}' \end{bmatrix}, \qquad \mathcal{F}^2 = \begin{bmatrix} F_0 \\ T^+F_0 \end{bmatrix}$$

Note that the operator $\mathcal{A}^2 : H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is bounded.

15.6 Equivalence and Invertibility Theorems

In the following theorem we shall see the equivalence of the original Direchlet boundary value problem to the boundary domain integral equation systems.

Theorem 6. *Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$.*

(i) If some $u \in H^1(\Omega)$ solves the BVP(15.2)-(15.3), then the pair (u, ψ),where

$$\psi = T^+u \in H^{-\frac{1}{2}}(\partial\Omega), \tag{15.36}$$

solves BDIE systems (D1) and (D2).

(ii) If a pair $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D1), and $\operatorname{diam}(\Omega) < 1$, then u solves BDIE system (D2) and BVP(15.2)-(15.3), this solution is unique, and ψ satisfies (15.36).

(iii) If a pair $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D2), then u solves BDIE system (D1) and BVP(15.2)-(15.3), this solution is unique, and ψ satisfies (15.36).

Proof. (i) Let $u \in H^1(\Omega)$ be solution of the BVP(15.2)-(15.3). Since $f \in L_2(\Omega)$, we have that $u \in H^{1,0}(\Omega; A)$. Setting ψ by (15.36) and recalling how BDIE systems (D1) and (D2) were constructed, we obtain that (u, ψ) solve them.

Let now a pair $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves system (D1) or (D2). Due to the first equations in the BDIE systems, the hypotheses of Lemma (1) are satisfied implying that u belongs to $H^{1,0}(\Omega; A)$ and solves PDE (15.2) in Ω, while the following equation also holds,

$$V(\psi - T^+u)(y) - W(\varphi_0 - \gamma^+u)(y) = 0, \qquad y \in \Omega. \tag{15.37}$$

(ii) Let $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system (D1). Taking the trace of the first equation in (D1) and subtracting the second equation from it, we get $\gamma^+u = \varphi_0$ on $\partial\Omega$. Thus, the Dirichlet boundary condition is satisfied, and using it in (15.37), we have $V(\psi - T^+u)(y) = 0,\ y \in \Omega$. Lemma 2(i) then implies $\psi = T^+u$.

(iii) Let now $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system (D2). Taking the conormal derivative of the first equation in (D2) and subtracting the second equation from it, we get $\psi = T^+u$ on $\partial\Omega$. Then inserting this in (15.37) gives $W(\varphi_0 - \gamma^+u)(y) = 0,\ y \in \Omega$ and Lemma 2(ii) implies $\varphi_0 = \gamma^+u$ on $\partial\Omega$.

The uniqueness of the BDIE system solutions follows form the fact that the corresponding homogeneous BDIE systems can be associated with the homogeneous Dirichlet problem, which has only the trivial solution. Then paragraphs (ii) and (iii) above imply that the homogeneous BDIE systems also have only the trivial solutions. □

Theorem 7. *If* $\operatorname{diam}(\Omega) < 1$*, then the following operators are invertible,*

$$\mathcal{A}^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{15.38}$$

$$\mathcal{A}^1 : H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega). \tag{15.39}$$

Proof. Theorem 6(ii) implies that operators (15.38) and (15.39) are injective.

Let us denote $\mathcal{A}_0^1 := \begin{bmatrix} I & -V \\ 0 & -\mathcal{V} \end{bmatrix}$. Then $\mathcal{A}_0^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators $I : H^1(\Omega) \to H^1(\Omega)$ and $-\mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ (see Theorem 5). By Corollary 2 the operator

$$\mathcal{A}^1 - \mathcal{A}_0^1 = \begin{bmatrix} R & 0 \\ \gamma^+R & 0 \end{bmatrix} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is compact, implying that operator (15.38) is a Fredholm operator with zero index, see e.g. [McL00, Theorem 2.26]. Then the injectivity of operator (15.38) implies its invertibility, see e.g. [McL00, Theorem 2.27].

To prove invertibility of operator (15.39), we remark that for any $\mathcal{F}^1 \in H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega)$, a solution of the equation $\mathcal{A}^1\mathcal{U} = \mathcal{F}^1$ can be written as $\mathcal{U} = (\mathcal{A}^1)^{-1}\mathcal{F}^1$, where $(\mathcal{A}^1)^{-1} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is the continuous inverse to operator (15.38). But due to Lemma 1 the first equation of system (D1) implies that $\mathcal{U} = (\mathcal{A}^1)^{-1}\mathcal{F}^1 \in H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega)$ and moreover, the operator $(\mathcal{A}^1)^{-1} : H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous, which implies invertibility of operator (15.39). □

The following similar assertion for the operator $\mathcal{A}^2$ holds without the limitation on the diameter of Ω.

Theorem 8. *The following operators are invertible.*

$$\mathcal{A}^2 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{15.40}$$
$$\mathcal{A}^2 : H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega). \tag{15.41}$$

Proof. Theorem 6(iii) implies that operators (15.40) and (15.41) are injective.

Let us denote $\mathcal{A}_0^2 = \begin{bmatrix} I & -V \\ 0 & \frac{1}{2}I \end{bmatrix}$. Then $\mathcal{A}_0^2 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators $I : H^1(\Omega) \to H^1(\Omega)$ and $I : H^{-\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$. By Corollaries 1 and 2 the operator

$$\mathcal{A}^2 - \mathcal{A}_0^2 = \begin{bmatrix} R & 0 \\ T^+R & -\mathcal{W}' \end{bmatrix} : H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$$

is compact. This implies that operator (15.40) is a Fredholm operator with zero index and then the injectivity of operator (15.40) implies its invertibility.

The invertibility of operator (15.41) is then proved similar to the last paragraph of the proof of Theorem 7. □

15.7 Conclusion

In this paper, we have considered the interior Dirichlet problem for variable coefficient PDE in a two-dimensional domain, where the right hand side function is from $L_2(\Omega)$ and the Dirichlet data from the space $H^{\frac{1}{2}}(\partial\Omega)$. The BVP was reduced to two systems of Boundary-Domain Integral Equations and their equivalence to the original BVP was shown. The invertibility of the associated operators in the corresponding Sobolev spaces was also proved.

In a similar way one can consider also the 2D versions of the BDIEs for the Neumann problem, mixed problem in interior and exterior domains, united BDIEs as well as the localised BDIEs, which were analysied for 3D case in [CMN09a, CMN13, Mik06, CMN09b].

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