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ANALYSIS OF SOME LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS

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ABSTRACT. Some direct segregated localized boundary-domain integral equation (LBDIE) systems associated with the Dirichlet and Neumann boundary value problems (BVP) for a scalar "Laplace" PDE with *variable* coefficient are formulated and analysed. The parametrix is localized by multiplication with a radial localizing function. Mapping and jump properties of surface and volume integral potentials based on a localized parametrix and constituting the LBDIE systems are studied in a scale of Sobolev (Bessel potential) spaces. The main results established in the paper are the LBDIEs equivalence to the original variable-coefficient BVPs and the invertibility of the LBDIE operators in the corresponding Sobolev spaces.

1. Introduction. Partial Differential Equations (PDEs) with variable coefficients arise naturally in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermo-conductivity, fluid flows trough porous media, and other areas of physics and engineering.

The Boundary Integral Equation Method/Boundary Element Method (BIEM/BEM) is a well established tool for solution Boundary Value Problems (BVPs) with constant coefficients. The main ingredient for reducing a BVP for a PDE to a BIE is a fundamental solution to the original PDE. However, it is generally not available in an analytical and/or cheaply calculated form for PDEs with variable coefficients.

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Following Levi and Hilbert, one can use in this case a parametrix (Levi function) as a substitute for the fundamental solution. Parametrices are usually much wider available than fundamental solutions. They correctly describes the main part of the fundamental solution although do not have to satisfy the original PDE. This reduces the problem not to a boundary integral equation but to a system of Boundary-Domain Integral Equations (BDIEs), see e.g. [**17, 18**]. A discretization of the BDIE leads then to a system of algebraic equations of the similar size as in the FEM, however the matrix of the system is not sparse as in the FEM but dense and thus less efficient for numerical solution.

The Localized Boundary-Domain Integral Equation Method (LBDIEM) emerged recently [**12, 19, 21, 23, 24**] addressing this issue and making the BDIE competitive with the FEM for such problems. The LBDIEM employs specially constructed localized parametrices to reduce BVPs with variable coefficients to Localized Boundary-Domain Integral or Integro-Differential Equations. After a locally-supported mesh-based or mesh-less discretization this ends up in sparse systems of algebraic equations. Further advancing the LBDIEM requires a deeper analytical insight on properties of the corresponding integral operators, particularly on LBDIE solvability, uniqueness of solution, equivalence to original BVPs and invertibility of the LBDIEs. Analysis of non-localized segregated BDIEs is presented in [**3**] and of united BDIDEs in [**14**]. This paper develops analysis of some direct segregated *localized* BDIEs for the Dirichlet and Neumann problems, based on a parametrix localized by multiplying with a cut-off function, [**12**]. Some results on analysis of two LBDIE systems were presented in [**4**] for smooth localizing functions with compact support. Here we provide complete proofs of the results for four LBDIE systems associated with Dirichlet and Neumann BVPs, in the case of not necessarily compact and smooth localization.

The paper is organized as follows. After introducing basic notations in Section 2, we define classes of localizing functions and derive localized boundary-domain integral identities in Section 3. In Section 4 we give the localized boundary-domain integral equation formulations for the Dirichlet and Neumann BVPs and formulate the main theorems of the paper describing (i) equivalence of the LBDIEs to the original BVPs and (ii) invertibility of the corresponding localized boundary-domain integral operators in the appropriate Sobolev spaces. Section

5 is devoted to the study of properties of localized single layer, double layer and volume potentials, depending on the smoothness of the localizing function. Section 6 deals with inverse to the localized Newton potential and some boundary value problem for the localized Newton, single and double layer potentials. The pseudo-differential operator techniques used in Sections 5 and 6, although close to the standard ones (see [**1, 2, 7, 8**]), are complicated by the limited smoothness of the localizing function and thus of the operator kernels, which needed a special consideration. Finally, in Section 7 we prove the main theorems formulated in Section 4.

2. Basic notions and notations. Let Ω^+ be a bounded open three–dimensional region of $\mathbb{R}^3$ and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. For simplicity, we assume that the boundary $\partial\Omega := \partial\Omega^+$ is a simply connected, closed, infinitely smooth surface. Let $a \in C^\infty(\mathbb{R}^3)$, $0 < a(x) < C$ for $x \in \mathbb{R}^3$. Let also $\partial_j = \partial_{x_j} := \partial/\partial x_j$ $(j = 1, 2, 3)$, $\partial_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

We consider below localized boundary-domain integral equations associated with the following scalar elliptic differential equation

$$
\begin{aligned}
(2.1) \qquad Lu(x) := L(x, \partial_x)\, u(x) &:= \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \Big(a(x) \frac{\partial u(x)}{\partial x_i} \Big) \\
&= f(x), \quad x \in \Omega^+,
\end{aligned}
$$

where u is an unknown function and f is a given function in Ω^+.

In what follows, $H^s(\Omega^+) = H_2^s(\Omega^+)$, $H_{loc}^s(\Omega^-) = H_{2,loc}^s(\Omega^-)$, $H^s(\partial\Omega) = H_2^s(\partial\Omega)$ denote the Bessel potential spaces (coinciding with the Sobolev–Slobodetski spaces if $s \geq 0$), $H_{\partial\Omega}^s := \{g : g \in H^s(\mathbb{R}^3), \operatorname{supp} g \subset \partial\Omega\}$. For an open set Ω, we, as usual, denote $\mathcal{D}(\Omega) = C_{comp}^\infty(\Omega)$ endowed with sequential continuity, $\mathcal{D}^*(\Omega)$ is the Schwartz space of sequentially continuous functionals on $\mathcal{D}(\Omega)$, while $\mathcal{D}(\bar{\Omega})$ is the set of restrictions on $\bar{\Omega}$ of functions from $\mathcal{D}(\mathbb{R}^3)$.

From the trace theorem (see, e.g., [**10**]) for $u \in H^1(\Omega^+)$ ($u \in H_{loc}^1(\Omega^-)$) it follows that $u|_{\partial\Omega}^{\pm} := \gamma^{\pm} u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^{\pm} = \gamma_{\partial\Omega}^{\pm}$ is the trace operator on $\partial\Omega$ from $\Omega^{\pm}$. We will use γ for $\gamma^{\pm}$ if $\gamma^+ = \gamma^-$. We will use also notations $u^{\pm}$ for the traces $u|_{\partial\Omega}^{\pm}$, when this will cause no confusion.

For the linear operator L, we introduce the following subspace of $H^s(\Omega)$, c.f. [**9, 5, 14**],

$$H^{s,0}(\Omega;L) := \{g : \ g \in H^s(\Omega), \ Lg \in L_2(\Omega)\}$$

endowed with the norm

$$\|g\|_{H^{s,0}(\Omega;L)} := \|g\|_{H^s(\Omega)} + \|Lg\|_{L_2(\Omega)}.$$

For $u \in H^1(\Omega^\pm)$ the co–normal derivative operators on $\partial\Omega$ do not generally exist in the trace sense. However if $u \in H^{1,0}(\Omega^\pm;L)$, one can correctly define the generalized (canonical) co–normal derivative $T^\pm u = [Tu]^\pm \in H^{-\frac{1}{2}}(\partial\Omega)$ with the help of the first Green identity (cf., for example, [**5**, Lemma 3.2], [**11**, Lemma 4.3]),

$$\big\langle T^\pm u \, , \, w \big\rangle_{\partial\Omega} := \pm \int\limits_{\Omega^\pm} \big[\, (ew)Lu + E(u,ew) \, \big] \, dx, \quad \forall \, w \in H^{\frac{1}{2}}(\partial\Omega), \tag{2.2}$$

where e is a right inverse to the trace operator,

$$E(u,v) := \sum_{i=1}^{3} a(x) \, \frac{\partial u(x)}{\partial x_i} \, \frac{\partial v(x)}{\partial x_i},$$

and the symbol $\langle g_1, g_2\rangle_{\partial\Omega}$ denotes the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, coinciding with $\int_\Omega g_1(x)g_2(x)dx$ if $g_1, g_2 \in L_2(\partial\Omega)$.

3. Localized parametrix and Green identities. Denote by $P_1(x,y)$ the parametrix (Levi function) of the operator $L(x,\partial_x)$ considered in [**3, 12**],

$$P_1(x,y) = -\frac{1}{4\pi \, a(y) \, |x-y|} \, , \quad x,y \in \mathbb{R}^3, \quad x \neq y, \tag{3.1}$$

with the property

$$L(x,\partial_x) \, P_1(x,y) = \delta(x-y) + R_1(x,y), \tag{3.2}$$

where $\delta(\cdot)$ is the Dirac distribution, and the remainder

$$R_1(x,y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi\, a(y)\, |x-y|^3}\, \frac{\partial a(x)}{\partial x_i}\,, \quad x, y \in \mathbb{R}^3, \quad x \neq y, \tag{3.3}$$

possesses a weak singularity of type $\mathcal{O}(|x-y|^{-2})$ for small $|x-y|$.

Let, as usual, $W_1^k(a,b)$ denote the Sobolev space of functions belonging along with their $k-$th derivative to the space $L_1(a,b)$ of absolutely integrable functions on the interval (a,b). Note that if $g \in W_1^k(0,\infty)$, $k \geq 1$, then $g \in C^{k-1}([0,\infty))$ and $d^j g(t)/dt^j \to 0$ as $t \to \infty$ for $j = 0, ..., k-1$.

Further, let us introduce three classes for localizing functions.

Definition 3.1. We say $\chi \in X^k$ for integer $k \geq 0$ if $\chi(x) = \breve{\chi}(|x|)$, $\breve{\chi} \in W_1^k(0,\infty)$ and $\varrho\breve{\chi}(\varrho) \in L_1(0,\infty)$.

We say $\chi \in X_+^k$ for integer $k \geq 1$ if $\chi \in X^k$, $\chi(0) > 0$ and

$$\sigma_\chi(\omega) := \frac{\hat{\chi}_s(\omega)}{\omega} > 0, \quad \forall\, \omega \in \mathbb{R}, \tag{3.4}$$

where

$$\hat{\chi}_s(\omega) := \int_0^\infty \breve{\chi}(\varrho) \sin(\varrho\, \omega) \partial\varrho.$$

We say $\chi \in X_{1+}^k$ for integer $k \geq 1$ if $\chi \in X_+^k$ and

$$\omega \hat{\chi}_s(\omega) \leq \chi(0), \qquad \forall\, \omega \in \mathbb{R}. \tag{3.5}$$

Evidently, we have the following imbeddings: $X^{k_1} \subset X^{k_2}$, $X_+^{k_1} \subset X_+^{k_2}$, $X_{1+}^{k_1} \subset X_{1+}^{k_2}$ for $k_1 > k_2$.

The class X_+^k is defined in terms of the sine-transform. The following lemma provides an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class.

Lemma 3.2. *Let $k \geq 1$. If $\chi \in X^k$, $\chi(0) > 0$, $\breve{\chi}(\varrho) \geq 0$ for all $\varrho \in (0,\infty)$, and $\breve{\chi}$ is a non-increasing function on $[0,+\infty)$, then $\chi \in X_+^k$.*

Proof. We have to check (3.4). Let us first consider $\omega > 0$ and rewrite the left hand side of (3.4) as

$$\sigma_\chi(\omega) = \frac{1}{\omega}\int_0^\infty \breve{\chi}(\varrho)\,\sin(\varrho\,\omega)\,d\varrho = \frac{1}{\omega^2}\int_0^\infty \breve{\chi}\left(\frac{\gamma}{\omega}\right)\,\sin\gamma\,d\gamma = \frac{1}{\omega^2}\sum_{m=0}^{\infty}\int_{2m\pi}^{2\pi m+2\pi}\breve{\chi}\left(\frac{\gamma}{\omega}\right)\,\sin\gamma\,d\gamma. \tag{3.6}$$

Taking into account that $\breve{\chi}$ is nonnegative and non-increasing we can easily check the following inequalities for $m = 0, 1, 2, \ldots,$

$$\begin{aligned}\int_{2m\pi}^{2m\pi+2\pi}\breve{\chi}\left(\frac{\gamma}{\omega}\right)\,\sin\gamma\,d\gamma &= \int_{2m\pi}^{2m\pi+\pi}\breve{\chi}\left(\frac{\gamma}{\omega}\right)\,\sin\gamma\,d\gamma + \int_{2m\pi+\pi}^{2m\pi+2\pi}\breve{\chi}\left(\frac{\gamma}{\omega}\right)\,\sin\gamma\,d\gamma \\ &= \int_0^\pi\left[\breve{\chi}\left(\frac{2m\pi+\gamma}{\omega}\right) - \breve{\chi}\left(\frac{2m\pi+\gamma+\pi}{\omega}\right)\right]\,\sin\gamma\,d\gamma \geq 0.\end{aligned} \tag{3.7}$$

These inequalities imply that $\sigma_\chi(\omega) \geq 0$ for any $\omega > 0$. Actually, we have here a strict inequality. Indeed, if $\sigma_\chi(\omega) = 0$ for some $\omega > 0$, then due to continuity and nonnegativity of the integrand in the third line of (3.7), we get

$$\breve{\chi}\Big(\frac{2m\pi+\gamma}{\omega}\Big) = \breve{\chi}\Big(\frac{2m\pi+\gamma+\pi}{\omega}\Big), \quad m = 0, 1, 2....; \quad \gamma \in [0,\,\pi]. \tag{3.8}$$

Taking into account the monotonicity and continuity of $\breve{\chi}$, equality (3.8) implies $\breve{\chi}(\varrho) = \breve{\chi}(0) > 0$, $\forall\ \varrho > 0$, which contradicts to the condition $\breve{\chi} \in W_1^0(0,\infty) = L_1(0,\infty)$. Thus $\sigma_\chi(\omega) > 0$ for any $\omega > 0$. Since $\sigma_\chi(\omega)$ is an even function, this implies $\sigma_\chi(\omega) > 0$ for any $\omega \in \mathbb{R}\backslash\{0\}$. On the other hand, by the Lebesgue convergence theorem,

$$\sigma_\chi(0) = \lim_{\omega\to 0}\sigma_\chi(\omega) = \int_0^\infty \rho\breve{\chi}(\varrho)\,\,d\varrho,$$

and, consequently, $\sigma_\chi(0) > 0$ since $\breve{\chi}(\varrho)$ is non-negative, continuous at $\varrho = 0$, and $\breve{\chi}(0) > 0$. □

Note that the classes X^k particularly include the localization functions χ with a compact support that are mostly interesting for applications, see e.g. [**12, 16**], and also χ with non-compact support that can be useful in applications for unbounded domains.

Some examples for χ,

$$\chi_{1k}(x) = \begin{cases} \left[1 - \dfrac{|x|}{\varepsilon}\right]^k & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| \geq \varepsilon, \end{cases} \qquad k = 1, 2, \dots \tag{3.9}$$

$$\chi_2(x) = \begin{cases} \exp\left[\dfrac{|x|^2}{|x|^2 - \varepsilon^2}\right] & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| \geq \varepsilon, \end{cases} \tag{3.10}$$

$$\chi_3(x) = \begin{cases} \left(1 - \dfrac{|x|}{\varepsilon}\right)^2 \left(1 - 2\dfrac{|x|}{\varepsilon}\right) & \text{for} \quad |x| < \varepsilon, \\ 0 & \text{for} \quad |x| \geq \varepsilon, \end{cases} \tag{3.11}$$

$$\chi_4(x) = e^{-|x|}\ . \tag{3.12}$$

One can observe that χ_{1k}, χ_2, χ_3, are compactly supported while χ_4 is not. On the other hand, $\chi_{1k} \in X_+^k$, $k \geq 1$, while $\chi_2, \chi_4 \in X_+^\infty$ due to Lemma 3.2. Evidently, $\chi_3 \in X^2$ is non-monotonous and negative on a part of its support, which prevents applying Lemma 3.2, however, the direct integration gives $s_{\chi_3}(\omega) = \{\varepsilon\omega[\varepsilon^2\omega^2 - 10 - 2\cos(\varepsilon\omega)] + 12\sin(\varepsilon\omega)\}\varepsilon^{-3}\omega^{-5} > 0$, $\omega \in \mathbb{R}$, implying $\chi_3 \in X_+^2$. Moreover, our analysis of condition (3.5) (analytical for all χ_{1k}, χ_2, χ_4, and numerical for χ_2) shown also that $\chi_{11} \notin X_{1+}^1$ and $\chi_2 \notin X_{1+}^\infty$, while $\chi_{12} \in X_{1+}^2$, $\chi_{13} \in X_{1+}^3$, $\chi_3 \in X_{1+}^2$ and $\chi_4 \in X_{1+}^\infty$.

Now we define a localized parametrix

$$P_\chi(x, y) := \chi(x - y)P_1(x, y), \quad x, y \in \mathbb{R}^3. \tag{3.13}$$

Evidently,

$$L(x, \partial_x)\, P_\chi(x, y) = \chi(0)\delta(x - y) + R_\chi(x, y), \tag{3.14}$$

$$L(y, \partial_y)\, P_\chi(x, y) = \chi(0)\delta(x - y) + R_{*\chi}(x, y), \tag{3.15}$$

where

$$(3.16)\quad R_\chi(x,y) = -\frac{1}{4\pi\, a(y)} \sum_{j=1}^{3} \Big\{ -\frac{\partial}{\partial y_j} \Big[\frac{\partial a(x)}{\partial x_j}\, \frac{\chi(x-y)}{|x-y|} + a(x) \frac{\partial \chi(x-y)}{\partial x_j} \frac{1}{|x-y|} \Big] + a(x) \frac{\partial \chi(x-y)}{\partial x_j} \frac{\partial}{\partial x_j} \frac{1}{|x-y|} \Big\},$$

$$(3.17)\quad R_{*\chi}(x,y) = -\frac{1}{4\pi} \sum_{j=1}^{3} \Big\{ -\frac{\partial}{\partial y_j} \Big[\frac{\partial a(y)}{\partial y_j}\, \frac{\chi(x-y)}{a(y)\,|x-y|} + \frac{\partial \chi(x-y)}{\partial x_j}\, \frac{1}{|x-y|} \Big] + \frac{\partial \chi(x-y)}{\partial x_j}\, \frac{\partial}{\partial x_j} \frac{1}{|x-y|} \Big\},$$
$$x, y \in \mathbb{R}^3, \quad x \neq y.$$

We see that the functions $R_\chi(x,y)$ and $R_{*\chi}(x,y)$ possess a weak singularity $\mathcal{O}(|x-y|^{-2})$ as $x \to y$ if χ is smooth enough, e.g., $\chi \in X^3$.

Let us introduce the surface and volume potentials, based on the localized parametrix P_χ, for $y \in \mathbb{R}^3$,

$$(3.18)\qquad V_\chi g(y) := -\int_{\partial\Omega} P_\chi(x,y)\, g(x)\, dS_x, \quad y \notin \partial\Omega,$$

$$(3.19)\qquad W_\chi\, g(y) := -\int_{\partial\Omega} \big[T(x, n(x), \partial_x)) P_\chi(x,y) \big] g(x)\, dS_x, \quad y \notin \partial\Omega,$$

$$(3.20)\qquad \mathcal{P}_\chi\, g(y) := \int_{\Omega^+} P_\chi(x,y)\, g(x)\, dx,$$

$$(3.21)\qquad \mathcal{R}_\chi g(y) := \int_{\Omega^+} R_\chi(x,y)\, g(x)\, dx,$$

$$(3.22)\qquad \mathcal{R}_{*\chi}\, g(y) := \int_{\Omega^+} R_{*\chi}(x,y)\, g(x)\, dx\,.$$

For the case $\chi = 1$ in $\mathbb{R}^3$, properties of the potentials (3.18 - 3.21) and the operators generated by them are studied in [**3**]. In the case of a non-constant localizing function χ, properties of these potentials are established in Section 5.

Let us also define the corresponding boundary operators, for $y \in \partial\Omega$,

$$\mathcal{V}_\chi\, g(y) := -\int\limits_{\partial\Omega} P_\chi(x,y)\, g(x)\, dS_x, \tag{3.23}$$

$$\mathcal{W}_\chi\, g(y) := -\int\limits_{\partial\Omega} \big[T(x,n(x),\partial_x))\, P_\chi(x,y)\big]\ \ g(x)\, dS_x, \tag{3.24}$$

$$\mathcal{W}'_\chi\, g(y) := -\int\limits_{\partial\Omega} \big[T(y,n(y),\partial_y))\, P_\chi(x,y)\big]\, g(x)\, dS_x, \tag{3.25}$$

$$\mathcal{L}^{\pm}_\chi\, g(y) := [T(y,n(y),\partial_y))\, W_\chi g(y)]^{\pm}. \tag{3.26}$$

Due to the results described in Section 5 these operators are well defined.

We remark that from (3.1, 3.13, 3.16 - 3.26), we have,

$$\mathcal{P}_\chi g = \frac{1}{a}\mathcal{P}_{\chi_\Delta} g, \quad V_\chi g = \frac{1}{a} V_{\chi_\Delta} g, \quad W_\chi g = \frac{1}{a} W_{\chi_\Delta}(ag), \tag{3.27}$$

$$\mathcal{R}_\chi g = -\frac{1}{a(y)}\sum_{j=1}^{3}\partial_j \mathcal{P}_{\chi_\Delta}(g\,\partial_j a) + \frac{1}{a(y)}\mathcal{R}_{\chi_\Delta}(ag), \tag{3.28}$$

$$\mathcal{R}_{*\chi} g = -\sum_{j=1}^{3}\partial_j\left[\frac{\partial_j a}{a}\mathcal{P}_{\chi_\Delta} g\right] + \mathcal{R}_{\chi_\Delta} g, \tag{3.29}$$

$$\mathcal{V}_\chi g = \frac{1}{a}\mathcal{V}_{\chi_\Delta} g, \quad \mathcal{W}_\chi g = \frac{1}{a}\mathcal{W}_{\chi_\Delta}(ag), \tag{3.30}$$

$$\mathcal{W}'_\chi g = \mathcal{W}'_{\chi_\Delta}(g) - \frac{1}{a}\left[\frac{\partial a}{\partial n}\right]\mathcal{V}_{\chi_\Delta} g, \tag{3.31}$$

$$\mathcal{L}^{\pm}_\chi g = \mathcal{L}^{\pm}_{\chi_\Delta}(ag) - \frac{1}{a}\left[\frac{\partial a}{\partial n}\right] W^{\pm}_{\chi_\Delta}(ag),$$

where the localized potentials $\mathcal{P}_{\chi_\Delta}$, $\mathcal{R}_{\chi_\Delta}$, V_{χ_Δ}, W_{χ_Δ}, $\mathcal{V}_{\chi_\Delta}$, $\mathcal{W}_{\chi_\Delta}$, $\mathcal{W}'_{\chi_\Delta}$, $\mathcal{L}^{\pm}_{\chi_\Delta}$ are associated with the operator L for $a = 1$, i.e., with the Laplace operator Δ.

Let us recall the second Green identity for the operator $L(x, \partial_x)$,

$$\int\limits_{\Omega^+} \big[v\, L(x,\partial_x)u - u\, L(x,\partial_x)v\big]dx = \big\langle T^+u, v^+\big\rangle_{\partial\Omega} - \big\langle T^+v, u^+\big\rangle_{\partial\Omega}, \tag{3.32}$$

where $u, v \in H^{1,0}(\Omega^+; L)$ are real functions.

Let $y \in \Omega$ and Ω_y^+ be the domain Ω^+ with a neighbourhood of y deleted. If $\chi \in X^3$, then $P_\chi(\cdot, y) \in H^{1,0}(\Omega_y^+; L)$ for any domain Ω_y^+ by Corollary 5.2, and for $v(x) := P_\chi(x, y)$ and $u \in H^{1,0}(\Omega^+; L)$, we obtain from (3.2) and (3.32) by standard limiting procedures (see, e.g., [**17**]) the third Green identity,

$$\chi(0)u + \mathcal{R}_\chi u - V_\chi T^+ u + W_\chi u^+ = \mathcal{P}_\chi L u \quad \text{in} \quad \Omega^+. \tag{3.33}$$

Then by the properties of the potentials presented in Section 5, taking trace and co-normal derivative of (3.33), we derive,

$$\frac{\chi(0)}{2} u^+ + \mathcal{R}_\chi^+ u - \mathcal{V}_\chi T^+ u + \mathcal{W}_\chi u^+ = \mathcal{P}_\chi^+ L u \qquad \text{on} \qquad \partial\Omega\,, \tag{3.34}$$

$$\frac{\chi(0)}{2} T^+ u + T^+ \mathcal{R}_\chi u - \mathcal{W}_\chi' T^+ u + \mathcal{L}_\chi^+ u^+ = T^+ \mathcal{P}_\chi L u \quad \text{on} \qquad \partial\Omega\,. \tag{3.35}$$

Here $\mathcal{R}_\chi^+ u := (\,\mathcal{R}_\chi u\,)^+$, $\mathcal{P}_\chi^+ f := (\,\mathcal{P}_\chi f\,)^+$.

4. Direct segregated LBDIEs for the Dirichlet and Neumann problems and main theorems.

To simplify the LBDIE form, we will assume in Section 4 that $\chi(0) = 1$.

4.1. *LBDIE formulations.* Let us consider the Dirichlet problem

$$L\,u = f \quad \text{in} \quad \Omega^+, \tag{4.1}$$

$$u^+ = \varphi_0 \quad \text{on} \quad \partial\Omega, \tag{4.2}$$

where equation (4.1) is understood in the distributional sense and condition (4.2) in the trace sense; $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in H^0(\Omega^+)$.

Denoting the unknown co-normal derivative T^+u as a new variable ψ and substituting (4.1, 4.2) in (3.33, 3.34), we arrive at the system of direct segregated LBDIE **(D1)**,

$$u + \mathcal{R}_\chi u - V_\chi \psi = \mathcal{P}_\chi f - W_\chi \varphi_0 \quad \text{in} \quad \Omega^+, \tag{4.3}$$

$$\mathcal{R}_\chi^+ u - \mathcal{V}_\chi \psi = [\mathcal{P}_\chi f]^+ - \frac{1}{2}\varphi_0 - \mathcal{W}_\chi \varphi_0 \quad \text{on} \quad \partial\Omega, \tag{4.4}$$

with the unknowns $u \in H^{1,0}(\Omega^+; L)$ and $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$.

Alternatively, substituting (4.1, 4.2) in (3.33, 3.35) and denoting again the co-normal derivative T^+u as a new variable ψ, we arrive for the Dirichlet BVP at another direct segregated LBDIE system of the second kind **(D2)**,

$$u + \mathcal{R}_\chi u - V_\chi \psi = \mathcal{P}_\chi f - W_\chi \varphi_0 \quad \text{in} \qquad \Omega^+, \tag{4.5}$$

$$T^+\mathcal{R}_\chi u + \frac{1}{2}\,\psi - \mathcal{W}'_\chi \psi = T^+\mathcal{P}_\chi f - \mathcal{L}^+_\chi \varphi_0 \quad \text{on} \qquad \partial\Omega \tag{4.6}$$

with the unknowns $u \in H^{1,0}(\Omega^+; L)$ and $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$.

Let us now consider the Neumann problem

$$L\,u = f \quad \text{in} \qquad \Omega^+, \tag{4.7}$$

$$T^+u = \psi_0 \quad \text{on} \qquad \partial\Omega, \tag{4.8}$$

where equation (4.7) is understood in the distributional sense, while equality (4.8) is understood in the functional sense in accordance with (2.2); $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f \in H^0(\Omega^+)$.

Denoting the unknown trace u^+ as a new variable φ and substituting (4.7, 4.8) in (3.33, 3.34), we arrive at the direct segregated LBDIE system of the second kind **(N2)**,

$$u + \mathcal{R}_\chi u + W_\chi \varphi = \mathcal{P}_\chi f + V_\chi \psi_0 \quad \text{in} \qquad \Omega^+, \tag{4.9}$$

$$\mathcal{R}^+_\chi u + \frac{1}{2}\,\varphi + \mathcal{W}_\chi \varphi = [\mathcal{P}_\chi f]^+ + \mathcal{V}_\chi \psi_0 \quad \text{on} \qquad \partial\Omega, \tag{4.10}$$

with the unknowns $u \in H^{1,0}(\Omega^+; L)$ and $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$.

Now let us go over to the alternative LBDIEs formulation for the Neumann BVP. Again, denoting the unknown trace u^+ as a new variable φ and substituting relations (4.7, 4.8) into (3.33, 3.35), we arrive at the LBDIE system **(N1)**,

$$u + \mathcal{R}_\chi u + W_\chi \varphi = \mathcal{P}_\chi f + V_\chi \psi_0 \quad \text{in} \qquad \Omega^+, \tag{4.11}$$

$$T^+\mathcal{R}_\chi u + \mathcal{L}^+_\chi \varphi = T^+\mathcal{P}_\chi f - \frac{1}{2}\,\psi_0 + \mathcal{W}'_\chi \psi_0 \quad \text{on} \qquad \partial\Omega. \tag{4.12}$$

with the unknowns $u \in H^{1,0}(\Omega^+; L)$ and $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$.

The digits 1 or 2 in the notations (D1), (D2), (N1), (N2) indicate, respectively, the first or the second kind of the boundary equation in these systems. We entitled the above LBDIE systems *segregated* to underline that the boundary unknown functions ψ and φ are treated in the equations as independent (segregated) of the unknown function u defined in the domain. If the unknown boundary traces and/or co-normal derivatives are not replaced by segregated unknown functions, one can arrive at some other systems of direct *united* localized boundary-domain integral or integro-differential equations for the Dirichlet, Neumann or mixed problems, cf. [**12, 14**], but in the present paper we confine ourselves with analysis of the LBDIE systems (D1), (D2), (N1), (N2) only.

4.2. *Main theorems.* We will prove in Section 7 the following equivalence and invertibility theorems.

Theorem 4.1. *Let* $\chi(0) = 1$, $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ *and* $f \in H^0(\Omega^+)$.

(i) *If a function* $u \in H^1(\Omega^+)$ *solves the Dirichlet problem* (4.1 - 4.2) *then the pair* (u, ψ) *with* $\psi = T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ *solves the LBDIEs* (D1) *and LBDIEs* (D2) *with any* $\chi \in X^3$.

(ii) *Vice versa, if a pair* $(u, \psi) \in H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega)$ *solves the LBDIEs* (D1) *with* $\chi \in X^3_+$ *or LBDIEs* (D2) *with* $\chi \in X^3_{1+}$, *then* u *solves the Dirichlet problem* (4.1 - 4.2), *and* $T^+u = \psi$.

(iii) *The Dirichlet problem* (4.1 - 4.2), *the LBDIEs* (D1) *with* $\chi \in X^3_+$ *and LBDIEs* (D2) *with* $\chi \in X^3_{1+}$ *are all uniquely solvable.*

Let us denote the localized boundary-domain integral operator generated by the left hand sides in LBDIE (D1) and (D2), respectively, as

$$\mathcal{A}^{D1}_\chi := \begin{bmatrix} I + \mathcal{R}_\chi & -V_\chi \\ \mathcal{R}^+_\chi & -\mathcal{V}_\chi \end{bmatrix}, \qquad \mathcal{A}^{D2}_\chi := \begin{bmatrix} I + \mathcal{R}_\chi & -V_\chi \\ T^+\mathcal{R}_\chi & \frac{1}{2}I - \mathcal{W}'_\chi \end{bmatrix}.$$

Theorem 4.2. *Let* $\chi(0) = 1$. *The following operators are continuous and continuously invertible,*

$$\mathcal{A}^{D1}_\chi : H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega) \quad \text{if } \chi \in X^3_+, \tag{4.13}$$

$$\mathcal{A}^{D2}_\chi : H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega) \quad \text{if } \chi \in X^3_{1+}. \tag{4.14}$$

Theorem 4.3. *Let $\chi(0) = 1$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f \in H^0(\Omega^+)$.*

(i) *If a function $u \in H^1(\Omega^+)$ solves the Neumann problem* (4.7 - 4.8) *then the pair (u, φ) with $\varphi = u^+ \in H^{\frac{1}{2}}(\partial\Omega)$ solves the LBDIEs* (N2) *and LBDIEs* (N1) *with any $\chi \in X^3$.*

(ii) *Vice versa, if a pair $(u, \varphi) \in H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the LBDIEs* (N2) *with $\chi \in X^3_+$ or LBDIEs* (N1) *with $\chi \in X^3_{1+}$, then u solves the Neumann problem* (4.7 - 4.8) *and $u^+ = \varphi$.*

(iii) *The homogeneous Neumann problem* (4.7 - 4.8) *admits only one linearly independent solution $u = 1$ in $H^1(\Omega^+)$, while the homogeneous LBDIEs* (N2) *with any $\chi \in X^3_+$ and LBDIEs* (N1) *with any $\chi \in X^3_{1+}$ admit only one linearly independent solution $(u, \varphi) = (1, 1)$ in $H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega)$.*

(iv) *The condition*

$$\langle f\,,\,1\rangle_{\Omega^+} - \langle \psi_0\,,\,1\rangle_{\partial\Omega} = 0 \tag{4.15}$$

is necessary and sufficient for solvability of the nonhomogeneous Neumann problem (4.7 - 4.8) *and nonhomogeneous LBDIEs* (N2) *with any $\chi \in X^3_+$ and LBDIEs* (N1) *with any $\chi \in X^3_{1+}$.*

Let us denote the localized boundary-domain integral operators generated by the left hand sides in LBDIEs (N2) and (N1), respectively, as

$$\mathcal{A}^{N2}_\chi =: \begin{bmatrix} I + \mathcal{R}_\chi & W_\chi \\ \mathcal{R}^+_\chi & \frac{1}{2}\,I + \mathcal{W}_\chi \end{bmatrix}, \qquad \mathcal{A}^{N1}_\chi := \begin{bmatrix} I + \mathcal{R}_\chi & W_\chi \\ T^+\mathcal{R}_\chi & \mathcal{L}^+_\chi \end{bmatrix}.$$

Theorem 4.4. *Let $\chi(0) = 1$. The following operators are continuous Fredholm operators with zero index,*

$$\mathcal{A}^{N2}_\chi : H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega) \quad \textit{if } \chi \in X^3_+, \tag{4.16}$$

$$\mathcal{A}^{N1}_\chi : H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega) \quad \textit{if } \chi \in X^3_{1+}. \tag{4.17}$$

They have one–dimensional null–spaces, $\ker \mathcal{A}^{N1}_\chi = \ker \mathcal{A}^{N2}_\chi$*, spanned over the element $(u, \varphi) = (1, 1)$.*

Remark that in Theorems 4.1 - 4.4 we needed χ from X^3_+ for LBDIEs (D1) and (N2), but χ from much more narrow class X^3_{1+} for LBDIEs (D2) and (N1).

Before proving these theorems in Section 7, we provide necessary tools for this, analysing in Section 5 mapping and jump properties of the potentials, and constructing in Section 6 the inverse to the localized volume potential operator.

5. Properties of localized potentials. We analyse here mapping and jump properties of the localized operators $\mathcal{P}_\chi$, $\mathcal{R}_\chi$, $\mathcal{R}_{*\chi}$, V_χ, W_χ, $\mathcal{V}_\chi$, $\mathcal{W}_\chi$, $\mathcal{W}'_\chi$ and $\mathcal{L}^\pm_\chi$, defined in Subsection 3.1.

Let

$$(5.1)\qquad \hat{P}_{\chi_\Delta}(\xi) := \mathcal{F}_{x\to\xi}\left[-\frac{1}{4\,\pi}\,\frac{\chi(x)}{|x|}\right] = -\frac{1}{4\,\pi}\int\limits_{\mathbb{R}^3}\frac{\chi(x)}{|x|}\,e^{-2\pi i\,x\cdot\xi}\,dx,$$

be a Fourier transform of the localized parametrix $P_{\chi_\Delta}(x)$ for the Laplace operator (i.e., corresponding to the case $a(x)=1$, see (3.1) and (3.13)). Here and in what follows $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ are the distributional direct and inverse Fourier transform operators, which on the integrable functions take the form

$$\hat{g}(\xi) = \mathcal{F}g(\xi) = \int\limits_{\mathbb{R}^3} g(x)e^{-2\pi i x\cdot\xi}dx, \quad \mathcal{F}^{-1}\hat{g}(x) = \int\limits_{\mathbb{R}^3}\hat{g}(\xi)e^{2\pi i x\cdot\xi}d\xi.$$

Denote by $\mathbf{P}_{\chi_\Delta}$ the pseudodifferential operator with the symbol $\hat{P}_{\chi_\Delta}$,

$$(5.2)\qquad \mathbf{P}_{\chi_\Delta}\,g := \mathcal{F}^{-1}\left[\,\hat{P}_{\chi_\Delta}(\xi)\,\mathcal{F}g\,\right], \quad g\in\mathcal{S}'(\mathbb{R}^3),$$

where $\mathcal{S}'(\mathbb{R}^3)$ is the space of tempered distributions (Schwartz space). For $v\in\mathcal{S}(\mathbb{R}^3)$, where $\mathcal{S}(\mathbb{R}^3)$ is the space of rapidly decreasing functions, we have,

$$(5.3)\quad \mathbf{P}_{\chi_\Delta}\,v(y) = \int\limits_{\mathbb{R}^3} P_{\chi_\Delta}(x-y)\,v(x)\,dx = -\frac{1}{4\,\pi}\int\limits_{\mathbb{R}^3}\frac{\chi(x-y)}{|x-y|}\,v(x)\,dx.$$

First of all we prove the following main lemma which is crucial in our further analysis.

Lemma 5.1. (i) *Let* $\chi \in X^k$*, with* $k \geq 0$*. Then* $\hat{P}_{\chi_\Delta} \in C(\mathbb{R}^3)$*,*

$$\hat{P}_{\chi_\Delta}(0) = -\int_0^\infty \breve{\chi}\,(\varrho)\;\varrho\,d\varrho, \tag{5.4}$$

and for $\xi \neq 0$ *the following equality holds*

$$\begin{aligned}\hat{P}_{\chi_\Delta}(\xi) = &\sum_{m=0}^{k^*} \frac{(-1)^{m+1}}{|2\pi\xi|^{2m+2}}\;\breve{\chi}^{(2m)}(0)\\ &- \frac{1}{|2\pi\xi|^{k+1}} \int_0^\infty \sin\Big(2\pi|\xi|\varrho + \frac{k\pi}{2}\Big)\breve{\chi}^{(k)}(\varrho)\,d\varrho\,,\end{aligned} \tag{5.5}$$

where k^* *is the integer part of* $(k-1)/2$ *and the sum disappears in* (5.5) *if* $k^* < 0$*, i.e., if* $k = 0$*.*

(ii) *If* $\chi \in X^0$ *and condition* (3.4) *is satisfied, then*

$$\hat{P}_{\chi_\Delta}(\xi) < 0 \;\; \textit{for all} \;\; \xi \in \mathbb{R}^3. \tag{5.6}$$

Proof. Let $(\varrho, \theta, \varphi)$ be coordinates of the point x in the spherical coordinate system with the azimuthal axis directed along ξ. Then

$$\begin{aligned}\hat{P}_{\chi_\Delta}(\xi) &= -\frac{1}{4\,\pi} \int_{\mathbb{R}^3} \frac{\chi(x)}{|x|}\, e^{-2\pi i\, x\cdot\xi}\, dx\\ &= -\frac{1}{4\,\pi} \int_0^\infty \int_0^\pi \int_0^{2\pi} \breve{\chi}\,(\varrho)\, e^{-2\pi i\,\varrho\,|\xi|\cos\theta}\varrho\,\sin\theta\, d\varphi\, d\theta\, d\varrho\\ &= -\frac{1}{2} \int_0^\infty \int_0^\pi \breve{\chi}\,(\varrho)\, e^{-2\pi i\,\varrho\,|\xi|\cos\theta}\varrho\,\sin\theta\, d\theta\, d\varrho\\ &= -\frac{1}{2\pi|\xi|} \int_0^\infty \breve{\chi}\,(\varrho)\;\sin(2\pi\varrho\,|\xi|)\, d\varrho\,.\end{aligned} \tag{5.7}$$

Integrating by parts, we have,

$$\hat{P}_{\chi_\Delta}(\xi) = -\frac{\breve{\chi}(0)}{|2\pi\xi|^2} - \frac{1}{|2\pi\xi|^2} \int\limits_0^\infty \breve{\chi}'(\varrho)\ \cos(2\pi\varrho\,|\xi|)\,d\varrho$$

$$= \frac{1}{|2\pi\xi|^2} \int\limits_0^\infty \breve{\chi}'(\varrho)\ [1-\cos(2\pi\varrho\,|\xi|)]\,d\varrho.$$

Further, successively integrating by parts, and taking into account that all derivatives of the localizing function $\breve{\chi}$ up to the order $k-1$ vanish as $\varrho \to \infty$, we easily derive (5.5). Equality (5.4) and continuity of $\hat{P}_{\chi_\Delta}(\xi)$ is obtained from penultimate equality in (5.7) by the Lebesgue convergence theorem. Item (ii) of the lemma immediately follows from the last inequality in (5.7). □

Lemma 5.1 implies the following important corollary.

Corollary 5.2. (i) *There exists a positive constant* c_1 *such that*

(5.8) $$|\hat{P}_{\chi_\Delta}(\xi)| \le c_1\ (1+|\xi|^2)^{-\frac{k+1}{2}} \quad \textit{for all} \quad \xi \in \mathbb{R}^3 \quad \textit{if}\ \ \chi \in X^k,\ k=0,1,$$

(5.9) $$|\hat{P}_{\chi_\Delta}(\xi)| \le c_1\ (1+|\xi|^2)^{-\frac{k+1}{2}} \quad \textit{for all} \quad \xi \in \mathbb{R}^3 \quad \textit{if}\ \chi \in X^k,\ k=2,3,\ \textit{and} \quad \chi(0)=0,$$

and the following operators are continuous,

(5.10) $$\mathbf{P}_{\chi_\Delta} : H^t(\mathbb{R}^3) \to H^{t+k+1}(\mathbb{R}^3)\ \ \forall\ \ t \in \mathbb{R} \quad \textit{if}\ \chi \in X^k, \quad k=0,1,$$

(5.11) $$: H^t(\mathbb{R}^3) \to H^{t+k+1}(\mathbb{R}^3)\ \ \forall\ \ t \in \mathbb{R} \quad \textit{if}\ \chi \in X^k, \quad k=2,3, \quad \textit{and} \quad \chi(0)=0.$$

(ii) *If* $\chi \in X^1_+$, *then there exist positive constants* c_1 *and* c_2 *such that*

(5.12) $$c_2\ (1+|\xi|^2)^{-1} \le |\hat{P}_{\chi_\Delta}(\xi)| \le c_1\ (1+|\xi|^2)^{-1}\ \ \textit{for all}\ \ \xi \in \mathbb{R}^3,$$

and the following operator is continuously invertible,

(5.13) $$\mathbf{P}_{\chi_\Delta}\ :\ H^t(\mathbb{R}^3) \to H^{t+2}(\mathbb{R}^3) \quad \forall\ t \in \mathbb{R}.$$

Proof. Item (i) is implied by ansatz (5.5) and continuity of $\hat{P}_{\chi_\Delta}(\xi)$ at $\xi = 0$.

Consider item (ii). The second inequality in (5.12) is given by (5.8). Properties (5.4) and (5.6) imply the first inequality in (5.12) on any finite interval of ξ. Since in (5.5) $\chi(0) \neq 0$, while the integral tends to zero as $|\xi| \to \infty$ due to the Lebesgue theorem, we obtain the left inequality in (5.12) at all $\xi \in \mathbb{R}^3$. This implies the continuous invertibility of operator (5.13). □

Let us now analyse properties of the operator $\mathcal{R}_{\chi_\Delta}$ involved in the expressions of the operators $\mathcal{R}_\chi$ and $\mathcal{R}_{*\chi}$ defined by (3.28) and (3.29). Let us denote,

$$\mathbf{R}_{\chi_\Delta} g := \int_{\mathbb{R}^3} R_{\chi_\Delta}(x-y) g(x)\, dx = \mathcal{F}^{-1}\left(\widehat{R}_{\chi_\Delta}\, \mathcal{F} g\right),$$

where $\widehat{R}_{\chi_\Delta} = \mathcal{F} R_{\chi_\Delta}$ and

$$\begin{aligned} R_{\chi_\Delta}(x-y) &= -\frac{1}{4\pi} \sum_{j=1}^{3} \left\{ \frac{\partial}{\partial x_j} \left[\frac{\partial \chi(x-y)}{\partial x_j} \frac{1}{|x-y|} \right] \right. \\ &\qquad \left. + \frac{\partial \chi(x-y)}{\partial x_j}\, \frac{\partial}{\partial x_j} \frac{1}{|x-y|} \right\} \\ &= \Delta\, P_{\chi_\Delta}(x-y) - \chi(0)\delta(x-y), \end{aligned} \tag{5.14}$$

cf. (3.16), (3.17) with $a = 1$ and (5.3).

Lemma 5.3. *Let $s \in \mathbb{R}$, $\chi \in X^k$, $k = 1, 2, 3$. The following operators are continuous,*

$$\mathbf{R}_{\chi_\Delta} : H^s(\mathbb{R}^3) \to H^{s+k-1}(\mathbb{R}^3), \quad \chi \in X^k. \tag{5.15}$$

Proof. Let $\chi \in X^k$, $k \geq 1$. By definition (5.14) we have, $\hat{R}_{\chi_\Delta} = -|2\pi\xi|^2 \hat{P}_{\chi_\Delta} - \breve{\chi}(0)$ and then by Lemma 5.1,

$$\begin{aligned} \hat{R}_{\chi_\Delta}(\xi) &= \sum_{m=1}^{k^*} \frac{(-1)^m}{|2\pi\xi|^{2m}}\, \breve{\chi}^{(2m)}(0) \\ &\quad + \frac{1}{|2\pi\xi|^{k-1}} \int_0^\infty \sin\left(2\pi\varrho|\xi| + \frac{k\pi}{2}\right) \breve{\chi}^{(k)}(\varrho)\, d\varrho\,, \end{aligned} \tag{5.16}$$

where k^* is the integer part of $(k-1)/2$, and the sum disappears in (5.16) if $k^* < 1$, i.e., $k < 3$. Equality (5.16) gives the estimates,

$$|\hat{R}_{\chi_\Delta}(\xi)| \le c\,(1+|\xi|^2)^{-\frac{k-1}{2}} \quad \text{for all } \xi \in \mathbb{R}^3 \quad \text{if } \chi \in X^k, \quad k = 1,2,3,$$

which imply (5.15). □

Taking into account that

$$\mathcal{P}_{\chi_\Delta} f = \mathbf{P}_{\chi_\Delta} f, \quad \mathcal{R}_{\chi_\Delta} f = \mathbf{R}_{\chi_\Delta} f \text{ for } f \in \widetilde{H}^s(\Omega^+), \quad s \in \mathbb{R}, \tag{5.17}$$

we can write down the mapping properties for $\mathcal{P}_\chi$ and $\mathcal{R}_\chi$ and $\mathcal{R}_{*\chi}$.

Theorem 5.4. *The following operators are continuous*

$$\begin{aligned}
&(5.18) && \mathcal{P}_\chi : \widetilde{H}^s(\Omega^+) \to H^{s+2}(\Omega^+), && s \in \mathbb{R}, \quad \chi \in X^1,\\
&(5.19) && : H^s(\Omega^+) \to H^{s+2}(\Omega^+), && -\tfrac{1}{2} < s < \tfrac{1}{2}, \quad \chi \in X^1;\\
&(5.20) && \mathcal{R}_\chi,\ \mathcal{R}_{*\chi} : \widetilde{H}^s(\Omega^+) \to H^s(\Omega^+), && s \in \mathbb{R}, \quad \chi \in X^1,\\
&(5.21) && : H^s(\Omega^+) \to H^s(\Omega^+), && -\tfrac{1}{2} < s < \tfrac{1}{2}, \quad \chi \in X^1,\\
&(5.22) && : \widetilde{H}^s(\Omega^+) \to H^{s+1}(\Omega^+), && s \in \mathbb{R}, \quad \chi \in X^2,\\
&(5.23) && : H^s(\Omega^+) \to H^{s+1}(\Omega^+), && -\tfrac{1}{2} < s < \tfrac{1}{2}, \quad \chi \in X^2.
\end{aligned}$$

Proof. The mapping property (5.18) is implied by (5.17, 3.27) and Corollary 5.2. Then (5.19) follows since $H^s(\Omega^+) = \widetilde{H}^s(\Omega^+)$ for $-\frac{1}{2} < s < \frac{1}{2}$. Similarly, (5.20) and (5.22) are implied by (3.28) and (3.29) if we take into account Lemma 5.3 and property (5.18). Then again (5.21) and (5.23) follow from (5.20) and (5.22). □

Now we can prove also some mapping properties of the above operators to subspaces $H^{s,t}(\Omega; L) \subset H^s(\Omega)$ for a range of t. For $t = 0$, the space $H^{s,t}(\Omega; L)$ is described in Section 2, for other t we present it following [**15**].

Definition 5.5. Let $s \in \mathbb{R}$ and $L_* : H^s(\Omega^\pm) \to \mathcal{D}^*(\Omega^\pm)$ be a linear operator. For $t \geq -\frac{1}{2}$, we introduce the space

$$H^{s,t}(\Omega^\pm; L_*) := \{g : \ g \in H^s(\Omega^\pm), \ L_* g|_{\Omega^\pm} = \tilde{f}_g|_{\Omega^\pm}, \ \tilde{f}_g \in \widetilde{H}^t(\Omega^\pm)\}$$

endowed with the norm $\|g\|_{H^{s,t}(\Omega^\pm;L_*)} := \|g\|_{H^s(\Omega^\pm)} + \|\tilde{f}_g\|_{\widetilde{H}^t(\Omega^\pm)}$.

The distribution $\tilde{f}_g \in \widetilde{H}^t(\Omega^\pm)$, $t \geq -\frac{1}{2}$, in the above definition is an extension of the distribution $L_* g|_{\Omega^\pm} \in H^t(\Omega^\pm)$, and the extension is unique (if it does exist) since any distribution from the space $H^t(\mathbb{R}^3)$ with a support in $\partial\Omega$ is identical zero if $t \geq -1/2$ (see e.g. [**11**, Lemma 3.39], [**15**, Lemma 4]). The uniqueness implies that the norm $\|g\|_{H^{s,t}(\Omega^\pm;L_*)}$ is well defined.

Theorem 5.6. *The following operators are continuous*

$$\mathcal{P}_\chi : \widetilde{H}^s(\Omega^+) \to H^{s+2,s}(\Omega^+; L), \qquad -\frac{1}{2} < s < \frac{1}{2}, \quad \chi \in X^1, \tag{5.24}$$

$$: H^s(\Omega^+) \to H^{s+2,s}(\Omega^+; L), \qquad -\frac{1}{2} < s < \frac{1}{2}, \quad \chi \in X^1, \tag{5.25}$$

$$: H^s(\Omega^+) \to H^{\frac{5}{2}-\varepsilon,\frac{1}{2}-\varepsilon}(\Omega^+; L), \qquad \frac{1}{2} \leq s < \frac{3}{2}, \quad \chi \in X^1 \quad \forall \varepsilon \in (0,1), \tag{5.26}$$

$$\mathcal{R}_\chi : \widetilde{H}^s(\Omega^+) \to H^{s+1,s-1}(\Omega^+; L), \qquad \frac{1}{2} < s < \frac{3}{2}, \quad \chi \in X^2, \tag{5.27}$$

$$: H^s(\Omega^+) \to H^{\frac{3}{2}-\varepsilon,s-1}(\Omega^+; L), \qquad \frac{1}{2} < s < \frac{3}{2}, \quad \chi \in X^3, \quad \forall \varepsilon > 0. \tag{5.28}$$

Proof. Mapping properties (5.24, 5.25) directly follow from (5.18, 5.19). To prove (5.26) we take into account the imbedding $H^s(\Omega^+) \subset H^{\frac{1}{2}-\varepsilon}(\Omega^+)$, for $\frac{1}{2} \leq s$, $\varepsilon > 0$, and continuity of (5.25).

Continuity of the operator (5.27) follows directly from continuity of (5.22) in Theorem 5.4. To deal with (5.28), let us first of all remark that for $\frac{1}{2} \leq s < \frac{3}{2}$ and any $\varepsilon > 0$ the operator $\mathcal{R}_\chi : H^s(\Omega^+) \subset$

$H^{\frac{1}{2}-\varepsilon}(\Omega^+) \to H^{\frac{3}{2}-\varepsilon}(\Omega^+)$ is continuous due to property (5.23) from Theorem 5.4.

Further, for any $u \in H^s(\Omega^+)$, $\frac{1}{2} < s < \frac{3}{2}$,

$$Lu - \Delta u = \sum_{i=1}^{3} \frac{\partial a}{\partial x_i} \frac{\partial u}{\partial x_i} \in H^{s-1}(\Omega^+) = \widetilde{H}^{s-1}(\Omega^+),$$

that is, the spaces $H^{s,s-1}(\Omega^+; L)$ and $H^{s,s-1}(\Omega^+; \Delta)$ coincide.

Let now $\frac{1}{2} < s < \frac{3}{2}$. Applying the Laplace operator to (3.28), we have,

$$\Delta \mathcal{R}_\chi g = \frac{1}{a(y)} \left\{ -\sum_{j=1}^{3} \partial_j \Delta \mathcal{P}_{\chi_\Delta}(g\, \partial_j a) + \Delta \mathcal{R}_{\chi_\Delta}(ag) \right\} + Qg, \tag{5.29}$$

$$\begin{aligned} Qg := \left(\Delta \frac{1}{a(y)}\right) &\left\{ -\sum_{j=1}^{3} \partial_j \mathcal{P}_{\chi_\Delta}(g\, \partial_j a) + \mathcal{R}_{\chi_\Delta}(ag) \right\} \\ &+ 2\sum_{k=1}^{3} \left(\partial_k \frac{1}{a(y)}\right) \partial_k \left\{ -\sum_{j=1}^{3} \partial_j \mathcal{P}_{\chi_\Delta}(g\, \partial_j a) + \mathcal{R}_{\chi_\Delta}(ag) \right\}. \end{aligned} \tag{5.30}$$

Due to properties (5.19) and (5.23) of Theorem 5.4, and imbedding $H^s(\Omega^+) \subset H^{s-1}(\Omega^+)$, the operator $Q : H^s(\Omega^+) \to H^{s-1}(\Omega^+)$ is continuous if $\chi \in X^2$.

Further $\Delta \mathcal{P}_{\chi_\Delta} h = h + \mathcal{R}_{*\chi_\Delta} h$ in Ω^+ for any $h \in H^s(\Omega^+)$, where $\mathcal{R}_{*\chi_\Delta} = \mathcal{R}_{\chi_\Delta}$ is defined by (3.15, 3.22) with $a = 1$. Due to Theorem 5.4, the operator $\mathcal{R}_{*\chi_\Delta} : H^s(\Omega^+) \subset H^{s-1}(\Omega^+) \to H^s(\Omega^+)$ is continuous and thus $\partial_j \Delta \mathcal{P}_{\chi_\Delta} : H^s(\Omega^+) \to H^{s-1}(\Omega^+)$ is continuous as well.

On the other hand, the operator $\Delta \mathcal{R}_{\chi_\Delta} : H^s(\Omega^+) \subset H^{s-1}(\Omega^+) = \tilde{H}^{s-1}(\Omega^+) \to H^{s-1}(\Omega^+)$ is continuous due to Lemma 5.3 if $\chi \in X^3$. Thus the operator $\Delta \mathcal{R}_\chi : H^s(\Omega^+) \to H^{s-1}(\Omega^+)$ is continuous, implying (5.28). $\square$

Before proving mapping properties of co-normal derivatives of potentials, we define following [**15**] the *canonical* co-normal derivative operator acting on functions from $H^{s,t}(\Omega; L)$, extending to a range of Sobolev spaces the definition of co-normal derivative given by (2.2). By definition, for $u \in H^{s,t}(\Omega; L)$, $s \in \mathbb{R}$, $t \geq -\frac{1}{2}$, the distribution $Lu \in H^t(\Omega)$ can be uniquely extended to a distribution in $\widetilde{H}^t(\Omega)$, which we will call *the canonical extension* and denote by $L^0 u$.

For $u \in H^s(\Omega^\pm)$, $v \in H^{2-s}(\Omega^\pm)$, $1/2 < s < 3/2$, let us define a bilinear form,

$$\mathcal{E}^\pm(u,v) := \sum_{i=1}^{3} \langle a\partial_i u, \partial_i v\rangle_{\Omega^\pm} ,$$

where $\langle \cdot\,,\,\cdot\rangle_{\Omega^\pm}$ are the duality brackets between the spaces $H^{s-1}(\Omega^\pm)$ and $\widetilde{H}^{1-s}(\Omega^\pm)$, and we took into account that $\widetilde{H}^{1-s}(\Omega^\pm) = H^{1-s}(\Omega^\pm)$ when $1/2 < s < 3/2$.

Definition 5.7. For $u \in H^{s,-\frac{1}{2}}(\Omega^\pm; L)$, $\frac{1}{2} < s < \frac{3}{2}$, we define the *canonical co-normal derivative* $T^\pm u \in H^{s-\frac{3}{2}}(\partial\Omega)$ as

$$\text{(5.31)}\quad \left\langle T^\pm u\,,\, w\right\rangle_{\partial\Omega} := \pm\langle L^0 u, ew\rangle_{\Omega^\pm} \pm \mathcal{E}^\pm(u, ew) \quad \forall\, w \in H^{\frac{3}{2}-s}(\partial\Omega),$$

where $e : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\mathbb{R}^3)$ is a right inverse to the trace operator. The canonical co-normal derivative $T^\pm u$ is independent of e, the operator $T^\pm : H^{s,-\frac{1}{2}}(\Omega^\pm; L) \to H^{s-\frac{3}{2}}(\partial\Omega)$ is continuous, and the first Green identity holds in the following form,

$$\text{(5.32)}\quad \pm\left\langle T^\pm u\,,\, v^\pm\right\rangle_{\partial\Omega} = \langle L^0 u, v\rangle_{\Omega^\pm} \;+\; \mathcal{E}^\pm(u,v) \quad \forall\, v \in H^{2-s}(\Omega^\pm).$$

Since $H^{s,t}(\Omega^\pm; L) \subset H^{s,-\frac{1}{2}}(\Omega^\pm; L)$ for $t > -\frac{1}{2}$, Definition 5.7 defines the continuous operators $T^\pm : H^{s,t}(\Omega^\pm; L) \to H^{s-\frac{3}{2}}(\partial\Omega)$ for any $t \geq -\frac{1}{2}$ and $\frac{1}{2} < s < \frac{3}{2}$.

Corollary 5.8. *The operators*

$$\text{(5.33)}\qquad \mathcal{R}_\chi : H^s(\Omega^+) \to H^s(\Omega^+), \qquad \frac{1}{2} < s < \frac{3}{2}, \qquad \chi \in X^2,$$

$$\text{(5.34)}\qquad \mathcal{R}^+_\chi : H^s(\Omega^+) \to H^{s-\frac{1}{2}}(\partial\Omega), \qquad \frac{1}{2} < s < \frac{3}{2} \qquad \chi \in X^2,$$

$$\text{(5.35)}\quad T^+\mathcal{R}_\chi : H^s(\Omega^+) \to H^{s-\frac{3}{2}}(\partial\Omega), \qquad \frac{1}{2} < s < \frac{3}{2} \qquad \chi \in X^3,$$

are compact.

Proof. Compactness of (5.33) and (5.34) is implied by property (5.23) of Theorem 5.4 along with the Rellich compact imbedding theorem and the trace theorem. Continuity of the operator $\mathcal{R}_\chi : H^s(\Omega^+) \to H^{\frac{3}{2}-\varepsilon,s-1}(\Omega^+;L)$, $\frac{1}{2} < s < \frac{3}{2}$, for any $\varepsilon > 0$ due to property (5.28) of Theorem 5.6, implies continuity of the operator $T^+\mathcal{R}_\chi : H^s(\Omega^+) \to H^{-\varepsilon}(\partial\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, for sufficiently small $\varepsilon > 0$, and thus compactness of operator (5.35) due to the Rellich compact imbedding theorem. □

To consider properties of the surface potentials, we remark that they can be presented as

$$V_\chi = \chi(0)V_1 + V_{\chi-\chi(0)}, \quad W_\chi = \chi(0)W_1 + W_{\chi-\chi(0)}. \tag{5.36}$$

Here V_1 and W_1 are the surface potentials based on the non-localized parametrix (3.1) and studied in detail in [**3**], while $V_{\chi-\chi(0)}$ and $W_{\chi-\chi(0)}$ are given by (3.18, 3.19) with $P_{\chi-\chi(0)}(x,y) = P_\chi(x,y) - \chi(0)P_1(x,y)$. Due to (3.27), the mapping and jump properties of $V_{\chi-\chi(0)}$ and $W_{\chi-\chi(0)}$ are defined by those of their counterparts for the Laplace operator, $V_{\chi-\chi(0),\Delta}$ and $W_{\chi-\chi(0),\Delta}$, based on the corresponding function $P_{\chi-\chi(0),\Delta}(x-y) = P_{\chi_\Delta}(x-y) - \chi(0)P_{1\Delta}(x-y)$.

Let us state mapping properties of the operator

$$\begin{aligned}\mathcal{P}_{\chi-\chi(0),\Delta}\, g(y) &= \int_\Omega P_{\chi-\chi(0),\Delta}(x-y)\, g(x)\, dx \\ &= -\frac{1}{4\,\pi}\int_\Omega \frac{\chi(x-y)-\chi(0)}{|x-y|}\, g(x)\, dx.\end{aligned}$$

Lemma 5.9. *Let $\chi(x) \in X^k$, k=1,2,3. The operator*

$$\mu\mathcal{P}_{\chi-\chi(0),\Delta} : \widetilde{H}^t(\Omega) \to H^{t+k+1}(\mathbb{R}^3) \ \ \forall \ \ t \in \mathbb{R} \tag{5.37}$$

is continuous for any $\mu \in \mathcal{D}(\mathbb{R}^3)$.

Proof. Let $g \in \widetilde{H}^t(\Omega)$. Then $\hat{g}(\xi) := \mathcal{F}g(\xi)$ belongs to $C^\infty(\mathbb{R}^3)$ since Ω is bounded. Moreover,

$$\mathcal{P}_{\chi-\chi(0),\Delta}\ g(y) = \mathbf{P}_{\chi-\chi(0),\Delta}\ g(y) := \int_{\mathbb{R}^3} P_{\chi-\chi(0),\Delta}\ g(x)dx, \quad y \in \mathbb{R}^3.$$

Let first $\chi \in X^k$, $k = 1, 2$. Taking into account that $\hat{P}_{\chi-\chi(0),\Delta} = \hat{P}_{\chi_\Delta} - \chi(0)\hat{P}_{1\Delta}$ and $\hat{P}_{1\Delta}(\xi) = -|2\pi\xi|^{-2}$ as the symbol of the volume potential operator for the Laplace operator, Lemma 5.1 leads to the following expression,

$$\hat{P}_{\chi-\chi(0),\Delta}(\xi) = -\frac{1}{|2\pi\xi|^{k+1}} \int_0^\infty \sin\Big(2\pi\varrho|\xi| + \frac{k\pi}{2}\Big)\breve{\chi}^{(k)}(\varrho)\, d\varrho\ . \tag{5.38}$$

Further we have,

$$\begin{aligned} \mathcal{F}[\mathbf{P}_{\chi-\chi(0),\Delta}\ g](\xi) &= \hat{P}_{\chi-\chi(0),\Delta}(\xi)\hat{g}(\xi) \\ &= \hat{P}_{\chi-\chi(0),\Delta}(\xi)\mu_1(\xi)\hat{g}(\xi) \\ &\quad + \hat{P}_{\chi-\chi(0),\Delta}(\xi)[1-\mu_1(\xi)]\hat{g}(\xi), \end{aligned} \tag{5.39}$$

where the cut-off function $\mu_1 \in \mathcal{D}(\mathbb{R}^3)$ and $\mu_1(\xi) = 1$ for $|\xi| \le 1$. The first term in the right hand side of (5.39) is integrable and compactly supported, which implies its inverse Fourier transform is infinitely smooth in $\mathbb{R}^3$. For the second term we have due to (5.38),

$$|\hat{P}_{\chi-\chi(0),\Delta}(\xi)[1-\mu_1(\xi)]| \le c_1\ (1+|\xi|^2)^{-\frac{k+1}{2}} \qquad \text{for all} \qquad \xi \in \mathbb{R}^3,$$

which implies continuity of the operator with the symbol $\hat{P}_{\chi-\chi(0),\Delta}(\xi)[1-\mu_1(\xi)]$ from $H^t(\mathbb{R}^3)$ to $H^{t+k+1}(\mathbb{R}^3)$ $\forall\ t \in \mathbb{R}$. Combining these statements we obtain continuity of (5.37) for $k = 1, 2$.

If $\chi \in X^3$, then integrating (5.38) with $k = 2$ by parts, we have,

$$\hat{P}_{\chi-\chi(0),\Delta}(\xi) = -\frac{1}{|2\pi\xi|^4} \int_0^\infty [1-\cos\Big(2\pi\varrho|\xi|\Big)]\,\breve{\chi}^{(3)}(\varrho)\, d\varrho\ . \tag{5.40}$$

This means

$$|\hat{P}_{\chi-\chi(0),\Delta}(\xi)[1-\mu_1(\xi)]| \le c_1\ (1+|\xi|^2)^{-2}$$
$$\text{for all}\ \ \xi \in \mathbb{R}^3 \quad \text{if}\ \chi \in X^3,$$

which by the same arguments as above implies continuity of (5.37) for $k = 3$. □

Let us introduce the distributions $\psi\,\delta_{\partial\Omega}$ and $\partial_n(\varphi\,\delta_{\partial\Omega})$ defined by the relations

$$(5.41)\qquad \begin{aligned}\langle\psi\,\delta_{\partial\Omega}\,,\,h\rangle &:= \langle\psi\,,\,\gamma h\rangle_{\partial\Omega},\\ \langle\partial_n(\varphi\,\delta_{\partial\Omega})\,,\,h\rangle &:= \langle\varphi\,,\,-\partial_n h\rangle_{\partial\Omega}\quad \text{for all}\quad h\in\mathcal{D}(\mathbb{R}^3).\end{aligned}$$

For $\psi,\ \phi\in H^{s-\frac32}(\partial\Omega),\ s<\frac32$, one can observe from the right hand sides of (5.41) (where $\partial_n h$ is understood in the trace sense) that $\psi\,\delta_{\partial\Omega}$ and $\partial_n(\varphi\,\delta_{\partial\Omega})$ are actually continuous functionals on $h\in H^{2-s}(\mathbb{R}^3)$ and $h\in H^{3-s}(\mathbb{R}^3)$, respectively. Moreover,

$$\begin{aligned}|\,\langle\psi\,\delta_{\partial\Omega}\,,\,h\rangle\,| = |\,\langle\psi\,,\,\gamma h\rangle_{\partial\Omega}\,| &\le c_1\,||\psi||_{H^{s-\frac32}(\partial\Omega)}\,||\gamma h||_{H^{\frac32-s}(\partial\Omega)}\\ &\le c_2\,||\psi||_{H^{s-\frac32}(\partial\Omega)}\,||h||_{H^{2-s}(\mathbb{R}^3)},\\ |\,\langle\partial_n(\varphi\,\delta_{\partial\Omega})\,,\,h\rangle\,| = |\langle\varphi\,,\,\partial_n h\rangle_{\partial\Omega}| &\le c_1\,\|\varphi\|_{H^{s-\frac32}(\partial\Omega)}\,\|\partial_n h\|_{H^{\frac32-s}(\partial\Omega)}\\ &\le c_3\,||\varphi||_{H^{s-\frac32}(\partial\Omega)}\,||h||_{H^{3-s}(\mathbb{R}^3)},\end{aligned}$$

due to the usual duality estimation and the trace theorem. This shows that $\psi\,\delta_{\partial\Omega}\in H^{s-2}(\mathbb{R}^3),\ \partial_n(\varphi\,\delta_{\partial\Omega})\in H^{s-3}(\mathbb{R}^3)$. Evidently, $\operatorname{supp}[\psi\delta_{\partial\Omega}]\subset\partial\Omega$ and $\operatorname{supp}[\partial_n(\varphi\,\delta_{\partial\Omega})]\subset\partial\Omega$, which implies $\psi\,\delta_{\partial\Omega}\in H^{s-2}_{\partial\Omega}(\Omega)\subset\widetilde{H}^{s-2}(\Omega)$,$\partial_n(\varphi\,\delta_{\partial\Omega})\in H^{s-3}_{\partial\Omega}(\Omega)\subset\widetilde{H}^{s-3}(\Omega)$. Thus the following linear mappings are continuous,

$$(5.42)\qquad \psi\longmapsto\psi\,\delta_{\partial\Omega}: H^{s-\frac32}(\partial\Omega)\to\widetilde{H}^{s-2}(\Omega)\subset H^{s-2}(\mathbb{R}^3),\quad s<\frac32,$$

$$(5.43)\qquad \varphi\longmapsto\partial_n(\varphi\,\delta_{\partial\Omega}): H^{s-\frac32}(\partial\Omega)\to\widetilde{H}^{s-3}(\Omega^+)\subset H^{s-3}(\mathbb{R}^3),\quad s<\frac32.$$

It is well-known that the single layer, double layer and volume potentials can be represented as convolutions (see, e.g., [**22**] for harmonic potentials):

$$(5.44)\quad V_{\chi_\Delta}\,\psi(y) = \frac{1}{4\pi}\int\limits_{\partial\Omega}\frac{\chi(x-y)}{|x-y|}\,\psi(x)\,dS_x = \frac{1}{4\,\pi}\,\left[\frac{\chi(x)}{|x|}\,*\,(\psi\,\delta_{\partial\Omega})\right],$$

$$(5.45)\quad \begin{aligned}W_{\chi_\Delta}\,\varphi(y) &= \frac{1}{4\pi}\int\limits_{\partial\Omega}\frac{\partial}{\partial n(x)}\frac{\chi(x-y)}{|x-y|}\,\varphi(x)\,dS_x\\ &= \frac{1}{4\,\pi}\,\left[\frac{\chi(x)}{|x|}\,*\,[-\partial_n(\varphi\,\delta_{\partial\Omega})]\right](y),\end{aligned}$$

$$(5.46)\quad \mathbf{P}_{\chi_\Delta}\,v(y) = -\frac{1}{4\,\pi}\int\limits_{\mathbb{R}^3}\frac{\chi(x-y)}{|x-y|}\,v(x)\,dx = -\frac{1}{4\,\pi}\left[\frac{\chi(x)}{|x|}\,*\,v\right](y),$$

where the symbol $*$ denotes the generalized convolution operation in $\mathbb{R}^3$. This means that the potentials can be written as pseudodifferential operators,

$$V_{\chi_\Delta}\,\psi = -\mathcal{F}^{-1}\{\,\hat{P}_{\chi_\Delta}(\xi)\,\mathcal{F}(\psi\,\delta_{\partial\Omega})\,\} = -\mathbf{P}_{\chi_\Delta}(\psi\,\delta_{\partial\Omega})\,, \tag{5.47}$$

$$W_{\chi_\Delta}\,\varphi = -\mathcal{F}^{-1}\{\,\hat{P}_{\chi_\Delta}(\xi)\,\mathcal{F}[\,-\partial_n(\varphi\,\delta_{\partial\Omega})\,]\} = \mathbf{P}_{\chi_\Delta}[\partial_n(\varphi\,\delta_{\partial\Omega})\,], \tag{5.48}$$

$$\mathcal{P}_{\chi_\Delta}\,f = \mathcal{F}^{-1}\{\,\hat{P}_{\chi_\Delta}(\xi)\,\mathcal{F}\widetilde{f}\,\} = \mathbf{P}_{\chi_\Delta}\widetilde{f}, \tag{5.49}$$

where $\widetilde{f}$ is the extension by zero of the function f from Ω onto the whole of $\mathbb{R}^3$.

Theorem 5.10. *The following operators are continuous*

$$V_\chi : H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\mathbb{R}^3), \quad s < \frac{3}{2}\,, \quad \textit{if}\ \chi \in X^1, \tag{5.50}$$

$$: H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s,s-1}(\Omega^\pm; L), \quad \frac{1}{2} < s < \frac{3}{2}, \quad \textit{if}\ \chi \in X^2, \tag{5.51}$$

$$W_\chi : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega^\pm), \quad s < \frac{3}{2}\,, \quad \textit{if}\ \chi \in X^2, \tag{5.52}$$

$$: H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s,s-1}(\Omega^\pm; L), \quad \frac{1}{2} < s < \frac{3}{2}, \quad \textit{if}\ \chi \in X^3. \tag{5.53}$$

Proof. Due to (3.27), it suffices to show mapping properties (5.50 - 5.52) for V_{χ_Δ}, W_{χ_Δ}, respectively. In accordance with formula (5.47), $V_{\chi_\Delta}\,\psi = -\mathbf{P}_{\chi_\Delta}(\psi\,\delta_{\partial\Omega})$. Then (5.50) holds due to (5.10) from Corollary 5.2 since $\psi\,\delta_{\partial\Omega} \in H^{s-2}(\mathbb{R}^3)$.

For the double layer potential, let $\mu_0 \in \mathcal{D}(\mathbb{R}^3)$ such that $\mu_0(0) = 1$ and present $\chi = \chi_0 + \chi_\infty$, where $\chi_0 = \mu_0\chi$, $\chi_\infty = (1-\mu_0)\chi$, then evidently, $W_{\chi_\Delta}\ \varphi = W_{\chi_0,\Delta}\ \varphi + W_{\chi_\infty,\Delta}\ \varphi$. For $\chi \in X^2$ we obtain that $\chi_0 \in X^2$ and is compactly supported, while $\chi_\infty \in X^2$ and $\chi_\infty(0) = 0$. Then

$$W_{\chi_\infty,\Delta} : H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\mathbb{R}^3), \quad s < \frac{3}{2}, \tag{5.54}$$

is continuous due to ansatz (5.48), continuity of operator (5.11) from Corollary 5.2 and continuity of mapping (5.43).

On the other hand, since $\partial\Omega$ is closed, $W_{\chi_0,\Delta}\,\varphi$ has a compact support independent of φ, and in accordance with formulas (5.36, 5.48),

$$W_{\chi_0,\Delta}\,\varphi = \chi(0)W_{1\Delta}\,\varphi + W_{\chi_0-\chi(0),\Delta}\,\varphi,$$
$$W_{\chi_0-\chi(0),\Delta}\,\varphi = \mathbf{P}_{\chi_0-\chi(0),\Delta}[\,\partial_n(\varphi\,\delta_{\partial\Omega})\,]$$

where the latter equality follows from (5.48). Taking into account that $\chi(0)=\chi_0(0)$, then Lemma 5.9 and continuity of mapping (5.43) imply continuity of the operator

$$\mu W_{\chi_0-\chi(0),\Delta} : H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\mathbb{R}^3), \quad s<\frac{3}{2}, \tag{5.55}$$

for any $\mu\in\mathcal{D}(\mathbb{R}^3)$ and $\chi\in X^2$.

In addition, for $W_{1\Delta}$ the following mapping properties are well known,

$$W_{1\Delta} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega^+),$$
$$\mu W_{1\Delta} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega^-) \quad \forall\,\mu\in\mathcal{D}(\mathbb{R}^3), \qquad s\in\mathbb{R},$$

which along with (5.55) proves continuity of the operator

$$\begin{aligned} W_{\chi_0,\Delta} &: H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega^+), \\ \mu W_{\chi_0,\Delta} &: H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega^-) \quad \forall\,\mu\in\mathcal{D}(\mathbb{R}^3), \qquad s<\frac{3}{2}\,. \end{aligned} \tag{5.56}$$

Taking μ such that $\mu(x)=1$ on the support of $V_{\chi_0,\Delta}\varphi$ and $W_{\chi_0,\Delta}\varphi$, we have, $\mu W_{\chi_0,\Delta} = W_{\chi_0,\Delta}$, i.e., μ can be dropped in the second line of (5.56), and its combining with continuity of (5.54) leads to continuity of (5.52).

To prove (5.51) and (5.53), we remark that these mappings are evident for $V_{1\Delta}$ and $W_{1\Delta}$, respectively, since $\Delta V_{1\Delta}=0$, $\Delta W_{1\Delta}=0$ in $\Omega^\pm$; thus (5.51) and (5.53) hold for V_1, and W_1. On the other hand,

$$\Delta V_{\chi_0-\chi(0)}\psi = -\Delta\mathbf{P}_{\chi_0-\chi(0),\Delta}(\psi\,\delta_{\partial\Omega}),$$
$$\Delta W_{\chi_0-\chi(0)}\varphi = \Delta\mathbf{P}_{\chi_0-\chi(0),\Delta}[\,\partial_n(\varphi\,\delta_{\partial\Omega})\,],$$

which implies continuity of the operators

$$\mu\Delta V_{\chi_0-\chi(0)} \;:\; H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-1}(\mathbb{R}^3) \quad \text{if} \quad \chi\in X^2, \tag{5.57}$$

$$\mu\Delta W_{\chi_0-\chi(0)} \;:\; H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-1}(\mathbb{R}^3) \quad \text{if} \quad \chi\in X^3, \tag{5.58}$$

for any $\mu \in \mathcal{D}(\mathbb{R}^3)$ and $s < \frac{3}{2}$ due to Lemma 5.9 and continuity of mappings (5.42, 5.43). Taking again μ such that $\mu(x) = 1$ on the support of $V_{\chi_0,\Delta}\psi$ and $W_{\chi_0,\Delta}\varphi$, we have, $\mu\Delta V_{\chi_0,\Delta} = \Delta V_{\chi_0,\Delta}$, $\mu\Delta W_{\chi_0,\Delta} = \Delta W_{\chi_0,\Delta}$, i.e., μ can be dropped in (5.57, 5.58).

Taking into account that the operators

$$\Delta V_{\chi_\infty} : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-1}(\mathbb{R}^3) \quad \text{if} \quad \chi \in X^2, \tag{5.59}$$

$$\Delta W_{\chi_\infty} : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-1}(\mathbb{R}^3) \quad \text{if} \quad \chi \in X^3 \tag{5.60}$$

are continuous for $s < \frac{3}{2}$, due to (5.47, 5.48), continuity of operators (5.11) from Corollary 5.2 and continuity of mappings (5.42, 5.43), then (5.51) and (5.53) for V_χ and W_χ follow. □

Lemma 5.11. *For any $\mu \in \mathcal{D}(\mathbb{R})$, the following operators are compact*

$$\mu V_{\chi-\chi(0)} : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s}(\mathbb{R}^3), \quad s < \frac{5}{2}, \quad \textit{if}\ \chi \in X^2, \tag{5.61}$$

$$: H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s,s-1}(\Omega^\pm; L), \quad \frac{1}{2} < s < \frac{3}{2}, \quad \textit{if}\ \chi \in X^3, \tag{5.62}$$

$$\mu W_{\chi-\chi(0)} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s}(\mathbb{R}^3), \quad s < \frac{3}{2}, \quad \textit{if}\ \chi \in X^2, \tag{5.63}$$

$$: H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s,s-1}(\Omega^\pm; L), \quad \frac{1}{2} < s < \frac{3}{2}, \quad \textit{if}\ \chi \in X^3. \tag{5.64}$$

Proof. We have,

$$V_{\chi-\chi(0),\Delta}\ \psi = -\mathbf{P}_{\chi-\chi(0),\Delta}(\psi\,\delta_{\partial\Omega}),$$
$$W_{\chi-\chi(0),\Delta}\ \varphi = \mathbf{P}_{\chi-\chi(0),\Delta}[\partial_n(\varphi\,\delta_{\partial\Omega})\,].$$

Then continuity of mappings (5.42, 5.43) and Lemma 5.9 imply that the following operators are continuous,

$$\begin{aligned}
\mu V_{\chi-\chi(0),\Delta} &: H^{s-\frac{5}{2}}(\partial\Omega) \to H^{s}(\mathbb{R}^3), \quad s < \frac{5}{2}, \quad \text{if } \chi \in X^2, \\
&: H^{s-\frac{5}{2}}(\partial\Omega) \to H^{s+1}(\mathbb{R}^3), \quad s < \frac{5}{2}, \quad \text{if } \chi \in X^3, \\
\mu W_{\chi-\chi(0)} &: H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s}(\mathbb{R}^3), \quad s < \frac{3}{2}, \quad \text{if } \chi \in X^2, \\
&: H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s+1}(\mathbb{R}^3), \quad s < \frac{3}{2}, \quad \text{if } \chi \in X^3.
\end{aligned}$$

Since the imbeddings $H^{s-\frac{3}{2}}(\partial\Omega) \subset H^{s-\frac{5}{2}}(\partial\Omega)$, $H^{s-\frac{1}{2}}(\partial\Omega) \subset H^{s-\frac{3}{2}}(\partial\Omega)$ are compact due to the Rellich theorem, operators (5.61 - 5.64) are compact. ◻

Theorem 5.10, the trace theorem and Definition 5.7 of the canonical co-normal derivative imply the following boundary mapping properties for the surface potentials.

Corollary 5.12. *The following operators are continuous for* $\frac{1}{2} < s < \frac{3}{2}$,

$$V_\chi^{\pm} := \gamma^{\pm} V_\chi \ : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad \chi \in X^1, \tag{5.65}$$

$$W_\chi^{\pm} := \gamma^{\pm} W_\chi : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad \chi \in X^2, \tag{5.66}$$

$$T^{\pm} V_\chi : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad \chi \in X^2, \tag{5.67}$$

$$\mathcal{L}_\chi^{\pm} := T^{\pm} W_\chi : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad \chi \in X^3. \tag{5.68}$$

We will use further the evident representations similar to (5.36),

$$\begin{aligned}
\mathcal{V}_\chi &= \chi(0)\mathcal{V}_1 + \mathcal{V}_{\chi-\chi(0)}, \\
\mathcal{W}_\chi &= \chi(0)\mathcal{W}_1 + \mathcal{W}_{\chi-\chi(0)}, \\
\mathcal{W}'_\chi &= \chi(0)\mathcal{W}'_1 + \mathcal{W}'_{\chi-\chi(0)}.
\end{aligned} \tag{5.69}$$

Theorem 5.13. *Let $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$ and $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$. Then there hold the following jump relations on $\partial\Omega$*

$$V^+_\chi\psi - V^-_\chi\psi = 0, \quad \chi \in X^1, \tag{5.70}$$

$$W^+_\chi\varphi - W^-_\chi\varphi = -\chi(0)\varphi, \quad \chi \in X^2, \tag{5.71}$$

$$T^+V_\chi\psi - T^-V_\chi\psi = \chi(0)\psi, \quad \chi \in X^2, \tag{5.72}$$

$$T^+W_\chi\varphi - T^-W_\chi\varphi = \chi(0)\varphi\frac{\partial a}{\partial n}, \quad \chi \in X^3. \tag{5.73}$$

Moreover,

$$\mathcal{V}_\chi\psi = V^+_\chi\psi = V^-_\chi\psi, \quad \chi \in X^1, \tag{5.74}$$

$$\mathcal{W}_\chi\varphi = \frac{1}{2}(W^+_\chi\varphi + W^-_\chi\varphi), \quad \chi \in X^2, \tag{5.75}$$

$$\mathcal{W}'_\chi\psi = \frac{1}{2}(T^+V_\chi\psi + T^+V_\chi\psi), \quad \chi \in X^2. \tag{5.76}$$

Proof. Similar to the proof of Theorem 5.10, for any $\mu \in \mathcal{D}(\mathbb{R}^3)$ we have from Lemma 5.9 continuity of the following operators,

$$\mu V_{\chi-\chi(0),\Delta} : H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\mathbb{R}^3), \quad \chi \in X^1, \tag{5.77}$$

$$: H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s+1}(\mathbb{R}^3), \quad \chi \in X^2, \tag{5.78}$$

$$\mu W_{\chi-\chi(0),\Delta} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\mathbb{R}^3), \quad \chi \in X^2, \tag{5.79}$$

$$: H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s+1}(\mathbb{R}^3), \quad \chi \in X^3, \tag{5.80}$$

which (for appropriate memberships of χ) imply continuity across the boundary of corresponding traces and co-normal derivatives for $V_{\chi-\chi(0),\Delta}$ and $W_{\chi-\chi(0),\Delta}$ and thus for $V_{\chi-\chi(0)}$ and $W_{\chi-\chi(0)}$ due to (3.27). Because of (5.36) this means the jumps of V_χ and W_χ coincide, up to the multiplier $\chi(0)$, with these of V_1 and W_1 considered in [**3**], which give relations (5.70 - 5.73).

To prove (5.74 - 5.76), we use representation (5.69). For $\mathcal{V}_1$, $\mathcal{W}_1$, $\mathcal{W}'_1$ the equations corresponding to (5.74 - 5.76) follow from results in [**3**]. For $\mathcal{V}_{\chi-\chi(0)}$, $\mathcal{W}_{\chi-\chi(0)}$, $\mathcal{W}'_{\chi-\chi(0)}$ they are implied by the continuity across the boundary of corresponding traces and co-normal derivatives for $V_{\chi-\chi(0),\Delta}$ and $W_{\chi-\chi(0),\Delta}$, mentioned above. Remark that for $\mathcal{V}_{\chi-\chi(0)}$, $\mathcal{W}_{\chi-\chi(0)}$, $\mathcal{W}'_{\chi-\chi(0)}$ this can be also easily obtained from analysis of the kernel singularities. □

Note that unlike the case of constant coefficient, $a = const$, there is a non-zero jump of the co-normal derivative of the parametrix-based double layer potential, see (5.73).

Theorem 5.14. *Let $\frac{1}{2} < s < \frac{3}{2}$. The following operators*

$$\mathcal{V}_\chi : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad \chi \in X^1, \tag{5.81}$$

$$\mathcal{W}_\chi : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad \chi \in X^2, \tag{5.82}$$

$$\mathcal{W}'_\chi : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad \chi \in X^2, \tag{5.83}$$

$$\mathcal{L}^\pm_\chi : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad \chi \in X^3, \tag{5.84}$$

are continuous. Moreover, operators (5.82 - 5.83) *and*

$$\mathcal{V}_{\chi-\chi(0)} : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad \chi \in X^2, \tag{5.85}$$

$$\mathcal{L}^\pm_{\chi-\chi(0)} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad \chi \in X^3, \tag{5.86}$$

are compact.

Proof. The continuity immediately follows from relations (5.75, 5.76) from Theorem 5.13. Compactness for the potentials $\mathcal{W}_1$ and $\mathcal{W}'_1$ follows from the well-known corresponding results for potentials $\mathcal{W}_{1\Delta}$ and $\mathcal{W}'_{1\Delta}$. On the other hand, compactness for potentials $\mathcal{W}_{\chi-\chi(0)}$ and $\mathcal{W}'_{\chi-\chi(0)}$ is implied by Lemma 5.11 due to relations (5.75, 5.76) from Theorem 5.13. Then (5.36) completes the compactness proof for (5.82, 5.83). Compactness for (5.85 - 5.86) is implied by Lemma 5.11, relation (5.74) of Theorem 5.13, and the definition of $\mathcal{L}^\pm_{\chi-\chi(0)}$ similar to (5.68). □

6. Inverse to the localized Newton potential. Keeping in mind the properties of $\hat{P}_{\chi_\Delta}(\xi)$ and thus $\mathbf{P}_{\chi_\Delta}$ studied in Section 5, let us denote by $\mathbf{P}^{-1}_{\chi_\Delta}$ the pseudodifferential operator with symbol $1/\hat{P}_{\chi_\Delta}(\xi)$,

$$\mathbf{P}^{-1}_{\chi_\Delta}\, v := \mathcal{F}^{-1}\,[\,\hat{P}^{-1}_{\chi_\Delta}(\xi)\,\mathcal{F}v\,]\,. \tag{6.1}$$

Remark 6.1. Let $\chi \in X_+^1$. By Corollary 5.2, there exist positive constants c_1 and c_2, such that

$$c_2^{-1}\,(1+|\xi|^2) \le \left|\hat{P}_{\chi_\Delta}^{-1}(\xi)\right| \le c_1^{-1}\,(1+|\xi|^2) \quad \text{for all} \quad \xi \in \mathbb{R}^3, \tag{6.2}$$

which implies that the operator $\mathbf{P}_{\chi_\Delta}^{-1} : H^t(\mathbb{R}^3) \to H^{t-2}(\mathbb{R}^3)$ is the continuous inverse to the operator $\mathbf{P}_{\chi_\Delta} : H^t(\mathbb{R}^3) \to H^{t+2}(\mathbb{R}^3)$ for arbitrary $t \in \mathbb{R}$.

Note that (5.49) implies $\mathbf{P}_{\chi_\Delta}\widetilde{f} = \mathcal{P}_{\chi_\Delta} f$ for $f \in H^s(\Omega^+)$, $-\frac{1}{2} < s < \frac{1}{2}$, where $\widetilde{f}$ is the extension of f by zero from Ω^+ onto the whole of $\mathbb{R}^3$.

Now we can prove the following assertions for the localized potentials associated with the Laplace operator, cf. (3.27).

Lemma 6.2. *Let $\chi \in X_+^1$, $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in H^0(\Omega^+)$. Then*

$$\mathbf{P}_{\chi_\Delta}^{-1}\, V_{\chi_\Delta}\, \psi = -(\psi\, \delta_{\partial\Omega}), \tag{6.3}$$

$$\mathbf{P}_{\chi_\Delta}^{-1}\, W_{\chi_\Delta}\, \varphi = \partial_n(\varphi\, \delta_{\partial\Omega}), \tag{6.4}$$

$$\mathbf{P}_{\chi_\Delta}^{-1}\, \mathcal{P}_{\chi_\Delta}\, f = \widetilde{f}, \tag{6.5}$$

and thus $\operatorname{supp} \mathbf{P}_{\chi_\Delta}^{-1}\, V_{\chi_\Delta}\, \psi \subset \partial\Omega$, $\operatorname{supp} \mathbf{P}_{\chi_\Delta}^{-1}\, W_{\chi_\Delta}\, \varphi \subset \partial\Omega$, $\operatorname{supp} \mathbf{P}_{\chi_\Delta}^{-1}\, \mathcal{P}_{\chi_\Delta}\, f \subset \overline{\Omega^+}$.

Proof. Taking into consideration (6.1) and (5.47) we get

$$\begin{aligned}
\mathbf{P}_{\chi_\Delta}^{-1}\, V_{\chi_\Delta}\, \psi &= \mathcal{F}^{-1}\left\{ \hat{P}_{\chi_\Delta}^{-1}\, \mathcal{F}(V_{\chi_\Delta}\, \psi) \right\} \\
&= -\mathcal{F}^{-1}\left\{ \hat{P}_{\chi_\Delta}^{-1}\, \hat{P}_{\chi_\Delta}\, \mathcal{F}(\psi\, \delta_{\partial\Omega}) \right\} \\
&= -\mathcal{F}^{-1}\left\{ \mathcal{F}(\psi\, \delta_{\partial\Omega}) \right\} = -(\psi\, \delta_{\partial\Omega}).
\end{aligned}$$

Quite similarly we derive (6.4) and (6.5). ◻

Lemma 6.3. *Let $\chi \in X_+^1$, $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $f \in H^0(\Omega^+)$. If*

$$V_\chi\, \psi + \mathcal{P}_\chi\, f = 0 \quad in \qquad \Omega^+, \tag{6.6}$$

then $\psi = 0$ on $\partial\Omega$ and $f = 0$ in Ω^+.

Proof. By (3.27) it suffices to prove the lemma for the case $a = 1$, i.e., under assumption

$$V_{\chi_\Delta}\,\psi + \mathcal{P}_{\chi_\Delta}\, f = 0 \quad \text{in} \qquad \Omega^+, \tag{6.7}$$

instead of (6.6). Denote $U := V_{\chi_\Delta}\,\psi + \mathcal{P}_{\chi_\Delta}\, f$ in $\mathbb{R}^3$. Let us show that U is zero in $\mathbb{R}^3$. To this end, let us note that $U \in \widetilde{H}^1(\Omega^-)$ due to (6.7) and Theorems 5.6, 5.10. Therefore, there exists a sequence $U_n \in \mathcal{D}(\Omega^-)$, $n = \overline{1,\infty}$, converging to U in the space $\widetilde{H}^1(\Omega^-)$, i.e., $\lim\limits_{n\to\infty} ||U - U_n||_{H^1(\mathbb{R}^3)} = 0$. Due to Lemma 6.2, $\mathbf{P}^{-1}_{\chi_\Delta} U$ is a distribution with compact support,

$$\mathbf{P}^{-1}_{\chi_\Delta}\, U = \widetilde{f} - \psi\, \delta_{\partial\Omega}, \tag{6.8}$$

where $\widetilde{f}$ is the extension by zero of the function f from Ω^+ onto the whole of $\mathbb{R}^3$. Therefore, $\mathbf{P}^{-1}_{\chi_\Delta} U = 0$ in Ω^- in the distributional sense, i.e., $\langle \mathbf{P}^{-1}_{\chi_\Delta} U\, ,\, v\rangle = 0$ for all $v \in \mathcal{D}(\Omega^-)$. In particular,

$$\langle \mathbf{P}^{-1}_{\chi_\Delta}\, U\, ,\, U_n\rangle = 0, \quad n = \overline{1,\infty}.$$

Then we have

$$\begin{aligned} 0 &= \langle \mathbf{P}^{-1}_{\chi_\Delta}\, U\, ,\, U_n\rangle = \langle \mathcal{F}^{-1}\, [\hat{P}^{-1}_{\chi_\Delta}\, \mathcal{F}U\,]\, ,\, U_n\rangle \\ &= \langle \hat{P}^{-1}_{\chi_\Delta}\, \mathcal{F}U\, ,\, \overline{\mathcal{F}U_n}\,\rangle = \int\limits_{\mathbb{R}^3} \hat{P}^{-1}_{\chi_\Delta}\, \mathcal{F}U\ \overline{\mathcal{F}U_n}\, d\xi \\ &= \int\limits_{\mathbb{R}^3} \hat{P}^{-1}_{\chi_\Delta}\, |\mathcal{F}U|^2 d\xi + \int\limits_{\mathbb{R}^3} \hat{P}^{-1}_{\chi_\Delta}\, \mathcal{F}U\ \overline{[\mathcal{F}U_n - \mathcal{F}U]}\, d\xi. \end{aligned} \tag{6.9}$$

By (6.2), we get from (6.9),

$$\begin{aligned} \left| \int\limits_{\mathbb{R}^3} \hat{P}^{-1}_{\chi_\Delta}\, |\mathcal{F}U|^2 d\xi \right| &\le C \int\limits_{\mathbb{R}^3} (1 + |\xi|^2)\, |\mathcal{F}U|\ |\overline{\mathcal{F}(U_n - U)}|\, d\xi \\ &\le C||U||_{H^1(\mathbb{R}^3)}\, ||U_n - U||_{H^1(\mathbb{R}^3)} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus

$$\int\limits_{\mathbb{R}^3} \hat{P}^{-1}_{\chi_\Delta}\, |\mathcal{F}U|^2 d\xi = 0,$$

whence $\mathcal{F}U = 0$ due to the inequality (6.2) and negativity of $\hat{P}^{-1}_{\chi_\Delta}$, see (5.6). Consequently, $U = V_{\chi_\Delta}\,\psi + \mathcal{P}_{\chi_\Delta}\,f = 0$ in $\mathbb{R}^3$.

Now, from (6.8) it follows that $\widetilde{f} - \psi\,\delta_{\partial\Omega} = 0$ in the distributional sense in $\mathbb{R}^3$, which implies $f=0$ in Ω^+ and $\psi=0$ on $\partial\Omega$. □

Let us prove a counterpart of Lemma 6.3 for the double layer potential W_{χ_Δ} and its combination with the volume potential.

Lemma 6.4. *Let* $\chi \in X^3_{1+}$, $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$, $f \in H^0(\Omega^+)$. *If*

$$\mathcal{P}_\chi f + W_\chi\,\varphi = 0 \quad in \qquad \Omega^+, \tag{6.10}$$

then $f = 0$ *in* Ω^+ *and* $\varphi = 0$ *on* $\partial\Omega$.

Proof. Let us define

$$U := \mathcal{P}_{\chi_\Delta} f + W_{\chi_\Delta}\,\varphi \quad \text{in} \qquad \mathbb{R}^3, \tag{6.11}$$

which evidently belongs to $H^0(\mathbb{R}^3)$. By (3.27) it suffices to prove the lemma for the case $a = 1$, i.e., under assumption $U = 0$ in Ω^+ instead of (6.10). Our goal is to show that U is zero in Ω^- which immediately leads to the proof of the lemma due to the jump properties of W_{χ_Δ} and the invertibility of the operator $\mathbf{P}_{\chi_\Delta}$. Note that in accordance with Theorems 5.6 and 5.10 we have the inclusions,

$$\mathcal{P}_{\chi_\Delta} f \in H^2(\mathbb{R}^3), \quad (W_{\chi_\Delta}\varphi)|_{\Omega^\pm} \in H^{1,0}(\Omega^\pm;\Delta) \tag{6.12}$$

implying $U \in H^{1,0}(\Omega^\pm;\Delta)$, $U^\pm \in H^{\frac{1}{2}}(\partial\Omega)$ and $T^\pm_\Delta U \in H^{-\frac{1}{2}}(\partial\Omega)$. Therefore, by [**9**, Lemma 1.5.3.9] (see also [**15**, Lemma 9]) there exists a sequence $U_l \in \mathcal{D}(\mathbb{R}^3)$, $l = \overline{1,\infty}$, such that $\lim\limits_{l\to\infty} \|U - U_l\|_{H^{1,0}(\Omega^-;\Delta)} = 0$. Then due to the trace theorem and [**15**, Lemma 8]

$$\begin{aligned} \lim_{l\to\infty} \|U^- - U_l\|_{H^{\frac{1}{2}}(\partial\Omega)} &= 0, \\ \lim_{l\to\infty} \|T^-_\Delta U - \partial_n U_l\|_{H^{-\frac{1}{2}}(\partial\Omega)} &= 0. \end{aligned} \tag{6.13}$$

By Lemma 5.1 we easily derive

$$\hat{P}^{-1}_{\chi_\Delta}(\xi) = -\frac{|2\pi\xi|^2}{\breve{\chi}(0)} - \hat{m}(\xi)\,, \quad \hat{m}(\xi) := \frac{c_\chi(2\pi|\xi|)}{\breve{\chi}(0)\hat{P}_{\chi_\Delta}(\xi)}\,, \tag{6.14}$$

where

$$c_\chi(2\pi|\xi|) := \int\limits_0^\infty \breve{\chi}'(\varrho)\ \cos(2\pi|\xi|\varrho)\,d\varrho$$
$$= -\chi(0) + 2\pi|\xi| \int\limits_0^\infty \breve{\chi}(\varrho)\ \sin(2\pi|\xi|\varrho)\,d\varrho$$
$$= -\chi(0) + |2\pi\xi|^2\sigma_\chi(2\pi|\xi|) = -\chi(0) - |2\pi\xi|^2\hat{P}_{\chi_\Delta}(\xi).$$

Since $\chi \in X^3_{1+}$, condition (3.5) implies

$$c_\chi(2\pi|\xi|) \le 0 \quad \forall \xi \in \mathbb{R}^3. \tag{6.15}$$

and by Lemma 5.1 we get $c_\chi(2\pi|\xi|) = O(|\xi|^{-2})$ as $\xi \to \infty$, and thus

$$0 \le \hat{m}(\xi) \le C < \infty \quad \forall \xi \in \mathbb{R}^3 \tag{6.16}$$

with some positive constant C.

In accordance with equation (6.14) we can represent the pseudodifferential operator $\mathbf{P}^{-1}_{\chi_\Delta}$ in the following form

$$\mathbf{P}^{-1}_{\chi_\Delta} = \frac{1}{\chi(0)}\,\Delta - \mathbf{m}_{\chi_\Delta}, \tag{6.17}$$

where Δ is the Laplace operator in the distributional sense (it corresponds to the symbol $-|2\pi\xi|^2$) and $\mathbf{m}_{\chi_\Delta}$ is a pseudodifferential operator with the symbol $\hat{m}(\xi)$. Evidently, $\mathbf{m}_{\chi_\Delta}$ is a bounded operator from the space $H^0(\mathbb{R}^3)$ into $H^0(\mathbb{R}^3)$ due to (6.16). Inclusions (6.12) imply $\Delta U|_{\Omega^\pm} \in H^0(\Omega^\pm)$.

From (6.11) and Lemma 6.2 we have, $\mathbf{P}^{-1}_{\chi_\Delta} U = 0$ in Ω^-, i.e.,

$$\chi(0)\,\mathbf{P}^{-1}_{\chi_\Delta} U = \Delta\, U - \chi(0)\,\mathbf{m}_{\chi_\Delta} U = 0 \quad \text{in} \qquad \Omega^-. \tag{6.18}$$

By equation (6.11), $U = 0$ in Ω^+, and taking into account the jump properties of the localized volume and double layer potentials, we conclude $T^-_\Delta U = 0$ on $\partial\Omega$. Using Green's identity, we then obtain,

$$0 = \int\limits_{\Omega^-} [\,\chi(0)\,\mathbf{P}^{-1}_{\chi_\Delta} U\,]\, U_l\, dx$$
$$= \int\limits_{\Omega^-} \Delta U\ U_l\, dx - \chi(0) \int\limits_{\Omega^-} [\mathbf{m}_{\chi_\Delta} U]\, U_l\, dx$$
$$= -\int\limits_{\Omega^-} \nabla U\ \cdot\ \nabla U_l\, dx - \chi(0) \int\limits_{\Omega^-} [\mathbf{m}_{\chi_\Delta} U]\, U_l\, dx.$$

Whence, by standard limiting procedure and in view of equations (6.13–6.18), we conclude,

$$-\int_{\Omega^-} \nabla U \cdot \nabla U \, dx - \chi(0) \int_{\Omega^-} [\mathbf{m}_{\chi_\Delta} U] \, U \, dx = 0. \tag{6.19}$$

Evidently, $\mathbf{m}_{\chi_\Delta} U \in H^0(\mathbb{R}^3)$ since $U \in H^0(\mathbb{R}^3)$. Therefore, taking into account that U is supported in $\mathbb{R}^3 \setminus \Omega^+$ and using Plancherel's theorem we can rewrite (6.19) as follows

$$-\int_{\Omega^-} \nabla U \cdot \nabla U \, dx - \chi(0) \int_{\mathbb{R}^3} \hat{m}(\xi) \, | \, \mathcal{F} U \, |^2 \, d\xi = 0. \tag{6.20}$$

By inequalities $\chi(0) > 0$ and (6.16), and inclusion $U \in H^1(\Omega^-)$ we get $U = 0$ in Ω^- and thus, $U = 0$ in $\Omega^{\pm}$. Then the jump relations of the localized boundary potentials (see Theorem 5.13) give, $\chi(0)\, \varphi = U^- - U^+ = 0$ on $\partial\Omega$.

Therefore, $U = \mathcal{P}_{\chi_\Delta} f = 0$ in $\mathbb{R}^3$, and by Lemma 6.2,

$$0 = \mathbf{P}^{-1}_{\chi_\Delta} U = \mathbf{P}^{-1}_{\chi_\Delta} \mathcal{P}_{\chi_\Delta} f = f \quad \text{in} \qquad \Omega^+. \qquad \square$$

7. Proofs of main theorems. The uniqueness and existence results for the Dirichlet and Neumann boundary value problems provided in the following theorem are well known (see, e.g., [**10**]).

Theorem 7.1. *The Dirichlet problem* (4.1 - 4.2) *with* $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ *and* $f \in H^0(\Omega^+)$ *has a unique solution in* $H^{1,0}(\Omega^+; L)$.

The homogeneous Neumann problem (4.7 - 4.8) *admits a constant as a general solution in* $H^{1,0}(\Omega^+; L)$*, while condition* (4.15) *is necessary and sufficient for solvability in* $H^{1,0}(\Omega^+; L)$ *of the nonhomogeneous Neumann problem* (4.7 - 4.8) *with* $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ *and* $f \in H^0(\Omega^+)$.

Proof of Theorem 4.1. Let $u \in H^1(\Omega^+)$ be a solution of the Dirichlet problem (4.1 - 4.2). Then $u \in H^{1,0}(\Omega^+; L)$ since $f \in H^0(\Omega^+)$, and by (3.33 - 3.35) we see that the pair (u, ψ) with $\psi = T^+ u$ solves LBDIEs (D1) as well as LBDIEs (D2), which proves item (i).

Let a pair $(u,\psi) \in H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve LBDIEs (D1). From (4.3) and Theorems 5.6 and 5.10 it follows that $u \in H^{1,0}(\Omega^+; L)$. Taking trace of (4.3) on $\partial\Omega$ and comparing the result with (4.4), we easily derive that $u^+ = \varphi_0$ on $\partial\Omega$. Therefore, subtracting Green's identity (3.33) from (4.3) we obtain

$$\mathcal{P}_\chi(Lu - f) + V_\chi(T^+u - \psi) = 0 \quad \text{in} \qquad \Omega^+. \tag{7.1}$$

Since $\chi \in X^3_+$, Lemma 6.3 implies that $Lu - f = 0$ in Ω^+ and $T^+u - \psi = 0$ on $\partial\Omega$, which completes the proof of item (ii) for LBDIEs (D1).

Now, let a pair (u,ψ) solve (D2). From (4.5) we see that $u \in H^{1,0}(\Omega^+)$ by Theorems 5.6 and 5.10. Taking the co-normal derivative of (4.5) on $\partial\Omega$ and subtracting it from (4.6), we obtain $T^+u = \psi$ on $\partial\Omega$. Further, take the difference of (4.5) and (3.33) to get

$$\mathcal{P}_\chi(Lu - f) - W_\chi(u^+ - \varphi_0) = 0 \quad \text{in} \qquad \Omega^+. \tag{7.2}$$

Whence $Lu = f$ in Ω^+ and $u^+ = \varphi_0$ on $\partial\Omega$ follow from Lemma 6.4 if $\chi \in X^3_{1+}$, completing item (ii) also for LBDIEs (D2).

The claim of item (iii) for the Dirichlet problem is covered by Theorem 7.1. Along with items (i) and (ii) this implies the claim of item (iii) for LBDIEs (D1) and LBDIEs (D2). □

Proof of Theorem 4.2. Theorems 5.6, 5.10, 5.14 and Corollary 5.8 imply continuity of operators (4.13) and (4.14).

Let us introduce the operator

$$\mathcal{A}_0^{D1} := \begin{bmatrix} I & -V_\chi \\ 0 & -\mathcal{V}_1 \end{bmatrix} \;:\; H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega) \tag{7.3}$$

where $\mathcal{V}_1$ is the (non-localized) operator defined by (3.23) with $\chi(x,y) = 1$. The operator

$$\mathcal{A}_\chi^{D1} - \mathcal{A}_0^{D1} \;:\; H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega)$$

is compact due to Lemma 5.11 and Theorem 5.14. Note that the operator

$$\mathcal{V}_1 \;:\; H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

is invertible (see [**3**], Remark 3.7). Therefore, we conclude that the operator (7.3) is invertible too. Thus, operator (4.13) is a Fredholm operator with zero index. It is also injective by Theorem 4.1(iii), implying invertibility of operator (4.13).

Similarly, operator (4.14) is a Fredholm operator with zero index since it is a compact perturbation of the following triangular operator matrix with invertible diagonal operators,

$$\mathcal{A}_0^{D2} := \begin{bmatrix} I & -V_\chi \\ 0 & \frac{1}{2}I \end{bmatrix} : H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega).$$

Then operator (4.14) is invertible since it is injective by Theorem 4.1(iii). □

Proof of Theorem 4.3. Let $u \in H^1(\Omega^+)$ be a solution of the Neumann problem (4.9 - 4.10). Then $u \in H^{1,0}(\Omega^+; L)$ since $f \in H^0(\Omega^+)$, and by (3.33 - 3.35) we see that the pair (u, φ) with $\varphi = u^+$ solves LBDIEs (N2) as well as LBDIEs (N1), which proves item (i).

Let a pair $(u, \varphi) \in H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega)$ solve the LBDIEs (N2). From mapping properties of the operators participating in LBDIEs (4.9) (see Theorems 5.6 and 5.10) it follows that $u \in H^{1,0}(\Omega^+; L)$. Further, taking the trace of (4.9) on $\partial\Omega$ and comparing the result with (4.10), we easily derive that $u^+ = \varphi$ on $\partial\Omega$. Therefore, from Green's identity (3.33) for the function u we have

$$u + \mathcal{R}_\chi u - V_\chi T^+ u + W_\chi \varphi = \mathcal{P}_\chi L u \quad \text{in} \qquad \Omega^+. \tag{7.4}$$

Taking the difference of the equations (4.9) and (7.4) we arrive at the relation

$$\mathcal{P}_\chi(f - Lu) + V_\chi(\psi_0 - T^+ u) = 0 \quad \text{in} \qquad \Omega^+. \tag{7.5}$$

Since $\chi \in X_+^3$, Lemma 6.3 then implies that $Lu = f$ in Ω^+ and $T^+ u = \psi_0$ on $\partial\Omega$, i.e., u solves the Neumann problem (4.9 - 4.10), which completes the proof of item (ii) for LBDIEs (N2).

Now, let a pair $(u, \varphi) \in H^1(\Omega^+) \times H^{\frac{1}{2}}(\partial\Omega)$ solve LBDIEs (N1). Then $u \in H^{1,0}(\Omega^+)$ by Theorems 5.6 and 5.10. Taking the co-normal derivative of (4.11) on $\partial\Omega$ and subtracting it from (4.12), we obtain $T^+u = \psi_0$ on $\partial\Omega$. Further, from (4.11) and (3.33) we derive

$$\mathcal{P}_\chi(Lu - f) - W_\chi(u^+ - \varphi) = 0 \quad \text{in} \qquad \Omega^+.$$

Whence $Lu = f$ in Ω^+ and $u^+ = \varphi$ on $\partial\Omega$ follow by Lemma 6.4 if $\chi \in X^3_{1+}$, completing item (ii) also for LBDIEs (N1).

The claims of points (iii) and (iv) for the Neumann problem is covered by Theorem 7.1. Along with items (i) and (ii) they imply the claims of items (iii) and (iv) for LBDIEs (N2) and LBDIEs (N1). ◻

Proof of Theorem 4.4. Theorems 5.6, 5.10, 5.14 and Corollary 5.8 imply continuity of operators (4.16) and (4.17).

Further consider operator (4.17). Let us denote $\mathcal{L}_1 g = \mathcal{L}^+_{\chi(0)\Delta}(ag) = \mathcal{L}^+_{1\Delta}(ag)$. The operator $\mathcal{L}_1 : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is a Fredholm operator with zero index (cf. e.g. [**5**, Theorem 2], [**6**, Ch. XI, Part B, §3,]). Therefore the operator

$$(7.6)\ \ \mathcal{A}_0^{N1} := \begin{bmatrix} I & W_\chi \\ 0 & \mathcal{L}_1^+ \end{bmatrix} : H^1(\Omega^+)\times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega^+)\times H^{-\frac{1}{2}}(\partial\Omega).$$

is also Fredholm with zero index. Operator (4.17) is a compact perturbation of $\mathcal{A}_0^{N1}$ since the operators

$$\begin{aligned} \mathcal{R}_\chi &: H^1(\Omega) \to H^1(\Omega) \\ \mathcal{L}^+_\chi - \mathcal{L}^+_1 &: H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \\ T^+\mathcal{R}_\chi &: H^1(\Omega^+) \to H^{-\frac{1}{2}}(\partial\Omega) \end{aligned}$$

are compact, due to Corollary 5.8, relation (3.31) and Theorem 5.14. Thus operator (4.17) is Fredholm with zero index. The claims that $\ker \mathcal{A}^{N1}_\chi$ is one–dimensional and the pair $(u, \varphi) = (1, 1)$ belongs to $\ker \mathcal{A}^{N1}_\chi$ directly follow from Theorem 4.3(iii).

The proof for operator (4.16) is similar, cf. also the proof of Theorem 4.2 . ◻

Concluding remarks. Four *segregated* direct localized boundary-domain integral equation systems associated with the Dirichlet and

Neumann problems for a scalar "Laplace" PDE with *variable* coefficient were formulated and analysed in the paper. Mapping and jump properties of surface and volume integral potentials based on a localized parametrix were studied in a scale of Sobolev (Bessel potential) spaces for different smoothness of the localizing function. Equivalence of the LBDIEs to the original variable-coefficient BVPs was proved in the case when right-hand side of the PDE is from $L_2(\Omega^+)$, and the Dirichlet and the Neumann data from the spaces $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$, respectively. The invertibility of the LBDIE operators was proved in the corresponding Sobolev spaces.

The main theorems for LBDIEs (D1) and (N2) were proved under condition $\chi \in X^3_+$ on the localizing function, while for LBDIEs (D2) and (N1) under more restrictive condition $\chi \in X^3_{1+}$. This is an open question whether the latter condition can be relaxed.

By the same approach, the corresponding LBDIE systems for unbounded domains can be analysed as well. The approach can be extended also to more general PDEs and to systems of PDEs, while smoothness of the variable coefficients and the boundary can be essentially relaxed, and the PDE right hand side can be considered in more general spaces, c.f. [**13, 14**].

This study can serve as a basis for rigorous analysis of numerical, especially mesh-less methods for the LBDIEs that after discretization lead to sparsely populated systems of linear algebraic equations attractive for numerical computations (see e.g. [**12, 16**] for algorithm and implementation), but this issue deserves a separate consideration.

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