Proceedings of the 8th UK Conference on Boundary Integral Methods, (edited by D. Lesnic), Leeds University Press, Leeds, UK, ISBN 978 0 85316 2957, (2011) 119-126.

LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR DIRICHLET PROBLEM FOR SECOND ORDER ELLIPTIC EQUATIONS WITH MATRIX VARIABLE COEFFICIENTS

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Abstract. Employing a localized parametrix the Dirichlet boundary value problem for elliptic equations in the divergence form with general variable matrix coefficients is reduced to a *localized boundary-domain integral equation (LBDIE) system.* The equivalence between the Dirichlet problem and the LBDIE system is studied. It is established that the localized boundary domain integral operator obtained in the paper belongs to the Boutet de Monvel algebra and the operator Fredholm properties and invertibility is investigated by the Wiener-Hopf factorization method.

1. INTRODUCTION

We consider the Dirichlet boundary value problem (BVP) for second order elliptic partial differential equations in the divergence form with a general variable matrix of coefficients and develop the boundary-domain integral approach based on the *localized parametrices.* The BVP treated in the paper is well investigated in the literature by the variational and, when the corresponding fundamental solution is available in an explicit form, also by the usual classical potential methods (see, e.g., [6, 7, 8, 11]). Our goal here is to show that solutions of the problem can be represented by *localized potentials* and that the *localized boundary-domain integral operator (LBDIO)* corresponding to the Dirichlet problem is invertible, which is particularly important for constructing and analysis of efficient numerical method for the LBDIE solution. Some numerical algorithms and implementations of the LBDIEs can be found in [9, 10, 13, 14, 15].

In our case, the localized parametrix $P_\chi(x, y)$ is represented as the product of the corresponding Levi function $P_1(y, x-y)$ of the differential operator under consideration by an appropriately chosen cut-off function $\chi(x, y)$ supported on some neighbourhood of the origin. Clearly, the kernels of the corresponding localized potentials are supported in some neighbourhood of the reference point y (assuming that x is an integration variable) and they do not solve the original differential equation, while the localized potentials preserve almost all mapping properties of the usual non-localized ones (cf. [3]).

By the direct approach we reduce the BVP to the *localized boundary-domain integral equations (LBDIE) system.* First we establish the equivalence between the original boundary value problem and the corresponding LBDIEs system which proved to be a quite nontrivial problem and plays a crucial role in our analysis.

Afterwards we establish that the localized boundary domain integral operator obtained belongs to the Boutet de Monvel algebra of pseudodifferential operators [2]. With the help of the Vishik-Eskin theory [4], based on the factorization method (Wiener-Hopf method), we investigate Fredholm properties and prove invertibility of the corresponding localized boundary-domain operator in appropriate function spaces.

2. THE BOUNDARY VALUE PROBLEM LOCALIZED POTENTIALS AND GREEN'S THIRD IDENTITY

Consider a uniformly elliptic second order scalar partial differential operator

$$A(x,\partial_x)\,u=\frac{\partial}{\partial x_k}\Big(a_{kj}(x)\,\frac{\partial u}{\partial x_j}\Big),$$

where $\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial_{x_j}=\partial/\partial x_j$, $a_{kj}\in C^\infty$ and $a_{kj}=a_{jk}$, $j,k=1,2,3$. Moreover, due to the uniform ellipticity, there are positive constants c_1 and c_2 such that

$$c_1\,|\xi|^2\le a_{kj}(x)\,\xi_k\,\xi_j\le c_2\,|\xi|^2\qquad\forall\,x\in\mathbb{R}^3,\ \ \forall\,\xi\in\mathbb{R}^3.$$

Here and in what follows we assume summation from 1 to 3 over repeated indices if not otherwise stated.

Further, let Ω^+ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega=S\in C^\infty$, $\overline{\Omega^+}=\Omega^+\cup S$. Throughout the paper $n=(n_1,n_2,n_3)$ denotes the unit normal vector to S directed outward the domain Ω^+. Set $\Omega^-:=\mathbb{R}^3\setminus\overline{\Omega^+}$.

By $H^r(\Omega)=H^r_2(\Omega)$ and $H^r(S)=H^r_2(S)$, $r\in\mathbb{R}$, we denote the Bessel potential spaces on a domain Ω and on a closed manifold S without boundary, while $\mathcal{D}(\mathbb{R}^3)$ stands for C^∞ functions in $\mathbb{R}^3$ with compact support and $\mathcal{S}(\mathbb{R}^3)$ denotes the Schwartz space of rapidly decreasing functions in $\mathbb{R}^3$. Recall that $H^0(\Omega)=L_2(\Omega)$ is a space of square integrable functions in Ω. Let us denote $u^\pm:=\gamma^\pm u$, where $\gamma\equiv\gamma^+$ and γ^- are the trace operators on $\partial\Omega$ from the interior and exterior of Ω^+ respectively.

We also need the following subspace of $H^1(\Omega)$,

$$H^{1,0}(\Omega;A):=\{u\in H^1(\Omega)\ :\ A(x,\partial)u\in H^0(\Omega)\}\,.$$

The Dirichlet boundary value problem reads as follows: Find a function $u\in H^{1,0}(\Omega^+,A)$ satisfying the differential equation

$$A(x,\partial_x)u=f\quad\text{in}\quad\Omega^+\tag{1}$$

and the boundary condition

$$u^+=\varphi_0\quad\text{on}\quad S,\tag{2}$$

where $\varphi_0\in H^{1/2}(S)$, $f\in H^0(\Omega^+)$.

Equation (1) is understood in the distributional sense, while the Dirichlet boundary condition (2) is understood in the usual trace sense.

Note,that the co-normal derivative is understood in the generalized functional trace sense defined by the following Green's identity for a function $u\in H^{1,0}(\Omega^+;A)$:

$$\langle T^+\,u\,,\,g\rangle_S:=\int_{\Omega^+}A(x,\partial_x)u(x)\,v(x)\,dx+\int_{\Omega^+}a_{kj}(x)\,\partial_{x_j}u(x)\;\partial_{x_k}v(x)\,dx,\tag{3}$$

where $g\in H^{1/2}(S)$ is an arbitrary function and $v\in H^1(\Omega)$ is an extension of g from S onto the whole of Ω^+, i.e., $v^+=g$ on S, while $\langle\cdot\,,\,\cdot\rangle_S$ denotes the duality between the adjoint spaces $H^{-\frac12}(S)$ and $H^{\frac12}(S)$ which extends the usual bilinear $L_2(S)$ inner product.

Let us denote

$$P_1(y,x)=\frac{\alpha}{\big(\mathbf{a}^{-1}(y)\,x\,,\,x\big)^{\frac12}}\qquad\text{with}\quad\alpha=-\frac{1}{4\pi\,[\det\mathbf{a}(y)]^{\frac12}},\ \ \mathbf{a}(y)=[a_{kj}(y)]_{3\times3},$$

where $\mathbf{a}^{-1}$ is the matrix inverse to $\mathbf{a}$ and $(\cdot\,,\,\cdot)$ denotes the usual scalar product in $\mathbb{R}^3$. It is well known (see. e.g., [11]) that $P_1(y, x-y)$ is the Levi function (parametrix) of the operator $A(x,\partial_x)$ and satisfies the equation

$$A(y,\partial_x)P_1(y,x-y) = \delta(x-y),$$

where $\delta(\,\cdot\,)$ is the Dirac distribution.

Let us define the following class of cut-off functions, see [3].

DEFINITION 1 *We say $\chi_0 \in \mathcal{X}^\infty$ if $\chi_0(x) = \breve{\chi}(|x|)$, $\breve{\chi} \in C^\infty[0,\infty)$ and $\varrho^\alpha \breve{\chi}^{(k)}(\varrho) \to 0$ as $\varrho \to 0$ for any real α and any non-negative integer k.*

We say $\chi_0 \in \mathcal{X}^\infty_+$ if $\chi \in \mathcal{X}^\infty$, $\chi(0) = 1$ and $\sigma_{\chi_0}(\omega) > 0$ for all $\omega \in \mathbb{R}$, where

$$\sigma_{\chi_0}(\omega) := \begin{cases} \dfrac{\hat{\chi}_s(\omega)}{\omega} & \textit{for} \quad \omega \in \mathbb{R}\setminus\{0\}, \\ \displaystyle\int_0^\infty \varrho \breve{\chi}\,(\varrho)\, d\varrho & \textit{for} \quad \omega = 0, \end{cases}$$

and $\hat{\chi}_s(\omega)$ denotes the sine-transform of the function $\breve{\chi}$,

$$\hat{\chi}_s(\omega) := \int_0^\infty \breve{\chi}\,(\varrho)\ \sin(\varrho\,\omega)\, d\varrho.$$

Note that $\mathcal{X}^\infty$ and $\mathcal{X}^\infty_+$ are subsets, respectively, of the classes X^k and X^k_+, defined in [3], for any $k \geq 0$.

Since $\mathbf{a}^{-1}$ is symmetric and positive definite, there is a symmetric positive definite matrix $\mathbf{d}$ such that $\mathbf{a}^{-1} = \mathbf{d}^2$ and $(\mathbf{a}^{-1}x\,,\,x) = |\,\mathbf{d}\,x\,|^2$, $\quad \det\mathbf{d} = [\det\mathbf{a}]^{-\frac{1}{2}}$.

Throughout the paper we we will consider functions χ of the form $\chi(y,z) = \chi_0\big(\mathbf{d}(y)z\big)$, where $\chi_0 \in \mathcal{X}^\infty$ if not otherwise stated.

Introduce the localized parametrix $P(y,x-y)$,

$$P(y,x-y) = P_\chi(y,x-y) = \chi(y,x-y)\,P_1(y,x-y).$$

It is easy to check that $A(x,\partial_x)\,P(y,x-y) = \delta(x-y) + R(x,y)$, where

$$R(x,y) = R_\chi(x,y) = P_1(y,x-y)\,A(y,\partial_x)\chi(y,x-y) + 2\,a_{kj}(y)\,\frac{\partial\chi(y,x-y)}{\partial x_j}\,\frac{\partial \mathcal{P}_1(y,x-y)}{\partial x_k}$$

$$+\big[a_{kj}(x) - a_{kj}(y)\big]\frac{\partial^2\big[\chi(y,x-y)P_1(y,x-y)\big]}{\partial x_k \partial x_j} + \frac{\partial a_{kj}(x)}{\partial x_k}\,\frac{\partial\big[\chi(y,x-y)P_1(y,x-y)\big]}{\partial x_j}\,.$$

The function $R(x,y)$ possesses a weak singularity of type

$$R(x,y) = \mathcal{O}(|x-y|^{-2}) \quad \text{as} \quad x \to y.$$

Let us introduce the localized surface and volume potentials, based on the above defined localized parametrix P,

$$V\,g(y) := -\int_S P(y,x-y)\,g(x)\,dS_x, \quad W\,g(y) := -\int_S \big[\,T(x,\partial_x)\,P(y,x-y)\,\big]\ g(x)\,dS_x, \tag{4}$$

$$\mathcal{P}\,h(y) := \int_{\Omega^+} P(y,x-y)\,h(x)\,dx, \quad \mathcal{R}\,h(y) := \int_{\Omega^+} R(y,x-y)\,h(x)\,dx. \tag{5}$$

If the domain of integration in the Newtonian volume potential is the whole space $\mathbb{R}^3$, we employ the notation $\mathbf{P}\,h := \mathcal{P}\,h$.

Now we recall Green's second identity for the operator $A(x, \partial_x)$,

$$\int_{\Omega^+} \big[v\, A(x,\partial_x)u - u\, A(x,\partial_x)v\big]\, dx = \langle T^+u, v^+\rangle_S - \langle T^+v, u^+\rangle_S. \tag{6}$$

where $u, v \in H^{1,0}(\Omega^+, A)$.

Taking $v(x) = P(y, x-y)$, from (6) by standard limiting arguments we arrive at the following localized third Green's identity for $u \in H^{1,0}(\Omega^+, A)$,

$$u + \mathcal{R}u - V(T^+u) + W(u^+) = \mathcal{P}(Au) \quad \text{in} \quad \Omega^+. \tag{7}$$

3. LBDIE FORMULATION FOR THE DIRICHLET PROBLEM

To derive the equivalent LBDIE formulation of the Dirichlet problem we need some auxiliary material. Let us denote by E_0 the operator of extension of a function, defined in Ω^+, by zero outside Ω^+ to a function defined in $\mathbb{R}^3$. Therefore, for $f \in H^0(\Omega^+)$ we can rewrite the volume potential $\mathcal{P}f$ as a pseudodifferential operator

$$\begin{aligned}\mathcal{P}f(y) &= \int_{\Omega^+} P(y, x-y)\, f(x)\, dx = \int_{\mathbb{R}^3} P(y, x-y)\, E_0 f(x)\, dx \\ &= \mathbf{P}(\,E_0 f\,)(y) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathfrak{S}(\mathbf{P})(y,\xi)\, e^{-i(y,\xi)}\mathcal{F}(E_0 f\,)(\xi)\, d\xi,\end{aligned}$$

where the symbol $\mathcal{F}$ stands for the generalized Fourier transform and $\mathfrak{S}(\mathbf{P})(y,\xi) := \mathcal{F}_{z\to\xi}\big[P(y,z)\big]$ is the complete symbol of the operator $\mathbf{P}$.

One can prove the following lemma.

LEMMA 2 *Let $\chi_0 \in \mathcal{X}^\infty$. Then $\mathfrak{S}(\mathbf{P}) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, and the following expansion holds for $\xi \neq 0$ and any integer $k \geq 0$,*

$$\begin{aligned}\mathfrak{S}(\mathbf{P})(y,\xi) &= \sum_{m=0}^{k^*} \frac{(-1)^{m+1}}{(\mathbf{a}(y)\xi\,,\,\xi)^{m+1}}\; \breve{\chi}_0^{(2m)}(0) \\ &\quad -\frac{1}{(\mathbf{a}(y)\xi\,,\,\xi)^{(k+1)/2}} \int_0^\infty \sin\Big((\mathbf{a}(y)\xi\,,\,\xi)^{1/2}\varrho + \frac{k\pi}{2}\Big)\breve{\chi}_0^{(k)}(\varrho)\, d\varrho\,,\end{aligned} \tag{8}$$

where k^* is the integer part of $(k-1)/2$ and the sum disappears in (8) if $k^* < 0$, i.e., if $k = 0$. For any multi-indices α, β there are positive constants $c_{\alpha\beta}$ independent of ξ and y such that

$$|\partial_\xi^\alpha \partial_y^\beta \mathfrak{S}(\mathbf{P})(y,\xi)| < c_{\alpha\beta}(1 + |\xi|^2)^{-1-|\alpha|/2}$$

for any $\xi \in \mathbb{R}^3$, $y \in \mathbb{R}^3$.

Moreover, if $\chi_0 \in \mathcal{X}_+^\infty$, then there is a positive constant c independents of ξ and y such that

$$c(1 + |\xi|^2)^{-1} < -\mathfrak{S}(\mathbf{P})(y,\xi)$$

for any $\xi \in \mathbb{R}^3$, $y \in \mathbb{R}^3$.

We have the following corollary which follows from the above Lemma 2 and Theorem 18.1.13 in [5].

COROLLARY 3 *Let $\chi_0 \in \mathcal{X}^\infty$. Then $\mathfrak{S}(\mathbf{P})$ belongs to the Hörmander class of symbols $S^{-2}(\mathbb{R}^3 \times \mathbb{R}^3)$ and the operator $\mathbf{P} : H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3)$ is continuous for all $s \in \mathbb{R}$.*

First we introduce the notation $A_\infty(\xi) := -|\xi|^2$ for the symbol of the Laplace operator and take into account that $A(y,\xi) := -a_{kj}(y)\,\xi_k\,\xi_j$. Further, let

$$A'(y,\xi) := A(y,\xi) - A_\infty(\xi) = -a_{kj}(y)\,\xi_k\,\xi_j + |\xi|^2,.$$

Let us denote $P_\infty(z) = -\chi_0(z)/4\pi|z|$ the localized parametrix of the Laplace operator, while $P'(y,x-y) := P(y,x-y) - P_\infty(x-y)$. Introducing the corresponding pseudodifferential operators

$$\mathbf{P}_\infty h(y) := \int_{\mathbb{R}^3} P_\infty(x-y)\,h(x)\,dx, \quad \mathbf{P}'h(y) := \int_{\mathbb{R}^3} P'(y,x-y)\,h(x)\,dx,$$

we arrive at the decomposition $\mathbf{P} = \mathbf{P}' + \mathbf{P}_\infty$. For the corresponding symbols we then obtain, $\mathfrak{S}(\mathbf{P}_\infty)(\xi) := \mathcal{F}_{z\to\xi}\big[P_\infty(z)\big] = \mathfrak{S}(\mathbf{P})(y,\xi)|_{a_{ij}\equiv\delta_{ij}}$, $\mathfrak{S}(\mathbf{P}')(y,\xi) := \mathfrak{S}(\mathbf{P})(y,\xi) - \mathfrak{S}(\mathbf{P}_\infty)(\xi)$.

From Lemma 2 we have the following decomposition

$$\mathfrak{S}(\mathbf{P})(y,\xi) = \frac{1}{A(y,\xi)} + \mathcal{O}(|\xi|^{-4}),$$

and therefore the corresponding principal homogeneous symbols of the operators $\mathbf{P}, \mathbf{P}'$ and $\mathbf{P}_\infty$, read as

$$\overset{\circ}{\mathfrak{S}}(\mathbf{P})(y,\xi) = \frac{1}{A(y,\xi)} = -\frac{1}{a_{kj}(y)\,\xi_k\,\xi_j}\,, \quad \overset{\circ}{\mathfrak{S}}(\mathbf{P}_\infty)(\xi) = \frac{1}{A_\infty(\xi)} = -\frac{1}{|\xi|^2}\,,$$
$$\overset{\circ}{\mathfrak{S}}(\mathbf{P}')(y,\xi) = \frac{1}{A(y,\xi)} - \frac{1}{A_\infty(\xi)} = \frac{A_\infty(\xi) - A(y,\xi)}{A(y,\xi)A_\infty(\xi)} = -\frac{A'(y,\xi)}{A(y,\xi)A_\infty(\xi)}\,.$$

It is easy to see that all the above principal homogeneous symbols are rational functions in ξ and satisfy the so called transmission conditions. Consequently, the operators $\mathbf{P}, \mathbf{P}_\infty$ and $\mathbf{P}'$ are pseudodifferential operators of rational type, see [2, 4, 12].

Note that if $\chi_0 \in X^1_+$, from Lemma 2 it then follows that the operator $\mathbf{P}_\infty : H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3)$, $s \in \mathbb{R}$, is invertible.

Similar decompositions can be written also for the layer potentials

$$V\psi = -\mathbf{P}(\gamma^*\psi) = -\mathbf{P}'(\gamma^*\psi) - \mathbf{P}_\infty(\gamma^*\psi) = V'\psi + V_\infty\psi\,,$$
$$W\varphi = -\mathbf{P}(T^*\varphi) = -\mathbf{P}'(T^*\varphi) - \mathbf{P}_\infty(T^*\varphi) = W'\varphi + W_\infty\varphi\,,$$

where

$$V'\psi := -\mathbf{P}'(\gamma^*\psi), \quad V_\infty\psi = -\mathbf{P}_\infty(\gamma^*\psi), \quad W'\varphi = -\mathbf{P}'(T^*\varphi), \quad W_\infty\varphi = -\mathbf{P}_\infty(T^*\varphi) \tag{9}$$

and their explicit forms are given by expressions (4) after replacing there the kernel P with P' and P_∞, respectively. Here the operator $\gamma^* : H^{\frac{1}{2}-t}(S) \to H^{-t}(\mathbb{R}^3)$, $t > 1/2$ is adjoint to the trace operator $\gamma : H^t(\mathbb{R}^3) \to H^{t-\frac{1}{2}}(S)$, $t > 1/2$, i.e., is defined by the relation

$$\langle \gamma^*\psi\,,\, h\rangle_{\mathbb{R}^3} := \langle \psi\,,\, \gamma h\rangle_S \quad \text{for all} \quad h \in H^t(\mathbb{R}^3), \quad \psi \in H^{\frac{1}{2}-t}(S), \quad t > \frac{1}{2}. \tag{10}$$

Similarly, $T^* : H^{\frac{3}{2}-t}(S) \to H^{-t}(\mathbb{R}^3)$, $t > \frac{3}{2}$ is the operator adjoint to the classical (defined in terms of the trace) co-normal derivative operator $T = a_{kj}\,n_k(x)\,\gamma\partial_{x_j} : H^t(\mathbb{R}^3) \to H^{t-\frac{3}{2}}(S)$, that is continuous for $t > \frac{3}{2}$ (for the infinitely smooth S), i.e.,

$$\langle T^*\varphi\,,\, h\rangle_{\mathbb{R}^3} := \langle \varphi\,,\, Th\rangle_S \;\text{ for any }\; h \in H^t(\mathbb{R}^3), \quad \varphi \in H^{\frac{3}{2}-t}(S), \quad t > \frac{3}{2}. \tag{11}$$

Now we can rewrite (7) in the form

$$u + \mathcal{R}\,u - V\,'(T^+u) - V_\infty(T^+u) - \mathbf{P}\,'(E_0Au) = \mathbf{P}_\infty(E_0Au) - W(u^+) \quad \text{in} \quad \Omega^+. \tag{12}$$

Further, for $u \in H^{1,0}(\Omega^+, A)$ and $v \in \mathcal{D}(\mathbb{R}^3)$ we can rewrite Green's second identity in Ω^+ as

$$\int_{\mathbb{R}^3} \big[v\,E_0Au - (E_0u)Av\big]dx = \langle T^+u, \gamma v\rangle_S - \langle Tv, \gamma^+u\rangle_S,$$

which, in turn, in view of the notation (10)-(11) can be written in the form

$$\langle E_0Au\,,\,v\rangle_{\mathbb{R}^3} - \langle AE_0u\,,\,v\rangle_{\mathbb{R}^3} = \langle \gamma^*T^+u, v\rangle_{\mathbb{R}^3} - \langle T^*\gamma^+u,\,v\rangle_{\mathbb{R}^3}, \quad \forall\, v \in \mathcal{D}(\mathbb{R}^3).$$

Whence we get $AE_0u = E_0Au - \gamma^*T^+u + T^*\gamma^+u$.

Therefore in view of (9) we get

$$\mathbf{P}\,'(AE_0u) = \mathbf{P}\,'(E_0Au) + V\,'(T^+u) - W\,'(u^+) \quad \text{in} \quad \mathbb{R}^3 \tag{13}$$

which implies

$$\begin{aligned}
\{\mathbf{P}\,'(E_0Au)\}^+ + \mathcal{V}\,'(T^+u) &= \{\mathbf{P}\,'(AE_0u)\}^+ + \{W\,'(u^+)\}^+ \\
&= \{\mathbf{P}\,'(AE_0u)\}^+ + \{W(u^+)\}^+ - \{W_\infty(u^+)\}^+ \\
&= \{\mathbf{P}\,'(AE_0u)\}^+ - \frac{1}{2}u^+ + \mathcal{W}(u^+) + \frac{1}{2}\mu\,u^+ - \mathcal{W}_\infty(u^+) \\
&= \{\mathbf{P}\,'(AE_0u)\}^+ - \frac{1}{2}(1-\mu)u^+ + \mathcal{W}(u^+) - \mathcal{W}_\infty(u^+),
\end{aligned} \tag{14}$$

where $\mu = a_{kj}\,n_k\,n_j$.

Due to (13), third Green's identity (12) can be also rewritten as

$$u + \mathcal{R}\,u - \mathbf{P}\,'A(E_0u) - V_\infty(T^+u) = \mathbf{P}_\infty(E_0Au) - W_\infty(u^+) \quad \text{in} \quad \Omega^+. \tag{15}$$

Further we have,

$$\begin{aligned}
\mathbf{P}\,'(AE_0u)(y) &= \langle P'(y,\cdot - y), AE_0u\rangle_{\mathbb{R}^3} = \langle AP'(y,\cdot - y), E_0u\rangle_{\mathbb{R}^3} = \langle AP'(y,\cdot - y), u\rangle_{\Omega^+} \\
&= \langle AP(y,\cdot - y) - AP_\infty(\cdot - y), u\rangle_{\Omega^+} = u(y) + \mathcal{R}u(y) - \langle AP_\infty(\cdot - y), u\rangle_{\Omega^+} \\
&= u(y) + \mathcal{R}u(y) - \beta(y)u(y) - \mathcal{N}_\infty u(y),
\end{aligned} \tag{16}$$

where

$$\beta(y) := \frac{1}{3}\,[\,a_{11}(y) + a_{22}(y) + a_{33}(y)\,], \quad \mathcal{N}_\infty u(y) = \text{v.p.}\int_{\Omega^+} N_\infty(x,y)u(x)dx,$$

$$N_\infty(x,y) := A(x,\partial_x)P_\infty(x-y) = \Big[-\frac{a_{kj}(x)}{4\,\pi}\frac{\partial^2}{\partial x_k\,\partial x_j}\frac{1}{|x-y|}\Big] + R_\infty(x,y), \quad x \neq y,$$

$$\begin{aligned}
R_\infty(x,y) := -\frac{1}{4\,\pi}\Big\{\frac{\partial}{\partial x_k}\Big[\frac{\partial\chi(x-y)}{\partial x_j}\,\frac{a_{kj}(x)}{|x-y|}\Big] + \frac{\partial\big[a_{kj}(x)\,\chi(x-y)\big]}{\partial x_k}\,\frac{\partial}{\partial x_j}\frac{1}{|x-y|} \\
+ a_{kj}(x)\,\big[\chi(x-y) - 1\big]\,\frac{\partial^2}{\partial x_k\partial x_j}\frac{1}{|x-y|}\Big\}, \quad x \neq y,
\end{aligned}$$

Substituting (16) in (15), we arrive at the equivalent form of the third Green identity that does not involve the parametrix P (cf. the two-operator Green identities in [1]),

$$\beta u + \mathcal{N}_\infty\,u - V_\infty(T^+u) + W_\infty(\gamma^+u) = \mathcal{P}_\infty(Au) \quad \text{in} \quad \Omega^+. \tag{17}$$

Taking trace of (17) on S and using the jump properties of the layer potentials (cf. [3]), we get

$$\left(\beta - \frac{1}{2}\mu\right) u^+ + \gamma^+\mathcal{N}_\infty u - \mathcal{V}_\infty(T^+u) + \mathcal{W}_\infty(\gamma^+u) = \gamma^+\mathcal{P}_\infty(Au) \quad \text{on} \quad S, \tag{18}$$

where $\mathcal{V}_\infty g$ and $\mathcal{W}_\infty g$ are the direct values of the corresponding potentials $V_\infty g$ and $W_\infty g$ on the boundary S.

Denoting $\psi = T^+u$ and substituting equations (1) and (2) in (17) and (18), we obtain the LBDIE system with respect to the unknowns $u \in H^{1,0}(\Omega^+, A)$ and $\psi \in H^{-1/2}(S)$ considered as independent of each other (i.e. segregated),

$$\beta u + \mathcal{N}_\infty u - V_\infty\psi = \mathcal{P}_\infty f - W_\infty\varphi_0 \quad \text{in} \quad \Omega^+, \tag{19}$$

$$\gamma^+\mathcal{N}_\infty u - \mathcal{V}_\infty\psi = \gamma^+\mathcal{P}_\infty(f) - (\beta - \frac{1}{2}\mu)\,\varphi_0 - \mathcal{W}_\infty\varphi_0 \quad \text{on} \quad S, \tag{20}$$

where $\varphi_0 \in H^{1/2}(S)$ and $f \in H^0(\Omega^+)$.

4. FORMULATION OF THE BASIC RESULTS

First, the following equivalence theorem can be proved.

THEOREM 4 *Let* $\chi_0 \in \mathcal{X}_+^\infty$.

(i) If a function $u \in H^{1,0}(\Omega^+, A)$ *solves the Dirichlet BVP* (1)-(2)*, then it is unique and the pair* $(u, \psi) \in H^{1,0}(\Omega^+, A) \times H^{-1/2}(S)$ *with*

$$\psi = T^+u \tag{21}$$

solves the LBDIE system (19)-(20)*.*

(ii) Vice versa, if a pair $(u, \psi) \in H^{1,0}(\Omega^+, A) \times H^{-1/2}(S)$ *solves the LBDIE system* (19)-(20)*, then it is unique and the function* u *solves the Dirichlet BVP* (1)-(2) *and the equality* (21) *holds.*

From Theorem 4 it follows that the LBDIE system (19)-(20), which has a special right hand side, is uniquely solvable in the class $H^{1,0}(\Omega^+, A) \times H^{-1/2}(S)$.

Now we investigate the LBDIO generated by the left hand side expressions in (19)-(20) in appropriate function spaces. The LBDIE system (19)-(20) with an arbitrary right hand side functions from the space $H^1(\Omega^+) \times H^{1/2}(S)$ can be written as

$$(\beta\, I + \mathcal{N}_\infty)u - V_\infty\psi = F_1 \quad \text{in} \quad \Omega^+, \tag{22}$$

$$\gamma^+\mathcal{N}_\infty u - \mathcal{V}_\infty\psi = F_2 \quad \text{on} \quad S, \tag{23}$$

where I stands for the identity operator and $F_1 \in H^1(\Omega^+)$, $F_2 \in H^{1/2}(S)$.

Denote by $\mathcal{A}$ the LBDIO generated by the left hand side expressions in LBDIE system (22)-(23) as

$$\mathcal{A} := \begin{bmatrix} r_{\Omega^+}(\beta\, I + \mathcal{N}_\infty) & -r_{\Omega^+}V_\infty \\ \gamma^+\mathcal{N}_\infty & -\mathcal{V}_\infty \end{bmatrix}.$$

The following assertion holds.

THEOREM 5 *Let* $\chi_0 \in \mathcal{X}_+^\infty$*. Then the operator*

$$\mathcal{A}\ : H^{r+1}(\Omega^+) \times H^{r-1/2}(S) \to H^{r+1}(\Omega^+) \times H^{r+1/2}(S)$$

is continuous and continuously invertible for any $r \geq 0$*.*

Acknowledgements

This research was supported by the EPSRC grant No EP/H020497/1:"Mathematical Analysis of Localized Boundary-Domain Integral Equations for Variable-Coefficients Boundary Value Problems".

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