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# Application of a Non-local Failure Criterion to a Crack in Heterogeneous Media

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## Abstract

A plane problem is considered for an infinite elastic medium with a circular elastic inclusion and a crack reaching the interface. The Boundary Element Method implemented in BEASY software package is used to the stress field calculation. A modification of an average stress non-local fracture criterion [i,ii,iii], is applied in this work to predict the crack propagation and evaluate the possibility of its further reflection, refraction or interface delamination. The two criterion parameters are obtained from classical tensile test and standard fracture test results. A numerical integration of the singular stress field needed for implementation of the criterion is performed using the weighted trapezoidal rule. Several examples are examined for various crack lengths. Applications to some structural materials, particularly, to reinforced concrete are discussed.

## 1. Introduction

In the classical approach, the failure of a body at a point is characterised by some function of the stress tensor evaluated only at the investigated point. This kind of approach give good description of experimental data when the stresses do not present high gradients on dimensions of the material structure scale. However when the stresses vary abruptly on such dimension, *size effects* are observed. In fact the body survive higher stresses under these conditions than under the corresponding (accounting the stress concentration) uniform ones.

In a cracked body, principles of Linear Elastic Fracture Mechanics (L.E.F.M.) could be applied when cracks are sufficiently large. However when the crack length becomes comparable with one of the material structure scale, L.E.F.M. do not describe properly the experimental data and in the limit when the crack length tend to zero the strength predicted by L.E.F.M. becomes infinite. Moreover when the singularity of the stresses at the crack tip differs from the inverse of the square root, simple Stress Intensity Factor based failure criteria are not valid.

Non-local failure criteria overcome these difficulties and give a more general approach to the strength evaluation. In fact they are applicable to homogeneous or heterogeneous bodies, with or without cracks and with or without singular or smooth concentrators. Such criteria uses stresses and strain not only at the investigated point but also in some neighbourhood.

There are several non-local fracture criteria in literature. One of them, based on the average stress over a fixed interval, and its generalisation on bonded materials is considered in this work .

# 2. Non-local failure criteria

## 2.1 Safety and Loading factors

Let us consider a body $\Omega$ having a stress field $\sigma_{ij}(x)$ induced by some mass and boundary loading. To predict the strength at a point *y* under such loading, let suppose $\underline{\lambda}(\sigma_{ij})>0$ is the *safety factor*[ii,iv], that is a minimal positive number $\lambda'$ such as the stress field $\sigma_{ij}^*(x):=\lambda'\sigma_{ij}(x)$ is the fracturing one; if $\lambda'\sigma_{ij}$ is not fracturing at any positive $\lambda'$, then $\underline{\lambda}(\sigma_{ij})$ is equal to infinity. It is evident, that if

$$\underline{\lambda}\left(\sigma_{ij}\right) > 1, \qquad (1)$$

then the stress $\sigma_{ij}$ is not fracturing and vice versa. Therefore inequality (1) is a strength condition. The previous inequality can be rewritten in the form

$$\underline{\Lambda}\left(\sigma_{ij}\right) < 1 \qquad (2)$$

where $\underline{\Lambda}$ is defined by the formula.

$$\underline{\Lambda}\left(\sigma_{ij}\right) := \frac{1}{\underline{\lambda}\left(\sigma_{ij}\right)} \qquad (3)$$

The function $\underline{\Lambda}$ is the so called *loading factor* if the condition stated in (2) is satisfied or *overloading factor* if not. Generally each material has each own strength condition. The strength condition for whole body $\Omega$ take the form

$$\Lambda\left(\sigma_{ij}, \Omega\right) = sup_{\mathbf{y}\in\Omega}\, \Lambda\left(\sigma_{ij}, \mathbf{y}\right) < 1 \qquad (4)$$

## 2.2 Average Stress Criterion

One of the popular non-local failure criteria for two-dimensional problems is the Average Stress Criterion. This criterion was considered by Neuber [i], Novozhilov, and other authors. Some generalisations were presented by Mikhailov [ii].

Let $(\rho,\theta)$ be a local coordinate system with the centre at an investigated point $y$ of the body; let $\eta(\theta)$ be a unit vector making an angle $\theta$ with the coordinate axis; and let $\sigma_{\rho\rho}$, $\sigma_{\theta\theta}$, $\sigma_{\rho\theta}$ be the stress component in this coordinate system. The (over)loading factor for the average stress strength condition has the form [2],

$$\begin{aligned} \underline{\Lambda}\left(\sigma_{ij}\right) &:= \max\left[0, \underline{\Lambda}_{+}\left(\sigma_{ij}\right)\right], \\ \underline{\Lambda}_{+}\left(\sigma_{ij}\right) &:= \frac{1}{d_1 \cdot \sigma_c} \max_{-\pi<\vartheta<\pi} \int_0^{d_1} \sigma_{\theta\theta}\left(y + \rho\cdot\eta(\theta)\right) d\rho \end{aligned} \qquad (5)$$

where $\sigma_c$ is the ultimate tensile stress and the characteristic length $d_1$ is a material constant.

Condition (2), (5) can not be used when the integration path intersects an interface between two bonded materials. To extend the condition to such cases we present the overloading factor in the following modified form:

$$\underline{\Lambda}_{+}\left(\sigma_{ij}\right) := \max_{-\pi<\theta<\pi} \sum_{i=1}^{n} \frac{1}{d_1^{(i)} \cdot \sigma_c^{(i)}} \int_{l^{(i)}} \sigma_{\theta\theta}\left(y + \rho\cdot\eta(\theta)\right) d\rho \qquad (6)$$

where $n$ is the number of materials through that the integration path goes. The geometric parameters $l^{(i)}$ are the interval length between two intersection. Particularly $l^{(1)}$ is the distance in the considered direction between the investigated point and the first intersection found, $l^{(n)}$ is evaluated from the equation:

$$\sum_{i=1}^{n} \frac{l^{(i)}}{d_1^{(i)}} = 1 \qquad (7)$$

### 2.3 Material constants

The Average stress criterion depends on two material parameters. The ultimate tensile stress has to be determined experimentally by the classical tensile test. After this, the $d_1$ parameter can be determined by a standard fracture test performed on a specimen with a crack large in comparison with the material structure dimension. In fact, for large cracks both L.E.F.M. and non-local criteria predict properly the failure. Applying singular stress asymptotic at the crack tip for the critical state, we obtain that the circumferential stress $\sigma_{\theta\theta}$ reaches its maximum at the crack prolongation direction $\theta$=0, and for this direction we have,

$$\sigma_{\theta\theta}(\rho,\theta)\Big|_{\theta=0} = \frac{K_{Ic}}{\sqrt{2\pi \cdot \rho}}. \qquad (8)$$

Substituting (8) in (2), (5) and performing the integration, we achieve the following formula

$$d_1 = \frac{2}{\pi} \cdot \left( \frac{K_{Ic}}{\sigma_c} \right)^2. \qquad (9)$$

Expression (8) shows that the parameter $d_1$ increases with the toughness and decreases with the ultimate tensile stress incrise.

## 3. Stress field evaluation

The BEASY software package, that implements the Boundary Element Method, is used in these work to evaluate the stress field.

### 3.1 Numerical integration

It is known (see, e.g., Fenner[v] and references there) that the stress singularities

$$\sigma_{\theta\theta}\left(\rho,\theta\right) \sim \rho^{-\gamma}, \quad 0<\gamma<1, \tag{10}$$

occur at the crack tip reaching an interface. The numerical integration of the circumferential stresses that appear in the Average Stress Criterion are performed in this case using weighted trapezoidal rules. It means we use the approximation

$$\sigma_{\theta\theta}(\rho) \cong w(\rho)\cdot\left[\frac{\dfrac{\sigma_{\theta\theta}(\rho_i)}{w(\rho_i)}\left(\rho_{i+1}-\rho\right)+\dfrac{\sigma_{\theta\theta}(\rho_{i+1})}{w(\rho_{i+1})}\left(\rho-\rho_i\right)}{\left(\rho_{i+1}-\rho_i\right)}\right] \tag{11}$$

in a discretization subinterval $\rho_i<\rho<\rho_{i+1}$, and the weighted function is defined by the following formula

$$w(\rho)=\rho^{-\gamma}. \tag{12}$$

The integral is calculated by dividing the interval (0, $d_1$) on $n$ subintervals ($\rho_i$ , $\rho_{i+1}$), $i$=0...$n$-1, such that $\rho_0$=0, $\rho_n$=$d_1$, and summing the contribution from each subinterval,

$$\begin{aligned}\int_o^{d_1}\sigma_{\theta\theta}(\rho)d\rho \cong {} & \frac{\sigma_{\theta\theta}(\rho_1)}{w(\rho_1)}\left(\rho_2\cdot J_0^0-J_0^1\right)+\frac{\sigma_{\theta\theta}(\rho_2)}{w(\rho_2)}\left(J_0^1-\rho_1\cdot J_0^0\right)+ \\ & \sum_{i=1}^{n-1}\left[\frac{\sigma_{\theta\theta}(\rho_i)}{w(\rho_i)}\left(\rho_{i+1}\cdot J_i^0-J_i^1\right)+\frac{\sigma_{\theta\theta}(\rho_{i+1})}{w(\rho_{i+1})}\left(J_i^1-\rho_i\cdot J_i^0\right)\right],\end{aligned} \tag{13}$$

where

$$J_i^0=\frac{\rho_{i+1}^{1-\gamma}-\rho_i^{1-\gamma}}{(1-\gamma)(\rho_{i+1}-\rho_i)}\ , \quad J_i^1=\frac{\rho_{i+1}^{2-\gamma}-\rho_i^{2-\gamma}}{(2-\gamma)(\rho_{i+1}-\rho_i)} \tag{14}$$

The integration along the first subinterval including singularity is performed using (11) as extrapolation formula for $\rho\in(0,\rho_1)$ and $i$=1.

# 4. Algorithm implementation

In order to apply efficiently non-local criterion to two dimensional problem, some FORTRAN routines have been written. They interact with BEASY using some peculiar features of the Boundary Element technique. The flow chart of the algorithm used to calculate stress field and the (over)loading factors is shown in Figure 1 .

In a first step the user builds a boundary element model implementing the BEASY Interactive Modelling System and solves the main algebraic system in order to obtain the solution at the boundary. After this stage, a FORTRAN routine generates some integration points, used in (13), in a data file. They are placed on a radial path around the investigated points specified by the user and the appropriate material zone membership is assigned.

The material proprieties and the integration interval are written in an intermediate file. In the next step, BEASY runs in a restart mode in order to calculate the stress tensor only at the integration points without solving the system of linear algebraic equations. Then another FORTRAN routine reads the results and integrates them in order to obtain the (over)loading factor by (5)-(7) for each investigated point.

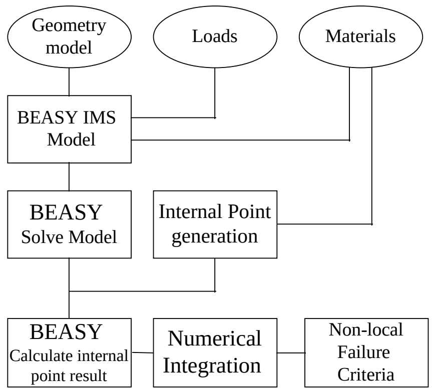

Figure 1: Algorithm implementation

A geometry of the model is written explicitly in a subroutine of the code. In this work, a plate with a circular inclusion has been analysed but the user can easily modify this geometry updating only the last subroutine.

# 5. Results

## 5.1 Plate with circular inclusion and crack

A reinforced concrete structure is considered as an example of heterogeneous media but the same consideration could be done also for other composites.

A plane strain problem for a concrete rectilinear plate with a circular steel inclusion is considered. The geometry and the boundary condition of the boundary element model is shown on Figure 2 were *r* and *L* have been taken respectively 1 cm and 15 m. The parameter 2*a* represents the crack length and varies from 0.065 cm to 0.39 cm.

The proprieties of the two materials are listed in Table 1 where $d_1$ was evaluated by (9).

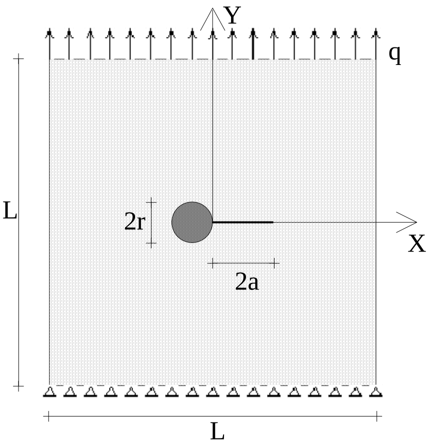

Figure 1 - Geometric Model and Boundary condition

If we take the Poisson ratios of the both material equal to 0.3, then the stress singularity is $\gamma \approx 0.3$ for the plane strain case according to Fenner [5].

The plate is loaded by a uniform tensile traction, which magnitude is equal to the fracturing one

$$q = \frac{K_{IC}}{\sqrt{\pi \cdot a}} = 2.775\,\text{MPa} \qquad (15)$$

| **Material Proprieties** | **Concrete** | **Steel** |
|---|---|---|
| Young Modulus E (MPa) | 34,300 | 210,000 |
| Ultimate tensile stress $\sigma_c$ (MPa) | 5.3 | 650 |
| Toughness $K_{Ic}\left(\text{MPa}\cdot\sqrt{\text{m}}\right)$ | 1.7 | 54 |
| $d_1$ (m) | 0.065 | 0.00439 |

Table 1 - Material Proprieties

for the infinite concrete plate without inclusion and with the crack of the length 2*a* = 0.39 cm.

The location of the internal points is shown in Figure 3. The integration path at $\theta \in [90^\circ, 92^\circ]$ is split according to (6), (7) into two intervals since the radial path intersects the interface in this case. (Angles $\theta < 90^\circ$ correspond to concrete, $\theta > 90^\circ$ correspond to inclusion.)

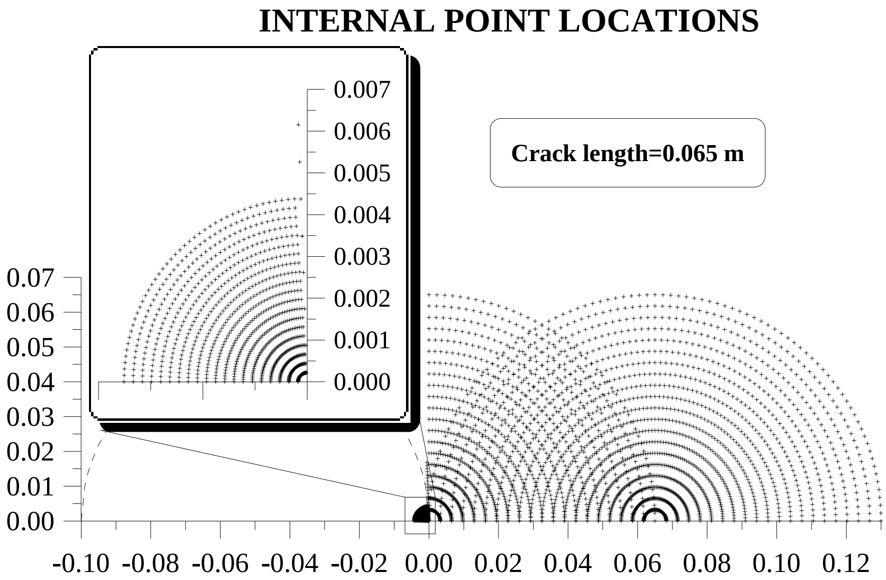


Figure 3 - Internal Point Locations

The resulting loading factors for different crack lengths are presented in Figures 4 and 5 respectively for the crack tips at the interface and far from it. The values of $\underline{\Lambda}_{+}(\sigma)<0$ (shadow zone) do not describe the strength since $\underline{\Lambda}(\sigma)=\max[0, \underline{\Lambda}_{+}(\sigma)]$ (see (5)). The results are not available in the case when the integration paths in (5), (6) lay near the interface, i.e., for angles $\theta$ between $80^\circ$ and $112^\circ$ (dashed zone), since numerical instabilities occur during the stress evaluation in internal points near the interface. The loading factor maximum for the crack tip at the interface is reached in inclusion at $180^\circ$, excluding the case of short crack (of the order of $d_1$). In the last case the integration include the field of the far crack tip and the maximum appears at small angles.

For the crack tip far from the interface the maximum is reached at zero angle. The maximum overloading factor is much greater then for the crack tip near the inclusion but is less the unity. Because of the choice of the load, this means that the steel inclusion increases the strength at the both crack tips.

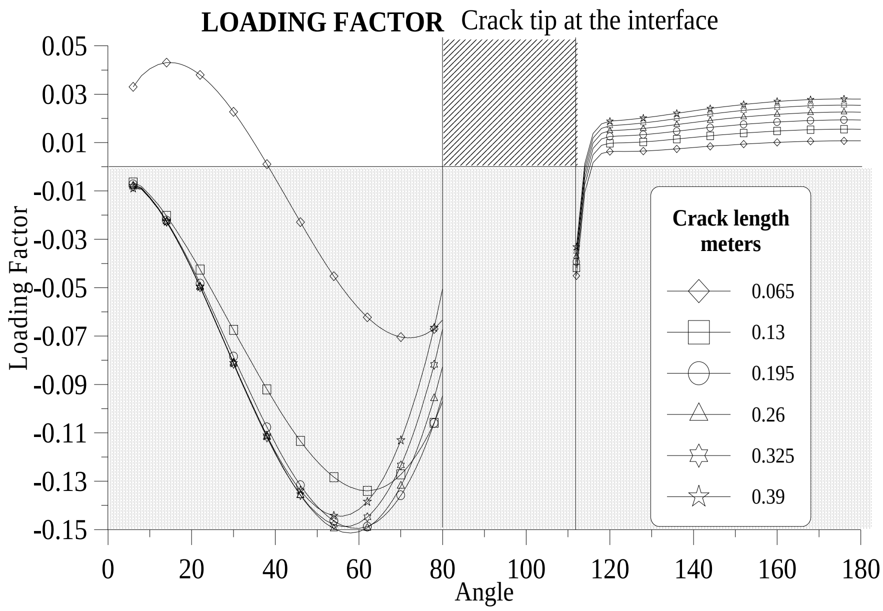


Figure 4 - Loading factor for the crack at the interface

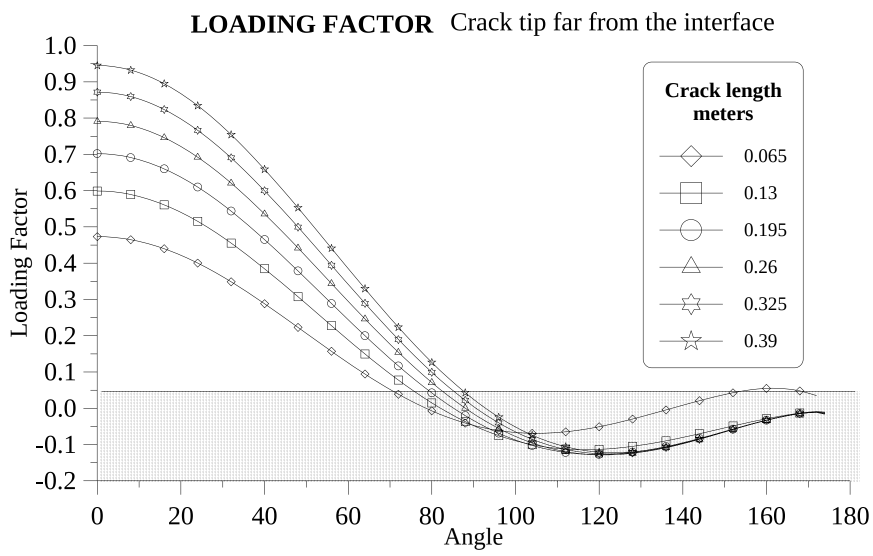


Figure 5 - Loading factor for the crack far from the interface

# 6. Conclusions

An application of a non-local criterion was presented to evaluate the strength of a concrete plate with a circular steel inclusion and a crack reaching the interface. Such approach allows to evaluate strength both at the far from inclusion crack tip, where the square root stress singularity occurs, and at the crack tip reaching the interface where another stress singularity occurs. The numerical examples presented show that the most loaded is the crack tip far from the interface, and the crack propagation will start there at the sufficiently high loading in the direction of crack prolongation. If the strength of this crack tip is ensured, the crack will propagate from the crack tip at the interface in the direction of crack prolongation (if the crack is not too short). It was supposed in the analysis that the interface strength coincides with the strength of one of the bonded materials and that the crack is normal to the interface and to the applied tractions. A change of these assumptions can change the analysis results too.

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