

Two-Operator Boundary-Domain Integral Equations for a Variable-Coefficient BVP

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4.1 Introduction

Partial Differential Equations (PDEs) with variable coefficients often arise in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermo-conductivity, fluid flows through porous media, and other areas of physics and engineering.

Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing formulation of explicit boundary integral equations for them, which can then be effectively solved numerically. Nevertheless, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) taken as a fundamental solution of corresponding frozen-coefficient PDEs, and reduce Boundary Value Problems (BVPs) for such PDEs to explicit systems of Boundary-Domain Integral Equations (BDIEs), see e.g. [Mi02, CMN09, Mi06] and references therein. However this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not available explicitly (as e.g. in the Lamé system of anisotropic elasticity).

To overcome this difficulty, one can apply the so-called two-operator approach, formulated in [Mi05] for some non-linear problems, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always choose it by such a way, that its parametrix is available explicitly. A simplest choice for the second PDE is the one with a fundamental solution explicitly available.

To analyse the two-operator approach we apply in this paper one of its linear versions to the mixed (Dirichlet-Neumann) BVP for a linear second-order scalar elliptic variable-coefficient PDE reducing it to four different BDIE systems. Although the considered BVP can be reduced to (other) BDIE systems also by the one-operator approach, it can be considered as a simple "toy"

model showing the main features of the two-operator approach arising also in reducing more general BVPs to BDIEs. The two-operator BDIE systems are nonstandard systems of equations containing integral operators defined on the domain under consideration and potential type and pseudo-differential operators defined on open sub-manifolds of the boundary. Using the results of [CMN09], we give a rigorous analysis of the two-operator BDIEs and show that the BDIE systems are *equivalent* to the mixed BVP and thus are *uniquely solvable*, while the corresponding boundary domain integral operators are *invertible* in appropriate Sobolev-Slobodetski (Bessel-potential) spaces.

4.2 Function spaces and BVP

Let $\Omega = \Omega^+$ be an open three-dimensional region of \mathbb{R}^3 , $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$ and the boundary $\partial\Omega$ be simply connected, closed, infinitely smooth surface. Moreover, $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$ where $\partial_D\Omega$ and $\partial_N\Omega$ are open, non-empty, non-intersecting, simply connected sub-manifolds of $\partial\Omega$ with an infinitely smooth boundary curve $\partial_D\Omega \cap \partial_N\Omega \in C^\infty$. Let $a \in C^\infty(\mathbb{R}^3)$, $a(x) > 0$ and also $\partial_j := \partial/\partial x_j$ ($j = 1, 2, 3$), $\partial_x = (\partial_1, \partial_2, \partial_3)$. We consider the following PDE with scalar variable coefficient,

$$L_a u(x) := L_a(x, \partial_x)u(x) := \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left[a(x) \frac{\partial u(x)}{\partial x_i} \right] = f(x), \quad x \in \Omega^\pm, \quad (4.1)$$

where u is unknown function and f is a given function in Ω^\pm .

In what follows, $H^s(\Omega^+) = H_2^s(\Omega^+)$, $H_{loc}^s(\Omega^-) = H_{2,loc}^s(\Omega^-)$, $H^s(\partial\Omega) = H_2^s(\partial\Omega)$ denote the Bessel potential spaces (coinciding with the Sobolev-Slobodetski spaces if $s \geq 0$). For $S_1 \subset \partial\Omega$, we will use the subspace $\tilde{H}^s(S_1) = \{g : g \in H^s(\partial\Omega), \text{supp}(g) \subset \overline{S_1}\}$ of $H^s(\partial\Omega)$, while $H^s(S_1) = \{r_{S_1} g : g \in H^s(\partial\Omega)\}$, where r_{S_1} denotes the restriction operator on S_1 .

From the trace theorem (see, e.g., [LiMa72]) for $u \in H^1(\Omega^\pm)$, it follows that $u|_{\partial\Omega}^\pm := \gamma^\pm u \in H^{\frac{1}{2}}(\partial\Omega)$, where γ^\pm is the trace operator on $\partial\Omega$ from Ω^\pm . We will use γ for γ^\pm if $\gamma^+ = \gamma^-$. We will use also notations u^\pm for the traces $u|_{\partial\Omega}^\pm$, when this will cause no confusion.

For a linear operator L_* we introduce the following subspace of $H^s(\Omega^\pm)$, [Gr85, Co88],

$$H^{s,0}(\Omega^\pm; L_*) := \{g \in H^s(\Omega^\pm) : L_* g \in L_2(\Omega^\pm)\},$$

$$\|g\|_{H^{s,0}(\Omega^\pm; L_*)}^2 := \|g\|_{H^s}^2 + \|L_* g\|_{H^0(\Omega^\pm)}^2 = \|g\|_{H^s}^2 + \|L_* g\|_{L_2(\Omega^\pm)}^2.$$

In this paper, we will particularly use the space $H^{1,0}(\Omega^\pm; L_*)$ for L_* being either the operator L_a from (4.1) or the Laplace operator Δ , and one can see that these spaces coincide.

For $u \in H^{1,0}(\Omega^\pm; \Delta)$, we can correctly define the (canonical) co-normal derivative $T_a^\pm u \in H^{-\frac{1}{2}}(\partial\Omega)$, cf. [Co88, McL00, Mi07], as

$$\langle T_a^\pm u, w \rangle_{\partial\Omega} := \pm \int_{\Omega^\pm} [\gamma_{-1}^\pm w \cdot L_a u + E_a(u, \gamma_{-1}^\pm w)] dx \quad \forall w \in H^{1/2}(\partial\Omega), \quad (4.2)$$

where $\gamma_{-1}^\pm : H^{1/2}(\partial\Omega) \rightarrow H^1(\Omega^\pm)$ is a right inverse to the trace operator γ^\pm ,

$$E_a(u, v) := \sum_{i=1}^3 a(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} = a(x) \nabla u(x) \cdot \nabla v(x)$$

and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, which extend the usual $L_2(\partial\Omega)$ inner product; to simplify notations we will also write sometimes the duality brackets as integral. Then for $u \in H^{1,0}(\Omega^\pm; \Delta)$, $v \in H^1(\Omega)$ the first Green identity holds, [Co88, Lemma3.4], [Mi07, Lemma 4.8],

$$\int_{\Omega^\pm} v(x) L_a u(x) dx = \pm \int_{\partial\Omega} v(x) T_a^\pm u(x) dS(x) - \int_{\Omega^\pm} E_a(u, v) dx. \quad (4.3)$$

If $u \in H^2(\Omega^\pm)$, the canonical co-normal derivative $T_a^\pm u$ defined by (4.2) reduces to its classical form

$$T_a^\pm u := \sum_{i=1}^3 a(x) n_i(x) \left[\frac{\partial u(x)}{\partial x_i} \right]^\pm = a(x) \left[\frac{\partial u(x)}{\partial n(x)} \right]^\pm, \quad (4.4)$$

where $n(x)$ is the exterior (to Ω^\pm) unit normal at the point $x \in \partial\Omega$.

We will derive and investigate the *two-operator boundary-domain integral equation systems* for the following mixed boundary value problem.

$$L_a u = f \quad \text{in } \Omega \quad (4.5)$$

$$u^+ = \varphi_0 \quad \text{on } \partial_D \Omega \quad (4.6)$$

$$T_a^+ u = \psi_0 \quad \text{on } \partial_N \Omega \quad (4.7)$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial_D \Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial_N \Omega)$ and $f \in L_2(\Omega)$. Equation (4.5) is understood in the distributional sense, condition (4.6) in the trace sense, while equality (4.7) in the functional sense (4.2).

Let us consider another auxiliary linear elliptic partial differential operator L_b such that

$$L_b u(x) := L_b(x, \partial_x) u(x) := \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left[b(x) \frac{\partial u(x)}{\partial x_i} \right], \quad (4.8)$$

where $b \in C^\infty(\mathbb{R}^3)$, $b(x) > 0$. Then for $u \in H^{1,0}(\Omega^\pm; \Delta) = H^{1,0}(\Omega^\pm; \Delta)$ the associate co-normal derivative operator T_b^\pm is defined by (4.2) (and for $u \in H^2(\Omega^\pm)$ by (4.4)) with a replaced by b . If $v \in H^{1,0}(\Omega^\pm; \Delta)$, $u \in H^1(\Omega)$ then for the operator L_b holds the first Green identity,

$$\int_{\Omega^\pm} u(x) L_b v(x) dx = \pm \int_{\partial\Omega} u(x) T_b^\pm v(x) dS - \int_{\Omega^\pm} E_b(u, v) dx. \quad (4.9)$$

If $u, v \in H^{1,0}(\Omega^\pm; \Delta)$, then subtracting (4.3) from (4.9), we obtain the *two-operator second Green identity*, cf. [Mi05],

$$\begin{aligned} \int_{\Omega^\pm} \{u(x) L_b v(x) - v(x) L_a u(x)\} dx = \\ \pm \int_{\partial\Omega} \{u(x) T_b^\pm v(x) - v(x) T_a^\pm u(x)\} dS \\ + \int_{\Omega^\pm} [a(x) - b(x)] \nabla v(x) \cdot \nabla u(x) dx \end{aligned} \quad (4.10)$$

Note that if $a = b$, then, the last domain integral disappears, and the two-operator Green identity degenerates into the classical second Green identity.

4.3 Parametrix and potential type operators

As follows from [Mir70, Mi02, CMN09], the function

$$P_b(x, y) = \frac{-1}{4\pi b(y)|x - y|}, \quad x, y \in \mathbb{R}^3 \quad (4.11)$$

is a parametrix (Levi function) for the operator $L_b(x; \partial_x)$ from (4.8), i.e., satisfies equation

$$L_b(x, \partial_x) P_b(x, y) = \delta(x - y) + R_b(x, y)$$

with

$$R_b(x, y) = \sum_{i=1}^3 \frac{x_i - y_i}{4\pi b(y)|x - y|^3} \frac{\partial b(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^3. \quad (4.12)$$

Evidently, the parametrix $P_b(x, y)$ is a fundamental solution to the operator $L_b(y, \partial_x) := b(y) \Delta(\partial_x)$ with "frozen" coefficient $b(x) = b(y)$, i.e.,

$$L_b(y, \partial_x) P_b(x, y) = \delta(x - y).$$

The parametrix-based Newtonian and the remainder volume potential operators, corresponding to the parametrix (4.11) and to remainder (4.12) are given, respectively, by

$$\mathcal{P}_b g(y) := \int_{\Omega} P_b(x, y) g(x) dx, \quad \mathcal{R}_b g(y) := \int_{\Omega} R_b(x, y) g(x) dx. \quad (4.13)$$

Let us introduce the single layer and the double layer surface potential operators, based on parametrix (4.11),

$$V_b g(y) := - \int_{\partial\Omega} P_b(x, y) g(x) dS_x, \quad y \notin \partial\Omega, \quad (4.14)$$

$$W_b g(y) := - \int_{\partial\Omega} [T_b(x, n(x), \partial_x) P_b(x, y)] g(x) dS_x, \quad y \notin \partial\Omega. \quad (4.15)$$

For $y \in \partial\Omega$, the corresponding boundary integral (pseudo-differential) operators of direct surface values of the simple layer potential \mathcal{V}_b and the double layer potential \mathcal{W}_b are

$$\mathcal{V}_b g(y) := - \int_{\partial\Omega} P_b(x, y) g(x) dS_x, \quad (4.16)$$

$$\mathcal{W}_b g(y) := - \int_{\partial\Omega} [T_b(x, n(x), \partial_x) P_b(x, y)] g(x) dS_x. \quad (4.17)$$

We can also calculate at $y \in \partial\Omega$ the co-normal derivatives, associated with the operator L_a , of the single layer potential and of the double layer potential,

$$T_a^\pm V_b g(y) = \frac{a(y)}{b(y)} T_b^\pm V_b g(y), \quad (4.18)$$

$$\mathcal{L}_{ab}^\pm g(y) := T_a^\pm W_b g(y) = \frac{a(y)}{b(y)} T_b^\pm W_b g(y) =: \frac{a(y)}{b(y)} \mathcal{L}_b^\pm g(y) \quad (4.19)$$

The direct value operators associated with (4.18) are

$$\mathcal{W}'_{ab} g(y) := - \int_{\partial\Omega} [T_a(y, n(y), \partial_y) P_b(x, y)] g(x) dS_x = \frac{a(y)}{b(y)} \mathcal{W}'_b g(y), \quad (4.20)$$

$$\mathcal{W}'_b g(y) := - \int_{\partial\Omega} [T_b(y, n(y), \partial_y) P_b(x, y)] g(x) dS_x. \quad (4.21)$$

From equations (4.13)-(4.21) we deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for $b = 1$, that is, associated with the fundamental solution $P_\Delta = -(4\pi|x-y|)^{-1}$ of the Laplace operator Δ .

$$\mathcal{P}_b g = \frac{1}{b} \mathcal{P}_\Delta g, \quad \mathcal{R}_b g = -\frac{1}{b} \sum_{j=1}^3 \partial_j \mathcal{P}_\Delta [g(\partial_j b)], \quad (4.22)$$

$$\frac{a}{b} V_a g = V_b g = \frac{1}{b} V_\Delta g; \quad \frac{a}{b} W_a \left(\frac{bg}{a} \right) = W_b g = \frac{1}{b} W_\Delta (bg), \quad (4.23)$$

$$\frac{a}{b} \mathcal{V}_a g = \mathcal{V}_b g = \frac{1}{b} \mathcal{V}_\Delta g; \quad \frac{a}{b} \mathcal{W}_a \left(\frac{bg}{a} \right) = \mathcal{W}_b g = \frac{1}{b} \mathcal{W}_\Delta (bg), \quad (4.24)$$

$$\mathcal{W}'_{ab} g = \frac{a}{b} \mathcal{W}'_b g = \frac{a}{b} \left\{ \mathcal{W}'_\Delta (bg) + \left[b \frac{\partial}{\partial n} \left(\frac{1}{b} \right) \right] \mathcal{V}_\Delta g \right\}, \quad (4.25)$$

$$\mathcal{L}_{ab}^\pm g := \frac{a}{b} \mathcal{L}_b^\pm g = \frac{a}{b} \left\{ \mathcal{L}_\Delta (bg) + \left[b \frac{\partial}{\partial n} \left(\frac{1}{b} \right) \right] W_\Delta^\pm (bg) \right\}. \quad (4.26)$$

It is taken into account that b and its derivatives are continuous in \mathbb{R}^3 and $\mathcal{L}_\Delta(bg) := \mathcal{L}_\Delta^+(bg) = \mathcal{L}_\Delta^-(bg)$ by the Liapunov-Tauber theorem.

The mapping properties of the volume and surface potentials are proved in [CMN09], see also Appendix A and B in [Mi06]. Similar to Theorems 3.3 and 3.6 in [CMN09] (see also Appendix A and B in [Mi06]), relations (4.23)-(4.26) imply the two following jump relation theorems.

Theorem 1. *Let $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$, and $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$. Then there hold the following relations on $\partial\Omega$,*

$$\begin{aligned} [V_b g_1]^\pm &= \mathcal{V}_b g_1, \\ [W_b g_2]^\pm &= \mp \frac{1}{2} g_2 + \mathcal{W}_b g_2, \\ T_a^\pm V_b g_1 &= \pm \frac{1}{2} \frac{a}{b} g_1 + \mathcal{W}'_{ab} g_1. \end{aligned}$$

Theorem 2. *Let S_1 and $\partial\Omega \setminus \overline{S_1}$ be nonempty, open, simply connected submanifolds of $\partial\Omega$ with an infinitely smooth boundary curve, and $0 < s < 1$. Then*

$$\mathcal{L}_{ab}^+ + \frac{a}{b} \frac{\partial b}{\partial n} \left(-\frac{1}{2} I + \mathcal{W}_b \right) = \mathcal{L}_{ab}^- + \frac{a}{b} \frac{\partial b}{\partial n} \left(\frac{1}{2} I + \mathcal{W}_b \right) \quad \text{on } \partial\Omega.$$

Moreover, the pseudo-differential operator $r_{S_1} \widehat{\mathcal{L}}_{ab} : \widetilde{H}^s(S_1) \rightarrow H^{s-1}(S_1)$, where

$$\widehat{\mathcal{L}}_{ab} g := \left[\frac{b}{a} \mathcal{L}_{ab}^\pm + \frac{\partial b}{\partial n} \left(\mp \frac{1}{2} I + \mathcal{W}_b \right) \right] g = \mathcal{L}_\Delta(bg) \quad \text{on } \partial\Omega,$$

is invertible, while the operators $r_{S_1} \left(\frac{b}{a} \mathcal{L}_{ab}^\pm - \widehat{\mathcal{L}}_{ab} \right) : \widetilde{H}^s(S_1) \rightarrow H^s(S_1)$ are bounded and the operators $r_{S_1} \left(\frac{b}{a} \mathcal{L}_{ab}^\pm - \widehat{\mathcal{L}}_{ab} \right) : \widetilde{H}^s(S_1) \rightarrow H^{s-1}(S_1)$ are compact.

For $v(x) := P_b(x, y)$ and $u \in H^{1,0}(\Omega; \Delta)$, we obtain from (4.10) by standard limiting procedures (cf. [Mir70]) the *two-operator third Green identity*,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b T_a^+ u + W_b u^+ = \mathcal{P}_b L_a u \quad \text{in } \Omega, \quad (4.27)$$

where

$$\begin{aligned} \mathcal{Z}_b u(y) &:= - \int_\Omega [a(x) - b(x)] \nabla_x P_b(x, y) \cdot \nabla u(x) dx \\ &= \frac{1}{b(y)} \sum_{j=1}^3 \partial_j \mathcal{P}_\Delta [(a - b) \partial_j u](y), \quad y \in \Omega. \end{aligned} \quad (4.28)$$

Using the Gauss divergence theorem, we can rewrite $\mathcal{Z}_b u(y)$ in the form that does not involve derivatives of u ,

$$\mathcal{Z}_b u(y) = \left[\frac{a(y)}{b(y)} - 1 \right] u(y) + \widehat{\mathcal{Z}}_b u(y), \quad (4.29)$$

$$\widehat{\mathcal{Z}}_b u(y) := \frac{a(y)}{b(y)} W_a u^+(y) - W_b u^+(y) + \frac{a(y)}{b(y)} \mathcal{R}_a u(y) - \mathcal{R}_b u(y), \quad (4.30)$$

which allows to call \mathcal{Z}_b integral operator in spite of its integro-differential ansatz (4.28).

Note that substituting (4.29)-(4.30) to (4.27) and multiplying by $b(y)/a(y)$ one reduces (4.27) to the one-operator parametrix-based third Green identity obtained in [CMN09],

$$u + \mathcal{R}_a u - V_a T_a^+ u + W_a u^+ = \mathcal{P}_a L_a u \quad \text{in } \Omega.$$

Relations (4.28)-(4.30) and the mapping properties of \mathcal{P}_Δ , \mathcal{R}_a , \mathcal{R}_b , W_a and W_b , given by Theorems 3.1, 3.8 in [CMN09], imply the following statement.

Theorem 3. *The operators*

$$\begin{aligned} \mathcal{Z}_b : H^s(\Omega) &\rightarrow H^s(\Omega), \quad s > \frac{1}{2}, \\ \widehat{\mathcal{Z}}_b : H^s(\Omega) &\rightarrow H^{s,0}(\Omega; \Delta), \quad s \geq 1, \end{aligned}$$

are continuous.

If $u \in H^{1,0}(\Omega; \Delta)$ is a solution of equation (4.5) with $f \in L_2(\Omega)$, then (4.27) gives

$$Gu := u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b T_a^+ u + W_b u^+ = \mathcal{P}_b f \quad \text{in } \Omega, \quad (4.31)$$

$$\mathcal{G}u := \frac{1}{2} u^+ + \mathcal{Z}_b^+ u + \mathcal{R}_b^+ u - \mathcal{V}_b T_a^+ u + \mathcal{W}_b u^+ = [\mathcal{P}_b f]^+ \quad \text{on } \partial\Omega, \quad (4.32)$$

$$\begin{aligned} \mathcal{T}u := \left(1 - \frac{a}{2b}\right) T_a^+ u + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} T_a^+ u \\ + \mathcal{L}_{ab}^+ u^+ = T_a^+ \mathcal{P}_b f \quad \text{on } \partial\Omega. \end{aligned} \quad (4.33)$$

where $\mathcal{Z}_b^+ u = [\mathcal{Z}_b u]^+$ and $\mathcal{R}_b^+ u = [\mathcal{R}_b u]^+$.

Note that if \mathcal{P}_b is not only the parametrix but also a fundamental solution of the operator L_b , then the remainder operator \mathcal{R}_b vanishes in (4.31)-(4.33) (and everywhere in the paper), while the operator \mathcal{Z}_b stays unless $L_a = L_b$.

For some functions f, Ψ, Φ , let us consider a more general "indirect" integral relation, associated with (4.31),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \Psi + W_b \Phi = \mathcal{P}_b f, \quad \text{in } \Omega \quad (4.34)$$

Similar to the proof of Lemma 4.1 in [CMN09], one can prove the following

Lemma 1. *Let $f \in L_2(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$, and $u \in H^1(\Omega)$ satisfy (4.34). Then $u \in H^{1,0}(\Omega; \Delta)$, $L_a u = f$ in Ω and*

$$V_b (\Psi - T_a^+ u) - W_b (\Phi - u^+) = 0 \quad \text{in } \Omega.$$

4.4 Two-operator boundary-domain integral equations

Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of the given data $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ from $\partial_D\Omega$ to $\partial\Omega$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ from $\partial_N\Omega$ to $\partial\Omega$, respectively. Let us also denote

$$F_0 := \mathcal{P}_b f + V_b \Psi_0 - W_b \Phi_0 \quad \text{in} \quad \Omega.$$

Note that for $f \in L_2(\Omega)$, $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we have the inclusion $F_0 \in H^{1,0}(\Omega, L_a)$ due to the mapping properties of the Newtonian (volume) and layer potentials (cf. Theorems 3.1 and 3.10 in [CMN09]).

To reduce BVP (4.5)-(4.7) to one or another two-operator BDIE system, we will use equation (4.31) in Ω , and restrictions of equation (4.32) or (4.33) on appropriate parts of the boundary. We will always substitute $\Phi_0 + \varphi$ for u^+ and $\Psi_0 + \psi$ for $T_a^+ u$, cf. [CMN09], where $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ are considered as known, while ψ belongs to $\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and φ to $\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ due to the boundary conditions (4.6)-(4.7) and are to be found along with $u \in H^{1,0}(\Omega; \Delta)$. This will lead us to *segregated* BDIE systems.

4.4.1 Boundary-domain integral equation system (\mathcal{GT})

Let us use equation (4.31) in Ω , the restriction of equation (4.32) on $\partial_D\Omega$ and the restriction of equation (4.33) on $\partial_N\Omega$. Then we arrive at the following two-operator segregated system of BDIEs,

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi + W_b \varphi = F_0 \quad \text{in} \quad \Omega, \quad (4.35)$$

$$\mathcal{Z}_b^+ u + \mathcal{R}_b^+ u - \mathcal{V}_b \psi + \mathcal{W}_b \varphi = F_0^+ - \varphi_0 \quad \text{on} \quad \partial_D\Omega, \quad (4.36)$$

$$T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} \psi + \mathcal{L}_{ab}^+ \varphi = T_a^+ F_0 - \psi_0 \quad \text{on} \quad \partial_N\Omega. \quad (4.37)$$

Note that due to Lemma 1, all terms of equation (4.35) belong to $H^{1,0}(\Omega; \Delta)$ and their co-normal derivatives are well defined. System (4.35)-(4.37) can be rewritten in the form

$$\mathcal{A}^{\mathcal{GT}} \mathcal{U} = \mathcal{F}^{\mathcal{GT}},$$

where

$$\begin{aligned} \mathcal{U}^\top &:= [u, \psi, \varphi] \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega), \\ \mathcal{F}^{\mathcal{GT}} &:= [F_0, r_{\partial_D\Omega} F_0^+ - \varphi_0, r_{\partial_N\Omega} T_a^+ F_0 - \psi_0]^\top, \\ \mathcal{A}^{\mathcal{GT}} &:= \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ r_{\partial_D\Omega} [\mathcal{Z}_b^+ + \mathcal{R}_b^+] & -r_{\partial_D\Omega} \mathcal{V}_b & r_{\partial_D\Omega} \mathcal{W}_b \\ r_{\partial_N\Omega} T_a^+ [\mathcal{Z}_b + \mathcal{R}_b] & -r_{\partial_N\Omega} \mathcal{W}'_{ab} & r_{\partial_N\Omega} \mathcal{L}_{ab}^+ \end{bmatrix}. \end{aligned}$$

4.4.2 Boundary-domain integral equation system (\mathcal{GG})

To obtain another system, we will use equation (4.31) in Ω and equation (4.32), associated with the operator \mathcal{G} on the whole boundary $\partial\Omega$, and arrive at the two-operator segregated BDIE system (\mathcal{GG}),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi + W_b \varphi = F_0 \quad \text{in } \Omega, \quad (4.38)$$

$$\frac{1}{2}\varphi + \mathcal{Z}_b^+ u + \mathcal{R}_b^+ u - \mathcal{V}_b \psi + \mathcal{W}_b \varphi = F_0^+ - \Phi_0 \quad \text{on } \partial\Omega. \quad (4.39)$$

System (4.38)-(4.39) can be written in the form

$$\mathcal{A}^{\mathcal{GG}} \mathcal{U} = \mathcal{F}^{\mathcal{GG}},$$

where

$$\begin{aligned} \mathcal{F}^{\mathcal{GG}} &:= [F_0, F_0^+ - \Phi_0]^\top, \\ \mathcal{U}^\top &:= [u, \psi, \varphi] \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega), \\ \mathcal{A}^{\mathcal{GG}} &:= \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ \mathcal{Z}_b^+ + \mathcal{R}_b^+ & -\mathcal{V}_b & \frac{1}{2}I + \mathcal{W}_b \end{bmatrix}. \end{aligned}$$

4.4.3 Boundary-domain integral equation system (\mathcal{TT})

To obtain one more system, we will use equation (4.31) in Ω and equation (4.33) on $\partial\Omega$ and arrive at the two-operator segregated BDIE system (\mathcal{TT}),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi + W_b \varphi = F_0 \quad \text{in } \Omega, \quad (4.40)$$

$$\begin{aligned} \left(1 - \frac{a}{2b}\right) \psi + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} \psi + \mathcal{L}_{ab}^+ \varphi = \\ T_a^+ F_0 - \Psi_0 \quad \text{on } \partial\Omega. \end{aligned} \quad (4.41)$$

System (4.40)-(4.41) can be written in the form

$$\mathcal{A}^{\mathcal{TT}} \mathcal{U} = \mathcal{F}^{\mathcal{TT}},$$

where

$$\begin{aligned} \mathcal{F}^{\mathcal{TT}} &:= [F_0, T_a^+ F_0 - \Psi_0]^\top, \\ \mathcal{U}^\top &:= [u, \psi, \varphi] \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega), \\ \mathcal{A}^{\mathcal{TT}} &:= \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ T_a^+ [\mathcal{Z}_b + \mathcal{R}_b] & (1 - \frac{a}{2b})I - \mathcal{W}'_{ab} & \mathcal{L}_{ab}^+ \end{bmatrix}. \end{aligned}$$

4.4.4 Boundary-domain integral equation system (\mathcal{TG})

To reduce BVP (4.5)-(4.7) to a BDIE system of "almost" the second kind (up to the spaces), we will use equation(4.31) in Ω , the restriction of equation (4.33) on $\partial_D\Omega$, and the restriction of equation (4.32) on $\partial_N\Omega$. Then we arrive at the following two-operator segregated BDIE system (\mathcal{TG}),

$$u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi + W_b \varphi = F_0 \quad \text{in } \Omega, \quad (4.42)$$

$$\begin{aligned} \left(1 - \frac{a}{2b}\right) T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}'_{ab} \psi + \mathcal{L}_{ab}^+ \varphi = \\ T_a^+ F_0 - \Psi_0 \quad \text{on } \partial_D \Omega, \end{aligned} \quad (4.43)$$

$$\frac{1}{2} \varphi + \mathcal{Z}_b^+ u + \mathcal{R}_b^+ u - \mathcal{V}_a \psi + \mathcal{W}_a \varphi = F_0^+ - \Phi_0 \quad \text{on } \partial_N \Omega. \quad (4.44)$$

System (4.42)-(4.44) can be rewritten in the form

$$\mathcal{A}^{\mathcal{TG}} \mathcal{U} = \mathcal{F}^{\mathcal{TG}},$$

where

$$\begin{aligned} \mathcal{F}^{\mathcal{TG}} &:= [F_0, r_{\partial_D \Omega}(T_a^+ F_0 - \Psi_0), r_{\partial_N \Omega}(F_0^+ - \Phi_0)]^\top, \\ \mathcal{U}^\top &:= [u, \psi, \varphi] \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega), \\ \mathcal{A}^{\mathcal{TG}} &:= \begin{bmatrix} I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\ r_{\partial_D \Omega} T_a^+ [\mathcal{Z}_b + \mathcal{R}_b] & (1 - \frac{a}{2b})I - r_{\partial_D \Omega} \mathcal{W}'_{ab} & r_{\partial_D \Omega} \mathcal{L}_{ab}^+ \\ r_{\partial_N \Omega} [\mathcal{Z}_b^+ + \mathcal{R}_b^+] & -r_{\partial_N \Omega} \mathcal{V}_b & \frac{1}{2}I + r_{\partial_N \Omega} \mathcal{W}_b \end{bmatrix}. \end{aligned}$$

4.4.5 Equivalence and invertibility

Using the arguments similar to the proofs of Theorems 5.2, 5.6, 5.9 and 5.12 in [CMN09], one can prove the following equivalence theorem.

Theorem 4. *Let $f \in L_2(\Omega)$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ and $\psi_0 \in H^{-\frac{1}{2}}$, respectively.*

(i) *If some $u \in H^1(\Omega)$ solves the mixed BVP (4.5)-(4.7) in Ω , then the solution is unique and the triple $(u, \psi, \varphi) \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$, where*

$$\psi = T_a^+ u - \Psi_0, \quad \varphi = u^+ - \Phi_0 \quad \text{on } \partial\Omega, \quad (4.45)$$

solves BDIE systems (\mathcal{GT}) , (\mathcal{GG}) , (\mathcal{TT}) and (\mathcal{TG}) .

(ii) *Vise versa, if a triple $(u, \psi, \varphi) \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ the solves BDIE system (\mathcal{GT}) or (\mathcal{GG}) or (\mathcal{TT}) or (\mathcal{TG}) , then the solution is unique, u solves BVP (4.5)-(4.7), and relations (4.45) hold.*

Application of the representation Lemma 5.13 and Corollary 5.14 as well as Corollary 5.16 about invertibility of the mixed BVP (4.5)-(4.7) operator, from [CMN09], along with the equivalence Theorem 4 above, lead to the following invertibility result.

Theorem 5. *The following operators are continuously invertible,*

$$\begin{aligned}\mathcal{A}^{\mathcal{G}\mathcal{G}} &: H^{1,0}(\Omega; \Delta) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega) \rightarrow H^{1,0}(\Omega; \Delta) \times H^{\frac{1}{2}}(\partial \Omega), \\ \mathcal{A}^{\mathcal{T}\mathcal{T}} &: H^{1,0}(\Omega; \Delta) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega) \rightarrow H^{1,0}(\Omega; \Delta) \times H^{-\frac{1}{2}}(\partial \Omega), \\ \mathcal{A}^{\mathcal{G}\mathcal{T}} &: H^{1,0}(\Omega; \Delta) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega) \rightarrow \\ &\quad H^{1,0}(\Omega; \Delta) \times H^{\frac{1}{2}}(\partial_D \Omega) \times H^{-\frac{1}{2}}(\partial_N \Omega), \\ \mathcal{A}^{\mathcal{T}\mathcal{G}} &: H^{1,0}(\Omega; \Delta) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega) \rightarrow \\ &\quad H^{1,0}(\Omega; \Delta) \times H^{-\frac{1}{2}}(\partial_D \Omega) \times H^{\frac{1}{2}}(\partial_N \Omega).\end{aligned}$$

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