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Chapter 3
Analysis of Boundary-Domain Integral Equations for Variable-Coefficient Neumann BVP in 2D

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3.1 Preliminaries

In this paper we will consider the Neumann Boundary Value problem for the "stationary heat transfer" partial differential equation with variable coefficient in a two-dimensional domain. This problem is reduced to some Boundary-Domain Integral Equations (BDIEs). The BDIEs in the two-dimensional case have special properties in comparison with the higher dimensions because of the logarithmic term in the parametrix for the associated partial differential equation. Consequently we need to set conditions on the domain or on the function spaces to insure invertibility of the layer potentials and hence the unique solvability of the BDIEs. Equivalence of the BDIEs to the original BVPs, BDIEs solvability, solution uniqueness/non uniqueness, as well as Fredholm property and invertibility of the BDIE operator are analysed. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

Let Ω be an interior domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$, and let $n(x)$ be the exterior unit normal vector defined on $\partial\Omega$. The set of all infinitely differentiable function on Ω with compact support is denoted by $\mathscr{D}(\Omega)$. The function space $\mathscr{D}'(\Omega)$ consists of all continuous linear functionals over $\mathscr{D}(\Omega)$. The space $H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, denotes the Bessel potential space. We define $H^s(\Omega) = \{u \in \mathscr{D}'(\Omega) : u = U|_\Omega$ for some $U \in H^s(\mathbb{R}^2)\}$. Note that $H^1(\Omega)$ coincides with the Sobolev space $W_2^1(\Omega)$, with equivalent norms. The space $\widetilde{H}^s(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^2)$, and for $s \in (-\frac{1}{2}, \frac{1}{2})$, the space $H^s(\Omega)$ can be identified with $\widetilde{H}^s(\Omega)$, see e.g. [McL00].

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We shall consider the scalar elliptic differential equation

$$Au(x) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left[a(x) \frac{\partial u(x)}{\partial x_i} \right] = f(x) \qquad \text{in } \Omega,$$

where u is unknown function and f is a given function in Ω. We assume that $a(x) \in C^\infty(\mathbb{R}^2)$, $a(x) > c > 0$.

From the well known theorem of Gauss and Ostrogradski, if $h \in C_0^1(\overline{\Omega})$, then

$$\int_\Omega \frac{\partial}{\partial x_i} h(x) dx = \int_{\partial\Omega} \gamma^+ h(x) n_i(x) ds_x, \quad (i = 1, 2) \tag{3.1}$$

where, $\gamma^+ h(x)$ is the interior boundary trace of $h(x)$. By the trace theorem (see, e.g., [McL00, Theorem 3.29, 3.38]), the integral relation (3.1) holds for any $h \in H^1(\Omega)$.

For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ if we put $h(x) = a(x) \frac{\partial u(x)}{\partial x_j} v(x)$ and applying the Gauss-Ostrogradski Theorem, we obtain the following *Green's first identity*

$$\mathscr{E}(u, v) = -\int_\Omega (Au)(x) v(x) dx + \int_{\partial\Omega} T^+ u(x) \gamma^+ v(x) ds_x, \tag{3.2}$$

where $\mathscr{E}(u, v) := \sum_{i=1}^{2} \int_\Omega a(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} dx$ is the symmetric bilinear form, and

$$T^+ u(x) := \sum_{i=1}^{2} n_i(x) \gamma^+ \left[a(x) \frac{\partial}{\partial x_i} u(x) \right] \text{ for } x \in \partial\Omega, \tag{3.3}$$

is the *interior conormal derivative*. Thus holds the following Lemma.

Lemma 1. *For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, $\mathscr{E}(u, v) = -(Au, v)_\Omega + (T^+ u, \gamma^+ v)_{\partial\Omega}$.*

Remark 1. For $v \in \mathscr{D}(\Omega)$, $\gamma^+ v = 0$. If $u \in H^1(\Omega)$, then we can define Au as a distribution on Ω by, $(Au, v) = -\mathscr{E}(u, v)$ for $v \in \mathscr{D}(\Omega)$.

The subspace $H^{1,0}(\Omega; A)$ is defined as in [Cos88](see also, [Mik11])

$$H^{1,0}(\Omega; A) := \{g \in H^1(\Omega) : Ag \in L_2(\Omega)\},$$

with the norm $\|g\|^2_{H^{1,0}(\Omega;A)} := \|g\|^2_{H^1(\Omega)} + \|Ag\|^2_{L_2(\Omega)}$.

For $u \in H^1(\Omega)$ the classical conormal derivative (3.3) is not well defined, but for $u \in H^{1,0}(\Omega; A)$, there exists the following continuous extension of this definition hinted by the first Green identity (3.2) (see, e.g.,[Cos88, Mik11] and the references therein).

Definition 1. For $u \in H^{1,0}(\Omega; A)$ the (canonical) co-normal derivative $T^+ u \in H^{-\frac{1}{2}}(\partial\Omega)$ is defined in the following weak form,

$$\langle T^+u,w\rangle_\Omega := \mathscr{E}(u,\gamma^+_{-1}w)+\int_\Omega (Au)\gamma^+_{-1}w dx \quad \text{for all } w\in H^{\frac{1}{2}}(\partial\Omega) \tag{3.4}$$

where $\gamma^+_{-1}: H^{\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)$ is a continuous right inverse of the interior trace operator γ^+, which maps $H^1(\Omega)\to H^{\frac{1}{2}}(\partial\Omega)$, while $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denote the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, which extend the usual $L_2(\partial\Omega)$ inner product.

Remark 2. The first Green identity (3.2) also holds for $u\in H^{1,0}(\Omega;A)$ and $v\in H^1(\Omega)$ ([Cos88, Mik11]).

By interchanging the role of u and v in the first Green identity and subtracting the result, we obtain *the Green second identity* for $u,v\in H^{1,0}(\Omega;A)$,

$$\int_\Omega (vAu-uAv)dx = \langle T^+u,\gamma^+v\rangle_{\partial\Omega}-\langle T^+v,\gamma^+u\rangle_{\partial\Omega}. \tag{3.5}$$

We will consider the following Neumann boundary value problem: *for $\psi_0\in H^{-\frac{1}{2}}(\partial\Omega)$, and $f\in L_2(\Omega)$ find a function $u\in H^1(\Omega)$ satisfying,*

$$Au = f \quad \text{in } \Omega, \tag{3.6}$$

$$T^+u = \psi_0 \quad \text{on } \partial\Omega. \tag{3.7}$$

Here equation (3.6) is understood in distributional sense as in Remark 1, and equation (3.7) is understood in functional sense (3.4).

The following assertion is well known, cf. e.g. [Ste08, Theorem 4.9].

Theorem 1. *The homogeneous problem corresponding to the BVP (3.6)-(3.7), admits solutions in $H^1(\Omega)$ spanned by $u^0=1$. The non-homogeneous problem (3.6)-(3.7) is solvable if and only if*

$$\langle f,u^0\rangle_\Omega-\langle\psi_0,\gamma^+u^0\rangle_{\partial\Omega}=0. \tag{3.8}$$

3.2 Parametrix-Based Potential Operators

Definition 2. A function $P(x,y)$ is a parametrix (Levi function) for the operator A if

$$A_xP(x,y)=\delta(x-y)+R(x,y)$$

where δ is the Dirac-delta distribution, while $R(x,y)$ is a remainder possessing at most a weak singularity at $x=y$.

The parametrix and the corresponding remainder can be chosen as in [Mik02],

$$P(x,y)=\frac{\log|x-y|}{2\pi a(y)}, \quad R(x,y)=\sum_{i=1}^{2}\frac{x_i-y_i}{2\pi a(y)|x-y|^2}\frac{\partial a(x)}{\partial x_i}, x,y\in\mathbb{R}^2.$$

Similar to [Mik02, CMN09a], we define the parametrix-based Newtonian and remained potential operators as

$$\mathscr{P}g(y) := \int_{\Omega} P(x,y)g(x)dx, \quad \mathscr{R}g(y) := \int_{\Omega} R(x,y)g(x)dx.$$

The single and double layer potential operators corresponding to the parametrix $P(x,y)$ are defined for $y \notin \partial\Omega$ as

$$Vg(y) := -\int_{\partial\Omega} P(x,y)g(x)ds_x, \qquad Wg(y) := -\int_{\partial\Omega} T_x^+ P(x,y)g(x)ds_x,$$

where g is some scalar density function. The following boundary integral (pseudo-differential) operators are also defined for $y \in \partial\Omega$,

$$\mathscr{V}g(y) := -\int_{\partial\Omega} P(x,y)g(x)ds_x, \qquad \mathscr{W}g(y) := -\int_{\partial\Omega} T_x^+ P(x,y)g(x)ds_x,$$

$$\mathscr{W}'g(y) := -\int_{\partial\Omega} T_y^+ P(x,y)g(x)ds_x, \qquad \mathscr{L}^+g(y) := T_y^+ Wg(y).$$

Let $V_\Delta, W_\Delta, \mathscr{V}_\Delta, \mathscr{W}_\Delta, \mathscr{L}_\Delta^+$ denote the potentials corresponding to the Laplace operator $A = \Delta$. Then the following relations hold (cf.[CMN09a] for 3D case),

$$Vg = \frac{1}{a}V_\Delta g, \qquad Wg = \frac{1}{a}W_\Delta(ag) \tag{3.9}$$

$$\mathscr{V}g = \frac{1}{a}\mathscr{V}_\Delta g, \qquad \mathscr{W}g = \frac{1}{a}\mathscr{W}_\Delta(ag), \tag{3.10}$$

$$\mathscr{W}'g = \mathscr{W}'_\Delta g + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]\mathscr{V}_\Delta g, \tag{3.11}$$

$$\mathscr{L}^+g = \mathscr{L}_\Delta^+(ag) + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]W_\Delta^+(ag). \tag{3.12}$$

If $u \in H^{1,0}(\Omega;A)$, then substituting $v(x)$ by $P(x,y)$ in the second Green identity (3.5) for $\Omega \setminus B(y,\varepsilon)$, where $B(y,\varepsilon)$ is a disc of radius ε centered at y, and taking the limit $\varepsilon \to 0$, we arrive at the following parametrix-based third Green identity (cf. e.g. [Mir70, Mik02, CMN09a]),

$$u + \mathscr{R}u - VT^+u + W\gamma^+u = \mathscr{P}Au \quad \text{in } \Omega. \tag{3.13}$$

Applying the *trace operator* to equation (3.13) and using the jump relation (see, e.g, [McL00, Theorem 6.11]), we have

$$\frac{1}{2}\gamma^+u + \gamma^+\mathscr{R}u - \mathscr{V}T^+u + \mathscr{W}\gamma^+u = \gamma^+\mathscr{P}Au \quad \text{on } \partial\Omega. \tag{3.14}$$

Similarly, applying *co-normal derivative operator* to equation (3.13), and using again the jump relation, we obtain.

$$\frac{1}{2}T^{+}u+T^{+}\mathscr{R}u-\mathscr{W}'T^{+}u+\mathscr{L}^{+}\gamma^{+}u=T^{+}\mathscr{P}Au \quad \text{on } \partial\Omega. \tag{3.15}$$

For some functions f,Ψ and Φ let us consider a more general, indirect integral relation associated with equation (3.13),

$$u+\mathscr{R}u-V\Psi+W\Phi=\mathscr{P}f \quad \text{in } \Omega. \tag{3.16}$$

Lemma 2. *Let $u\in H^{1}(\Omega), f\in L_{2}(\Omega), \Psi\in H^{-\frac{1}{2}}(\partial\Omega), \Phi\in H^{\frac{1}{2}}(\partial\Omega)$ satisfy equation (3.16). Then u belongs to $H^{1,0}(\Omega;A)$, and is a solution of PDE (3.6), i.e., $Au=f$ in Ω, and $V(\Psi-T^{+}u)(y)-W(\Phi-\gamma^{+}u)(y)=0, \quad y\in\Omega$.*

Proof. The proof follows in the similar way as in the corresponding proof in 3D case in [CMN09a, Lemma 4.1]. □

Let us define the subspaces $H^{s}_{**}(\partial\Omega)=\{g\in H^{s}(\partial\Omega):\langle g,1\rangle_{\partial\Omega}=0\}$ (see, e.g., [Ste08, p. 147]). The following result is proved in [DM15, Lemma 2].

Lemma 3. *(i) Let either $\Psi^{*}\in H^{-\frac{1}{2}}(\partial\Omega)$ and $\operatorname{diam}(\Omega)<1$, or $\Psi^{*}\in H^{-\frac{1}{2}}_{**}(\partial\Omega)$. If $V\Psi^{*}(y)=0$, in Ω , then $\Psi^{*}=0$ on $\partial\Omega$.*
(ii) Let $\Phi^{*}\in H^{\frac{1}{2}}(\partial\Omega)$. If $W\Phi^{*}(y)=0$, in Ω, then $\Phi^{*}=0$ on $\partial\Omega$.

Let $\mathscr{L}^{+}_{\Delta}$ denote the operator $\mathscr{L}^{+}$ for the constant-coefficient case $a\equiv 1$. Its null-space in $H^{\frac{1}{2}}(\partial\Omega)$ includes non-zero functions. One can see this by taking $u(y)\equiv 1$ in Ω in the trace of the third Green identity (3.15) for the case $a\equiv 1$. Let us introduce the operator, $\hat{\mathscr{L}}g:=\left[\mathscr{L}^{+}+\frac{\partial a}{\partial n}(-\frac{1}{2}I+\mathscr{W})\right]g=\mathscr{L}^{+}_{\Delta}(ag)$ on $\partial\Omega$.

Theorem 2. *Let $\partial\Omega$ be an infinitely smooth boundary curve .*
(i) The pseudo-differential operator

$$\hat{\mathscr{L}}:H^{\frac{1}{2}}_{**}(\partial\Omega)\to H^{-\frac{1}{2}}(\partial\Omega), \tag{3.17}$$

is invertible.
(ii) The operator

$$\mathscr{L}^{+}-\hat{\mathscr{L}}:H^{\frac{1}{2}}(\partial\Omega)\to H^{-\frac{1}{2}}(\partial\Omega), \tag{3.18}$$

is bounded and compact.

Proof. For $g\in H^{\frac{1}{2}}(\partial\Omega)$ using the jump relation, one can obtain the relation, $\mathscr{L}^{+}g=\mathscr{L}_{\Delta}(ag)-\frac{\partial a}{\partial n}(-\frac{1}{2}I+\mathscr{W})g$, or $\hat{\mathscr{L}}g=\mathscr{L}_{\Delta}(ag)=\mathscr{L}^{+}g+\frac{\partial a}{\partial n}(-\frac{1}{2}I+\mathscr{W})g$. The hypersingular boundary integral operator $\mathscr{L}_{\Delta}:H^{\frac{1}{2}}(\partial\Omega)\to H^{-\frac{1}{2}}(\partial\Omega)$ is bounded (see, [DM15, Theorem 1 for $s=\frac{1}{2}$] and the references therein). Moreover, it is $H^{\frac{1}{2}}_{**}(\partial\Omega)$-elliptic (cf. [Ste08, Eq. (6.38)]). Then the Lax-Milgram lemma implies the $H^{\frac{1}{2}}_{**}(\partial\Omega)-$invertibility of $\mathscr{L}_\Delta$. Hence the invertibility of (3.17) follows. The operator $\mathscr{W}: H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{3}{2}}(\partial\Omega)$ is continuous (see e.g, [DM15, Theorem 1, for $s=\frac{1}{2}$]), and since $H^{\frac{3}{2}}(\partial\Omega)$ is continuously embedded in $H^{\frac{1}{2}}(\partial\Omega)$, using the relation $\mathscr{L}^+ - \hat{\mathscr{L}} = -\frac{\partial a}{\partial n}(\frac{1}{2}I - \mathscr{W})$, we obtain continuity of the operator $\mathscr{L}^+ - \hat{\mathscr{L}}: H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$. The embedding $H^{\frac{1}{2}}(\partial\Omega) \subset H^{-\frac{1}{2}}(\partial\Omega)$ is compact, which implies that the operator $\mathscr{L}^+ - \hat{\mathscr{L}}: H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is compact. □

Corollary 1. *The operator* $\mathscr{L}^+ : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ *is Fredholm operator of index zero.*

Proof. The operator $\mathscr{L}_\Delta : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is Fredholm of index zero (see e.g.,[McL00, Theorem 7.8]). Thus, the operator $\hat{\mathscr{L}}: H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is Fredholm of index zero. Since $\mathscr{L}^+ = (\mathscr{L}^+ - \hat{\mathscr{L}}) + \hat{\mathscr{L}}$, it is the sum of a compact operator and a Fredholm operator of index zero and hence is also a Fredholm operator of index zero (cf. eg. [McL00, Theorem 2.26]). □

3.3 BDIEs for Neumann BVP

To reduce the variable-coefficient Neumann BVP (3.6)-(3.7) to a *segregated* boundary-domain integral equation system, let us denote the unknown trace by $\varphi := \gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$ and further consider φ as formally independent of u.

Assuming that the function u satisfies the PDE $Au = f$, by substituting the Neumann condition into the third Green identity (3.13) and either into its trace (3.14) or into its co-normal derivative (3.15) on $\partial\Omega$, we can reduce the BVP (3.6)-(3.7) to two different systems of Boundary-Domain Integral equations for the unknown functions $u \in H^{1,0}(\Omega;A)$ and $\varphi := \gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$.

BDIE system (N1), obtained from the third Green identity (3.13) and its conormal derivative (3.15), is

$$\begin{aligned} u + \mathscr{R}u + W\varphi &= G_0 \quad \text{in } \Omega, \\ T^+\mathscr{R}u + \mathscr{L}^+\varphi &= T^+G_0 - \psi_0 \quad \text{on } \partial\Omega, \end{aligned}$$

where

$$G_0 := \mathscr{P}f + V\psi_0 \quad \text{in } \Omega. \tag{3.19}$$

Also note that, $T^+G_0 = T^+\mathscr{P}f + \mathscr{W}'\psi_0 + \frac{1}{2}\psi_0$. The system can be written in matrix operator form as $\mathfrak{N}^1\mathscr{U} = \mathscr{G}^1$ where $\mathscr{U} := [u,\varphi]^t \in H^{1,0}(\Omega;A) \times H^{\frac{1}{2}}(\partial\Omega)$ and

$$\mathfrak{N}^1 = \begin{bmatrix} I+\mathscr{R} & W \\ T^+\mathscr{R} & \mathscr{L}^+ \end{bmatrix}, \qquad \mathscr{G}^1 = \begin{bmatrix} G_0 \\ T^+G_0 - \psi_0 \end{bmatrix}. \tag{3.20}$$

From the mapping properties of the operators V and $\mathscr{P}$ in [DM15, Theorems 1 and 3], we get the inclusion $G_0 \in H^{1,0}(\Omega;A)$, and Definition 1 implies $T^+G_0 \in H^{-\frac{1}{2}}(\partial\Omega)$. Therefore, $\mathscr{G}^1$ belongs to $H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved in $\mathfrak{N}^1$, the operator $\mathfrak{N}^1 : H^{1,0}(\Omega;A)\times H^{\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ is bounded.

BDIE system (N2), obtained from the third Green identity (3.13) and its trace (3.14), is

$$u+\mathscr{R}u+W\varphi = G_0 \quad \text{in } \Omega,$$
$$\gamma^+\mathscr{R}u+\frac{1}{2}\varphi+\mathscr{W}\varphi = \gamma^+G_0 \quad \text{on } \partial\Omega,$$

where G_0 is given by the relation (3.19). In a matrix form it can be written as $\mathfrak{N}^2\mathscr{U} = \mathscr{G}^2$ where

$$\mathfrak{N}^2 = \begin{bmatrix} I+\mathscr{R} & W \\ \gamma^+\mathscr{R} & \frac{1}{2}I+\mathscr{W}\end{bmatrix}, \qquad \mathscr{G}^2 = \begin{bmatrix} G_0 \\ \gamma^+G_0\end{bmatrix}. \tag{3.21}$$

By the trace theorem $\gamma^+G_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Therefore, $\mathscr{G}^2 \in H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$. The mapping properties of operators involved in $\mathfrak{N}^2$ imply the operator $\mathfrak{N}^2 : H^{1,0}(\Omega;A)\times H^{\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ is bounded.

Remark 3. Let $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\operatorname{diam}(\Omega)<1$, or $\psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$. Then $\mathscr{G}^2=0$ if and only if $(f,\psi_0)=0$. Indeed, the latter equality evidently implies the former. Inversely, if $\mathscr{G}^2=0$, then $G_0=0$ and $\gamma^+G_0=0$. Then, $G_0=0$ implies $\mathscr{P}f+V\psi_0 = 0$ in Ω. Multiplying by a, taking into consideration that $aV=V_\Delta$ is harmonic and applying Laplace operator, we get $f=0$. And hence $V\psi_0=0$ in Ω. Then by Lemma 3(i), $\psi_0=0$ on $\partial\Omega$.

3.4 Equivalence and Invertibility Theorems

In the following theorem we shall see the equivalence of the original boundary value problem with the boundary-domain integral equation systems.

Theorem 3. *Let $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f\in L_2(\Omega)$ satisfy the solvability condition (3.8).*

(i) If some $u\in H^1(\Omega)$ solves the Neumann BVP (3.6)-(3.7), then the pair (u,φ), where

$$\varphi = \gamma^+u \in H^{\frac{1}{2}}(\partial\Omega), \tag{3.22}$$

solves the BDIE systems (N1) and (N2).

(ii) If a pair $(u,\varphi)\in H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE systems (N1), then u solves BDIE system (N2) and Neumann BVP (3.6)-(3.7), and φ satisfies (3.22).

(iii) If a pair $(u,\varphi)\in H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ *solves the BDIE systems (N2), and* $\mathrm{diam}(\Omega)<1$*, then u solves BDIE system (N1) and Neumann BVP (3.6)-(3.7), and* φ *satisfies (3.22).*

(iv) The homogeneous BDIE systems (N1) and (N2) have linearly independent solutions spanned by $\mathscr{U}^0=(u^0,\varphi^0)^T=(1,1)^T$ *in* $H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$*. Condition (3.8) is necessary and sufficient for solvability of the nonhomogeneous BDIE system (N1) and, if* $\mathrm{diam}(\Omega)<1$*, also of the system (N2), in* $H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$*.*

Proof. (i) Let $u\in H^1(\Omega)$ be a solution of the Neumann BVP (3.6)-(3.7). Since $f\in L_2(\Omega)$, then $u\in H^{1,0}(\Omega;A)$. Setting $\varphi=\gamma^+u$, and recalling how BDIE systems (N1) and (N2) were constructed, we obtain that (u,φ) solves them.

(ii) Let now a pair $(u,\varphi)\in H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ solve the system (N1) or (N2). Due to the first equations in the BDIE systems, the hypotheses of Lemma 2 are satisfied implying that u belongs to $H^{1,0}(\Omega;A)$ and solves PDE (3.6) in Ω, while the following equation also holds,

$$V(\psi_0-T^+u)(y)-W(\varphi-\gamma^+u)(y)=0,\quad y\in\Omega. \tag{3.23}$$

If a pair $(u,\varphi)\in H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ solves the system (N1) then taking the co-normal derivatives of the first equation in (N1) and subtracting the second from it, we get $T^+u=\psi_0$ on $\partial\Omega$. Thus the Neumann condition is satisfied, and using it in (3.23) we get $W(\varphi-\gamma^+u)(y)=0$ on $y\in\partial\Omega$. Lemma 3(ii) implies $\varphi=\gamma^+u$ on $\partial\Omega$.

(iii) Let now a pair $(u,\varphi)\in H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ solve the system (N2). Taking the trace of the first equation in (N2) and subtracting the second from it, we get $\varphi=\gamma^+u$ on $\partial\Omega$. Then inserting it in (3.23) gives $V(\psi_0-T^+u)(y)=0$ on $y\in\partial\Omega$. Lemma 3(i) implies, $\psi_0=T^+u$ on $\partial\Omega$. Hence the Neumann condition is satisfied.

(iv) Theorem 1 along with items (i)-(iii) implies the claims of item (iv). □

Note that Theorem 3(iv) implies that the operators

$$\mathfrak{N}^1: H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega), \tag{3.24}$$

and

$$\mathfrak{N}^2: H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega), \tag{3.25}$$

are not injective.

Theorem 4. *Operators* (3.24) *and* (3.25) *are Fredholm operator with zero index. Moreover, the kernels (null-spaces) of these operator are spanned by the element* $(u^0,\varphi^0)=(1,1)$ *and thus the kernels and co-kernels of the operators are one-dimensional.*

Proof. (i) By Corollary 1, the operator $\hat{\mathscr{L}}: H^{\frac{1}{2}}(\partial\Omega)\to H^{-\frac{1}{2}}(\partial\Omega)$ is a Fredholm operator with zero index. Therefore, the operator

$$\mathfrak{N}^1_0:=\begin{bmatrix} I & W\\ 0 & \hat{\mathscr{L}}\end{bmatrix}: H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega),$$

is also Fredholm with zero index. By the properties of operators $\mathscr{R}$ and $T^+\mathscr{R}$ (see, e.g, [DM15, Corollary 2], and the reference therein) and Theorem 2(ii), the operator

$$\mathfrak{N}^1 - \mathfrak{N}^1_0 = \begin{bmatrix} \mathscr{R} & 0 \\ T^+\mathscr{R} & \mathscr{L}^+ - \hat{\mathscr{L}} \end{bmatrix} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega),$$

is compact, implying that operator (3.24) is Fredholm with zero index ([McL00, Theorem 2.26]).

(ii) Let us consider the operator

$$\mathfrak{N}^2_0 = \begin{bmatrix} I & W \\ 0 & \frac{1}{2}I \end{bmatrix}$$

Then, $\mathfrak{N}^2_0 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators, $I : H^1(\Omega) \to H^1(\Omega)$, and $I : H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$. Due to the compactness properties of $\mathscr{R}, \gamma^+\mathscr{R}$ and $\mathscr{W}$ (see, e.g, [DM15, Corollary 1 and 2], the operator

$$\mathfrak{N}^2 - \mathfrak{N}^2_0 = \begin{bmatrix} \mathscr{R} & 0 \\ \gamma^+\mathscr{R} & \mathscr{W} \end{bmatrix} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega),$$

is compact. This implies that the operator (3.25) is a Fredholm operator with zero index.

(iii) The kernels of the operators are spanned by the element $(u^0, \varphi^0) = (1, 1)$ due to Theorem 3(iv). □

The following theorem describes the co-kernels of these operators. The proof is similar to the proofs of the corresponding assertions for 3D case in [Mik15]. Let $\gamma^* : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-2}_{\partial\Omega}$ denote the operator adjoint to the trace operator $\gamma : H^{2-s}(\mathbb{R}^2) \to H^{\frac{3}{2}-s}(\partial\Omega)$, for $s < \frac{3}{2}$.

Theorem 5. *Let* $\operatorname{diam}(\Omega) < 1$ *and* $u^0(x) = 1$.

(i) The co-kernel of operator (3.24) is spanned over the functional $g^{*1} \in \widetilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ *defined as*

$$g^{*1} := \begin{pmatrix} -a\gamma^* \mathscr{V}_\Delta^{-1} \gamma^+ u^0 \\ 0 \end{pmatrix}. \tag{3.26}$$

(ii) The co-kernel of operator (3.25) is spanned over the functional $g^{*2} \in \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ *defined as*

$$g^{*2} = \begin{pmatrix} -a\gamma^* (\frac{1}{2} + \mathscr{W}'_\Delta) \mathscr{V}_\Delta^{-1} \gamma^+ u^0 \\ -a(\frac{1}{2} - \mathscr{W}'_\Delta) \mathscr{V}_\Delta^{-1} \gamma^+ u^0 \end{pmatrix}. \tag{3.27}$$

3.5 Perturbed BDIE systems for the Neumann problem

Theorem 3 implies that even when the solvability condition (3.8) is satisfied, the solutions of both BDIE systems, (N1) and (N2), are not unique. By Theorem 4, in turn, the BDIE left-hand side operators, $\mathfrak{N}^1$ and $\mathfrak{N}^2$, have non-zero kernels and thus are not invertible. To find a solution (u,φ) from uniquely solvable BDIE systems with continuously invertible left-hand side operators, let us consider, following [Mik99], some BDIE systems obtained form (N1) and (N2) by finite-dimensional operator perturbations, cf.[Mik15] for the three-dimensional case. Below we use the notations $\mathscr{U}=(u,\varphi)^T$ and $|\partial\Omega| := \int_{\partial\Omega} ds$.

Perturbation of BDIE system (N1): Let us introduce the perturbed counterparts of the BDIE system (N1),

$$\hat{\mathfrak{N}}^1\mathscr{U}=\mathscr{G}^1, \tag{3.28}$$

$$\hat{\mathfrak{N}}^1 := \mathfrak{N}^1+\mathring{\mathfrak{N}}^1 \text{ and } \mathring{\mathfrak{N}}^1\mathscr{U}(y) := g^0(\mathscr{U})\mathscr{Z}^1(y)=\frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)ds\begin{pmatrix}a^{-1}(y)\\0\end{pmatrix},$$

that is,

$$g^0(\mathscr{U}) := \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)ds, \quad \mathscr{Z}^1(y) := \begin{pmatrix}a^{-1}(y)u^0(y)\\0\end{pmatrix}.$$

For the functional g^{*1} given by (3.26) in Theorem 5, since the operator $\mathscr{V}_\Delta^{-1}: H^{\frac{1}{2}}(\partial\Omega)\to H^{-\frac{1}{2}}(\partial\Omega)$ is positive definite (with additional condition $\operatorname{diam}(\Omega)<1$) and $u^0(x)=1$, there exists a positive constant C such that

$$\begin{aligned}g^{*1}(\mathscr{Z}^1) &= \langle -a\gamma^{+*}\mathscr{V}_\Delta^{-1}\gamma^+u^0, a^{-1}u^0\rangle_\Omega = -\langle\mathscr{V}_\Delta^{-1}\gamma^+u^0,\gamma^+u^0\rangle_{\partial\Omega}\\ &\le -C\|\gamma^+u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \le -C\|\gamma^+u^0\|^2_{L_2(\partial\Omega)} = -C|\partial\Omega|^2<0.\end{aligned} \tag{3.29}$$

Further, for $\mathscr{U}^0=(u^0,\varphi^0)^T=(1,1)^T$ in $H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$,

$$g^0(\mathscr{U}^0)=\frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma^+u^0ds=1. \tag{3.30}$$

Due to (3.29) and (3.30), [Mik99, Lemma 2], implies the following assertion.

Theorem 6. *Let* $\operatorname{diam}(\Omega)<1$*, then*

(i) The operator $\hat{\mathfrak{N}}^1: H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ *is continuous and continuously invertible.*
(ii) If $g^{*1}(\mathscr{G}^1)=0$ *(or condition (3.8) for* $\mathscr{G}^1$ *in form (3.20) is satisfied), then the unique solution of the perturbed BDIDE system (3.28) gives a solution of the original BDIE system (N1) such that*

$$g^0(\mathscr{U})=\frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi ds=0.$$

Perturbation of BDIE system (N2): Let us introduce the perturbed counterparts of the BDIE system (N2)

$$\hat{\mathfrak{N}}^2\mathscr{U} = \mathscr{G}^2, \tag{3.31}$$

where

$$\hat{\mathfrak{N}}^2 := \mathfrak{N}^2 + \mathring{\mathfrak{N}}^2 \text{ and } \mathring{\mathfrak{N}}^2\mathscr{U}(y) := g^0(\mathscr{U})\mathscr{F}^2(y) = \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)ds\begin{pmatrix} a^{-1}(y) \\ \gamma^+ a^{-1}(y)\end{pmatrix},$$

that is,

$$g^0(\mathscr{U}) := \frac{1}{|\partial\Omega|}\int_{|\partial\Omega|}\varphi(x)ds, \quad \mathscr{F}^2(y) := \begin{pmatrix} a^{-1}(y)u^0(y) \\ \gamma^+[a^{-1}u^0](y)\end{pmatrix}.$$

For the functional g^{*2} given by (3.27) in Theorem 5(ii), since the operator $\mathscr{V}_\Delta^{-1}: H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is positive definite (with additional condition $\mathrm{diam}(\Omega) < 1$) and $u^0(x) = 1$, there exists a positive constant C such that

$$\begin{aligned} g^{*2}(\mathscr{F}^2) &= \langle -a\gamma^{+*}(\frac{1}{2} + \mathscr{W}_\Delta')\mathscr{V}_\Delta^{-1}\gamma^+u^0, a^{-1}u^0\rangle_\Omega \\ &\quad + \langle -a(\frac{1}{2} - \mathscr{W}_\Delta')\mathscr{V}_\Delta^{-1}\gamma^+u^0, \gamma^+(a^{-1}u^0)\rangle_{\partial\Omega} \\ &= -\langle(\frac{1}{2} + \mathscr{W}_\Delta')\mathscr{V}_\Delta^{-1}\gamma^+u^0 + (\frac{1}{2} - \mathscr{W}')\mathscr{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\rangle_{\partial\Omega} = -\langle\mathscr{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\rangle_{\partial\Omega} \\ &\le -C\|\gamma^+u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \le -C\|\gamma^+u^0\|^2_{L_2(\partial\Omega)} = -C|\partial\Omega|^2 < 0. \end{aligned} \tag{3.32}$$

Due to (3.32) and (3.30), [Mik99, Lemma 2], implies the following assertion.

Theorem 7. *Let* $\mathrm{diam}(\Omega) < 1$*, then*

(i) The operator $\hat{\mathfrak{N}}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ *is continuous and continuously invertible.*

(ii) If $g^{*2}(\mathscr{G}^2) = 0$ *(or condition (3.8) for* $\mathscr{G}^2$ *in form (3.21) is satisfied), then the unique solution of the perturbed BDIDE system (3.31) gives a solution of the original BDIE system (N2) such that*

$$g^0(\mathscr{U}) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma^+u\,ds = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi\,ds = 0.$$

3.6 Conclusion

In this paper, we have considered the interior Neumann boundary value problem for a variable-coefficient PDE in a 2D domain, where the right-hand side function is from $L_2(\Omega)$ and the Neumann data from the space $H^{-\frac{1}{2}}(\partial\Omega)$. The BVP was reduced to two systems of Boundary-Domain Integral Equations and their equivalence to the original BVP was shown.

The null-spaces of the corresponding BDIE systems are not trivial. Moreover, the BDIE systems are neither uniquely nor unconditionally solvable. The BDIE operators for the Neumann BVP are bounded but only Fredholm with zero index. The kernels and co-kernels of these operators were analysed, and appropriate finite-dimensional perturbations were constructed to make the perturbed operators invertible and provide a solution of the original BDIE systems and of the Neumann BVP.

In a similar way one can consider also the 2D versions of the BDIEs for other BVP problems in interior and exterior domains, united BDIEs as well as the localised BDIEs, which were analysied for 3D case in [CMN09a, CMN13, Mik06, CMN09b].

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References

[CMN09a] Chkadua, O., Mikhailov, S.E., and Natroshvili, D.: Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficent, I: Equivalence and Invertibility. *J. Integral Equations and Appl.*, **21**, 499–543 (2009).

[CMN09b] Chkadua, O. , Mikhailov, S.E., and Natroshvili, D.: Analysis of some localized boundary-domain integral equations. *J. Integral Equations and Appl.*, **21**, 405–445 (2009).

[CMN13] Chkadua, O., Mikhailov, S.E., and Natroshvili, D.: Analysis of direct segregated boundary-domain integral equations for variable-coefficient mixed BVPs in exterior domains. *Analysis and Appl.*, **11**, No.4, 1350006(1-33) (2013).

[Cos88] Costabel, M.: Boundary integral operators on Lipschiz domains: Elementary results. *SIAM J. Mathematical Analysis*, **19**, 613–626 (1988).

[DM15] Dufera, T.T. and Mikhailov, S.E.: Analysis of Boundary-Domain Integral Equations for Variable-Coefficient Drichlet BVP in 2D. In: *Integral Methods in Science and Engineering: Computational and Analytic Aspects*, C. Constanda and A. Kirsch (eds.), Birkhäuser, Boston (2015), pp. 163-175.

[McL00] McLean, W.: *Strongly Elliptic Systems and Boundary Integral Equations.* Cambrige University Press, Cambrige (2000).

[Mik99] Mikhailov, S.E.: Finite-dimensional perturbations of linear operators and some applications to boundary integral equations. *Int. J. Engineering Analysis with Boundary Elements*, **23**, 805–813 (1999).

[Mik02] Mikhailov, S.E.: Localized boundary-domain integral formulations for problems with variable coefficients. *Int. J. Engineering Analysis with Boundary Elements*, **26**, 681–690 (2002).

[Mik06] Mikhailov, S.E.: Analysis of united boundary-domain integral and integro-differential equations for a mixed BVP with variable coefficients. *Math. Meth. Appl. Sci.*, **29**, 715–739 (2006).

[Mik11] Mikhailov, S.E.: Traces, extensions and co-normal derivatives for elliptic systems on Lipschiz domains. *Journal of Mathematical Analysis and Applications*, **378**, 324–342 (2011).

[Mik15] Mikhailov, S.E.: Analysis of Segregated Boundary-Domain Integral Equations for Variable-Coefficient Dirichlet and Neumann Problems with General Data. *ArXiv:1509.03501*, 1-32 (2015).

[Mir70] Miranda, C.: *Partial Differential Equations of Elliptic Type.* Springer, Berlin - Heidelberg - New York, 2nd edition, (1970).

[Ste08] Steinbach, O.: *Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements.* Springer, New York, NY (2008).