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BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR VARIABLE-COEFFICIENT HELMHOLTZ BVPs IN 2D

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Abstract
In this paper, we construct boundary-domain integral equations (BDIEs) of the Dirichlet and mixed boundary value problems for a two-dimensional variable-coefficient Helmholtz equation. Using an appropriate parametrix, these problems are reduced to several BDIE systems. It is shown that the BVPs and the formulated BDIE systems are equivalent. Fredholm properties and unique solvability and invertibility of BDIE systems are investigated in appropriate Sobolev spaces.

Keywords Helmholtz equation · Dirichlet problem · Mixed problem · Parametrix · Boundary-domain integral equations · Equivalence · Fredholm properties

1 Introduction

Many problems of mathematical physics and engineering such as the ones associated with steady-state oscillations (mechanical, acoustic, electromagnetic, etc.) lead to the Helmholtz equation. Since the fundamental solution of the constant-coefficient Helmholtz equation is known explicitly, the boundary value problems (BVPs) for this equation can be reduced to the boundary integral equations (BIEs), which have the advantage that the dimension of the problem is reduced by one and the BIEs could be effectively solved numerically.

In applications, such as seismic or medical imaging, the coefficients in the Helmholtz equation become variable [26]. For such partial differential equations (PDEs) with variable coefficients, a fundamental solution is generally not available in explicit form, preventing the reduction of BVPs for such PDEs to explicit BIEs. Instead, one can use a parametrix (Levi function), which is more widely available, to reduce the variable-coefficient BVPs to either segregated or united direct boundary-domain integral or integro-differential equations [19], BDIEs or BDIDEs. These equations are well studied for Dirichlet, Neumann, and mixed (Dirichlet-Neumann) BVPs for variable-coefficient second-order scalar elliptic PDE

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$$Au(x) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x), \qquad x \in \Omega \tag{1.1}$$

in 3D [6–9, 20, 23, 24] as well as in 2D [4, 5, 13].
However, this is not the case for the parametrix-based system of BDIEs for variable-coefficient Helmholtz equation

$$Au(x) + k^2(x)u(x) = f(x), \qquad x \in \Omega \tag{1.2}$$

where $k(x)$ is a real function of x, $a(x)$ is a known variable coefficient, u is an unknown function, and $f \in L^2(\Omega)$ is a given function. Note that when $\Omega = \mathbb{R}^n$ and $k(x)$ is constant outside a bounded domain, (1.2) can be reduced to the Lippmann-Schwinger type integral equation; see, e.g., [15, Section 8] for the case when $a(x)$ is a constant in $\mathbb{R}^n$, and [11, 16, 17] for the case when $a(x)$ is a constant only outside a bounded domain in $\mathbb{R}^n$. In both cases, the integral equations can be considered as special cases of BDIEs. We also mention [1], where the numerical solutions of BDIE and BDIDE of the mixed problem for PDE (1.2) are given (without analysis of the equivalence to the BVP or the solution existence and uniqueness).
Applying the previously developed techniques for the operator A in (1.1), in this paper, we shall construct and investigate BDIE systems for the Dirichlet and mixed (Dirichlet-Neumann) BVPs associated with PDE (1.2) in appropriate function spaces in the two-dimensional case. The BDIEs in the n–dimensional cases with $n \geq 3$ can also be analyzed in a similar way, although the scaling with the parameter r_0 in the parametrix will not be needed in such cases because the invertibility of the standard single layer potential operator will not depend on the domain size then.

2 Preliminaries

Let Ω be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$. The set of all infinitely differentiable functions on Ω with compact support is denoted by $\mathcal{D}(\Omega)$. The function space $\mathcal{D}'(\Omega)$ consists of all continuous linear functionals over $\mathcal{D}(\Omega)$. The space of restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^2)$ is denoted by $\mathcal{D}(\overline{\Omega})$. The space $H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, denotes the Bessel potential space, and $H^{-s}(\mathbb{R}^2)$ is the dual space to $H^s(\mathbb{R}^2)$. We define $H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) : u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^2)\}$, and $H^1_0(\Omega)$ is the space of functions in $H^1(\Omega)$ with zero traces on $\partial\Omega$. By $H^s(\partial\Omega)$, we denote the Bessel potential spaces on the boundary $\partial\Omega$ (cf., e.g., [18]).
For the scalar elliptic differential operator A given by

$$A = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial}{\partial x_i}\right], \tag{2.1}$$

we consider the Helmholtz equation

$$Au(x) + k^2(x)u(x) = f(x), \qquad x \in \Omega$$

where $k(x)$ is a real function of x, $a(x)$ is a known variable coefficient, u is an unknown function, and f is a given function in Ω. We assume that $a, k \in C^\infty(\overline{\Omega})$ and $0 < a_0 < a(x) < a_1 < \infty$ for some constants a_0 and a_1, for all $x \in \Omega$.
Let us denote $A_k := A + k^2$. Following the definition given, e.g., in [10, 14, 21], for $s \in \mathbb{R}$ the subspace $H^{s,0}(\Omega;A_k)$ of $H^s(\Omega)$ is defined as

$$H^{s,0}(\Omega;A_k) := \{g \in H^s(\Omega) : \ A_k g \in L^2(\Omega)\}, \tag{2.2}$$

with the norm $\|g\|^2_{H^{s,0}(\Omega;A_k)} := \|g\|^2_{H^s(\Omega)} + \|A_k g\|^2_{L^2(\Omega)}$. Since $A_k u - Au = k^2 u \in L^2(\Omega)$ for $u \in H^1(\Omega)$, we get $H^{1,0}(\Omega;A_k) = H^{1,0}(\Omega;A)$. Moreover, if $s_2 \leq s_1$, then we have the embedding $H^{s_1,0}(\Omega;A_k) \subset H^{s_2,0}(\Omega;A_k)$.
For $u \in H^s(\Omega)$, $s > 3/2$, the corresponding classical co-normal derivative operator on $\partial\Omega$ in the sense of traces denoted by T^{c+} is given by

$$T^{c+}u(x) = \sum_{i=1}^{2} a(x)n_i(x)\gamma^+ \frac{\partial u(x)}{\partial x_i}, \tag{2.3}$$

where $n(x)$ is the outward (to Ω) unit normal vector at the point $x \in \partial\Omega$, and γ^+ is the trace operator.
For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, from the Gauss-Ostrogradsky theorem, we get

$$\int_\Omega v(x)Au(x)dx = -\sum_{i=1}^{2}\int_\Omega a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}dx + \int_{\partial\Omega} T^{c+}u(x)\gamma^+ v(x)dS_x.$$

From this, we obtain the first Green identity:

$$\mathcal{E}_k(u,v) = -\int_\Omega v(x)A_k u(x)dx + \int_{\partial\Omega} T^{c+}u(x)\gamma^+ v(x)dS_x, \tag{2.4}$$

where

$$\mathcal{E}_k(u,v) := \int_\Omega a(x)\nabla u(x)\cdot\nabla v(x)dx - \int_\Omega k^2(x)u(x)v(x)dx$$

is the symmetric bilinear form.

Even though the classical co-normal derivative is, generally, not defined for $u \in H^s(\Omega)$, $s < 3/2$ (some examples are provided in [23, Appendix A]), there exists the following continuous extension of this definition of the classical co-normal derivative hinted by the first Green identity (2.4), for $u \in H^{s,0}(\Omega;A_k)$, $1/2 < s < 3/2$ (see, e.g., [10], [18, Lemma 4.3],[21, 22]).

Definition 2.1 For $u \in H^{s,0}(\Omega;A_k)$, $1/2 < s < 3/2$, the (canonical) co-normal derivative $T^+u \in H^{s-\frac{3}{2}}(\partial\Omega)$ is defined in the following weak form:

$$\begin{aligned}\langle T^+u, w\rangle_{\partial\Omega} &:= \langle A_k u, \gamma^{-1}w\rangle_\Omega + \mathcal{E}_k(u,\gamma^{-1}w)\\ &= \langle Au, \gamma^{-1}w\rangle_\Omega + \mathcal{E}_0(u,\gamma^{-1}w), \qquad \forall w \in H^{\frac{3}{2}-s}(\partial\Omega).\end{aligned} \tag{2.5}$$

In (2.5) and further on, $\gamma^{-1} : H^{\frac{3}{2}-s}(\partial\Omega) \to H^{2-s}(\Omega)$ is a bounded right inverse to the trace operator $\gamma : H^{2-s}(\Omega) \to H^{\frac{3}{2}-s}(\partial\Omega)$, the notation $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality brackets between the spaces $H^{s-\frac{3}{2}}(\partial\Omega)$ and $H^{\frac{3}{2}-s}(\partial\Omega)$, while $\langle\cdot,\cdot\rangle_\Omega$ denotes the duality brackets between the spaces $H^{s-1}(\Omega)$ and $H^{1-s}(\Omega)$, extending the usual L^2-inner products.
The operator $T^+ : H^{s,0}(\Omega;A_k) \to H^{s-3/2}(\partial\Omega)$ is continuous for $s > 1/2$. Moreover, as we observe from [21, Corollary 3.14],

$$T^+u = T^{c+}u \text{ for } u \in H^s(\Omega), \quad s > 3/2. \tag{2.6}$$

By [10, Lemma 3.4], the first Green identity (2.4) in the form

$$\langle T^+u, \gamma^+v\rangle_{\partial\Omega} = \mathcal{E}_k(u,v) + \langle A_k u, v\rangle_\Omega. \tag{2.7}$$

holds for $u \in H^{1,0}(\Omega;A_k)$ and $v \in H^1(\Omega)$.
Interchanging the roles of u and v in the first Green identity (2.7) for $u \in H^1(\Omega)$ and $v \in H^{1,0}(\Omega;A_k)$, we obtain the first Green identity for v,

$$\langle T^+v, \gamma^+u\rangle_{\partial\Omega} = \mathcal{E}_k(v,u) + \langle A_k v, u\rangle_\Omega. \tag{2.8}$$

Then, subtracting (2.8) from (2.7), we obtain the *second Green identity* for $u, v \in H^{1,0}(\Omega;A_k)$,

$$\langle A_k u, v\rangle_\Omega - \langle A_k v, u\rangle_\Omega = \langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T^+v, \gamma^+u\rangle_{\partial\Omega}. \tag{2.9}$$

3 Parametrix-based potential operators

Definition 3.1 A function $P(x, y)$ is a *parametrix* for the operator A_k if

$$(A_k)_x P(x, y) = \delta(x - y) + R_k(x, y),$$

where δ is the Dirac-delta distribution, while $R_k(x, y)$ is a remainder possessing at most a weak singularity at $x = y$.

Based on [19], the function

$$P(x, y) = \frac{1}{a(y)} P_\Delta(x, y) = \frac{1}{2\pi a(y)} \ln\left(\frac{|x - y|}{r_0}\right), \quad x, y \in \mathbb{R}^2,$$

where $r_0 > 0$ is a constant parameter, is a parametrix for the operator A. Note that

$$P_\Delta(x, y) = \frac{1}{2\pi} \ln\left(\frac{|x - y|}{r_0}\right), \quad r_0 > 0, \quad x, y \in \mathbb{R}^2 \tag{3.1}$$

is a fundamental solution of the Laplace operator, Δ (cf., e.g., [18, Theorem 8.1]). We can also take $P(x, y)$ as a parametrix for the operator A_k. Then, the corresponding remainder function becomes

$$R_k(x, y) = k^2(x) P(x, y) + R(x, y), \qquad x, y \in \mathbb{R}^2, \tag{3.2}$$

where

$$R(x, y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x - y|^2} \frac{\partial a(x)}{\partial x_i}, \qquad x, y \in \mathbb{R}^2,$$

is the remainder function for the operator A and is weakly singular due to the smoothness of the function $a(x)$. Hence, $R_k(x, y)$ is also weakly singular, and thus, $P(x, y)$ is, indeed, a parametrix for the operator A_k.

3.1 Surface potentials

The single and the double layer surface potential operators corresponding to the parametrix $P(x, y)$ are defined for $y \notin \partial\Omega$ as

$$Vg(y) := -\int_{\partial\Omega} P(x, y) g(x) dS_x, \qquad Wg(y) := -\int_{\partial\Omega} \left[T_x^+ P(x, y)\right] g(x) dS_x$$

where the integrals are understood as the appropriate dual products if the scalar density function g is not integrable.

The corresponding boundary integral (pseudodifferential) operators of direct surface values of the single layer potential $\mathcal{V}$ and of the double layer potential $\mathcal{W}$, and the co-normal derivatives of the single layer potential $\mathcal{W}'$, and of the double layer potential $\mathcal{L}^+$, for $y \in \partial\Omega$, are

$$\begin{aligned} \mathcal{V}g(y) &:= -\int_{\partial\Omega} P(x, y) g(x) dS_x, & \mathcal{W}g(y) &:= -\int_{\partial\Omega} \left[T_x^+ P(x, y)\right] g(x) dS_x, \\ \mathcal{W}'g(y) &:= -\int_{\partial\Omega} \left[T_y^+ P(x, y)\right] g(x) dS_x, & \mathcal{L}^+ g(y) &:= T^+ W g(y). \end{aligned} \tag{3.3}$$

Let $V_\Delta, W_\Delta, \mathcal{V}_\Delta, \mathcal{W}_\Delta$ and $\mathcal{L}_\Delta^+$ denote the potentials and the boundary operators corresponding to the Laplace operator Δ. That is, the subscript Δ means that the corresponding surface potentials are constructed by means of the fundamental solution (3.1) of the Laplace operator Δ. Then, the following relations hold in 2D (cf. [13]).

$$Vg = \frac{1}{a} V_\Delta g, \qquad Wg = \frac{1}{a} W_\Delta(ag) \tag{3.4}$$

$$\mathcal{V}g = \frac{1}{a}\mathcal{V}_\Delta g, \qquad \mathcal{W}g = \frac{1}{a}\mathcal{W}_\Delta(ag), \tag{3.5}$$

$$\mathcal{W}'g = \mathcal{W}'_\Delta g + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]\mathcal{V}_\Delta g, \tag{3.6}$$

$$\mathcal{L}^+g = \mathcal{L}^+_\Delta(ag) + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]\gamma^+W_\Delta(ag). \tag{3.7}$$

The following two theorems are proved in [13, Theorem 1 and Theorem 2].

Theorem 3.2 *Let $u \in H^{-\frac{1}{2}}(\partial\Omega)$ and $v \in H^{\frac{1}{2}}(\partial\Omega)$. Then, the following relations hold for $y \in \partial\Omega$,*

$$\gamma^+Vu(y) = \mathcal{V}u(y), \tag{3.8}$$

$$\gamma^+Wv(y) = -\frac{1}{2}v(y) + \mathcal{W}v(y), \tag{3.9}$$

$$T^+Vu(y) = \frac{1}{2}u(y) + \mathcal{W}'u(y). \tag{3.10}$$

Theorem 3.3 *For $s \in \mathbb{R}$, the following operators are continuous,*

$$\begin{aligned} V &: H^s(\partial\Omega) \to H^{s+\frac{3}{2}}(\Omega), \\ W &: H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega), \\ \mathcal{V}, \mathcal{W}, \mathcal{W}' &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega). \end{aligned}$$

These theorems imply the following assertion.

Corollary 3.4 *The following operators are continuous,*

$$\begin{aligned} V &: H^s(\partial\Omega) \to H^{s+\frac{3}{2},0}(\Omega; A_k), \qquad s \geq -\frac{1}{2}, \\ W &: H^s(\partial\Omega) \to H^{s+\frac{1}{2},0}(\Omega; A_k), \qquad s \geq \frac{1}{2}. \end{aligned}$$

Proof For $g \in H^s(\partial\Omega)$, from Theorem 3.3, we get $Vg \in H^{s+\frac{3}{2}}(\Omega)$. Then,

$$\begin{aligned} A(Vg) &= \Delta(aVg) - \sum_{i=1}^{2}\partial_i(Vg\partial_i a) \\ &= \Delta(V_\Delta g) - \sum_{i=1}^{2}\partial_i(Vg\partial_i a) = -\sum_{i=1}^{2}\partial_i(Vg\partial_i a) \end{aligned}$$

belongs to $L^2(\Omega)$ if $s \geq -\frac{1}{2}$. A similar proof holds for the operator W as well. □

The compactness of the following surface potential operators in Corollary 3.5 follows directly from Theorem 3.3 and Rellich compact embedding theorem, see, e.g., [18, Theorem 3.27].

Corollary 3.5 *For $s \in \mathbb{R}$, the following operators are compact,*

$$\mathcal{V}, \mathcal{W}, \mathcal{W}' : H^s(\partial\Omega) \to H^s(\partial\Omega).$$

For $s \in \mathbb{R}$, $\Gamma_1 \subset \partial\Omega$, let us define the following subspaces of the space $H^s(\partial\Omega)$, (see, e.g., [27, pp 147):

$$\widetilde{H}^s(\Gamma_1) := \{\psi \in H^s(\partial\Omega) : \operatorname{supp}\psi \subset \overline{\Gamma}_1\},$$
$$H^s_{**}(\partial\Omega) := \{\psi \in H^s(\partial\Omega) : \langle\psi, 1\rangle_{\partial\Omega} = 0\},$$
$$\widetilde{H}^s_{**}(\Gamma_1) := \{\psi \in \widetilde{H}^s(\Gamma_1) : \langle\psi, 1\rangle_{\partial\Omega} = 0\}.$$

Corollary 3.5 implies the following assertion.

Theorem 3.6 *Let Γ_1 and Γ_2 be non-empty smooth pieces of a curve $\partial\Omega$. Then the operators*

$$r_{\Gamma_2}\mathcal{V}, r_{\Gamma_2}\mathcal{W}, r_{\Gamma_2}\mathcal{W}' : \widetilde{H}^s(\Gamma_1) \longrightarrow H^s(\Gamma_2). \tag{3.11}$$

are compact for $s \in \mathbb{R}$.

In (3.11) and further on, r_{Γ_1}, r_{Γ_2}, etc. denote the corresponding restriction operators.

3.1.1 Invertibility of single layer potential operator on $\partial\Omega$

It is well known that the kernel of the operator

$$\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \tag{3.12}$$

with the parameter $r_0 = 1$ in (3.1), is non-zero for some domains in 2D (see, e.g., [27, Theorem 6.22 proof]). Then, the first relation in (3.5) and scaling imply a non-zero kernel also for $\mathcal{V}$ with $r_0 > 0$, for some domains Ω.
The following result is proved in [13, Theorem 4].

Theorem 3.7 *Let $\psi \in H^{-1/2}_{**}(\partial\Omega)$. If $\mathcal{V}\psi = 0$ on $\partial\Omega$, then $\psi = 0$.*

On the other hand, choosing for a given Ω an appropriate parameter r_0, one can get the zero kernel for $\mathcal{V}$ not only on the subspace $H^{-1/2}_{**}(\partial\Omega)$ but also on the entire space $H^{-1/2}(\partial\Omega)$ and then prove the following invertibility assertion.

Theorem 3.8 *Let $\Omega \subset \mathbb{R}^2$ with $r_0 > \operatorname{diam}(\Omega)$. Then, the operator*

$$\mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega) \tag{3.13}$$

is invertible.

Proof For $r_0 = 1$, the assertion is available in [13, Theorem 5]. For arbitrary $r_0 > \operatorname{diam}(\Omega)$, the invertibility of operator (3.12) can be obtained by scaling the result for $r_0 = 1$, e.g., from Theorem 6.23 and reasoning following it in [27]. Then, the first relation in (3.5) implies the invertibility of operator (3.13) as well. (Cf. also [2, Theorem 5.2] and [3, Theorem 6].) □

Similarly to [5, Corollary 2.7], we obtain the following assertion.

Corollary 3.9 *Let Γ_1 be non-empty relatively open connected part of a curve $\partial\Omega$. Then, the operator*

$$r_{\Gamma_1}\mathcal{V} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$$

is bounded and Fredholm of index zero.

Theorem 3.10 *Let Γ_1 be a non-empty relatively open connected part of the boundary curve $\partial\Omega$ with $r_0 > \mathrm{diam}(\Gamma_1)$. Then, the operator $r_{_{\Gamma_1}}\mathcal{V} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ has a bounded inverse.*

Proof Taking into account the condition $r_0 > \mathrm{diam}(\Gamma_1)$, we can follow the proof of [5, Corollary 2.9]. □

Due to (3.9) and the second relation in (3.4), relation (3.7) can also be written as

$$\widehat{\mathcal{L}}g = \Big[\mathcal{L}^{+} + \frac{\partial a}{\partial n}\Big(-\frac{1}{2}I + \mathcal{W}\Big)\Big]g, \quad \text{on } \partial\Omega, \tag{3.14}$$

where $\widehat{\mathcal{L}}g := \mathcal{L}^{+}_{\Delta}(ag)$.

The following assertion is available, e.g., in [5, Theorem 2.10] (cf. [6, Theorem 3.6] in the 3D case).

Theorem 3.11 *Let Γ_1 be a non-empty open smooth part of $\partial\Omega$.*

(i) *Then, the operator*

$$r_{_{\Gamma_1}}\widehat{\mathcal{L}} : \widetilde{H}^{\frac{1}{2}}(\Gamma_1) \to H^{-\frac{1}{2}}(\Gamma_1)$$

is continuously invertible.

(ii) *Moreover, the operator*

$$r_{_{\Gamma_1}}(\mathcal{L}^{+} - \widehat{\mathcal{L}}) : \widetilde{H}^{\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$$

is bounded, and the operator

$$r_{_{\Gamma_1}}(\mathcal{L}^{+} - \widehat{\mathcal{L}}) : \widetilde{H}^{\frac{1}{2}}(\Gamma_1) \to H^{-\frac{1}{2}}(\Gamma_1)$$

is compact.

3.2 Volume potentials

Similar to [4, 6, 19], we define the parametrix-based logarithmic and remainder volume potential operators, respectively, as

$$\mathcal{P}g(y) := \int_{\Omega} P(x,y)g(x)dx, \qquad \mathcal{R}_k g(y) := \int_{\Omega} R_k(x,y)g(x)dx, \quad y \in \mathbb{R}^2.$$

Remark 3.12 As for the layer potentials, let $\mathcal{P}_{\Delta}$ denote the logarithmic potential for the operator Δ, that is,

$$\mathcal{P}_{\Delta}g(y) := \int_{\Omega} P_{\Delta}(x,y)g(x)dx, \quad x, y \in \mathbb{R}^2,$$

where P_{Δ} is the fundamental solution (3.1). Then,

$$\mathcal{P}g = \frac{1}{a}\mathcal{P}_{\Delta}g, \quad \mathcal{R}_k g = \mathcal{P}(k^2 g) + \mathcal{R}g, \tag{3.15}$$

where $\mathcal{R}$ is the parametrix-based remainder volume potential operator for the remainder function $R(x, y)$ and, see [13, 19],

$$\mathcal{R}g = -\frac{1}{a}\sum_{i=1}^{2}\partial_i[\mathcal{P}_\Delta(g\partial_i a)],$$

where $\partial_i = \partial/\partial x_i$.

Theorem 3.13 *Let Ω be a bounded open region in $\mathbb{R}^2$ with closed, infinitely smooth boundary $\partial\Omega$. The following operators are continuous.*

$$\mathcal{P} : H^s(\Omega) \longrightarrow H^{s+2}(\Omega), \qquad s > -\frac{1}{2}; \tag{3.16}$$

$$\mathcal{R} : H^s(\Omega) \longrightarrow H^{s+1}(\Omega), \qquad s > -\frac{1}{2}; \tag{3.17}$$

$$\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s+1}(\Omega), \qquad s > -\frac{1}{2}; \tag{3.18}$$

$$\gamma^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > -\frac{1}{2}; \tag{3.19}$$

$$T^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\partial\Omega), \qquad s > \frac{1}{2}. \tag{3.20}$$

Proof For (3.16) and (3.17), we refer to [13, Theorem 3]. From the second relation in (3.15), together with (3.16) and (3.17), we obtain the continuity of (3.18). The continuity of the operators (3.19) and (3.20) is the direct consequence of the trace theorem, Definition 2.1 of the co-normal derivative and relation (2.6). □

Corollary 3.14 *The following operators are continuous.*

$$\mathcal{P} : H^s(\Omega) \longrightarrow H^{s+2,0}(\Omega;A_k), \qquad s \geq 0; \tag{3.21}$$

$$\mathcal{R} : H^s(\Omega) \longrightarrow H^{s+1,0}(\Omega;A_k), \qquad s \geq 1; \tag{3.22}$$

$$\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s+1,0}(\Omega;A_k), \qquad s \geq 1. \tag{3.23}$$

Proof Using the continuity of operators (3.16), (3.17), and (3.18) and the space definition (2.2), we obtain the continuity of operators (3.21), (3.22), and (3.23). □

Corollary 3.15 *The following operators are compact.*

$$\mathcal{R}_k : H^s(\Omega) \longrightarrow H^s(\Omega), \quad s > -\frac{1}{2}; \tag{3.24}$$

$$\gamma^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}; \tag{3.25}$$

$$T^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{3}{2}}(\partial\Omega) \quad s > \frac{1}{2}. \tag{3.26}$$

Proof The compactness of operators (3.24), (3.25), and (3.26) follows from (3.18), (3.19), and (3.20) and the Rellich compact embedding theorem. □

Corollary 3.16 *The operator*

$$\mathcal{R}_k - \mathcal{R} : H^s(\Omega) \longrightarrow H^{s,0}(\Omega;A_k), \quad s > 0, \tag{3.27}$$

is compact.

Proof From the second equation in (3.15), we see that $\mathcal{R}_k g - \mathcal{R} g = \mathcal{P}(k^2 g)$. Then, by (3.16) for $s > -1/2$, the operator $\mathcal{R}_k - \mathcal{R} : H^s(\Omega) \to H^{s+2}(\Omega)$ is continuous, and the operator $\mathcal{R}_k - \mathcal{R} : H^s(\Omega) \to H^s(\Omega)$ is compact. Hence, the operator $\Delta(\mathcal{R}_k - \mathcal{R}) : H^s(\Omega) \to H^s(\Omega)$ is also continuous for $s > -1/2$, and the operator $\Delta(\mathcal{R}_k - \mathcal{R}) : H^s(\Omega) \to L^2(\Omega)$ is compact for $s > 0$.

Further, $A_k(\mathcal{R}_k - \mathcal{R}) = a\Delta(\mathcal{R}_k - \mathcal{R}) + \sum_{j=1}^2 (\partial_j a)\partial_j(\mathcal{R}_k - \mathcal{R}) + k^2(\mathcal{R}_k - \mathcal{R})$. The operator $\partial_j(\mathcal{R}_k - \mathcal{R}) : H^s(\Omega) \to H^{s+1}(\Omega)$ is continuous, and hence, the operator $\partial_j(\mathcal{R}_k - \mathcal{R}) : H^s(\Omega) \to H^0(\Omega)$ is compact for $s > -1/2$. Thus, the operator $A_k(\mathcal{R}_k - \mathcal{R}) : H^s(\Omega) \to L^2(\Omega)$ is compact for the operator A_k with infinitely smooth coefficients, for $s > 0$. Hence, the compactness of operator (3.27) follows from the space definition (2.2). □

Corollary 3.17 *Let Γ_1 and Γ_2 be non-empty, non-intersecting parts of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. Then, the operators*

$$r_{\Gamma_1}\gamma^+\mathcal{R}, r_{\Gamma_1}\gamma^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma_1),$$

$$r_{\Gamma_1}T^+\mathcal{R}, r_{\Gamma_1}T^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{3}{2}}(\Gamma_1),$$

are compact for $s > \frac{1}{2}$.

Proof Theorem 3.13 implies that the following operators are continuous for $s > \frac{1}{2}$:

$$r_{\Gamma_1}\gamma^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s+\frac{1}{2}}(\Gamma_1),$$

$$r_{\Gamma_1}T^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma_1).$$

Then, the proof follows from the compactness of the embeddings $H^{s+\frac{1}{2}}(\Gamma_1) \subset H^{s-\frac{1}{2}}(\Gamma_1)$ and $H^{s-\frac{1}{2}}(\Gamma_1) \subset H^{s-\frac{3}{2}}(\Gamma_1)$. The proof holds true also for $k = 0$. □

4 The third Green identity

As, e.g., in [4–6, 13], for $u \in H^{1,0}(\Omega;A_k)$, we substitute $P(x, y)$ for $v(x)$ in Green's second identity (2.9) for $\Omega \setminus \overline{B_\epsilon(y)}$, where $B_\epsilon(y)$ is a disc of radius ϵ centered at y and take the limit as $\epsilon \to 0$ to arrive at the parametrix-based third Green identity

$$u + \mathcal{R}_k u - VT^+u + W\gamma^+u = \mathcal{P}A_k u \quad \text{in} \quad \Omega. \tag{4.1}$$

Taking the trace of (4.1) and using relations (3.8) and (3.9), we obtain

$$\frac{1}{2}\gamma^+u + \gamma^+\mathcal{R}_k u - \mathcal{V}T^+u + \mathcal{W}\gamma^+u = \gamma^+\mathcal{P}A_k u \quad \text{on} \quad \partial\Omega. \tag{4.2}$$

From Corollaries 3.4 and 3.14, we see that each term of (4.1) belongs to $H^{1,0}(\Omega;A_k)$. Now, taking the co-normal derivative of (4.1) and using relation (3.10), we get

$$\frac{1}{2}T^+u + T^+\mathcal{R}_k u - \mathcal{W}'T^+u + T^+W\gamma^+u = T^+\mathcal{P}A_k u \quad \text{on} \quad \partial\Omega. \tag{4.3}$$

If $u \in H^1(\Omega)$ is a solution of equation $A_k u = f$ in Ω, where $f \in L^2(\Omega)$, then (4.1) becomes

$$u + \mathcal{R}_k u - VT^+u + W\gamma^+u = \mathcal{P}f \quad \text{in} \quad \Omega. \tag{4.4}$$

For some functions f, Ψ, and Φ, let us consider a more general indirect integral relation associated with (4.4),

$$u + \mathcal{R}_k u - V\Psi + W\Phi = \mathcal{P}f \quad \text{in } \Omega. \tag{4.5}$$

Lemma 4.1 *Let $u \in H^1(\Omega), f \in L^2(\Omega), \Psi \in H^{-\frac{1}{2}}(\partial\Omega), \Phi \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy (4.5). Then, u belongs to $H^{1,0}(\Omega;A_k)$ and is a solution of PDE $A_k u = f$ in Ω, and*

$$V(\Psi - T^+u)(y) - W(\Phi - \gamma^+u)(y) = 0, \quad y \in \Omega. \tag{4.6}$$

Proof As in [6, Lemma 4.1] in the 3D case for $k = 0$, from Corollaries 3.4 and 3.14, we conclude that all terms in (4.5) except u belong to $H^{1,0}(\Omega;A_k)$. Then, (4.5) implies that u belongs to $H^{1,0}(\Omega;A_k)$ as well. Now, let us prove the remaining results.

Subtracting (4.5) from (4.1), we obtain

$$V\Psi^* - W\Phi^* = \mathcal{P}[A_k u - f] \quad \text{in } \Omega, \tag{4.7}$$

where $\Psi^* := T^+u - \Psi$ and $\Phi^* := \gamma^+u - \Phi$. Multiplying equality (4.7) by $a(y)$ and using relation (3.4) and (3.15), we get

$$V_\Delta \Psi^* - W_\Delta(a\Phi^*) = \mathcal{P}_\Delta[A_k u - f], \quad \text{in } \Omega. \tag{4.8}$$

The application of the Laplace operator Δ to (4.8) gives

$$A_k u - f = 0 \quad \text{in } \Omega. \tag{4.9}$$

This shows that u solves the differential equation $A_k u = f$ in Ω.

Substituting (4.9) into (4.7) leads to (4.6). □

Lemma 4.2

(i) *Let $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \operatorname{diam}(\Omega)$. If $V\Psi^* = 0$ in Ω, then $\Psi^* = 0$ on $\partial\Omega$.*
(ii) *Let $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$ and $r_0 > 0$. If $W\Phi^* = 0$ in Ω, then $\Phi^* = 0$ on $\partial\Omega$.*

Proof The assertion was proved in [13, Lemma 2] for $r_0 = 1$. Taking into account Theorem 3.8, we follow the proof of [13, Lemma 2] almost word for word to obtain the assertion for arbitrary $r_0 > 0$. □

Lemma 4.3 *Let $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where Γ_1 and Γ_2 are non-empty, non-intersecting relatively open parts of the boundary curve $\partial\Omega$. Let $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\Gamma_2)$ and $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_1)$ with $r_0 > \operatorname{diam}(\Gamma_1)$. If*

$$V\Psi^*(y) - W\Phi^*(y) = 0, \quad y \in \Omega, \tag{4.10}$$

then $\Psi^* = 0$ and $\Phi^* = 0$ on $\partial\Omega$.

Proof Keeping in mind [18, Theorem 8.16], we follow the proof of [6, Lemma 4.2 (iii)] (See also [5, Lemma 2.12], [2, Lemma 5.8], [3, Lemma 3]). □

Remark 4.4 The results of Lemmas 4.2 and 4.3 with no restriction on the parameter r_0 can be similarly obtained if $\Psi^* \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ and $\Psi^* \in \widetilde{H}_{**}^{-\frac{1}{2}}(\Gamma_1)$, respectively.

5 Boundary-domain integral equations of the Dirichlet BVP

Consider the Dirichlet BVP

$$\begin{aligned} A_k u &= f \qquad \text{in } \Omega, \\ \gamma^+ u &= \varphi_0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{5.1}$$

for unknown function $u \in H^1(\Omega)$, where $f \in L^2(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ are given functions. The first equation is understood in the distribution sense.

Let us derive and analyze BDIE systems for the Dirichlet BVP (5.1).

To reduce the variable-coefficient Dirichlet BVP (5.1) to segregated BDIE systems, we denote the unknown co-normal derivative as $\psi := T^+ u$ and further consider ψ as formally independent of u.

5.1 BDIE system (D1)

We substitute $A_k u$ and $\gamma^+ u$ from the Dirichlet BVP (5.1) into (4.1) and into its trace (4.2) to reduce the Dirichlet BVP (5.1) to the BDIE system (D1) with the unknowns u and ψ:

$$\begin{aligned} u + \mathcal{R}_k u - V\psi &= F_0 \qquad \text{in } \Omega, \\ \gamma^+ \mathcal{R}_k u - \mathcal{V}\psi &= \gamma^+ F_0 - \varphi_0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{D1}$$

where

$$F_0 = \mathcal{P} f - W\varphi_0 \qquad \text{in } \Omega. \tag{5.2}$$

The matrix form of system (D1) is $\mathcal{A}_k^1 \mathcal{U} = \mathcal{F}^1$, where $\mathcal{U} = (u, \psi)^t \in H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega)$,

$$\mathcal{A}_k^1 = \begin{bmatrix} I + \mathcal{R}_k & -V \\ \gamma^+ \mathcal{R}_k & -\mathcal{V} \end{bmatrix}, \qquad \mathcal{F}^1 = \begin{bmatrix} F_0 \\ \gamma^+ F_0 - \varphi_0 \end{bmatrix}. \tag{5.3}$$

From the mapping properties of $\mathcal{P}$ and W provided in Section 3, we get $F_0 \in H^{1,0}(\Omega; A_k)$. Moreover, the trace theorem implies that $\gamma^+ F_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Therefore, $\mathcal{F}^1 \in H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved in (5.3) (see Section 3), the following operators are bounded:

$$\mathcal{A}_k^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{5.4}$$

$$\mathcal{A}_k^1 : H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega). \tag{5.5}$$

Remark 5.1 $\mathcal{F}^1 = \mathbf{0}$ if and only if $(f, \varphi_0) = \mathbf{0}$.

Proof If $\mathcal{F}^1 = \mathbf{0}$, then $F_0 = 0$ and $\gamma^+ F_0 + \varphi_0 = 0$. Consequently, $\varphi_0 = 0$ on $\partial\Omega$. From this and $F_0 = 0$, we obtain that $\mathcal{P} f = 0$ in Ω, and hence, $f = 0$ in Ω. The reverse implication is trivial. □

5.2 BDIE system (D2)

This system is obtained by substituting $A_k u$ and $\gamma^+ u$ from the Dirichlet BVP (5.1) into (4.1) and into its co-normal derivative (4.3), with the unknowns u and ψ:

$$\begin{aligned} u + \mathcal{R}_k u - V\psi &= F_0 && \text{in } \Omega, \\ \frac{1}{2}\psi + T^+\mathcal{R}_k u - \mathcal{W}'\psi &= T^+ F_0 && \text{on } \partial\Omega, \end{aligned} \tag{D2}$$

where F_0 is the relation (5.2). The system (D2) can be written in matrix form as

$$\mathcal{A}_k^2 \mathcal{U} = \mathcal{F}^2,$$

where

$$\mathcal{A}_k^2 := \begin{bmatrix} I + \mathcal{R}_k & -V \\ T^+\mathcal{R}_k & \frac{1}{2}I - \mathcal{W}' \end{bmatrix}, \qquad \mathcal{F}^2 = \begin{bmatrix} F_0 \\ T^+ F_0 \end{bmatrix},$$

and F_0 is given by (5.2). The following operators are bounded:

$$\mathcal{A}_k^2 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{5.6}$$

$$\mathcal{A}_k^2 : H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega). \tag{5.7}$$

Remark 5.2 $\mathcal{F}^2 = \mathbf{0}$ if and only if $(f, \varphi_0) = \mathbf{0}$.

Proof If $\mathcal{F}^2 = \mathbf{0}$, then $F_0 = 0$. From which we get

$$0 = \Delta(aF_0) = \Delta(\mathcal{P}_\Delta f) + \Delta W_\Delta(\varphi_0) = f \quad \text{in } \Omega.$$

Then, the condition $F_0 = 0$ gives $W_\Delta(\varphi_0) = 0$ and Lemma 4.2*(ii)* implies that $\varphi_0 = 0$ on $\partial\Omega$. The reverse implication is trivial. □

6 Equivalence, Fredholm properties, and invertibility for BDIEs of the Dirichlet BVP

In this section, we first prove the equivalence of the Dirichlet BVP (5.1) to the BDIE systems (D1) and (D2), and then we show the necessary conditions for the invertibility of the two corresponding operators to the BDIE systems.

Theorem 6.1 *Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L^2(\Omega)$.*

(i) *If some $u \in H^1(\Omega)$ solves the BVP (5.1), then the pair $(u, \psi)^t$, where*

$$\psi = T^+ u \in H^{-\frac{1}{2}}(\partial\Omega), \tag{6.1}$$

solves BDIE systems (D1) and (D2).

(ii) *Let $r_0 >$ diam(Ω). If a pair $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D1), then u solves BVP (5.1) and ψ satisfies (6.1).*

(iii) *Let $r_0 > 0$. If a pair $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D2), then u solves BVP (5.1), and ψ satisfies (6.1).*

Proof To prove *(i)*, we let $u \in H^1(\Omega)$ be a solution of the BVP (5.1). Since $A_k u = f \in L^2(\Omega)$, we get $u \in H^{1,0}(\Omega; A_k)$. Setting $\psi = T^+ u$ and recalling how BDIE system (D1) and (D2) are constructed, we obtain that the couple $(u, \psi)^t$ solves the systems.

To prove *(ii)* and *(iii)*, let us assume first that a pair $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves system (D1) or (D2). Due to the first equation in the BDIE systems, the hypotheses of Lemma 4.1 are satisfied implying that u belongs to $H^{1,0}(\Omega;A_k)$ and solves the PDE in the BVP (5.1) in Ω. Moreover, the equation

$$W(\varphi_0 - \gamma^+ u)(y) - V(\psi - T^+ u)(y) = 0, \quad y \in \Omega, \tag{6.2}$$

holds.

To prove the remaining parts of *(ii)*, we let $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system (D1). Taking the trace of the first equation in (D1) and subtracting the second equation from it, we get the Dirichlet boundary condition

$$\gamma^+ u = \varphi_0 \quad \text{on } \partial\Omega,$$

and substituting this in equation (6.2) we obtain

$$V(\psi - T^+ u)(y) = 0, \qquad y \in \Omega.$$

Since $r_0 > \operatorname{diam}(\Omega)$, from Lemma 4.2 *(i)*, we get $\psi = T^+ u$.

To complete *(iii)*, we let $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system (D2). It is already shown that $u \in H^{1,0}(\Omega;A_k)$. Moreover, all the remaining terms in the first equation of (D2) belong to $H^{1,0}(\Omega;A_k)$ due to the mapping properties of the operators involved (see Section 3). Then, taking the co-normal derivative of the first equation in (D2) and subtracting the second one from it, we get

$$\psi = T^+ u \quad \text{on } \partial\Omega.$$

Then, inserting this in (6.2) gives

$$W(\varphi_0 - \gamma^+ u)(y) = 0, \quad y \in \Omega,$$

and Lemma 4.2 *(ii)* implies $\varphi_0 = \gamma^+ u$ on $\partial\Omega$. □

Theorem 6.1 implies the following two corollaries.

Corollary 6.2 *Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L^2(\Omega)$.*

(i) *Let $r_0 > \operatorname{diam}(\Omega)$. If a pair $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D1), it solves BDIE system (D2).*
(ii) *Let $r_0 > 0$. If a pair $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D2), it solves BDIE system (D1).*

Corollary 6.3

(i) *Let $r_0 > \operatorname{diam}(\Omega)$. The homogeneous counterpart of BDIE system (D1) has a non-trivial solution in $H^1 \times H^{-\frac{1}{2}}(\partial\Omega)$ if and only if the homogeneous counterpart of the Dirichlet problem (5.1) has a non-trivial solution in $H^1(\Omega)$.*
(ii) *Let $r_0 > 0$. The homogeneous counterpart of BDIE system (D2) has a non-trivial solution in $H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ if and only if the homogeneous counterpart of the Dirichlet problem (5.1) has a non-trivial solution in $H^1(\Omega)$.*

Let us now analyze the Fredholm properties of operators (5.4), (5.5), (5.6), and (5.7). As a bi-product, we also prove the invertibility of the corresponding operators for $k = 0$.

Theorem 6.4

(i) *If $r_0 > \operatorname{diam}(\Omega)$, then operator (5.4) is Fredholm with zero index.*
(ii) *If $r_0 > 0$, then operator (5.6) is Fredholm with zero index.*

Proof **(i)** Let $r_0 > \mathrm{diam}(\Omega)$. Let us consider the auxiliary operator

$$\mathcal{A}^1_* := \begin{bmatrix} I & -V \\ 0 & -\mathcal{V} \end{bmatrix}.$$

Then, the operator $\mathcal{A}^1_* : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded. It is invertible due to the invertibility of its diagonal operators

$$I : H^1(\Omega) \to H^1(\Omega) \quad \text{and} \quad \mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega),$$

see Theorem 3.8. Due to the mapping properties of the operators involved, the operator $\mathcal{A}^1_k - \mathcal{A}^1_* : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ where

$$\mathcal{A}^1_k - \mathcal{A}^1_* = \begin{bmatrix} \mathcal{R}_k & 0 \\ \gamma^+\mathcal{R}_k & 0 \end{bmatrix},$$

is compact. Thus, operator (5.4) is Fredholm with index zero.

(ii) The operator $\mathcal{A}^2_* : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, where

$$\mathcal{A}^2_* = \begin{bmatrix} I & -V \\ 0 & \frac{1}{2}I \end{bmatrix}$$

is bounded. It is also invertible due to the invertibility of its diagonal operators

$$I : H^1(\Omega) \longrightarrow H^1(\Omega) \quad \text{and} \quad I : H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega).$$

By Corollaries 3.5 and 3.15, the operator

$$\mathcal{A}^2_k - \mathcal{A}^2_* : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega),$$

where

$$\mathcal{A}^2_k - \mathcal{A}^2_* = \begin{bmatrix} \mathcal{R}_k & 0 \\ T^+\mathcal{R}_k & -\mathcal{W}' \end{bmatrix},$$

is compact. This implies that operator (5.6) is a Fredholm operator of index zero. □

Let us consider the particular cases of operators (5.4), (5.5), (5.6), and (5.7), for $k = 0$, that is,

$$\mathcal{A}^1_0 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{6.3}$$

$$\mathcal{A}^1_0 : H^{1,0}(\Omega;A) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{1,0}(\Omega;A) \times H^{\frac{1}{2}}(\partial\Omega), \tag{6.4}$$

$$\mathcal{A}^2_0 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{6.5}$$

$$\mathcal{A}^2_0 : H^{1,0}(\Omega;A) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;A) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{6.6}$$

where

$$\mathcal{A}^1_0 = \begin{bmatrix} I + \mathcal{R} & -V \\ \gamma^+\mathcal{R} & -\mathcal{V} \end{bmatrix}, \quad \mathcal{A}^2_0 = \begin{bmatrix} I + \mathcal{R} & -V \\ T^+\mathcal{R} & \frac{1}{2}I - \mathcal{W}' \end{bmatrix}.$$

Theorem 6.5

(i) *If* $r_0 > \mathrm{diam}(\Omega)$, *then operators* (6.3) *and* (6.4) *are invertible.*
(ii) *If* $r_0 > 0$, *then operators* (6.5) *and* (6.6) *are invertible.*

Proof The theorem for $r_0 = 1$ was proved in [13, Theorems 7 and 8]. Here, we update the proof for arbitrary $r_0 > 0$.

It is well known that the homogeneous Dirichlet problem (5.1) with $k = 0$, that is, with $A_k = A$, where the operator A is given by (2.1) and $0 < a_0 < a(x) < a_1 < \infty$, has only the trivial solution in $H^{1,0}(\Omega;A)$ and $H^1(\Omega)$. This can be obtained, e.g., from the first Green identity (2.7). Then, the equivalence Theorem 6.1 implies that operators (6.3), (6.4), (6.5), and (6.6) are injective. By Theorem 6.4, operators (6.3) and (6.5) are Fredholm operators with zero index. Then, the injectivity of operators (6.3) and (6.5) implies their invertibility (see, e.g., [18, Theorem 2.27]).

To prove invertibility of operator (6.4), we remark that for any $\mathcal{F}^1 \in H^{1,0}(\Omega;A) \times H^{\frac{1}{2}}(\partial\Omega)$, a solution of the equation $\mathcal{A}_0^1\mathcal{U} = \mathcal{F}^1$ can be written as $\mathcal{U} = (\mathcal{A}_0^1)^{-1}\mathcal{F}^1$, where $(\mathcal{A}_0^1)^{-1} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is the continuous inverse to operator (6.3). But due to Lemma 4.1 the first equation of system (D1) with $k = 0$ implies that $\mathcal{U} = (\mathcal{A}_0^1)^{-1}\mathcal{F}^1 \in H^{1,0}(\Omega;A) \times H^{-\frac{1}{2}}(\partial\Omega)$ and moreover, the operator $(\mathcal{A}_0^1)^{-1} : H^{1,0}(\Omega;A) \times H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;A) \times H^{-\frac{1}{2}}(\partial\Omega)$ is continuous, which implies invertibility of operator (6.4).

The invertibility of operator (6.6) is proved in a similar fashion. □

Now, we are in the position to prove an analog of Theorem 6.4 for operators (5.5) and (5.7).

Theorem 6.6

(i) *If* $r_0 > \mathrm{diam}(\Omega)$, *then operator* (5.5) *is Fredholm with zero index.*
(ii) *If* $r_0 > 0$, *then operator* (5.7) *is Fredholm with zero index.*

Proof By Theorem 6.5, we see that operators (6.4) and (6.6) are invertible. Due to Corollary 3.16, the operators

$$\mathcal{A}_k^1 - \mathcal{A}_0^1 : H^{1,0}(\Omega;A_k) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;A_k) \times H^{\frac{1}{2}}(\partial\Omega),$$
$$\mathcal{A}_k^2 - \mathcal{A}_0^2 : H^{1,0}(\Omega;A_k) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega;A_k) \times H^{-\frac{1}{2}}(\partial\Omega),$$

where

$$\mathcal{A}_k^1 - \mathcal{A}_0^1 = \begin{bmatrix} \mathcal{R}_k - \mathcal{R} & 0 \\ \gamma^+(\mathcal{R}_k - \mathcal{R}) & 0 \end{bmatrix}, \quad \mathcal{A}_k^2 - \mathcal{A}_0^2 = \begin{bmatrix} \mathcal{R}_k - \mathcal{R} & 0 \\ T^+(\mathcal{R}_k - \mathcal{R}) & 0 \end{bmatrix},$$

are compact, implying that operators (5.5) and (5.7) are Fredholm operators with index zero. □

Corollary 6.7

(i) *Let* $r_0 > \mathrm{diam}(\Omega)$. *The homogeneous counterpart of the Dirichlet problem* (5.1) *has only the trivial solution in* $H^1(\Omega)$ *if and only if operators* (5.4) *and* (5.5) *are invertible.*
(ii) *Let* $r_0 > 0$. *The homogeneous counterpart of the Dirichlet problem* (5.1) *has only the trivial solution in* $H^1(\Omega)$ *if and only if operators* (5.6) *and* (5.7) *are invertible.*

Proof If the homogeneous counterpart of the Dirichlet problem (5.1) has only the trivial solution in $H^1(\Omega)$, by Corollary 6.3(i), the operators (5.4) and (5.5) will be injective. Hence, these operators become invertible due to Theorem 6.4.

Conversely, if the operator (5.4) or (5.5) is invertible, the homogeneous counterpart of BDIE system (D1) can have only the trivial solution in $H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, and hence, the result follows from Corollary 6.3 (i).

For operators (5.6) and (5.7), the proof is similar. □

7 Boundary-domain integral equations of the mixed BVP

Let $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$, where $\partial\Omega_D$ and $\partial\Omega_N$ are non-empty, relatively open, non-intersecting parts of $\partial\Omega$. We will derive and analyze the system of BDIEs for the following mixed BVP

$$\begin{aligned} A_k u &= f && \text{in} \quad \Omega, \\ \gamma^+ u &= \varphi_0 && \text{on} \quad \partial\Omega_D, \\ T^+ u &= \psi_0 && \text{on} \quad \partial\Omega_N, \end{aligned} \tag{7.1}$$

for unknown function $u \in H^1(\Omega)$, where $f \in L^2(\Omega)$, $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ are given functions. Similar to the 3D case in [6] and the 2D case with $k = 0$ in [5], we let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of the given function φ_0 from $\partial\Omega_D$ to $\partial\Omega$ and ψ_0 from $\partial\Omega_N$ to $\partial\Omega$, respectively. Then, an arbitrary extension $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$ preserving the function space can be represented as $\Phi = \Phi_0 + \varphi$ with $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$; and $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$ as $\Psi = \Psi_0 + \psi$ with $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$.

Considering (4.1), and restrictions of either (4.2) or (4.3) on the appropriate parts of $\partial\Omega$, we reduce the BVP (7.1) to four different BDIE systems. In each case, we substitute f for $A_k u$, $\Phi = \Phi_0 + \varphi$ for the boundary trace $\gamma^+ u$ and $\Psi = \Psi_0 + \psi$ for the co-normal derivative $T^+ u$, where Φ_0 and Ψ_0 are considered known while the triple $(u, \psi, \varphi) \in H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ is to be found.

7.1 BDIE system (M11)

This system is obtained by considering the third Green identity (4.1) in Ω, the restriction of its trace (4.2) on $\partial\Omega_D$, and the restriction of its co-normal derivative (4.3) on $\partial\Omega_N$, with respect to the unknowns u, ψ, and φ:

$$\begin{aligned} u + \mathcal{R}_k u - V\psi + W\varphi &= F_0, && \text{in} \quad \Omega, \\ \gamma^+ \mathcal{R}_k u - \mathcal{V}\psi + \mathcal{W}\varphi &= \gamma^+ F_0 - \varphi_0, && \text{on} \quad \partial\Omega_D, \\ T^+ \mathcal{R}_k u - \mathcal{W}'\psi + \mathcal{L}^+ \varphi &= T^+ F_0 - \psi_0, && \text{on} \quad \partial\Omega_N, \end{aligned} \tag{M11}$$

where

$$F_0 = \mathcal{P} f + V\Psi_0 - W\Phi_0 \qquad \text{in} \quad \Omega. \tag{7.2}$$

The BDIE system (M11) can be rewritten in matrix form as

$$\mathcal{M}_k^{11} \mathcal{U} = \mathcal{F}^{11}, \tag{7.3}$$

where $\mathcal{U} = (u, \psi, \varphi)^t \in H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and

$$\mathcal{M}_k^{11} = \begin{bmatrix} I + \mathcal{R}_k & -V & W \\ r_{\partial\Omega_D} \gamma^+ \mathcal{R}_k & -r_{\partial\Omega_D} \mathcal{V} & r_{\partial\Omega_D} \mathcal{W} \\ r_{\partial\Omega_N} T^+ \mathcal{R}_k & -r_{\partial\Omega_N} \mathcal{W}' & r_{\partial\Omega_N} \mathcal{L}^+ \end{bmatrix}, \quad \mathcal{F}^{11} = \begin{bmatrix} F_0 \\ r_{\partial\Omega_D} \gamma^+ F_0 - \varphi_0 \\ r_{\partial\Omega_N} T^+ F_0 - \psi_0 \end{bmatrix}.$$

Due to Corollaries 3.4 and 3.14, we get $F_0 \in H^{1,0}(\Omega; A_k)$. Then we have $\mathcal{F}^{11} \in H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N)$ and the operators

$$\mathcal{M}_k^{11} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N), \tag{7.4}$$

$$\mathcal{M}_k^{11} : H^{1,0}(\Omega;A_k)\times\widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)\times\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)\longrightarrow H^{1,0}(\Omega;A_k)\times H^{\frac{1}{2}}(\partial\Omega_D)\times H^{-\frac{1}{2}}(\partial\Omega_N) \tag{7.5}$$

are bounded.

Taking into account Lemma 4.3, we prove the following Remark in the same way as [6, Remark 5.1].

Remark 7.1 Let $r_0 > \operatorname{diam}(\Omega)$. $\mathcal{F}^{11} = \mathbf{0}$ if and only if $(f, \Phi_0, \Psi_0) = \mathbf{0}$.

7.2 BDIE system (M12)

By taking the third Green identity (4.1) in Ω and its trace (4.2) on the whole boundary $\partial\Omega$, we arrive at the system (M12):

$$\begin{aligned} u + \mathcal{R}_k u - V\psi + W\varphi &= F_0 \qquad \text{in} \ \ \Omega, \\ \frac{1}{2}\varphi + \gamma^+\mathcal{R}_k u - \mathcal{V}\psi + \mathcal{W}\varphi &= \gamma^+F_0 - \Phi_0, \quad \text{on} \ \ \partial\Omega, \end{aligned} \tag{M12}$$

where F_0 is given by the relation (7.2). System (M12) can be rewritten in matrix form as

$$\mathcal{M}_k^{12}\mathcal{U} = \mathcal{F}^{12}, \tag{7.6}$$

where $\mathcal{U} = (u, \psi, \varphi)^t \in H^{1,0}(\Omega;A_k)\times\widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)\times\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and

$$\mathcal{M}_k^{12} = \begin{bmatrix} I+\mathcal{R}_k & -V & W \\ \gamma^+\mathcal{R}_k & -\mathcal{V} & \frac{1}{2}I+\mathcal{W} \end{bmatrix}, \qquad \mathcal{F}^{12} = \begin{bmatrix} F_0 \\ \gamma^+F_0 - \Phi_0 \end{bmatrix}.$$

Note that $\mathcal{F}^{12} \in H^{1,0}(\Omega;A_k)\times H^{\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved (see Corollaries 3.4 and 3.14, Theorem 3.13 and [13, Theorem 1]), we see that the operators

$$\mathcal{M}_k^{12} : H^1(\Omega)\times\widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)\times\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)\longrightarrow H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega), \tag{7.7}$$

$$\mathcal{M}_k^{12} : H^{1,0}(\Omega;A_k)\times\widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)\times\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)\longrightarrow H^{1,0}(\Omega;A_k)\times H^{\frac{1}{2}}(\partial\Omega) \tag{7.8}$$

are bounded.

Remark 7.2 Let $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ with $r_0 > \operatorname{diam}(\Omega)$. Then, $\mathcal{F}^{12} = \mathbf{0}$ if and only if $(f, \Phi_0, \Psi_0) = \mathbf{0}$.

Indeed, the latter obviously implies the former. Conversely, let $\mathcal{F}^{12} = (F_0, \gamma^+F_0 - \Phi_0) = \mathbf{0}$. From $F_0 = 0$, we get $f = 0$ and $V\Psi_0 - W\Phi_0 = 0$ in Ω. Again from $\gamma^+F_0 - \Phi_0 = 0$, we get $\Phi_0 = 0$ on $\partial\Omega$. Hence, we obtain $V\Psi_0 = 0$ in Ω, and the result follows from Lemma 4.2 *(i)*.

7.3 BDIE system (M21)

We obtain this system by using the third Green identity (4.1) on Ω and its co-normal derivative (4.3) on the whole boundary $\partial\Omega$:

$$\begin{aligned} u + \mathcal{R}_k u - V\psi + W\varphi &= F_0 \quad \text{in} \ \ \Omega, \\ \frac{1}{2}\psi + T^+\mathcal{R}_k u - \mathcal{W}'\psi + \mathcal{L}^+\varphi &= T^+F_0 - \Psi_0 \quad \text{on} \ \ \partial\Omega, \end{aligned} \tag{M21}$$

where F_0 is given by (7.2). We rewrite the system (M21) in matrix form as

$$\mathcal{M}_k^{21}\mathcal{U} = \mathcal{F}^{21},$$

where $\mathcal{U} = (u, \psi, \varphi)^t \in H^{1,0}(\Omega;A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and

$$\mathcal{M}_k^{21} = \begin{bmatrix} I + \mathcal{R}_k & -V & W \\ T^+\mathcal{R}_k & \frac{1}{2}I - \mathcal{W}' & \mathcal{L}^+ \end{bmatrix}, \qquad \mathcal{F}^{21} = \begin{bmatrix} F_0 \\ T^+F_0 - \Psi_0 \end{bmatrix}.$$

Here, $\mathcal{F}^{21} \in H^{1,0}(\Omega;A_k) \times H^{-\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved in $\mathcal{M}_k^{21}$, the following operators are bounded.

$$\mathcal{M}_k^{21} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{7.9}$$

$$\mathcal{M}_k^{21} : H^{1,0}(\Omega;A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega;A_k) \times H^{-\frac{1}{2}}(\partial\Omega). \tag{7.10}$$

Remark 7.3 Let $r_0 > 0$. $\mathcal{F}^{21} = \mathbf{0}$ if and only if $(f, \Phi_0, \Psi_0) = \mathbf{0}$.

We prove this remark in the same way as Remark 7.2.

7.4 BDIE system (M22)

Here, we use the third Green identity (4.1) in Ω, the restriction of its trace (4.2) on $\partial\Omega_N$ and the restriction of its co-normal derivative (4.3) on $\partial\Omega_D$ to get the system (M22),

$$\begin{aligned} u + \mathcal{R}_k u - V\psi + W\varphi &= F_0 \quad \text{in} \quad \Omega, \\ \frac{1}{2}\psi + T^+\mathcal{R}_k u - \mathcal{W}'\psi + \mathcal{L}^+\varphi &= T^+F_0 - \Psi_0 \quad \text{on} \quad \partial\Omega_D, \\ \frac{1}{2}\varphi + \gamma^+\mathcal{R}_k u - \mathcal{V}\psi + \mathcal{W}\varphi &= \gamma^+F_0 - \Phi_0 \quad \text{on} \quad \partial\Omega_N, \end{aligned} \tag{M22}$$

where F_0 is given by (7.2). Let us write the system (M22) in matrix form as

$$\mathcal{M}_k^{22}\mathcal{U} = \mathcal{F}^{22},$$

where $\mathcal{U} = (u, \psi, \varphi)^t \in H^{1,0}(\Omega;A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$, and

$$\mathcal{M}_k^{22} = \begin{bmatrix} I + \mathcal{R}_k & -V & W \\ r_{\partial\Omega_D} T^+\mathcal{R}_k & r_{\partial\Omega_D}\left(\frac{1}{2}I - \mathcal{W}'\right) & r_{\partial\Omega_D}\mathcal{L}^+ \\ r_{\partial\Omega_N}\gamma^+\mathcal{R}_k & -r_{\partial\Omega_N}\mathcal{V} & r_{\partial\Omega_N}\left(\frac{1}{2}I + \mathcal{W}\right) \end{bmatrix},$$

$$\mathcal{F}^{22} = \begin{bmatrix} F_0 \\ r_{\partial\Omega_D}(T^+F_0 - \Psi_0) \\ r_{\partial\Omega_N}(\gamma^+F_0 - \Phi_0) \end{bmatrix}.$$

From the mapping properties of the operators involved, $\mathcal{F}^{22} \in H^{1,0}(\Omega;A_k) \times H^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N)$ and the following operators are bounded.

$$\mathcal{M}_k^{22} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N), \tag{7.11}$$

$$\mathcal{M}_k^{22} : H^{1,0}(\Omega;A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega;A_k) \times H^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N). \tag{7.12}$$

Taking into account Lemma 4.3, we prove the following remark in the same way as [6, Remark 5.11].

Remark 7.4 Let $r_0 > \mathrm{diam}(\Omega)$. $\mathcal{F}^{22} = \mathbf{0}$ if and only if $(f, \Phi_0, \Psi_0) = \mathbf{0}$.

8 Equivalence, Fredholm properties, and invertibility for BDIE operators of the mixed BVP

Let us prove that the mixed BVP (7.1) is equivalent to the BDIE systems (M11), (M12), (M21), and (M22).

Theorem 8.1 *Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$, respectively, and let $f \in L^2(\Omega)$.*

(i) *If some $u \in H^1(\Omega)$ solves the mixed BVP (7.1), then the triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$, where*

$$\psi = T^+u - \Psi_0, \qquad \varphi = \gamma^+u - \Phi_0 \qquad \text{on } \partial\Omega, \tag{8.1}$$

solves the BDIE systems (M11), (M12), (M21) and (M22).

(ii) *Let $r_0 > \mathrm{diam}(\Omega)$. If a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves one of the BDIE systems (M11) or (M12) or (M22), then u solves BVP (7.1), and relations (8.1) hold.*

(iii) *Let $r_0 > 0$. If a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves the BDIE system (M21), then u solves BVP (7.1), and relations (8.1) hold.*

Proof To prove (i), we let $u \in H^1(\Omega)$ be a solution to BVP (7.1). Then, for ψ and φ defined by (8.1), we get $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$. Recalling how the four BDIE systems were constructed, the result immediately follows from relations (4.1)–(4.3).

To prove (ii) and (iii), let us first assume that a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves either the BDIE system (M11) or (M12) or (M21) or (M22). The first equation of each system and Lemma 4.1 with $\Psi = \psi + \Psi_0$ and $\Phi = \varphi + \Phi_0$ imply that u solves the PDE $A_k u = f$ on Ω and the relation

$$V\Psi^* - W\Phi^* = 0 \qquad \text{in } \Omega \tag{8.2}$$

holds for

$$\Psi^* = \Psi_0 + \psi - T^+u \quad \text{and} \quad \Phi^* = \Phi_0 + \varphi - \gamma^+u. \tag{8.3}$$

Whenever in the remaining proof we take the trace or co-normal derivative of the first equation of each system, we make use of relations (3.8)–(3.10) and the last equation in (3.3).

Proof for (M11). Let a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solve the BDIE system (M11). Taking the trace of the first equation in (M11) on $\partial\Omega_D$ and subtracting the second equation from it, we obtain

$$\gamma^+u = \varphi_0 \quad \text{on } \partial\Omega_D, \tag{8.4}$$

i.e., u satisfies the Dirichlet condition in (7.1). We now take the co-normal derivative of the first equation in (M11) on $\partial\Omega_N$ and subtract the third equation from it to get

$$T^+u = \psi_0 \quad \text{on } \partial\Omega_N, \tag{8.5}$$

i.e., u satisfies the Neumann condition in (7.1). Taking into account that $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$ and $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$, (8.4) and (8.5) imply that the first equation in (8.1) is satisfied on $\partial\Omega_N$ and the second equation in (8.1) on $\partial\Omega_D$.

From this and relation (8.3), we have $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$, $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$. Since relation (8.2) holds and $r_0 > \text{diam}(\partial\Omega_D)$, from Lemma 4.3, we get $\Psi^* = \Phi^* = 0$, which completes the proof of conditions (8.1).

Proof for (M12). Now, let a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solve BDIE system (M12). Taking trace of the first equation in (M12) on $\partial\Omega$ and subtracting the second one from it, we obtain

$$\gamma^+ u = \Phi_0 + \varphi \quad \text{on } \partial\Omega, \tag{8.6}$$

which means that the second equation in (8.1) holds. Since $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$, we see that the Dirichlet condition in (7.1) is satisfied.

Due to (8.6), the second term in (8.2) vanishes and by Lemma 4.2(i), we obtain

$$\Psi_0 + \psi - T^+ u = 0 \quad \text{on } \partial\Omega, \tag{8.7}$$

which shows that the first equation of (8.1) is satisfied as well. Since $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$, (8.7) implies that u satisfies the Neumann boundary condition in (7.1).

Proof for (M22). Now, let a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solve the BDIE system (M22). Taking the co-normal derivative of the first equation in (M22) on $\partial\Omega_D$ and subtracting it from the second equation, we obtain

$$\psi = T^+ u - \Psi_0 \quad \text{on } \partial\Omega_D. \tag{8.8}$$

Taking the trace of the first equation in (M22) on $\partial\Omega_N$ and subtracting it from the third equation yields

$$\varphi = \gamma^+ u - \Phi_0 \quad \text{on } \partial\Omega_N. \tag{8.9}$$

Equations 8.8 and 8.9 imply that the first equation in (8.1) is satisfied on $\partial\Omega_D$ and the second one on $\partial\Omega_N$. Due to (8.8) and (8.9), we have $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_N)$, $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_D)$ in (8.2) and (8.3). Then, Lemma (4.3) with $\Gamma_1 = \partial\Omega_N$ and $\Gamma_2 = \partial\Omega_D$ implies that $\Psi^* = \Phi^* = 0$, which completes the proof of conditions (8.1) on the whole boundary $\partial\Omega$. Taking into account that $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$ and $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$, (8.1) implies the boundary conditions in the mixed BVP (7.1).

Proof for (M21). Let a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solve the BDIE system (M21). We take the co-normal derivative of the first equation in (M21) on $\partial\Omega$ and subtract the second equation from it to obtain

$$\psi + \Psi_0 - T^+ u = 0 \quad \text{on } \partial\Omega, \tag{8.10}$$

which is the first equation of (8.1). Since $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$, we see that u satisfies the Neumann condition in (7.1).

Due to (8.10), the first term in (8.2) vanishes and, by Lemma 4.2(ii), we obtain

$$\Phi_0 + \varphi - \gamma^+ u = 0 \quad \text{on } \partial\Omega, \tag{8.11}$$

which means that the second condition in (8.1) holds as well. Since $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$, from (8.11), we see that u satisfies the Dirichlet boundary condition in (7.1).

Corollary 8.2 *Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$, respectively, and let $f \in L^2(\Omega)$.*

(i) *Let $r_0 > \text{diam}(\Omega)$. If a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves the BDIE system (M11) or (M12) or (M22), then it solves all the other three BDIE systems.*

(ii) *Let $r_0 > 0$. If a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves the BDIE system (M21), then it solves (M11), (M12) and (M22).*

Corollary 8.3

(i) *Let* $r_0 > \operatorname{diam}(\Omega)$. *The homogeneous counterpart of BDIE system* (*M*11) *or* (*M*12) *or* (*M*22) *has a non-trivial solution in* $H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ *if and only if the homogeneous counterpart of the mixed problem* (7.1) *has a non-trivial solution in* $H^1(\Omega)$.

(ii) *Let* $r_0 > 0$. *The homogeneous counterpart of BDIE system* (*M*21) *has a non-trivial solution in* $H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ *if and only if the homogeneous counterpart of the mixed problem* (7.1) *has a non-trivial solution in* $H^1(\Omega)$.

Now, we prove the Fredholm property of the corresponding operators of the BDIE system (M11), (M12), and (M21).

Theorem 8.4

(i) *If* $r_0 > \operatorname{diam}(\Omega)$, *operators* (7.4) *and* (7.7) *are Fredholm with index zero.*

(ii) *If* $r_0 > 0$, *operator* (7.9) *is Fredholm with index zero.*

Proof Here, we follow the arguments similar to the ones used in [6, for 3D case].

Operator (7.4). To prove the Fredholm property of operator (7.4), let us consider the operator

$$\mathcal{M}_*^{11} := \begin{bmatrix} I & -V & W \\ 0 & -r_{_{\partial\Omega_D}}\mathcal{V} & 0 \\ 0 & 0 & r_{_{\partial\Omega_N}}\widehat{\mathcal{L}} \end{bmatrix},$$

where $\widehat{\mathcal{L}}$ is given by (3.14).

The operator $\mathcal{M}_*^{11}$ is an upper triangular matrix operator with the following scalar diagonal operators,

$$I : H^1(\Omega) \longrightarrow H^1(\Omega),$$
$$r_{_{\partial\Omega_D}}\mathcal{V} : \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \longrightarrow H^{\frac{1}{2}}(\partial\Omega_D),$$
$$r_{_{\partial\Omega_N}}\widehat{\mathcal{L}} : \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega_N),$$

that are invertible (due to Theorems 3.10 and 3.11(i) for the second and third operators). Along with the mapping properties of the operators V and W (see Theorem 3.3), the operator

$$\mathcal{M}_*^{11} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N)$$

is invertible. The operator

$$\mathcal{M}_k^{11} - \mathcal{M}_*^{11} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N),$$

where

$$\mathcal{M}_k^{11} - \mathcal{M}_*^{11} := \begin{bmatrix} \mathcal{R}_k & 0 & 0 \\ r_{_{\partial\Omega_D}}\gamma^+\mathcal{R}_k & 0 & r_{_{\partial\Omega_D}}\mathcal{W} \\ r_{_{\partial\Omega_N}}T^+\mathcal{R}_k & -r_{_{\partial\Omega_N}}\mathcal{W}' & r_{_{\partial\Omega_N}}\left(\mathcal{L}^+ - \widehat{\mathcal{L}}\right) \end{bmatrix}.$$

is compact due to Corollaries 3.15 and 3.17 as well as Theorems 3.6 and 3.11(ii). Hence, (7.4) is a Fredholm operator with zero index.

Operator (7.7). Let us denote

$$\mathcal{M}_*^{12} := \begin{bmatrix} I & -V & W \\ 0 & -\mathcal{V} & \frac{1}{2}I \end{bmatrix}.$$

Then,

$$\mathcal{M}_*^{12} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is bounded. To show the invertibility of $\mathcal{M}_*^{12}$, taking into account Theorem 3.10, we follow the proof for 3D case in [6]. Consider the equation

$$\mathcal{M}_*^{12}\mathcal{U} = \widetilde{F} \tag{8.12}$$

with an unknown vector $\mathcal{U} = (u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and a given vector $\widetilde{F} := (\widetilde{F}_1, \widetilde{F}_2)^t \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Rewrite (7.9) componentwise as

$$u - V\psi + W\varphi = \widetilde{F}_1 \quad \text{in} \;\; \Omega, \tag{8.13}$$

$$\frac{1}{2}\varphi - \mathcal{V}\psi = \widetilde{F}_2 \quad \text{on} \;\; \partial\Omega. \tag{8.14}$$

The restriction of (8.14) on $\partial\Omega_D$ gives

$$-r_{\partial\Omega_D}\mathcal{V}\psi = r_{\partial\Omega_D}\widetilde{F}_2. \tag{8.15}$$

Due to Theorem 3.10, (8.15) is uniquely solvable, i.e., for arbitrary $\widetilde{F}_2 \in H^{\frac{1}{2}}(\partial\Omega)$ there exists a unique $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ satisfying (8.15). Moreover,

$$\left[\mathcal{V}\psi + \widetilde{F}_2\right] \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N). \tag{8.16}$$

Then, (8.14) along with (8.16) yields that φ is defined also uniquely as

$$\varphi = 2\left[\mathcal{V}\psi + \widetilde{F}_2\right] \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N).$$

Hence, (8.14) with arbitrary $\widetilde{F}_2 \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega)$ defines $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ uniquely. Since $V\psi, W\varphi \in H^1(\Omega)$, from (8.13) we obtain that

$$u = V\psi - W\varphi + \widetilde{F}_1 \quad \text{in} \;\; \Omega,$$

showing that the function $u \in H^1(\Omega)$ is also defined uniquely. The above arguments show that operator $\mathcal{M}_*^{12}$ is invertible.

Due to Corollaries 3.5 and 3.15, the operator

$$\mathcal{M}_k^{12} - \mathcal{M}_*^{12} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

where

$$\mathcal{M}_k^{12} - \mathcal{M}_*^{12} := \begin{bmatrix} \mathcal{R}_k & 0 & 0 \\ \gamma^+\mathcal{R}_k & 0 & \mathcal{W} \end{bmatrix},$$

is compact. Then, operator (7.7) is Fredholm of index zero.

Operator (7.9). The proof for operator (7.9) follows by the arguments similar to those in the proof for operator (7.7). Let

$$\mathcal{M}_*^{21} := \begin{bmatrix} I & -V & W \\ 0 & \frac{1}{2}I & \widehat{\mathcal{L}} \end{bmatrix}.$$

Then,

$$\mathcal{M}_*^{21} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$$

is bounded. Since the operators $I : H^1(\Omega) \to H^1(\Omega)$ and $\widehat{\mathcal{L}} : \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \to H^{-\frac{1}{2}}(\partial\Omega)$ are invertible, using similar arguments as in the proof of the operator (7.7), we can show that $\mathcal{M}_*^{21}$ is invertible.

Due to the mapping properties of the operators involved, the operator

$$\mathcal{M}_k^{21} - \mathcal{A}_*^{21} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega),$$

where

$$\mathcal{M}_k^{21} - \mathcal{M}_*^{21} := \begin{bmatrix} \mathcal{R}_k & 0 & 0 \\ T^+\mathcal{R}_k & -\mathcal{W}' & \left(\mathcal{L}^+ - \widehat{\mathcal{L}}\right) \end{bmatrix}$$

is compact implying that $\mathcal{M}_k^{21}$ is Fredholm operator of index zero. □

Let us consider the particular cases of operators (7.4), (7.5), (7.7), (7.8), (7.9), and (7.10), for $k = 0$, that is,

$$\mathcal{M}_0^{11} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N), \tag{8.17}$$

$$\mathcal{M}_0^{11} : H^{1,0}(\Omega;A) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega;A) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N), \tag{8.18}$$

$$\mathcal{M}_0^{12} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{8.19}$$

$$\mathcal{M}_0^{12} : H^{1,0}(\Omega;A) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega;A) \times H^{\frac{1}{2}}(\partial\Omega), \tag{8.20}$$

$$\mathcal{M}_0^{21} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{8.21}$$

$$\mathcal{M}_0^{21} : H^{1,0}(\Omega;A) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega;A) \times H^{-\frac{1}{2}}(\partial\Omega). \tag{8.22}$$

where

$$\mathcal{M}_0^{11} = \begin{bmatrix} I + \mathcal{R} & -V & W \\ r_{_{\partial\Omega_D}}\gamma^+\mathcal{R} & -r_{_{\partial\Omega_D}}\mathcal{V} & r_{_{\partial\Omega_D}}\mathcal{W} \\ r_{_{\partial\Omega_N}}T^+\mathcal{R} & -r_{_{\partial\Omega_N}}\mathcal{W}' & r_{_{\partial\Omega_N}}\mathcal{L}^+ \end{bmatrix},$$

$$\mathcal{M}_0^{12} = \begin{bmatrix} I + \mathcal{R} & -V & W \\ \gamma^+\mathcal{R} & -\mathcal{V} & \frac{1}{2}I + \mathcal{W} \end{bmatrix}, \quad \mathcal{M}_0^{21} = \begin{bmatrix} I + \mathcal{R} & -V & W \\ T^+\mathcal{R} & \frac{1}{2}I - \mathcal{W}' & \mathcal{L}^+ \end{bmatrix}.$$

Theorem 8.5

(i) *If* $r_0 > \operatorname{diam}(\Omega)$, *then operators* (8.17), (8.18), (8.19), *and* (8.20) *are invertible.*
(ii) *If* $r_0 > 0$, *then operators* (8.21) *and* (8.22) *are invertible.*

Proof This theorem for $r_0 = 1$ was proved in [12, Theorem 3.25]. Here, we update the proof for arbitrary $r_0 > 0$ similar to Theorem 6.5 for the BDIE system of the Dirichlet problem.

It is well known that the homogeneous mixed problem (7.1) with $k = 0$, that is, with $A_k = A$, where the operator A is given by (2.1) and $0 < a_0 < a(x) < a_1 < \infty$, has only the trivial solution in $H^{1,0}(\Omega;A)$ and $H^1(\Omega)$. This can be obtained, e.g., from the first Green identity (2.7). Then, the equivalence Theorem 8.1 implies that all operators (8.17)–(8.22) are injective. By Theorem 8.4, operators (8.17), (8.19), and (8.21) are Fredholm with zero index. Then, the injectivity of operators (8.17), (8.19), and (8.21) implies their invertibility (see, e.g., [18, Theorem 2.27]).

To prove the invertibility of operator (8.18), we remark that for any $\mathcal{F}^{11} \in H^{1,0}(\Omega;A) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N)$, a solution of the equation $\mathcal{M}_0^{11}\mathcal{U} = \mathcal{F}^{11}$ can be written as $\mathcal{U} = (\mathcal{M}_0^{11})^{-1}\mathcal{F}^{11}$, where $(\mathcal{M}_0^{11})^{-1} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N) \to H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ is the continuous inverse to operator (8.17). But due to Lemma 4.1, the first equation of system (M11) with $k = 0$ implies that $\mathcal{U} = (\mathcal{M}_0^{11})^{-1}\mathcal{F}^{11} \in H^{1,0}(\Omega;A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$, and moreover, the operator $(\mathcal{M}_0^{11})^{-1} : H^{1,0}(\Omega;A) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N) \to H^{1,0}(\Omega;A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ is continuous, which implies the invertibility of operator (8.18).

The invertibility of operators (8.20) and (8.22) is proved in a similar fashion. □

Now, we are in the position to prove an analog of Theorem 8.4 for operators (7.5), (7.8), and (7.10).

Theorem 8.6

(i) *If* $r_0 > \operatorname{diam}(\Omega)$, *operators* (7.5) *and* (7.8) *are Fredholm with index zero.*
(ii) *If* $r_0 > 0$, *operator* (7.10) *is Fredholm with index zero.*

Proof By Theorem 8.5, we see that operators (8.18), (8.20), and (8.22) are invertible. Due to Corollaries 3.16, the operators

$$\begin{aligned}
\mathcal{M}_k^{11} - \mathcal{M}_0^{11} &: H^{1,0}(\Omega;A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega;A_k) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N),\\
\mathcal{M}_k^{12} - \mathcal{M}_0^{12} &: H^{1,0}(\Omega;A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega;A_k) \times H^{\frac{1}{2}}(\partial\Omega),\\
\mathcal{M}_k^{21} - \mathcal{M}_0^{21} &: H^{1,0}(\Omega;A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega;A_k) \times H^{-\frac{1}{2}}(\partial\Omega).
\end{aligned}$$

where

$$\mathcal{M}_k^{11} - \mathcal{M}_0^{11} = \begin{bmatrix} \mathcal{R}_k - \mathcal{R} & 0 & 0 \\ r_{_{\partial\Omega_D}}\gamma^+(\mathcal{R}_k - \mathcal{R}) & 0 & 0 \\ r_{_{\partial\Omega_N}}T^+(\mathcal{R}_k - \mathcal{R}) & 0 & 0 \end{bmatrix},$$

$$\mathcal{M}_k^{12} - \mathcal{M}_0^{12} = \begin{bmatrix} \mathcal{R}_k - \mathcal{R} & 0 & 0 \\ \gamma^+(\mathcal{R}_k - \mathcal{R}) & 0 & 0 \end{bmatrix}, \quad \mathcal{M}_k^{21} - \mathcal{M}_0^{21} = \begin{bmatrix} \mathcal{R}_k - \mathcal{R} & 0 & 0 \\ T^+(\mathcal{R}_k - \mathcal{R}) & 0 & 0 \end{bmatrix}.$$

are compact, implying that operators (7.5), (7.8), and (7.10) are Fredholm operators with index zero. □

Due to Corollary 8.3 and Theorem 8.4, we obtain the following assertion.

Corollary 8.7

(i) *Let* $r_0 > \mathrm{diam}(\Omega)$. *The homogeneous counterpart of the mixed problem* (7.1) *has only the trivial solution in* $H^1(\Omega)$ *if and only if the operators* (7.4), (7.5), (7.7), *and* (7.8) *are invertible*.

(ii) *Let* $r_0 > 0$. *The homogeneous counterpart of the mixed problem* (7.1) *has only the trivial solution in* $H^1(\Omega)$ *if and only if the operators* (7.9) *and* (7.10) *are invertible*.

Remark 8.8 Equivalence, Fredholm properties, and invertibility for BDIE operators (7.11) and (7.12), for $\mathcal{M}_k^{22}$, are not analyzed in Section 8. Note that they can be considered using a different approach similar to [9, Theorem 7.1], [12, Theorem 3.31], cf. also [6, Theorems 5.15, 5.19].

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Declarations

Conflict of interest The authors declare no competing interests.

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