B.O.N.E.
Book of Numerical Experiments

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1 Preface

The intention of this book is to cover the numerical experiments possible with the core *maiprogs* system, without user-supplied code. It will give all bcl-scripts necessary for the computation and graphical conversion of all examples presented. BCL (batch control language) is the script language of the maiprogs system, specially designed for controlling large batch jobs in a flexible way and passing information from one program of the system to an other program.

Currently the book covers only a subset of all experiments done so far. Mostly neglected are experiments done for the Helmholtz and Lamé-equations. After installing *maiprogs* and deleting all '*.dat'-files in the subdirectories of bone/, a simple make will recompute all results. But be cautious, depending on your hardware the computation will take some days or weeks.

New versions of this documentation you can download from my home page:
`http://people.brunel.ac.uk/~mastmmm`

The package *maiprogs* is not public domain but it is free for scientific use (if you acknowledge its use in a corresponding publication) and can be obtained from the author.

In the following, we have separate sections, for 1d, 2d and 3d-problems, using FEM, BEM or a coupling method. Subsections are for examples showing the convergence of methods and the efficiency of solvers.

Caution: This book serves mainly as an internal documentation and is also used to test whether old experiments can still be computed by newer versions of the code. It always can happen that in the figures and tables some methods show strange results. If you notice this and you need to use the corresponding method please contact the author.

Remark: *maiprogs* consists of the programs maicoup3, maicoup2, maifem1 and maigraf, which are actively developed. Sometimes there are still scripts which are using older programs, which are due to be replaced.

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Contents

2 Boundary Element Methods (2D)

2.1 Convergence

2.1.1 Laplace

Here we deal with the Dirichlet problem of the Laplacian, corresponding to a weakly singular integral equation

$$V\psi(x) = -\frac{1}{\pi}\int_\Gamma \log|x-y|\psi(y)\,ds_y = f(x) \tag{1}$$

and the Neumann problem of the Laplacian, corresponding to a hyper-singular integral equation

$$Wu = -\frac{1}{\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\log|x-y|\psi(y)\,ds_y = g(x) \tag{2}$$

Example 2.1. *The following command files, if executed by* maicoup2*, compute the energy norms and the errors in energy norm for the Laplace Dirichlet problem with boundary data equal to 1 on a slit [-1,1] using the uniform h-version with piecewise constant splines, the p-version with two boundary elements and increasing polynomial degrees, the algebraically graded mesh towards the endpoints of the slit and the geometrically refined hp-version. The exact solution is* $u(x) = \frac{1}{\log 2}\frac{1}{\sqrt{1-x^2}}$.

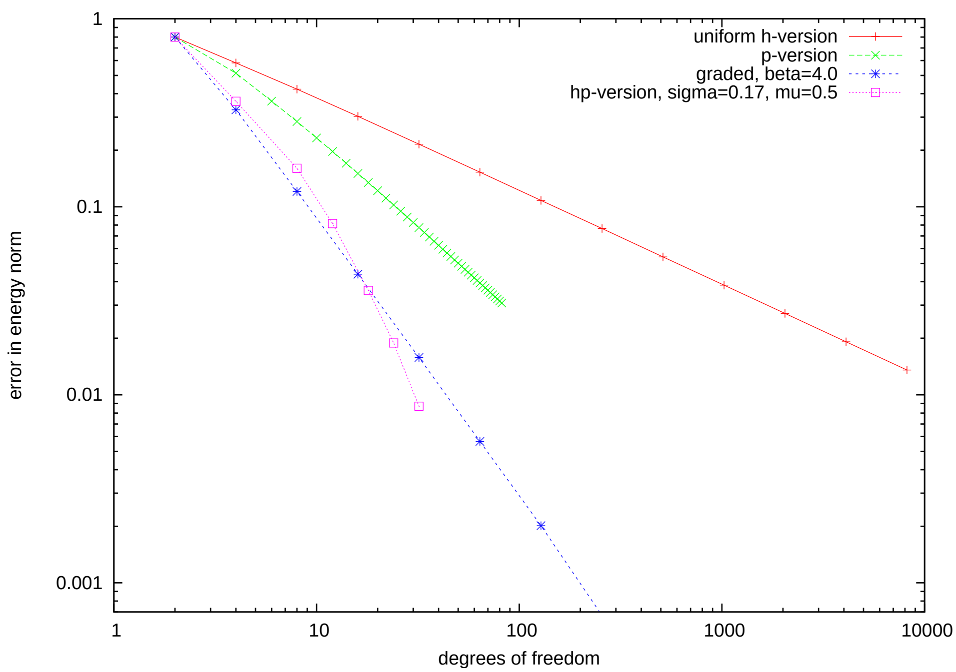


Figure 2.1: Weakly singular integral equation, convergence in energy norm

bem2/ex1hin

```
! h-Version, Laplace, Dirichlet, Slit [-1,1]
open(1) 'test' ; open(2) 'ex1hin.dat'
geometry('Slit',gm='Ng',bmode=(/1,2/)) ; #ti
problem('Laplace',nickname='VIGLSCR')
R=3
EPS=1.0d-10
```

```
J=2
do I=0,12
 mesh('uniform',n=J,p=0,spline='N')
 matrix('analytic'); TM=SEC
 lft 4 R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 #hno. 2.1289340389
 write(2) DOF, I, ENO, ENOERR, TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

bem2/ex1pin

```
! p-Version, Laplace, Dirichlet, Slit [-1,1]
open(1) 'test' ; open(2) 'ex1pin.dat'
geometry('Slit',gm='Ng',bmode=(/1,2/)) ; #ti
problem('Laplace',nickname='VIGLSCR')
R=3
EPS=1.0d-10
do I=0,40
 mesh('uniform',n=2,p=I,spline='N')
 matrix('numeric',ijn=6,sigma=0.17,mu=1.0,gqna=40,gqnb=40); TM=SEC
 lft 40 R 0 R; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 #hno. 2.1289340389
 write(2) DOF, I, ENO, ENOERR, TM, TL, TS, ITER, COND
continue
end
```

bem2/ex1g4in

```
! h-Version, graded, beta=4.0, Laplace, Dirichlet
open(1) 'test' ; open(2) 'ex1g4in.dat'
geometry('Slit',gm='Ng',bmode=(/1,2/)) ; #ti
problem('Laplace',nickname='VIGLSCR')
R=3
EPS=1.0d-10
J=1
do I=0,7
 mesh('graded',n=J,p=0,beta=4.0,spline='N')
 matrix('analytic'); TM=SEC
 lft 4 R 0 R; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='diag',mit='CG'); TS=SEC
 #hno. 2.1289340389
 write(2) DOF, I, ENO, ENOERR, TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

bem2/ex1hpin

```
! hp-Version, geometrical, sigma=0.17, mu=0.5, Laplace, Dirichlet
open(1) 'test' ; open(2) 'ex1hpin.dat'
geometry('Slit',gm='Ng',bmode=(/1,2/)) ; #ti
problem('Laplace',nickname='VIGLSCR')
```

```
R=3
EPS=1.0d-10
do I=1,7
 mesh('geometrical',n=I,p=0,sigma=0.17,mu=0.5,spline='N')
 matrix('analytic'); TM=SEC
 lft 16 R 0 R; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='diag',mit='CG'); TS=SEC
 #hno. 2.1289340389
 write(2) DOF, I, ENO, ENOERR, TM, TL, TS, ITER, COND
continue
end
```

Example 2.2. *Here we deal with the Dirichlet problem of the Laplacian on a closed polygon. The corresponding integral equation is*

$$V\psi(\vec{x}) = (I + K)f(\vec{x}) \tag{3}$$

The boundary data on the L-Shape are given by

$$f(r,\varphi) = u(r,\varphi)|_\Gamma, u(r,\varphi) = r^{2/3}\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$$

The exact solution is

$$\psi(r,\varphi) = \frac{\partial u}{\partial n}|_\Gamma = \frac{2}{3}r^{-1/3}(\sin(\frac{2}{3}(\varphi-\frac{\pi}{2}))(n_x\cos\varphi+n_y\sin\varphi)+\cos(\frac{2}{3}(\varphi-\frac{\pi}{2}))(-n_x\sin\varphi+n_y\cos\varphi))$$

and the energy norm of the exact solution can be extrapolated to

$$\|\psi\|_V = 1.6018674$$

bem2/ex2hin

```
! h-Version, Laplace, Dirichlet, L-Shape
open(1) 'test' ; open(2) 'ex2hin.dat'
geometry('L-Shape',gm='Ng',styp=1,bmode=(/1,2/)) ; #ti
problem('Laplace',nickname='VIGL')
R=1 ! right hand side
EPS=1.0d-10
J=2
do I=0,10
 mesh('uniform',n=J,p=0,spline='N')
 matrix('analytic'); TM=SEC
 lft 24 R 0 R
 solve(eps=EPS,mdi='x=0',mdc='no',mit='CG'); TS=SEC
 #hno. 1.6018674
 #err. 24 R 'L2' 0 'N'
 write(2) DOF, I, ENO, ENOERR, ERR, ITER, TM, TS
 J=J*2
continue
end
```

The following command file computes the stress intensity factor for the Laplace Dirichlet problem on a L-Shape where the boundary data are given by the first singularity function without volume data. Note that for every operator and boundary condition the index number ltyp=1 denotes the singularity function and ltyp=5 denotes the singularity function with negative index (see [5]).

bem2/ex2in

```
! stress coefficients, L-shape, Laplace, Dirichlet
open(1) 'test' ; open(2) 'ex2sin.dat'
geometry('L-Shape',styp=1) ; #ti
#pro 0 0 0 ; #pro. ; #pol 1
J=2
do I=0,8
mesh('uniform',n=J,p=0) ; #g.
#cnin 1 ! first singular function for origin
matrix('analytic')
```

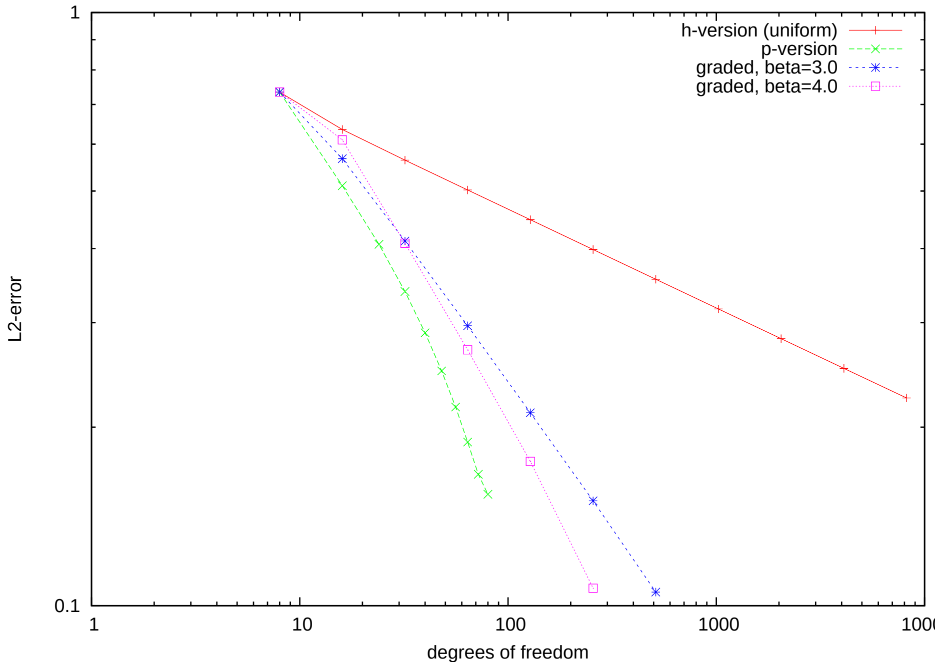


Figure 2.2: Weakly singular integral equation, convergence in L_2-norm

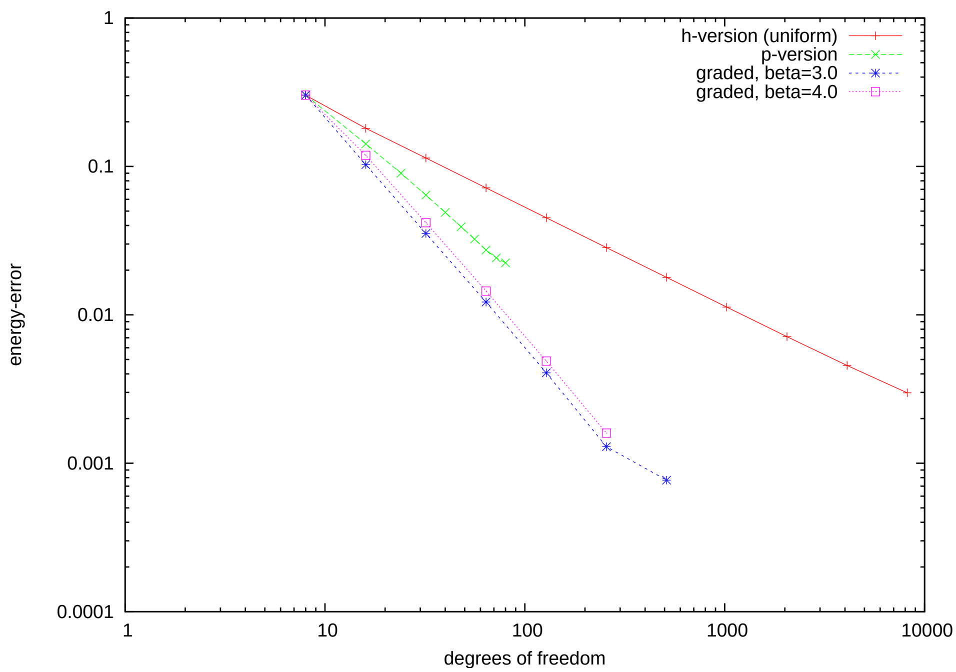


Figure 2.3: Weakly singular integral equation, convergence in energy-norm

```
lft 16 5 0 ; #l.
rlgs 1.0d-10 -1 0 0 1 1
#hno. ; #c.
#c1. 16 1 -1 ! postprocessing for stress coefficient
write(2) DOF, STRESS
```

```
J=J*2
continue
end
```

N	c_1	α_N
8	0.9436001	
16	0.9760774	1.237
32	0.9951636	2.306
64	0.9986983	1.894
128	0.9995756	1.617
256	0.9998388	1.397
512	0.9999320	1.245
1024	0.9999690	1.133
2048	0.9999850	1.047

Table 2.1: Approximation of stress coefficients

Example 2.3. *The following command files, if executed by* maicoup2*, compute the energy norms and the errors in energy norm for the Laplace Neumann problem with boundary data equal to 1 on a slit [-1,1] using the uniform h-version with continuous splines, the p-version with two boundary elements and increasing polynomial degrees, the algebraically graded mesh towards the endpoints of the slit and the geometrically refined hp-version. The exact solution is* $u(x) = \sqrt{1-x^2}$ *and the exact energy norm* $\sqrt{\pi/2} = 1.25331413732$.

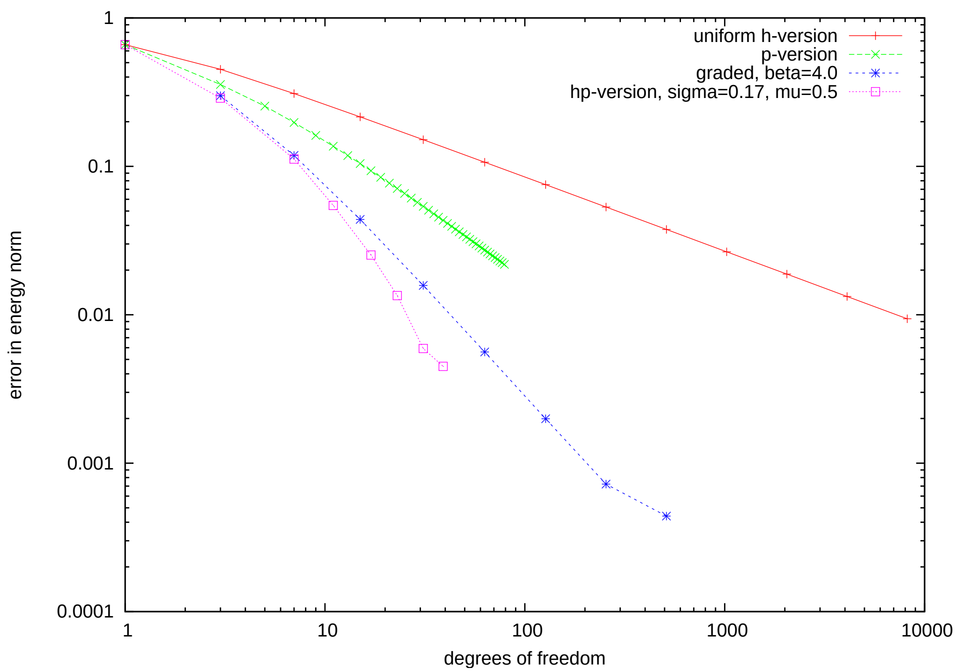


Figure 2.4: Hyper-singular integral equation, convergence in energy norm

bem2/ex3hin

```
! h-Version, Laplace, Neumann, Slit [-1,1]
open(1) 'test' ; open(2) 'ex3hin.dat'
geometry('Slit',gm='Dg',bmode=(/1,2/)) ; #ti
problem('Laplace',nickname='WIGLSCR')
R=16 ! right hand side
EPS=1.0d-10
J=2
do I=0,12
 mesh('uniform',n=J,p=1,spline='D')
 matrix('analytic'); TM=SEC
 lft 4 R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 #hno. 1.25331413732
 #err. 16 R 'L2' 0 'D'
 write(2) DOF, I, ENO, ENOERR, ERR, TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

Example 2.4. *Here we investigate the exterior Neumann Laplace problem*

$$-\Delta u = 0 \ in \ \mathbb{R}^2 \backslash \Omega$$
$$\frac{\partial u}{\partial n} = t_0 \ on \ \partial\Omega$$

with t_0 satisfying the compatibility condition

$$\int_\Gamma t_0(x)\, ds_x = 0.$$

The boundary integral formulation reads:

$$(W + P'P)u(x) = (\frac{1}{2}I - K')t(x) \ for \ x \in \Gamma$$

where P is the stabilization operator

$$Pu := \int_\Gamma u(x)\, ds_x.$$

We choose the model solution

$$u_0(x) = -\log|x - \bar{x}| - c$$
$$t_0(x) = -\frac{n_x(x - \bar{x})}{|x - \bar{x}|^2}$$

on the L-shaped domain Ω and $\bar{x} = (-0.125, -0.125)$, with $c \in \mathbb{R}$ choosen, such that $\int_\Gamma u_0 = 0$.

bem2/ex8hin

```
! h-Version, Laplace, Neumann, L-Shape, exterior
open(1) 'test' ; open(2) 'ex8hin.dat'
geometry('L-Shape',gm='Dg',bmode=(/1,2/),dim=(/0.5,0.5/) ) ; #ti
problem('Laplace',nickname='WIGLIRHSX')
R=2 ! right hand side
EPS=1.0d-10
J=2
Q=26
do I=0,12
 mesh('uniform',n=J,p=1,spline='D')
 setlaplaceu('D',R)
 matrix('analytic'); TM=SEC
 approx 0 R 'N' 't0'
 lft Q R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 #hno. 1.25331413732
 #err. Q R 'L2' 0 'D' 'u0' ; E[0]=ERR
 #err. Q R 'H1' 0 'D' 'u0' ; E[1]=ERR
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

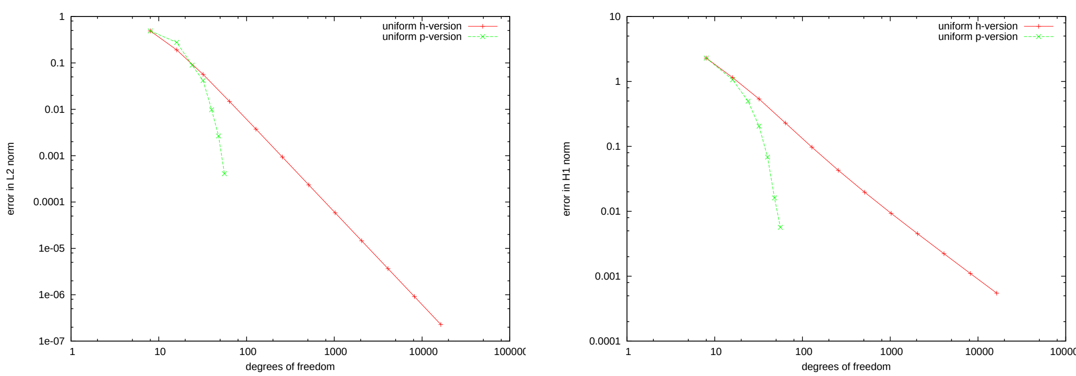


Figure 2.5: Laplace(Neumann): First kind integral equation, convergence in L_2-norm and H^1-norm

2.1.2 Lamé

Example 2.5. *Here we investigate the interior Dirichlet Lamé problem*

$$-\Delta^* \vec{u} = 0 \ \textit{in} \ \Omega$$
$$u = \vec{u}_0 \ \textit{on} \ \partial\Omega$$

where $\vec{u}_0$ *has to satisfy no compatability conditions. The boundary integral formulation reads: Find* $t \in [H^{-1/2}(\Gamma)]^2$ *such that*

$$Vt(x) = (\frac{1}{2}I + K)u(x) \ \textit{for} \ x \in \Gamma$$

Our choosen model solution is

$$\vec{u}_0 = \begin{pmatrix} -\frac{\lambda+3\mu}{4\mu(\lambda+2\mu)\pi}\log(r) + \frac{\lambda+\mu}{4\mu(\lambda+2\mu)\pi}\frac{(x_1-\bar{x}_1)^2}{r^2} \\ \frac{\lambda+\mu}{4\mu(\lambda+2\mu)\pi}\frac{(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{r^2} \end{pmatrix}$$
$$\vec{t} = T\vec{u}_0$$

on the L-shaped domain Ω*,* $E = 2000, \nu = 0.3$ *and* $\bar{x} = (0, 1.5)$*.*

bem2/ex21hin

```
! h-Version, Lame, Dirichlet, L-Shape, interior
open(1) 'test' ; open(2) 'ex21hin.dat'
geometry('L-Shape',gm='Ng',bmode=(/1,2/),dim=(/0.5,0.5/)) ; #ti
problem('Lame',nickname='VIGLIRHS')
#ep 2000. 0.3
setlamex( (/0.,1.5/) )
EPS=1.0d-10
R=12
J=2
do I=0,11
 mesh('uniform',n=J,p=0,spline='N')
 matrix('analytic') ; TM=SEC
 approx 0 R 'D' 'u0'
 lft 16 R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 #err. 16 R 'L2' 0 'N' 't0'; E[0]=ERR
 #hno. 1.
 write(2) DOF, I, ENO, ENOERR, E[0], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

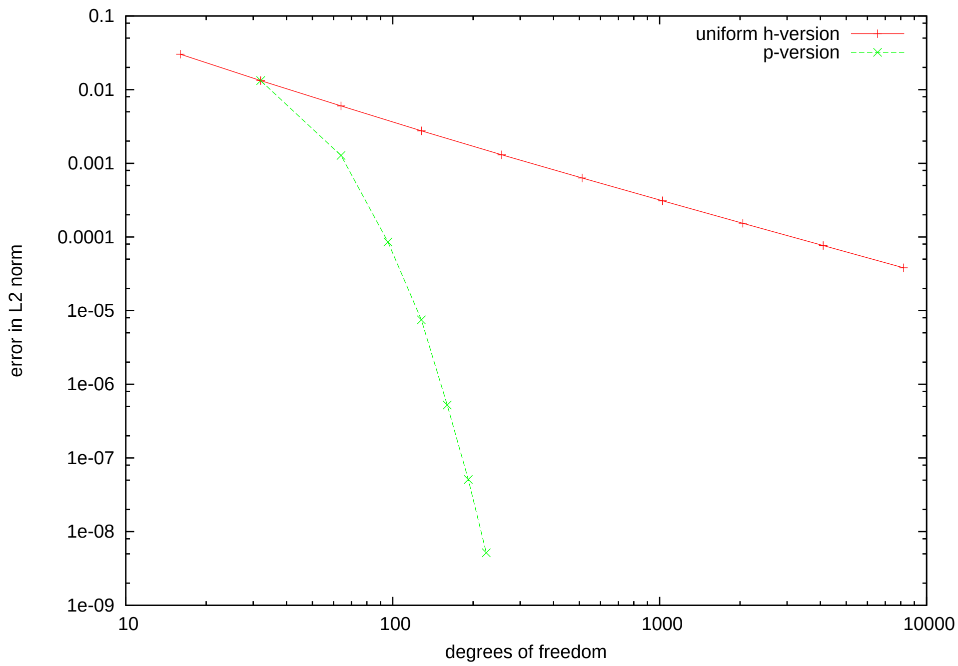


Figure 2.6: Lame(Dirichlet, interior): First kind integral equation, convergence in L_2-norm

Example 2.6. *Here we investigate the interior Neumann Lamé problem*

$$-\Delta^* \vec{u} = 0 \text{ in } \Omega$$
$$Tu = \vec{t_0} \text{ on } \partial\Omega$$

where $\vec{t_0}$ has to satisfy no compatability conditions. The boundary integral formulation reads: Find $u \in [H^{1/2}(\Gamma)]^2$ such that

$$Wu(x) = (\frac{1}{2}I - K)t_0(x) \text{ for } x \in \Gamma$$

Our choosen model solution is

$$\vec{u}_0 = \begin{pmatrix} -\frac{\lambda+3\mu}{4\mu(\lambda+2\mu)\pi}\log(r) + \frac{\lambda+\mu}{4\mu(\lambda+2\mu)\pi}\frac{(x_1-\bar{x}_1)^2}{r^2} \\ \frac{\lambda+\mu}{4\mu(\lambda+2\mu)\pi}\frac{(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{r^2} \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - b\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$$
$$\vec{t_0} = T\vec{u}_0$$

on the L-shaped domain Ω, $E = 2, \nu = 0.3$ and $\bar{x} = (0, 1.5)$, where $a_1, a_2, b \in \mathbb{R}$ are choosen such that, $\int_\Gamma \vec{u}_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \int_\Gamma \vec{u}_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \int_\Gamma \vec{u}_0 \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = 0.$

bem2/ex22hin

```
! h-Version, Lame, Neumann, L-Shape, interior
open(1) 'test' ; open(2) 'ex22hin.dat'
geometry('L-Shape',gm='Dg',bmode=(/1,2/),dim=(/0.5,0.5/)) ; #ti
problem('Lame',nickname='WIGLIRHS')
#ep 2. 0.3
EPS=1.0d-10
R=13
setlamex( (/0.,1.5/) )
J=2
do I=0,11
 mesh('uniform',n=J,p=1,spline='D')
 matrix('analytic') ; TM=SEC
 setlameu('D',12)
 approx 0 R 'N' 't0'
 lft 16 R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC ; #rno.
 lamephi('D')
 #hno. 1.
 #err. 16 R 'L2' 0 'D' 'u0'; E[0]=ERR
 #err. 16 R 'H1' 0 'D' 'u0'; E[1]=ERR
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

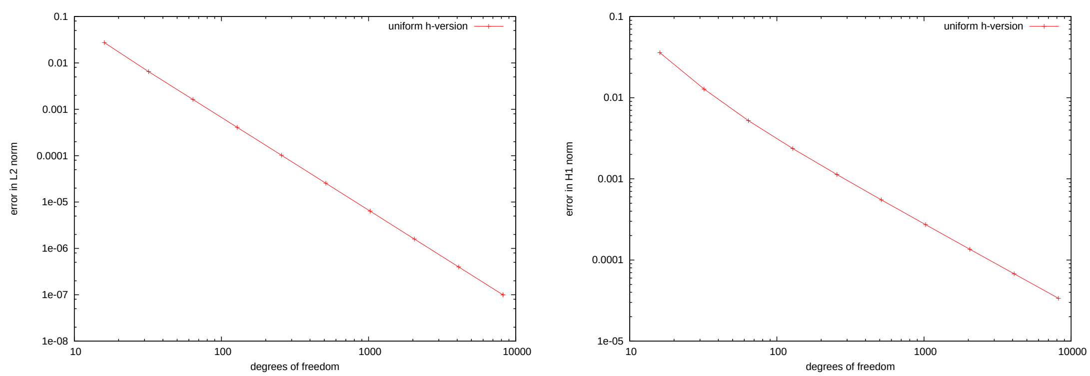


Figure 2.7: Lame(Neumann, interior): First kind integral equation, convergence in L_2-norm and H^1-norm

Example 2.7. *Here we investigate the exterior Dirichlet Lamé problem*

$$-\Delta^* \vec{u} = 0 \ in \ \mathbb{R}^2 \backslash \Omega$$
$$u = \vec{u}_0 \ on \ \partial\Omega$$

where $\vec{u}_0$ has to satisfy no compatability conditions. The boundary integral formulation reads: Find $t \in [H^{-1/2}(\Gamma)]^2$ such that

$$Vt(x) = (-\frac{1}{2}I + K)u(x) \ for \ x \in \Gamma$$

bem2/ex27hin

```
! h-Version, Lame, Dirichlet, L-Shape, exterior
open(1) 'test' ; open(2) 'ex27hin.dat'
geometry('L-Shape',gm='Ng',bmode=(/1,2/),dim=(/0.5,0.5/) ) ; #ti
problem('Lame',nickname='VIGLIRHSX')
R=12 ! right hand side
setlamex( (/-0.15,-0.15/) )
#ep 2. 0.3
EPS=1.0d-10
J=2
Q=26
do I=0,12
 mesh('uniform',n=J,p=0,spline='N')
 setlameu('N',12)
 matrix('analytic'); TM=SEC
 approx 0 R 'D' 'u0'
 lamephi('D')
 lft Q R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 lamephi('N')
 #hno. 1.25331413732
 #err. Q R 'L2' 1 'N' 't0' ; E[0]=ERR
 #err. Q R 'H1' 0 'N' 't0' ; E[1]=ERR
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

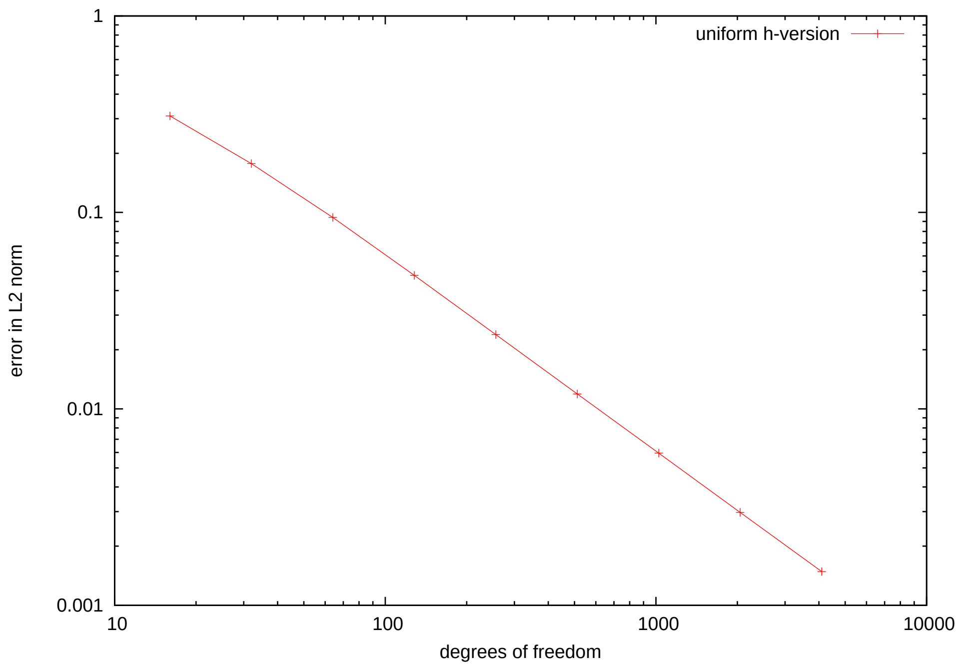


Figure 2.8: Lame(Dirichlet, exterior): First kind integral equation, convergence in L_2-norm

Example 2.8. *Here we investigate the exterior Neumann Lamé problem*

$$-\Delta^* \vec{u} = 0 \ \text{in } \mathbb{R}^2 \backslash \Omega$$
$$T\vec{u} = \vec{t}_0 \ \text{on } \partial\Omega$$

where $\vec{t}_0$ does not need to satisfy any compatibility condition.

The boundary integral formulation reads:

$$(W + P'P)u(x) = -(\frac{1}{2}I + K')t(x) \ \text{for } x \in \Gamma$$

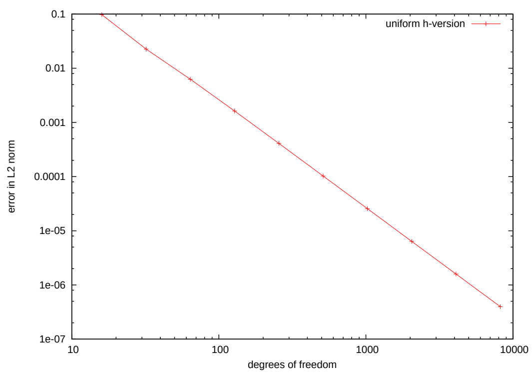
Our choosen model solution is

$$\vec{u}_0 = \begin{pmatrix} -\frac{\lambda+3\mu}{4\mu(\lambda+2\mu)\pi}\log(r) + \frac{\lambda+\mu}{4\mu(\lambda+2\mu)\pi}\frac{(x_1-\bar{x}_1)^2}{r^2} \\ \frac{\lambda+\mu}{4\mu(\lambda+2\mu)\pi}\frac{(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{r^2} \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - b\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$$
$$\vec{t}_0 = T\vec{u}_0$$

on the L-shaped domain Ω, $E = 2, \nu = 0.3$ and $\bar{x} = (-0.15, -0.15)$, where $a_1, a_2, b \in \mathbb{R}$ are choosen such that, $\int_\Gamma \vec{u}_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \int_\Gamma \vec{u}_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \int_\Gamma \vec{u}_0 \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = 0.$

bem2/ex28hin

```
! h-Version, Lame, Neumann, L-Shape, exterior
open(1) 'test' ; open(2) 'ex28hin.dat'
geometry('L-Shape',gm='Dg',bmode=(/1,2/),dim=(/0.5,0.5/) ) ; #ti
problem('Lame',nickname='WIGLIRHSX')
R=13 ! right hand side
setlamex( (/-0.15,-0.15/) )
#ep 2. 0.3
EPS=1.0d-10
J=2
Q=26
do I=0,12
 mesh('uniform',n=J,p=1,spline='D')
 setlameu('D',R)
 matrix('analytic'); TM=SEC
 approx 0 R 'N' 't0'
 lft Q R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 lamephi('D')
 #hno. 1.25331413732
 #err. Q R 'L2' 0 'D' 'u0' ; E[0]=ERR
 #err. Q R 'H1' 0 'D' 'u0' ; E[1]=ERR
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```



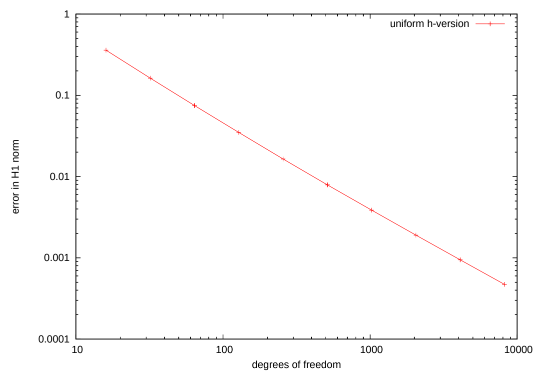


Figure 2.9: Lame(Neumann, exterior): First kind integral equation, convergence in L_2-norm and H^1-norm

Example 2.9. *Here we investigate the interior Neumann Lamé problem*

$$-\Delta^* \vec{u} = 0 \ \textit{in}\ \Omega$$
$$T\vec{u} = \vec{t}_0 \ \textit{on}\ \partial\Omega$$

where $\vec{t}_0$ does satisfy some compatibility condition. We use the Poincaré-Steklov formulation

$$(W + P'P)u + (\frac{1}{2}I + K)\phi = t_0$$
$$(\frac{1}{2}I + K)u - V\phi = 0$$

bem2/ex30hin

```
! h-Version, Lame, Neumann, L-Shape, interior, Steklov
open(1) 'test' ; open(2) 'ex30hin.dat'
geometry('L-Shape',gm='Dg',bmode=(/1,2/),dim=(/0.5,0.5/)) ; #ti
problem('Lame',nickname='WVIGL')
#ep 2. 0.3
EPS=1.0d-10
R=13
setlamex( (/0.,1.5/) )
J=2
do I=0,11
 mesh('uniform',n=J,p=1,spline='D')
 matrix('analytic') ; TM=SEC
 setlameu('D',12)
 approx 0 R 'N' 't0'
 lft 16 R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC ; #rno.
 lamephi('D')
 #hno. 1.
 #err. 16 R 'L2' 0 'D' 'u0'; E[0]=ERR
 #err. 16 R 'H1' 0 'D' 'u0'; E[1]=ERR
 #err. 16 R 'L2' 0 'N' 't0'; E[2]=ERR
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], E[2], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

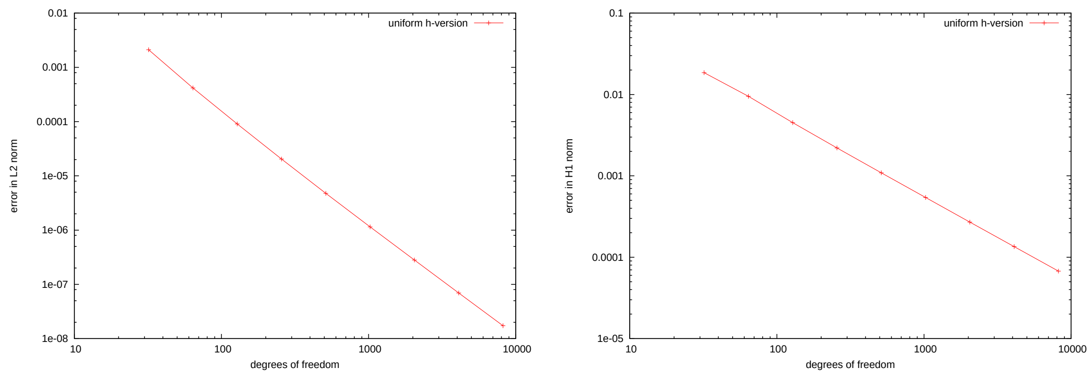


Figure 2.10: Lame(Neumann, interior, displacement): Steklov-Operator, convergence in L_2-norm and H^1-norm

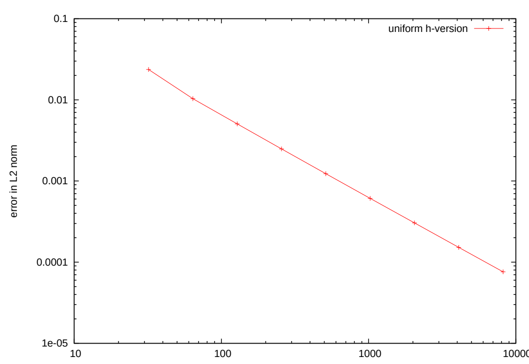


Figure 2.11: Lame(Neumann, interior, traction): Steklov-Operator, convergence in L_2-norm

Example 2.10. *Here we investigate the interior Dirichlet Lamé problem*

$$-\Delta^* \vec{u} = 0 \text{ in } \Omega$$
$$\vec{u} = \vec{u}_0 \text{ on } \partial\Omega$$

where $\vec{u}_0$ does satisfy some compatibility condition. We use the Poincaré-Steklov formulation

$$Vt + (\frac{1}{2}I - K)\psi = u_0$$
$$(\frac{1}{2}I - K)t - (W + P'P)\psi = 0$$

bem2/ex31hin

```
! h-Version, Lame, Dirichlet, L-Shape, interior, Steklov
open(1) 'test' ; open(2) 'ex31hin.dat'
geometry('L-Shape',gm='Ng',bmode=(/1,2/),dim=(/0.5,0.5/)) ; #ti
problem('Lame',nickname='VWIGL')
#ep 2. 0.3
EPS=1.0d-10
R=13
setlamex( (/0.,1.5/) )
J=2
do I=0,11
 mesh('uniform',n=J,p=0,spline='N')
 matrix('analytic') ; TM=SEC
 setlameu('N',12)
 lft 16 R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC ; #rno.
 lamephi('D')
 #hno. 1.
 #err. 16 R 'L2' 0 'N' 't0'; E[0]=ERR
 #err. 16 R 'L2' 0 'D' 'u0'; E[1]=ERR
 #err. 16 R 'H1' 0 'D' 'u0'; E[2]=ERR
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], E[2], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

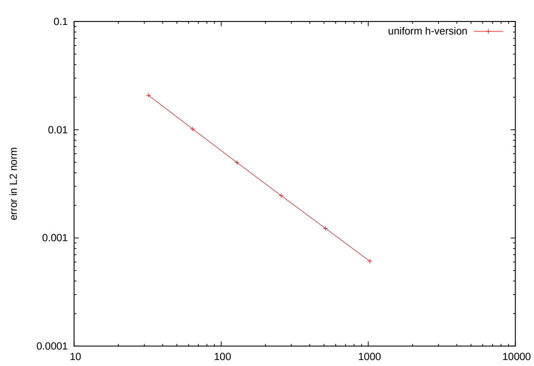


Figure 2.12: Lame(Dirichlet, interior, traction): Steklov-Operator, convergence in L_2-norm

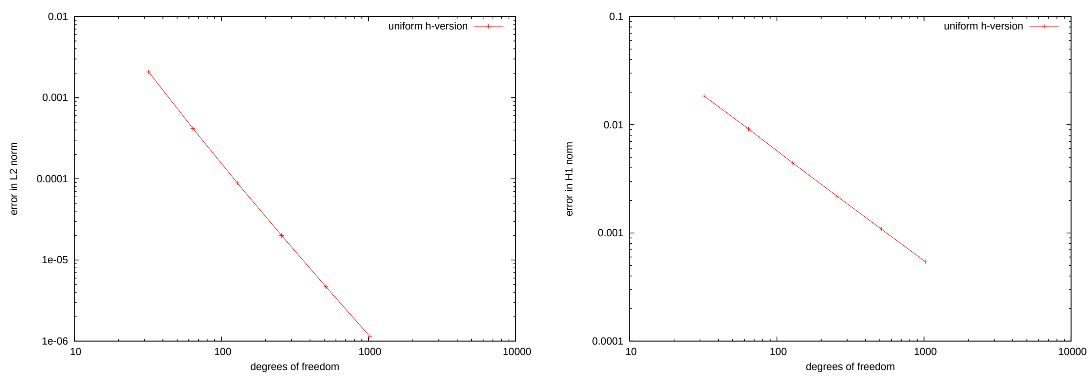


Figure 2.13: Lame(Dirichlet, interior, displacement): Steklov-Operator, convergence in L_2-norm and H^1-norm

Example 2.11. *Here we investigate the exterior Neumann Lamé problem*

$$-\Delta^* \vec{u} = 0 \ \text{in}\ \mathbb{R}^2 \backslash \Omega$$
$$T\vec{u} = \vec{t_0} \ \text{on}\ \partial\Omega$$

where $\vec{t_0}$ does not need to satisfy any compatibility condition. We use the Poincaré-Steklov formulation

$$(W + P'P)u + (-\frac{1}{2}I + K')\phi = -t_0$$
$$(-\frac{1}{2}I + K)u - V\phi = 0$$

bem2/ex29hin

```
! h-Version, Lame, Neumann, L-Shape, exterior  (Steklov Operator)
open(1) 'test' ; open(2) 'ex29hin.dat'
geometry('L-Shape',gm='Dg',bmode=(/1,2/),dim=(/0.5,0.5/) ) ; #ti
problem('Lame',nickname='WVIGLX')
R=13 ! right hand side
setlamex( (/-0.15,-0.15/) )
#ep 2. 0.3
EPS=1.0d-10
J=2
Q=26
do I=0,12
 mesh('uniform',n=J,p=1,spline='D')
 setlameu('D',12)
 matrix('analytic'); TM=SEC
 lft Q R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC; #rno.
 lamephi('D')
 #hno. 1.25331413732
 #err. Q R 'L2' 0 'D' 'u0' ; E[0]=ERR
 #err. Q R 'H1' 0 'D' 'u0' ; E[1]=ERR
 #err. Q R 'L2' 0 'N' 't0' ; E[2]=ERR
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], E[2], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

Example 2.12. *Here we investigate the exterior Dirichlet Lamé problem*

$$-\Delta^* \vec{u} = 0 \ \textit{in} \ \mathbb{R}^2 \backslash \Omega$$
$$u = \vec{u}_0 \ \textit{on} \ \partial\Omega$$

We use the Poincaré-Steklov formulation

$$Vt - (\frac{1}{2} + K)\psi = -u_0$$
$$-(\frac{1}{2}I + K')t - (W + P'P)\psi = 0$$

bem2/ex33hin

```
! h-Version, Lame, Dirichlet, L-Shape, exterior, Steklov
open(1) 'test' ; open(2) 'ex33hin.dat'
geometry('L-Shape',gm='Ng',bmode=(/1,2/),dim=(/0.5,0.5/)) ; #ti
problem('Lame',nickname='VWIGLX')
#ep 2. 0.3
EPS=1.0d-10
R=13
setlamex( (/-0.15,-0.15/) )
J=2
do I=0,11
 mesh('uniform',n=J,p=0,spline='N')
 matrix('analytic') ; TM=SEC
 setlameu('N',12)
 lft 16 R 0 R ; TL=SEC
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC ; #rno.
 lamephi('D')
 #hno. 1.
 #err. 16 R 'L2' 0 'N' 't0'; E[0]=ERR
 #err. 16 R 'L2' 0 'D' 'u0'; E[1]=ERR
 #err. 16 R 'H1' 0 'D' 'u0'; E[2]=ERR
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], E[2], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

2.1.3 Helmholtz

Example 2.13. *Here we deal with the Dirichlet problem of the Helmholtz equation on a closed polygon. The corresponding integral equation is*

$$V_k\psi(\vec{x}) = (I + K_k)u_0(\vec{x}) \tag{4}$$

The boundary data on the L-Shape are given by

$$u_0(x,y) = u(x,y) = \tilde{J}_{2/3}(kr)\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$$

The exact solution is

$$\psi(r,\varphi) = \frac{\partial u}{\partial n}|_\Gamma$$

$\tilde{J}_{2/3}(x) = \Gamma(2/3+1)J_{2/3}(x)$ *is a rescaled Bessel function.*

bem2/ex42hin

```
! h-Version, Helmholtz, Dirichlet, L-Shape
open(1) 'test' ; open(2) 'ex42hin.dat'
geometry('L-Shape',gm='Ng',styp=1,bmode=(/1,2/)) ; #ti
problem('Helmholtz',nickname='VIGLIRHS')
R=1 ! right hand side
EPS=1.0d-10; Q=24; ITER=0
do K=1,10
 KW=Real(K)/2.0; #kw KW
 J=2
 open(2) 'ex42hk'//KW:3//'in.dat'
 do I=0,7
  mesh('uniform',n=J,p=0,spline='N')
  matrix('analytic'); TM=SEC
  approx 0 R 'D' 'u0'
  lft Q R 0 R
  solve(eps=EPS,mdi='x=0',mdc='no',mit='GAUSS'); TS=SEC; #rno.
  norm('NO','L2','N')
  #hno. 1.6018674
  #err. Q R 'L2' 0 'N'
  write(2) DOF, I, ENO, ENOERR, ERR, ITER, TM, TS, NO
  J=J*2
 continue
 close(2)
continue
end
```

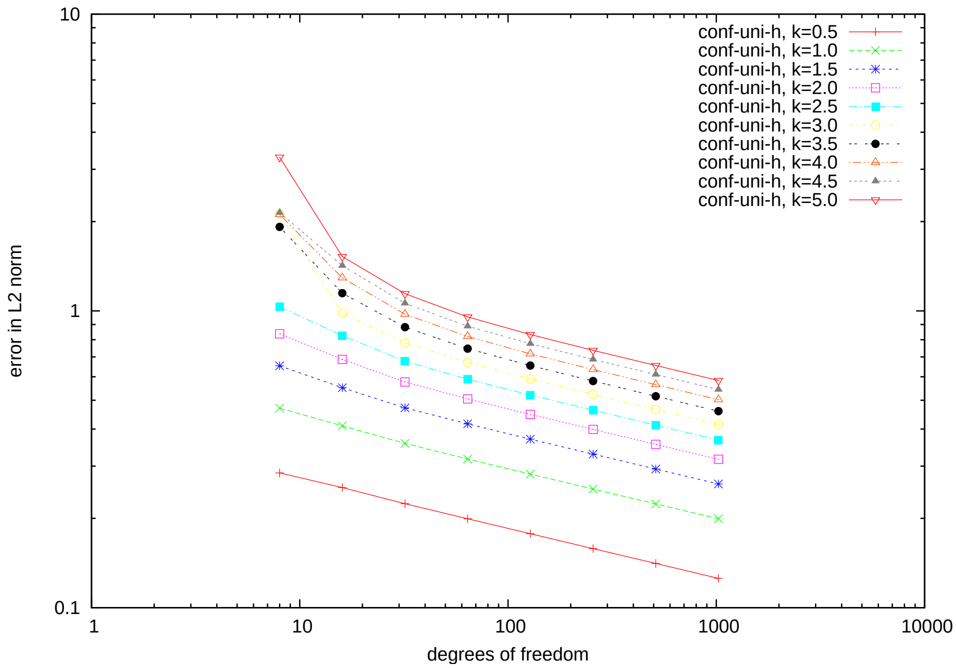


Figure 2.14: Helmholtz: Weakly singular integral equation, convergence in L_2-norm

Example 2.14. *Here we deal with the Neumann problem of the Helmholtz equation on a closed polygon. The corresponding integral equation is*

$$W_k v(\vec{x}) = (I - K_k')t_0(\vec{x}) \tag{5}$$

The boundary data on the L-Shape are given by

$$t_0(x,y) = \frac{\partial u(x,y)}{\partial n}, \quad u(x,y) = \tilde{J}_{2/3}(kr)\cos(\frac{2}{3}(\varphi - \frac{\pi}{2}))$$

The exact solution is $v(r,\varphi) = u(r,\varphi)|_\Gamma$. $\tilde{J}_{2/3}(x) = \Gamma(2/3+1)J_{2/3}(x)$ *is a rescaled Bessel function.*

bem2/ex43hin

```
! h-Version, Helmholtz, Neumann, L-Shape
open(1) 'test' ; open(2) 'ex43hin.dat'
geometry('L-Shape',gm='Dg',styp=1,bmode=(/1,2/)) ; #ti
problem('Helmholtz',nickname='WIGLIRHS')
R=21 ! right hand side
EPS=1.0d-10; Q=24; ITER=0
do K=1,10
 KW=Real(K)/2.0; #kw KW
 J=2
 open(2) 'ex43hk'//KW:3//'in.dat'
 do I=0,7
  mesh('uniform',n=J,p=1,spline='D')
  matrix('analytic'); TM=SEC
  approx 0 R 'N' 't0'
  lft Q R 0 R
  solve(eps=EPS,mdi='x=0',mdc='no',mit='GAUSS'); TS=SEC; #rno.
  norm('NO','L2','D')
  #hno. 1.6018674
  #err. Q R 'L2' 0 'D' ; E[0]=ERR
  #err. Q R 'H1' 0 'D' ; E[1]=ERR
  write(2) DOF, I, ENO, ENOERR, E[0], E[1], ITER, TM, TS, NO
  J=J*2
 continue
 close(2)
continue
end
```

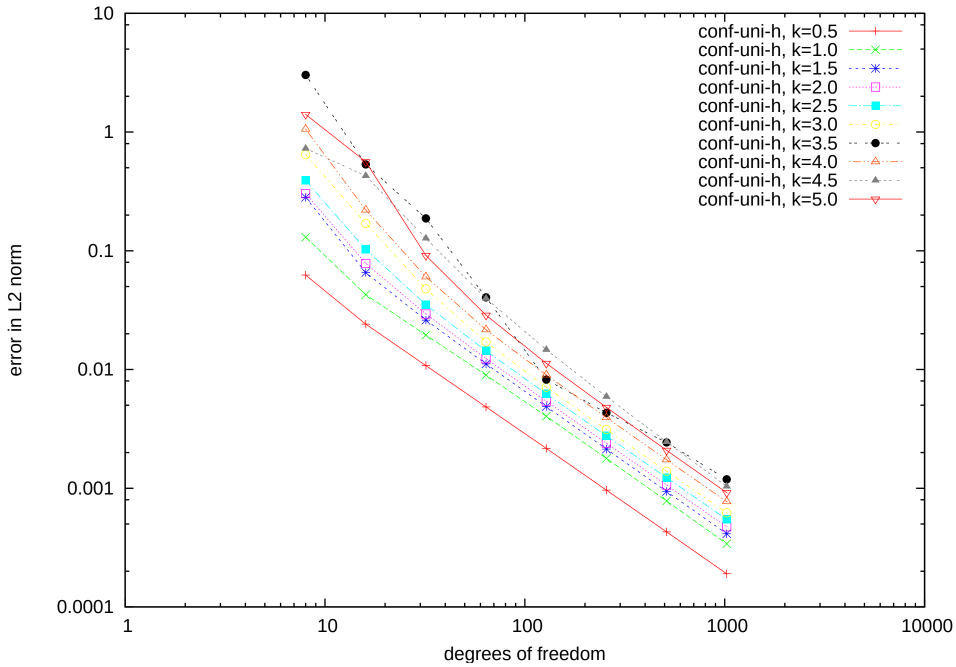


Figure 2.15: Helmholtz: Hypersingular integral equation, convergence in L_2-norm

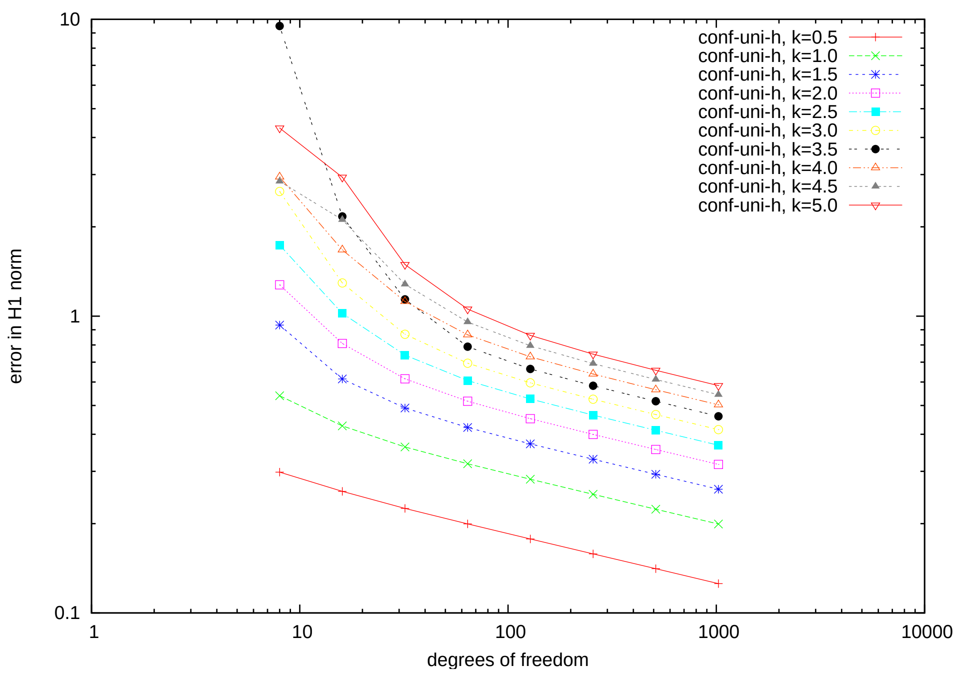


Figure 2.16: Helmholtz: Hypersingular integral equation, convergence in H^1-norm

Example 2.15. *Here we deal with the Neumann problem of the Helmholtz equation on a closed polygon. The corresponding second kind integral equation is*

$$(I + K_k)v(\vec{x}) = V_k t_0(\vec{x}) \tag{6}$$

The boundary data on the L-Shape are given by

$$t_0(x, y) = \frac{\partial u(x, y)}{\partial n}, \quad u(x, y) = \tilde{J}_{2/3}(kr)\cos(\frac{2}{3}(\varphi - \frac{\pi}{2}))$$

The exact solution is $v(r, \varphi) = u(r, \varphi)|_\Gamma$*.* $\tilde{J}_{2/3}(x) = \Gamma(2/3 + 1)J_{2/3}(x)$ *is a rescaled Bessel function.*

In this example we use continuous test- and ansatz functions.

bem2/ex44hin

```
! h-Version, Helmholtz, Neumann, Second kind, L-Shape
open(1) 'test' ; open(2) 'ex44hin.dat'
geometry('L-Shape',gm='Dg',styp=1,bmode=(/1,2/)) ; #ti
problem('Helmholtz',nickname='KIGLIRHS')
R=21 ! right hand side
EPS=1.0d-10; Q=24; ITER=0
do K=1,10
 KW=Real(K)/2.0; #kw KW; #kwc.
 J=2
 open(2) 'ex44hk'//KW:3//'in.dat'
 do I=0,7
  mesh('uniform',n=J,p=1,spline='D')
  matrix('analytic'); TM=SEC
  approx 0 R 'N' 't0'
  lft Q R 0 R
  solve(eps=EPS,mdi='x=0',mdc='no',mit='GAUSS'); TS=SEC; #rno.
  norm('NO','L2','D')
  #hno. 1.6018674
  #err. Q R 'L2' 0 'D' ; E[0]=ERR
  #err. Q R 'H1' 0 'D' ; E[1]=ERR
  write(2) DOF, I, ENO, ENOERR, E[0], E[1], ITER, TM, TS, NO
  J=J*2
 continue
 close(2)
continue
end
```

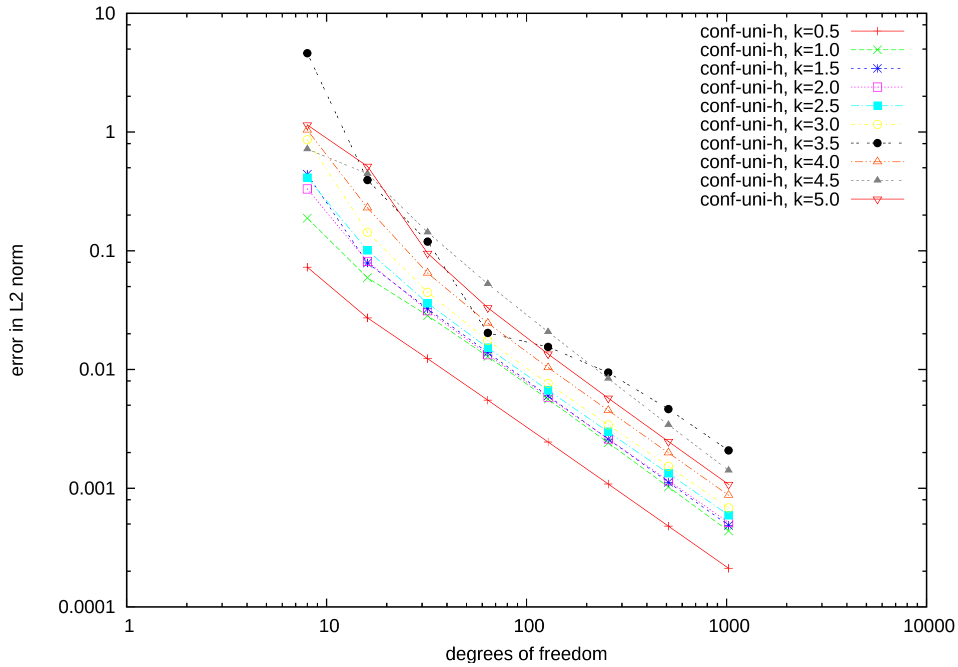


Figure 2.17: Helmholtz: Second kind integral equation, convergence in L_2-norm

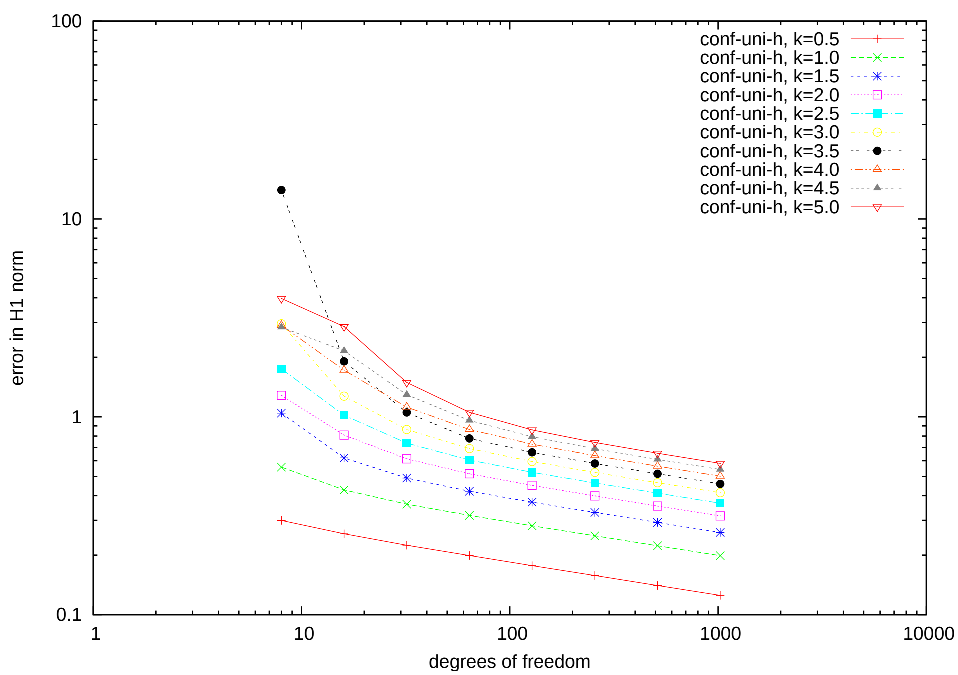


Figure 2.18: Helmholtz: Second kind integral equation, convergence in H^1-norm

2.1.4 Stokes

Example 2.16. *Here we investigate the interior Dirichlet Stokes problem*

$$\begin{aligned} -\nu\Delta\vec{u} + \nabla p &= 0 \ \textit{in } \mathbb{R}^2\backslash\Omega \\ \operatorname{div}\vec{u} &= 0 \ \textit{in } \mathbb{R}^2\backslash\Omega \\ \vec{u} &= u_0 \ \textit{on } \partial\Omega \end{aligned}$$

with u_0 satisfying the compatibility condition

$$\int_\Gamma u_0(x)n(x)\,ds_x = 0$$

The boundary integral formulation reads:

$$Vt(x) = (\frac{1}{2}I + K)u(x) \ \textit{for } x \in \Gamma$$

with

$$Vt(x) := \int_\Gamma U^*(x,y)t(y)\,ds_y, \quad Ku(x) := \int_\Gamma T^*(x,y)u(y)\,ds_y$$

and

$$\begin{aligned} U^*_{ij}(x,y) &= \frac{1}{4\cdot\pi\nu}\left[-\log|x-y|\delta_{ij} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right] \\ T^*_{ij}(x,y) &= \frac{2}{2\cdot\pi}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\frac{n_y\cdot(x-y)}{|x-y|^2} \end{aligned}$$

We choose the right hand side

$$u_0(x) = \frac{1}{\nu}\begin{pmatrix} -\log(r) + \frac{(x_1-\bar{x}_1)^2}{r^2} \\ \frac{(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{r^2}, \end{pmatrix}, \quad p_0(x) = 2\frac{x_1-\bar{x}_1}{r^2}, \quad r = |x-\bar{x}|_2$$

with the corresponding traction

$$t(x) = T(u_0,p_0) = \sigma(u_0,p_0)\vec{n} = -p_0\vec{n} + \nu(\nabla u_0 + \nabla u_0^T)\vec{n}$$

on the L-shaped domain Ω, $\nu = 4$ and $\bar{x} = (0, 1.5)$.

bem2/ex56hin

```
! h-Version, Stokes, Dirichlet, L-Shape, interior
open(1) 'test.h' ; open(2) 'ex56hin.dat'
geometry('L-Shape',gm='Ng',bmode=(/1,2/),dim=(/0.5,0.5/)) ; #ti
problem('Stokes',nickname='VIGLIRHS')
R=10
setstokesx( (/0.0,1.5/) )
NU=4.0
#stokes NU NU
EPS=1.0d-10
J=2
Q=26
do I=0,12
 mesh('uniform',n=J,p=0,spline='N')
 matrix('analytic',gqna=16,gqnb=16,ijn=8,mu=1); TM=SEC
 approx 0 R 'D' 'u0'
```

```
 lft Q R 0 R ; TL=SEC
 #lx. 'N'
 setstokesp('N')
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 stokestau('N')
 stokesphi('D')
 #err. Q R 'L2' 0 'N' 't0'; E[0]=ERR
 #hno. 0.1057060
 write(2) DOF, I, ENO, ENOERR, E[0], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

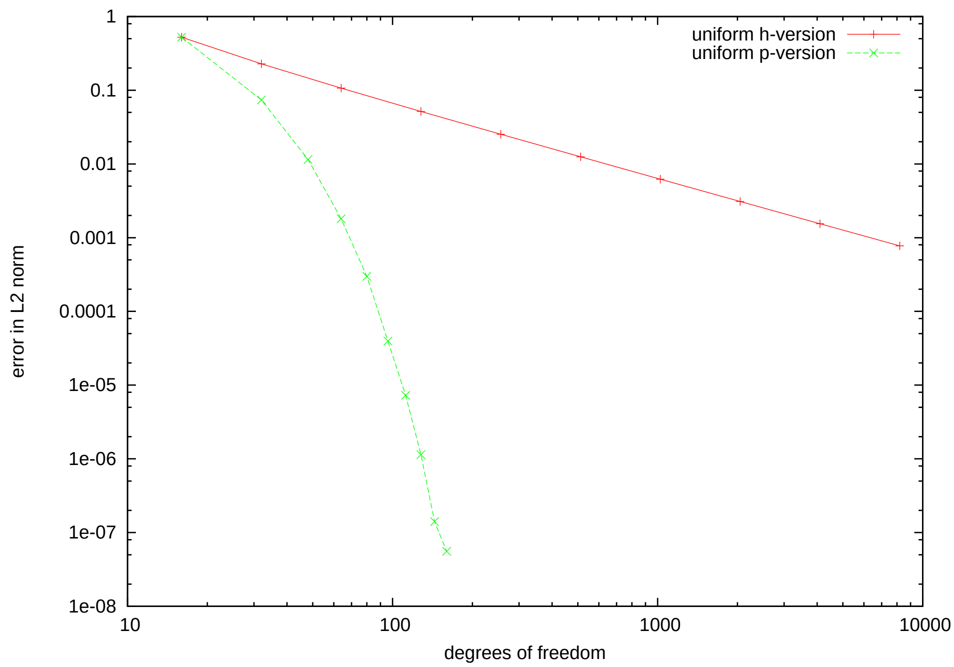


Figure 2.19: Stokes(Dirichlet): First kind integral equation, convergence in L_2-norm

Example 2.17. *Here we investigate the exterior Dirichlet Stokes problem*

$$\begin{aligned} -\nu\Delta\vec{u} + \nabla p &= 0 \;\; in \; \mathbb{R}^2\backslash\Omega \\ \operatorname{div}\vec{u} &= 0 \;\; in \; \mathbb{R}^2\backslash\Omega \\ \vec{u} &= u_0 \;\; on \; \partial\Omega \\ \vec{u}(x) &= \Sigma\log|x| + \mathcal{O}(1) \\ p(x) &= \mathcal{O}(|x|^{-1}) \;\; as \; |x| \to \infty \end{aligned}$$

where u_0 has to satisfy no compatibility condition. The boundary integral formulation reads: Find $t \in [H^{-1/2}(\Gamma)]^2, \omega \in \mathbb{R}^2$ such that

$$Vt(x) - \omega = (-\frac{1}{2}I + K)u(x) \;\; for \; x \in \Gamma, \qquad \int_\Gamma t(x)\,ds_x = \Sigma.$$

We choose the right hand side

$$u_0(x) = \frac{1}{\nu}\begin{pmatrix} -\log(r) + \frac{(x_1-\bar{x}_1)^2}{r^2} \\ \frac{(x_1-\bar{x}_1)(x_2-\bar{x}_2)}{r^2}, \end{pmatrix}, \quad p_0(x) = 2\frac{x_1-\bar{x}_1}{r^2}, \quad r = |x-\bar{x}|_2$$

with the corresponding traction

$$t(x) = T(u_0, p_0) = \sigma(u_0, p_0)\vec{n} = -p_0\vec{n} + \nu(\nabla u_0 + \nabla u_0^T)\vec{n}$$

on the L-shaped domain Ω, $\nu = 4$ and $\bar{x} = (0, -0.2)$.

bem2/ex58hin

```
! h-Version, Stokes, Dirichlet, L-Shape, exterior
open(1) 'test.h' ; open(2) 'ex58hin.dat'
geometry('L-Shape',gm='Ng',bmode=(/1,2/),dim=(/0.5,0.5/)) ; #ti
problem('Stokes',nickname='VIGLIRHSX')
R=10
setstokesx( (/0.0,-0.2/) )
NU=4.0
#stokes NU NU
EPS=1.0d-10
J=2
Q=26
do I=0,12
 mesh('uniform',n=J,p=0,spline='N')
 matrix('analytic',gqna=16,gqnb=16,ijn=8,mu=1); TM=SEC
 approx 0 R 'D' 'u0'
 lft Q R 0 R ; TL=SEC
 setstokesp('N')
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 stokestau('N')
 stokesphi('D')
 #err. Q R 'L2' 0 'N' 't0'; E[0]=ERR
 #hno. 0.1057060
 write(2) DOF, I, ENO, ENOERR, E[0], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

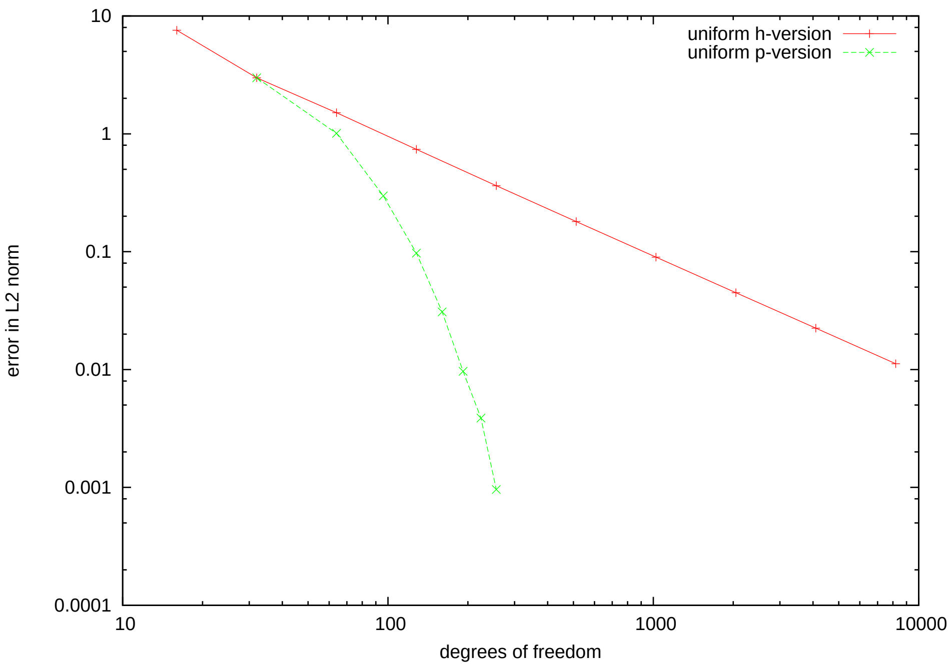


Figure 2.20: Stokes(Dirichlet, exterior): First kind integral equation, convergence in L_2-norm

Example 2.18. *Here we investigate the interior Neumann Stokes problem*

$$\begin{aligned} -\nu\Delta\vec{u} + \nabla p &= 0 \ \textit{in}\ \Omega \\ \operatorname{div}\vec{u} &= 0 \ \textit{in}\ \Omega \\ T\vec{u} &= t_0 \ \textit{on}\ \partial\Omega \end{aligned}$$

with t_0 satisfying the compatibility conditions

$$\int_\Gamma t_0(x)\begin{pmatrix}1\\0\end{pmatrix} ds_x = \int_\Gamma t_0(x)\begin{pmatrix}0\\1\end{pmatrix} ds_x = \int_\Gamma t_0(x)\begin{pmatrix}x_2\\-x_1\end{pmatrix} ds_x = 0$$

The boundary integral formulation reads:

$$Wu(x) = (\frac{1}{2}I - K')t(x) \ \textit{for}\ x \in \Gamma$$

on the L-shaped domain Ω, $\nu = 4$ and $\bar{x} = (0, 1.5)$.

bem2/ex57hin

```
! h-Version, Stokes, Neumann, L-Shape, interior
open(1) 'test.h' ; open(2) 'ex57hin.dat'
geometry('L-Shape',gm='Dg',bmode=(/1,2/),dim=(/0.5,0.5/)) ; #ti
problem('Stokes',nickname='WIGLIRHS')
R=10
setstokesx( (/0.0,1.5/) )
NU=4.0
#stokes NU NU
EPS=1.0d-10
J=2
Q=26
do I=0,12
 mesh('uniform',n=J,p=1,spline='D')
 matrix('analytic',gqna=16,gqnb=16,ijn=8,mu=1); TM=SEC
 approx 0 R 'N' 't0'
 lft Q R 0 R ; TL=SEC
 nmean('Rhs(D)')
 setstokesp('D')
 setstokesu('D')
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 stokestau('N')
 stokesphi('D')
 #err. Q 20 'L2' 0 'D' 'u0'; E[0]=ERR
 #err. Q 20 'H1' 0 'D' 'u0'; E[1]=ERR
 stokesphi('D')
 #hno. 0.1057060
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

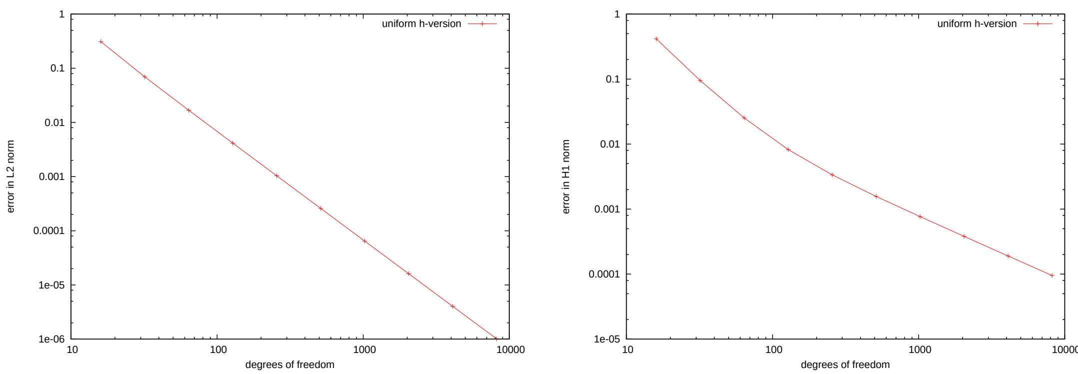


Figure 2.21: Stokes(Neumann): First kind integral equation, convergence in L_2-norm and H^1-norm

Example 2.19. *Here we investigate the exterior Neumann Stokes problem*

$$\begin{aligned} -\nu\Delta\vec{u} + \nabla p &= 0 \ \textit{in}\ \mathbb{R}^2\backslash\Omega \\ \operatorname{div}\vec{u} &= 0 \ \textit{in}\ \mathbb{R}^2\backslash\Omega \\ T\vec{u} &= t_0 \ \textit{on}\ \partial\Omega \end{aligned}$$

where t_0 does not need to satisfy any compatibility condition.

The boundary integral formulation reads:

$$(W + P'P)u(x) = -(\frac{1}{2}I + K')t(x) \ \textit{for}\ x \in \Gamma$$

The model solution is choosen as in Example 2.17 on the L-shaped domain Ω, $\nu = 4$ and $\bar{x} = (-0.15, -0.15)$.

bem2/ex60hin

```
! h-Version, Stokes, Neumann, L-Shape, exterior
open(1) 'test.h' ; open(2) 'ex60hin.dat'
geometry('L-Shape',gm='Dg',bmode=(/1,2/),dim=(/0.5,0.5/)) ; #ti
problem('Stokes',nickname='WIGLIRHSX')
R=20
setstokesx( (/-0.15,-0.15/) )
NU=4.0
#stokes NU NU
EPS=1.0d-10
J=2
Q=26
do I=0,12
 mesh('uniform',n=J,p=1,spline='D')
 matrix('analytic',gqna=16,gqnb=16,ijn=8,mu=1); TM=SEC
 setstokesp('D',10)
 setstokesu('D',10)
 approx 0 R 'N' 't0'
 lft Q R 0 R ; TL=SEC
 #lx. 'D'
 nmean('Rhs(D)')
 solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); TS=SEC
 stokestau('N')
 stokesphi('D')
 #err. Q R 'L2' 0 'D' 'u0'; E[0]=ERR
 #err. Q R 'H1' 0 'D' 'u0'; E[1]=ERR
 stokesphi('D')
 #hno. 0.1057060
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

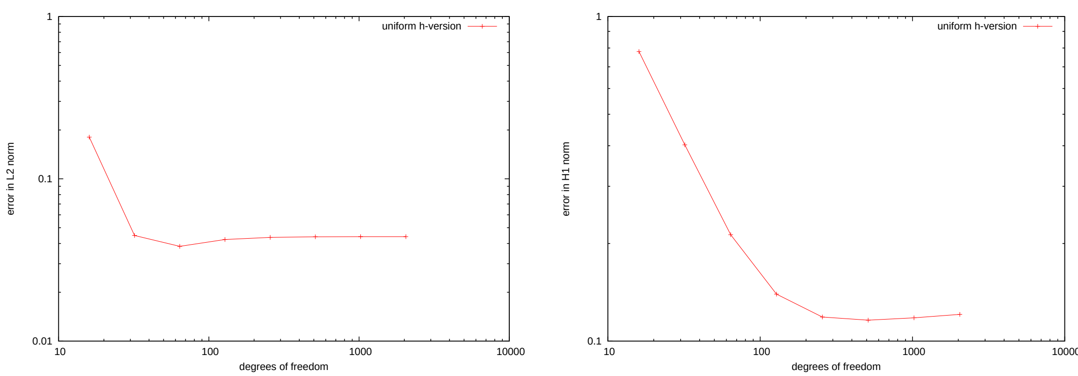


Figure 2.22: Stokes(Neumann): First kind integral equation, convergence in L_2-norm and H^1-norm

2.2 Solvers

The following examples demonstrate the Multiplicative Schwarz Method as an efficient solver for the p-Version of the weakly singular and the hyper-singular integral equation on an interval [23].

Example 2.20. ***bem2/ex13in***

```
! p-Version, Laplace, Neumann, Slit [-1,1], Multiplikativ Schwarz
open(1) 'test'
geometry('Slit',gm='Dg',bmode=(/1,2/)) ; #ti
problem('Laplace',nickname='WIGLSCR')
R=16 ! right hand side
EPS=1.0d-10
K=2; P=30

do J=1,7
 inquire(file='mspw'//J//'.dat',T)
 if (T.eq.0); then
  open(2) 'mspw'//J//'.dat'
  write(2) '# Neumann, p-version with coarse grid, n=',K
  write(2) '# p rate iter enorm time dof'
  do I=2,P
   mesh('uniform',n=K,p=I,spline='D')
   matrix('numeric',gqna=34,gqnb=34,ijn=6,mu=1.0,sigma=0.17); TM=SEC
   lft 40 R 0 R ; TL=SEC
   defprec(mode='MSM',spline='D',name='PD',mat='W',hpmodus=1,ddmodus=1,domains=0,omega=1.0)

   solve(eps=EPS,mdi='x=0',mdc='D.PD.D',mit='MSM'); S=SEC
   #hno. ; #rno.
   solve(eps=EPS,mdi='x=0',mdc='D.PD.D',mit='MSMRATE') ! Contraction rate
   write(2) I , LMAX,ITER,ENO,S,DOF
  continue
  write(2) ' '
 fi
 K=K*2
continue
end
```

Example 2.21. ***bem2/ex14in***

```
! p-Version,  Laplace, Dirichlet, Slit [-1,1], Multiplikativ Schwarz
open(1) 'test'
geometry('Slit',gm='Ng',bmode=(/1,2/)) ; #ti
problem('Laplace',nickname='VIGLSCR')
R=3 ! right hand side
EPS=1.0d-10
K=2; P=30

do J=1,7
 inquire(file='mspv'//J//'.dat',T)
 if (T.eq.0); then
  open(2) 'mspv'//J//'.dat'
  write(2) '# Dirichlet, p-version with coarse grid, n=',K
  write(2) '# p rate iter enorm time dof'
```

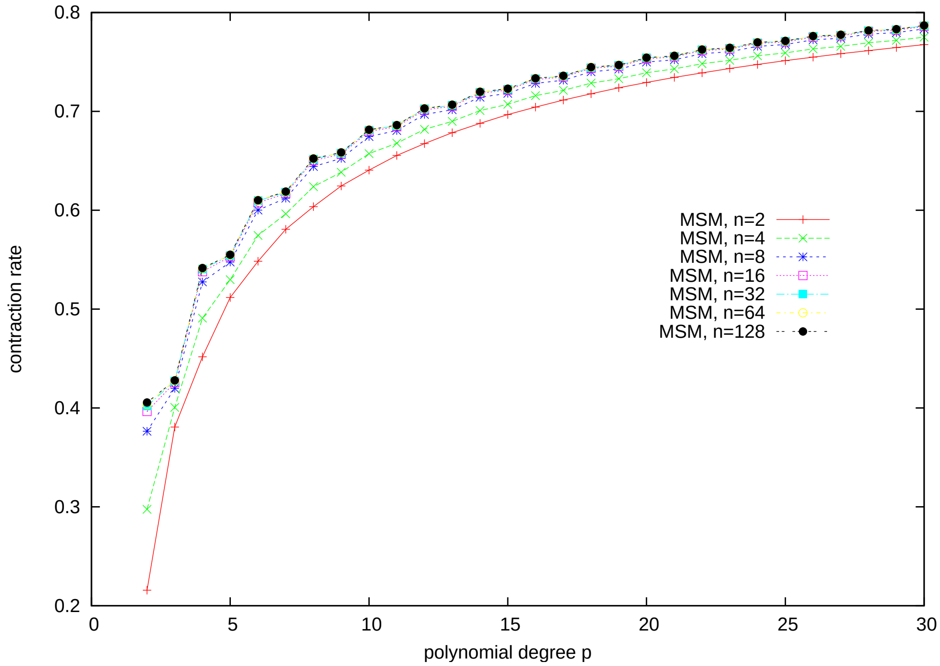


Figure 2.23: Hyper-singular integral equation, p-version, contraction rate

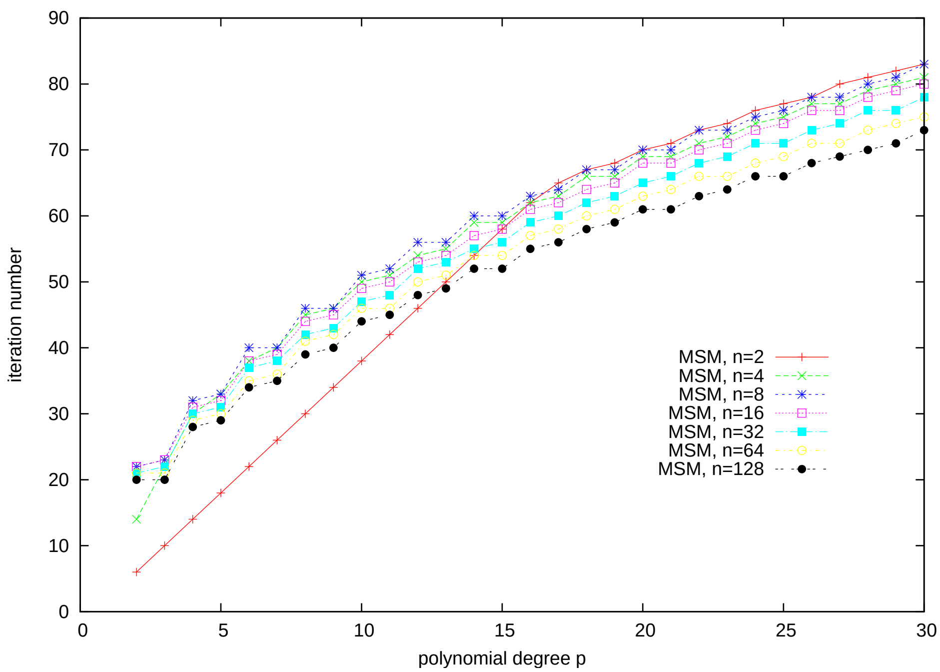


Figure 2.24: Hyper-singular integral equation, p-version, iteration numbers

```
do I=1,P
 mesh('uniform',n=K,p=I,spline='N')
 matrix('numeric',gqna=34,gqnb=34,ijn=6,mu=1.0,sigma=0.17); TM=SEC
 lft 40 R 0 R ; TL=SEC
 defprec(mode='MSM',spline='N',name='PN',mat='V',hpmodus=1,ddmodus=1,domains=0,omega=1.0)
```

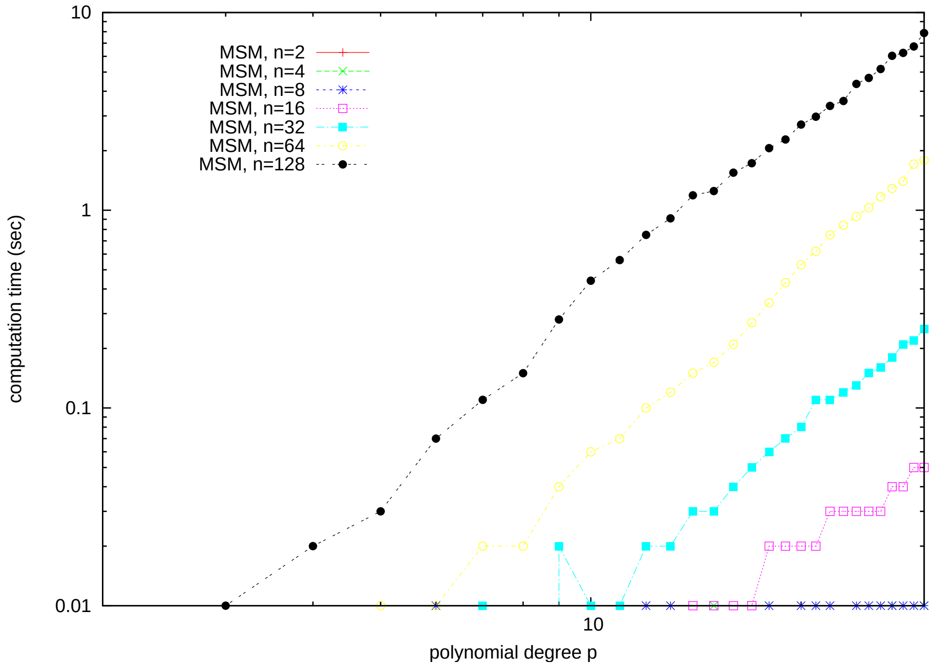


Figure 2.25: Hyper-singular integral equation, p-version, computing time

```
   solve(eps=EPS,mdi='x=0',mdc='N.PN.N',mit='MSM'); S=SEC
   #hno.; #rno.
   solve(eps=EPS,mdi='x=0',mdc='N.PN.N',mit='MSMRATE')

   write(2) I  , LMAX,ITER,ENO,S,DOF
  continue
  write(2) ' '
 fi
 K=K*2
continue
end
```

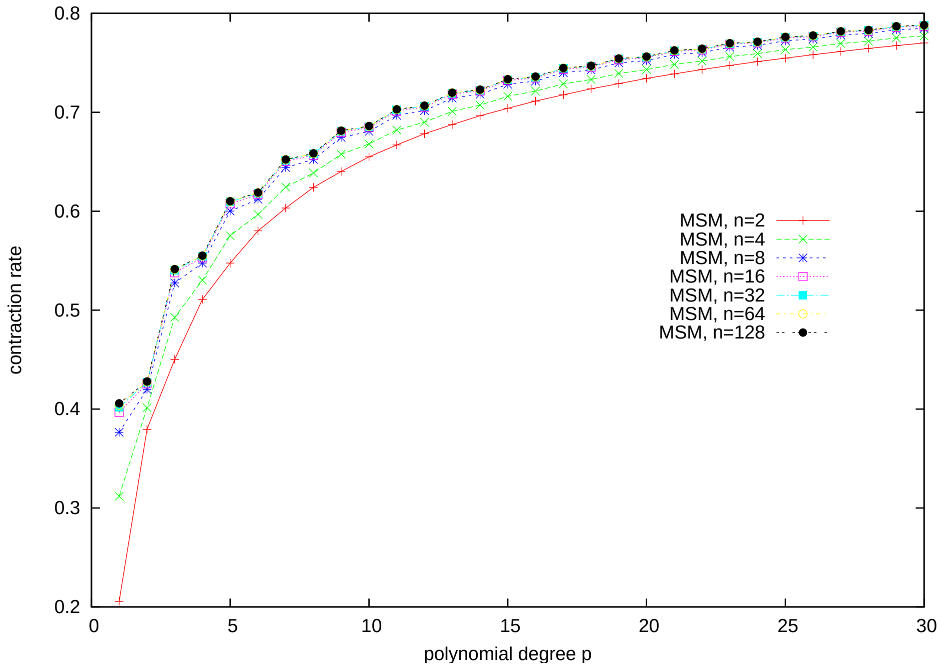


Figure 2.26: Weakly singular integral equation, p-version, contraction rate

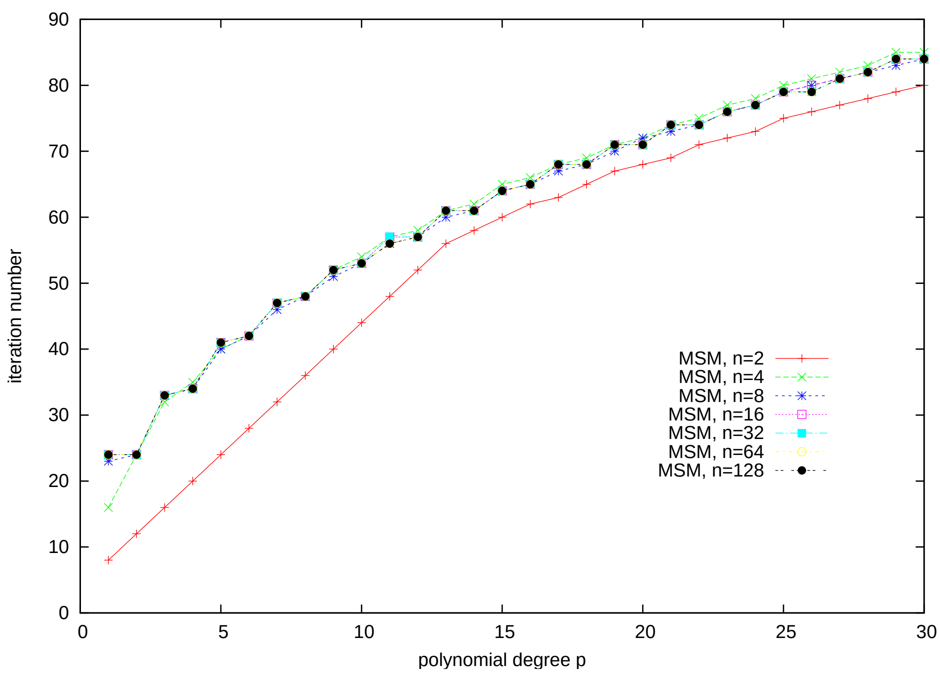


Figure 2.27: Weakly singular integral equation, p-version, iteration numbers

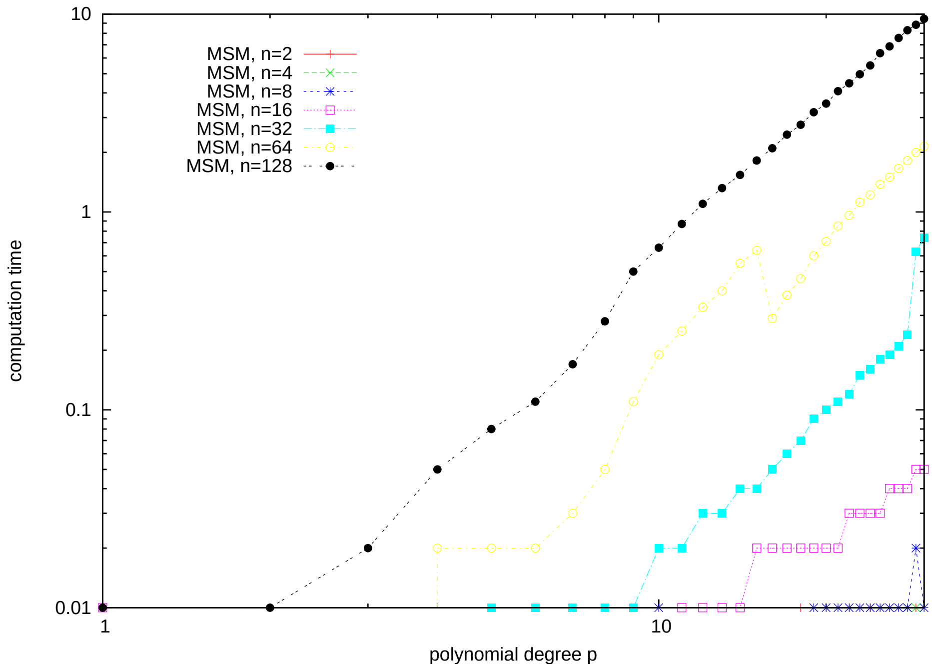


Figure 2.28: Weakly singular integral equation, p-version, computing time

Example 2.22. *Additive Schwarz algorithm for the Neumann problem on the slit.*

bem2/ex15in

```
! p-Version,  Laplace, Neumann, Slit [-1,1], Additive Schwarz
open(1) 'test.aspw'
geometry('Slit',gm='Dg',bmode=(/1,2/)) ; #ti
problem('Laplace',nickname='WIGLSCR')
R=16 ! right hand side
EPS=1.0d-10
K=2; P=30

do J=1,7
 inquire(file='aspw'//J//'.dat',T)
 if (T.eq.0); then
  open(2) 'aspw'//J//'.dat'
  write(2) '# Dirichlet, p-version with coarse grid, n=',K
  write(2) '# p lmin lmax cond iter enorm time dof'
  do I=1,P
   mesh('uniform',n=K,p=I,spline='D')
   matrix('numeric',gqna=34,gqnb=34,ijn=6,mu=1.0,sigma=0.17); TM=SEC
   lft 40 R 0 R ; TL=SEC
   defprec(mode='ASM',spline='D',name='PD',mat='W',hpmodus=1,ddmodus=1,domains=0,omega=1.0)

   solve(eps=EPS,mdi='x=0',mdc='D.PD.D',mit='CG'); S=SEC
   #hno.; #rno.
   solve(eps=EPS,mdi='x=0',mdc='D.PD.D',mit='EW')
   write(2) I , LMIN,LMAX,COND,ITER,ENO,S,DOF
  continue
  write(2) ' '
 fi
 K=K*2
continue
end
```

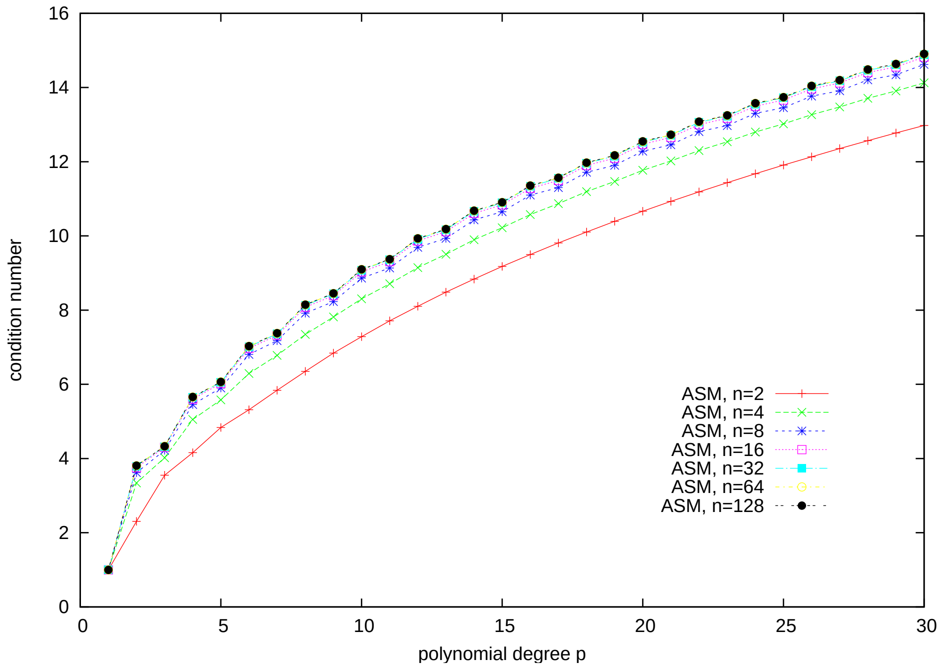


Figure 2.29: Hyper singular integral equation, p-version, condition number

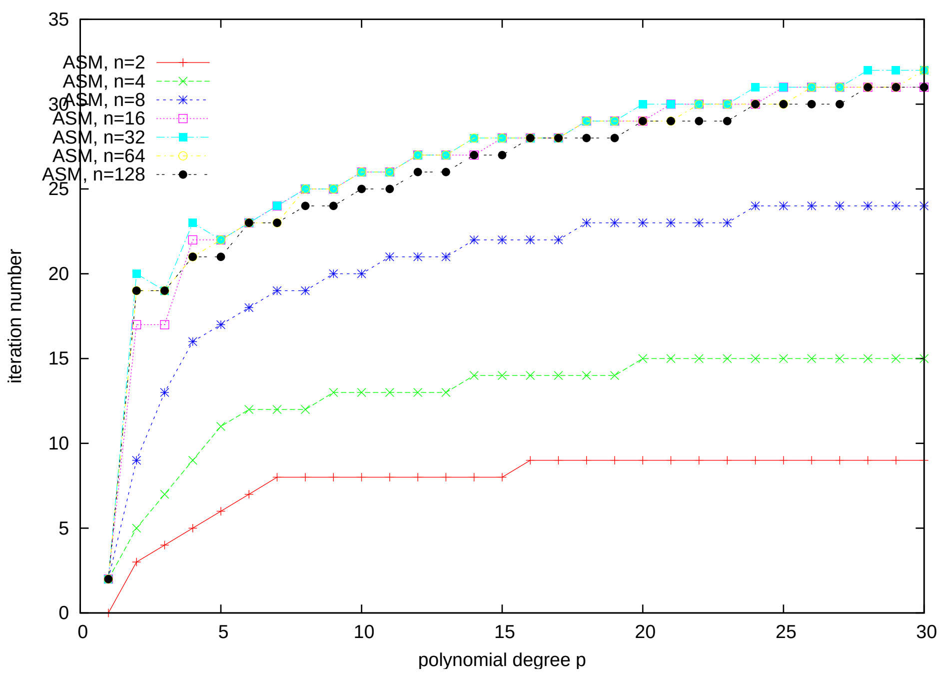


Figure 2.30: Hyper singular integral equation, p-version, iteration numbers

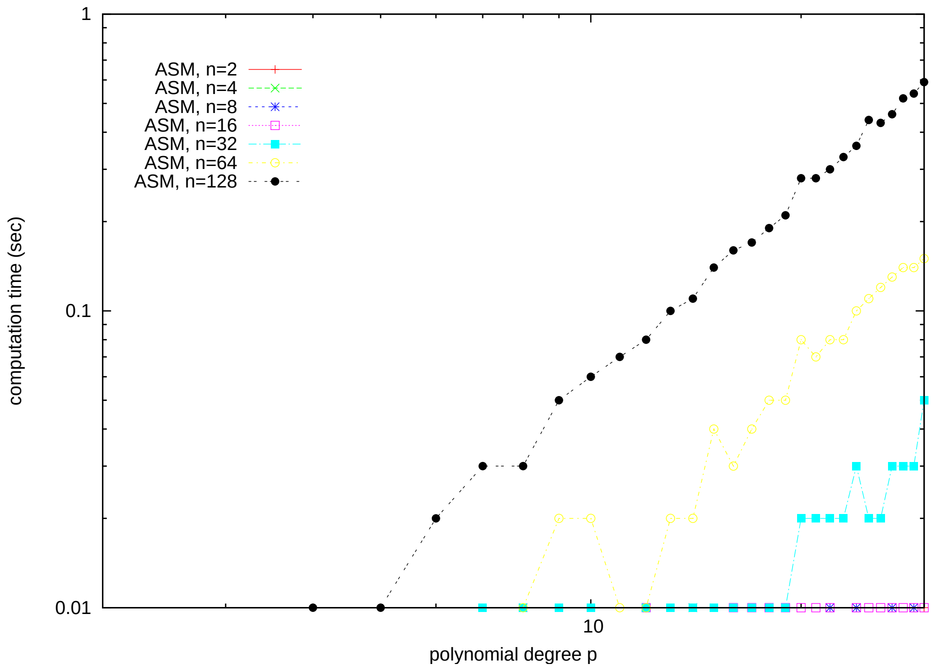


Figure 2.31: Hyper singular integral equation, p-version, computing time

Example 2.23. *Additive Schwarz algorithm for the Dirichlet problem on the slit.*

bem2/ex16in

```
! p-Version,  Laplace, Dirichlet, Additive Schwarz
open(1) 'test.aspv'
geometry('Slit',gm='Ng',bmode=(/1,2/)) ; #ti
problem('Laplace',nickname='VIGLSCR')
R=3 ! right hand side
EPS=1.0d-10
K=2; P=30

do J=1,7
 inquire(file='aspv'//J//'.dat',T)
 if (T.eq.0); then
  open(2) 'aspv'//J//'.dat'
  write(2) '# Dirichlet, p-version with coarse grid, n=',K
  write(2) '# p lmin lmax cond iter enorm time dof'
  do I=1,P
   mesh('uniform',n=K,p=I,spline='N')
   matrix('numeric',gqna=34,gqnb=34,ijn=6,sigma=0.17,mu=1.0); TM=SEC
   lft 40 R 0 R; TL=SEC
   defprec(mode='ASM',spline='N',name='PN',mat='V',hpmodus=1,ddmodus=1,domains=0,omega=1.0)

   solve(eps=EPS,mdi='x=0',mdc='N.PN.N',mit='CG'); S=SEC
   #hno. ; #rno.
   solve(eps=EPS,mdi='x=0',mdc='N.PN.N',mit='EW')
   write(2) I , LMIN,LMAX,COND,ITER,ENO,S,DOF
  continue
  write(2) ' '
 fi
 K=K*2
continue
end
```

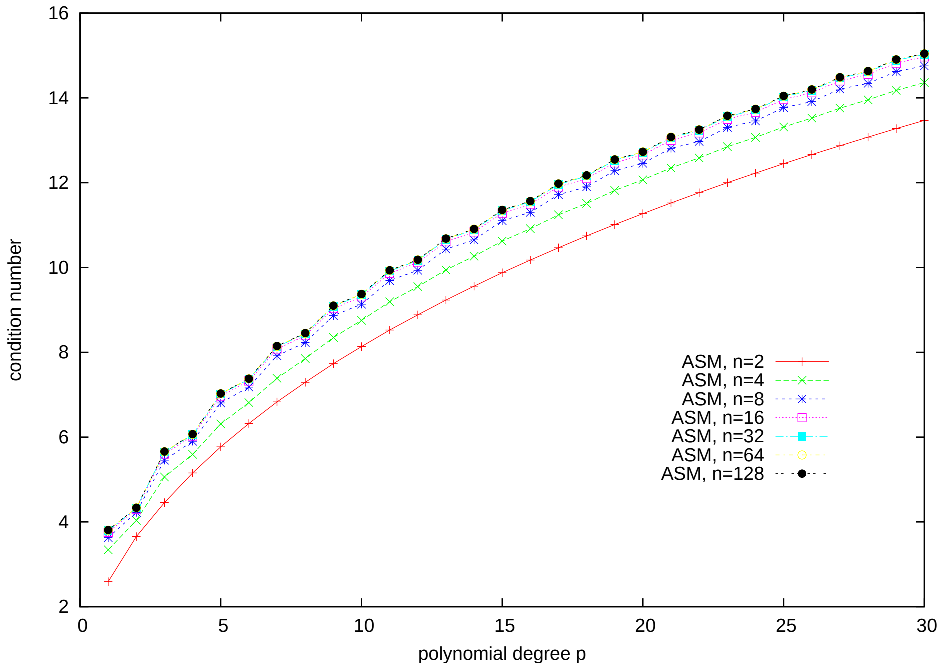


Figure 2.32: Weakly singular integral equation, p-version, condition number

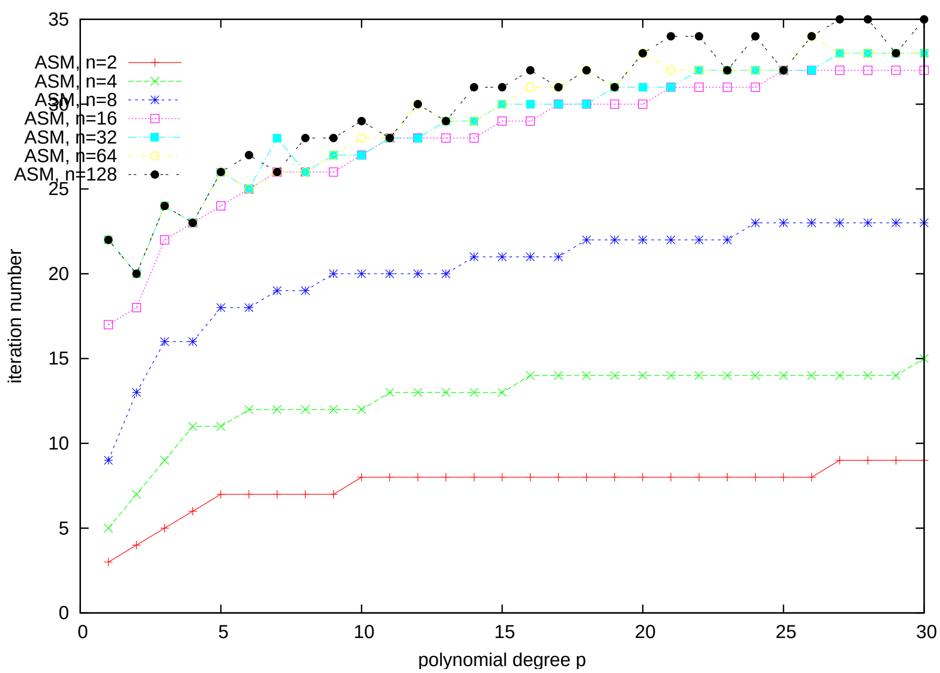


Figure 2.33: Weakly singular integral equation, p-version, iteration numbers

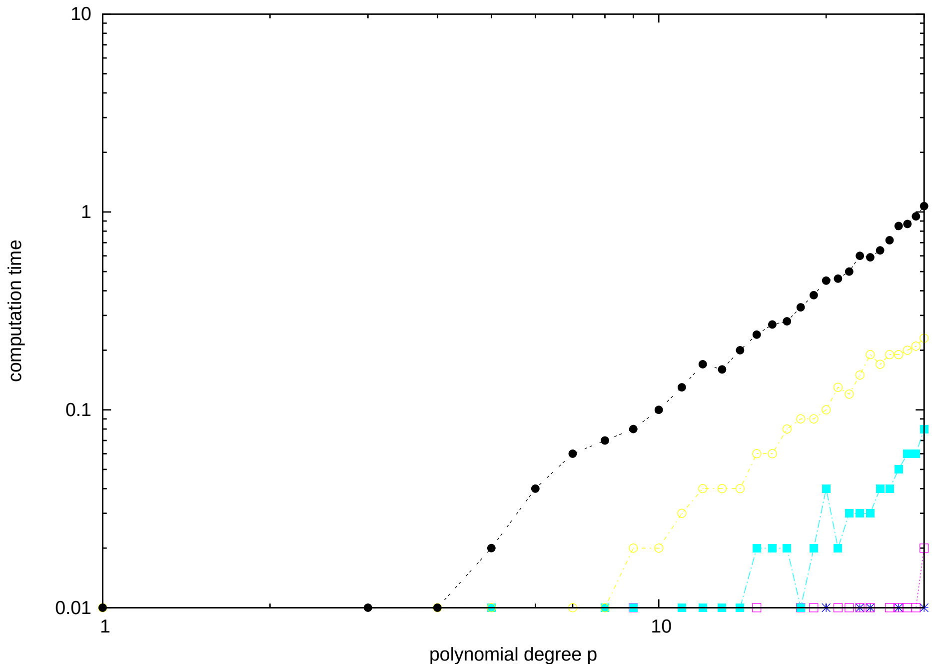


Figure 2.34: Weakly singular integral equation, p-version, computing time

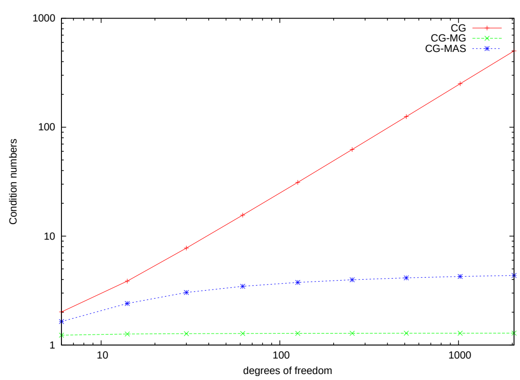
Example 2.24. *This is an example for multilevel preconditioners for the hypersingular Lamé operator on the interval* $[-1, 1]$ *using an uniform mesh, with* $E = 2000$ *and* $\nu = 0.3$.

$$W_{\text{Lame}} u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We give the condition number of the unpreconditioned system, and the condition numbers of the system preconditioned with the Multigrid operator and the Multilevel Additive Schwarz (BPX) operator. Additionally we give also the contraction rate of the Multigrid operator ϱ_{MG}, *when it is used as a solver and not as a preconditioner. In all cases we also give the iteration number to achieve a precision of* $\frac{\|\Delta x\|}{\|x\|} \leq 10^{-10}$.

	Condition number				Iteration number			
N	A	MG	MAS	ϱ_{MG}	cg	cgMG	cgMAS	MG
2	1.0000000	1.0000000	1.0000000	.1110E-15	2	0	2	2
6	2.0123071	1.2316028	1.6440217	0.1882123	3	3	3	15
14	3.8645690	1.2611600	2.4067249	0.2070792	5	5	5	15
30	7.7629377	1.2713449	3.0353968	0.2134314	9	8	9	15
62	15.573962	1.2767038	3.4613649	0.2168091	12	8	12	15
126	31.145753	1.2800999	3.7561476	0.2190700	19	8	15	15
254	62.432452	1.2821643	3.9714492	0.2207272	28	8	17	15
510	125.10275	1.2833512	4.1335332	0.2219960	40	8	18	15
1022	250.48025	1.2842944	4.2578242	0.2229916	57	8	18	15
2046	501.24442	1.2850525	4.3545915	0.2237851	80	8	18	15
4094	1002.7795	1.2861275	4.4309088	0.2244248	111	9	18	15

Table 2.2: Hypersingular Lamé integral equation, condition number and iteration numbers of multilevel preconditioners



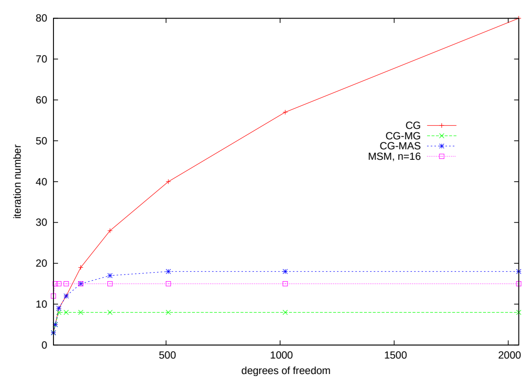
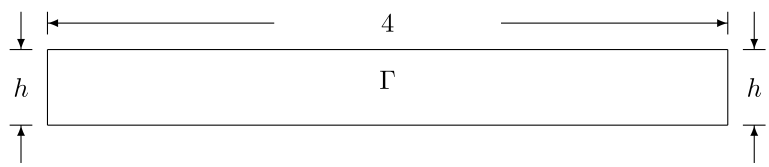


Figure 2.35: Hypersingular Lamé integral equation, condition number and iteration numbers of multilevel preconditioners

Example 2.25. *This example demonstrates the stability of the boundary element method with respect to the ratio of mesh lengths. We investigate the single layer potential equation (Dirichlet problem) for a domain, where one side length tends to zero.*

$$V\psi = (I + K)(x + y) \text{ on } \Gamma$$



mesh ratio h/4	$\|\psi - \psi_N\|_{L^2(\Gamma)}$
1	$8.34 \cdot 10^{-15}$
0.1	$6.40 \cdot 10^{-15}$
0.01	$1.96 \cdot 10^{-11}$
0.001	$2.61 \cdot 10^{-10}$
0.0001	$9.75 \cdot 10^{-08}$
0.00001	$1.19 \cdot 10^{-05}$

Table 2.3: Mesh ratio and error

Example 2.26. *This example demonstrates the Multilevel Additive Schwarz preconditioner applied to Hypersingular Integral Equation of the Laplacian on algebraically graded meshes on the Slit [18].*

$\beta\backslash N$	7	15	31	63	127	255	511
1.0	2.4067	3.0354	3.4614	3.7561	3.9714	4.1335	4.2578
2.0	2.8372	3.5537	4.0480	4.3829	4.6135	4.7748	4.8894
3.0	2.9789	3.7577	4.3090	4.6812	4.9295	5.0957	5.2923
4.0	3.0238	3.8515	4.4545	4.8629	5.1917	4.6646	718.45

Table 2.4: Hypersingular integral equation, algebraically graded h version with Multilevel additive Schwarz preconditioner, condition number $\kappa(P_{\mathrm{ASM}})$

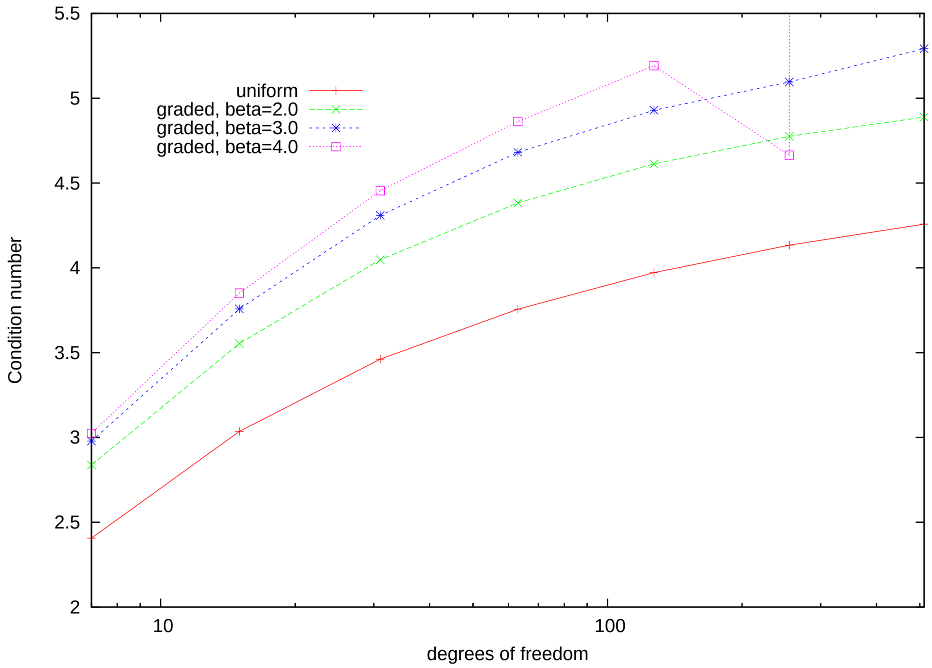


Figure 2.36: Hypersingular integral equation, algebraically graded h version with Multilevel additive Schwarz preconditioner, condition number $\kappa(P_{\mathrm{ASM}})$

$\beta\backslash N$	7	15	31	63	127	255	511
1.0	3.876520	7.762938	15.58174	31.24591	62.58732	125.2775	250.6621
2.0	2.746969	5.431018	10.83385	21.65727	43.30900	86.61535	173.2294
3.0	2.336708	4.464931	8.817385	17.57539	35.12010	70.23174	142.4378
4.0	2.229383	4.191197	8.240055	16.40540	33.11575	198.4693	.230E+05

Table 2.5: Hypersingular integral equation, algebraically graded h version without preconditioner, condition number $\kappa(A_N)$

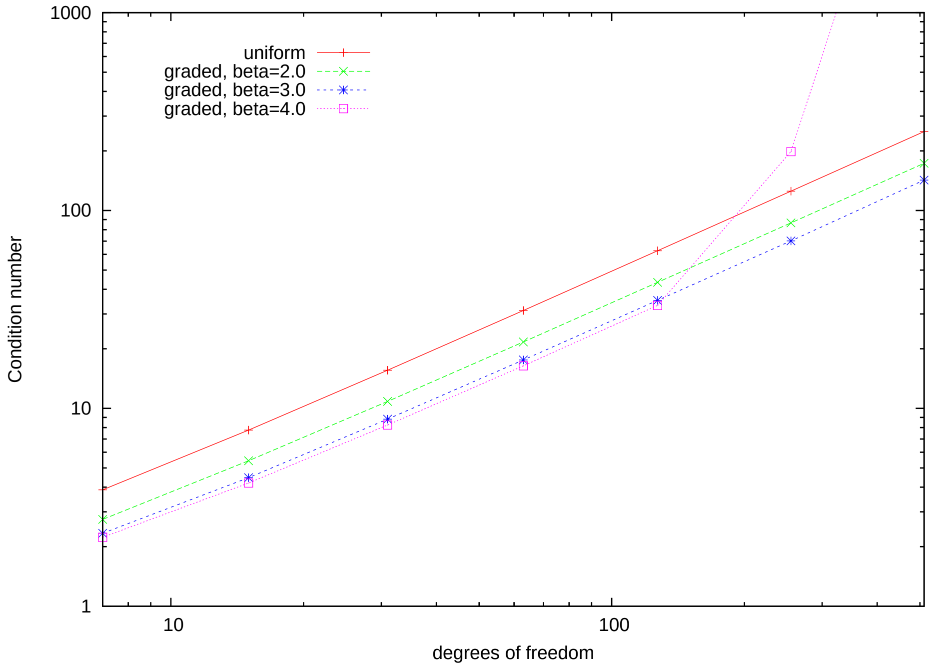


Figure 2.37: Hypersingular integral equation, algebraically graded h version without preconditioner, condition number $\kappa(A_N)$

3 Contact Problems (2D)

3.1 Signorini Problems (Lamé)

Example 3.1. *This example solves the Hertz contact problem, i.e. an elastic body under constant load pressed against a rigid obstacle, using a symmetric formulation [15] and successive over-relaxation for solving.*

Find $\mathbf{u} \in K_\Gamma := \{\mathbf{v} \in H^{1/2}(\Gamma) \,:\, \mathbf{v}_n \leq g|_{\Gamma_S}\}$ *such that*

$$\langle S\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle \geq \mathbf{l}(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K_\Gamma$$

which is equivalent to find $\mathbf{u}$ *such that*

$$J(\mathbf{u}) = \min_{\mathbf{v}\in K_\Gamma} J(\mathbf{v}) \text{ with } J(\mathbf{v}) = \frac{1}{2}\langle S\mathbf{v}, \mathbf{v}\rangle - \mathbf{l}(\mathbf{v})$$

cont2/ex1hin

```
! solve Hertz contact problem with sor, h-version
open(1) 'test.sor' ; open(2) 'ex1hin.dat'
#gm 1 1 8. 8. ; #ti
#pro 1
#ep 2000. 0.3
JUX=-3214.187328
K=30; L=20
do I=1,6
#rc K L
mat 0 ; TA=SEC
lft 16 0 1600.
schur 2 ; TSR=SEC
sor 1.e-8 1.7 ; TS=SEC ; IS=ITER; #rno.
#hno. JUX
test
mlev 1.e-8 1 0 1 I 1.7 0 1 300 3 2 0 1 0.0 1 ; TM=SEC ; IM=ITER; #rno.
#hno. JUX
mlev 1.e-8 1 0 1 I 1.7 1 1 300 3 2 0 1 0.0 1 ; TMK=SEC ; IMK=ITER; #rno.
#hno. JUX
rlgs 1.e-8 1 0 0 1 0 ; TP=SEC ; IP=ITER; #rno.
#hno. JUX
#no. 'L2' 'D'
#no. 'L2' 'N'
#c. 'D' ; #c. 'N'
#l. 'D'
#err. 16 0 'L2' 0 'N'
write(2) DOF,JU:14,ERRJU,IS,IM,IMK,IP,TA,TSR,TS,TM,TMK,TP
K=K*2; L=L*2
continue
end
```

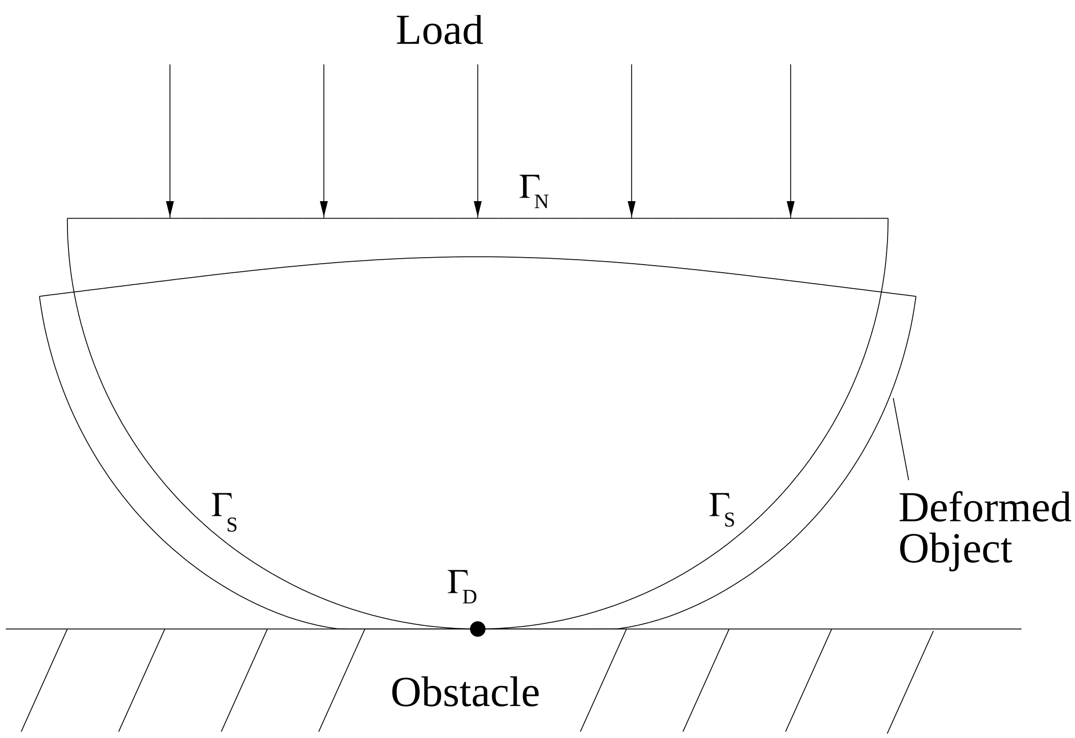


Figure 3.38: Hertz contact problem

Example 3.2. *Here we investigate the deformation of an elastic bar with an elasticity module* $E = 2000$ *and a Poisson number* $\nu = 0.3$*. The rigid obstacle partly supports the bar, see Figure 3.39, i.e. we have the obstacle function* $\mathbf{g} \equiv 0$*. We apply a constant load* $\mathcal{T}(\mathbf{u}) = (0, -160)$ *from above on* $\Gamma_{N,1}$*. The remaining part of the boundary is load free. The exact*

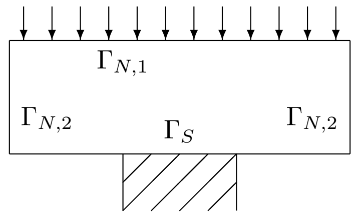


Figure 3.39: Model geometry

value of the potential $J(\mathbf{u}) := \frac{1}{2}\langle S\mathbf{u}, \mathbf{u}\rangle - l(\mathbf{u}) = -35.413$ *is obtained by extrapolation of the potential values of the uniform h-version.*

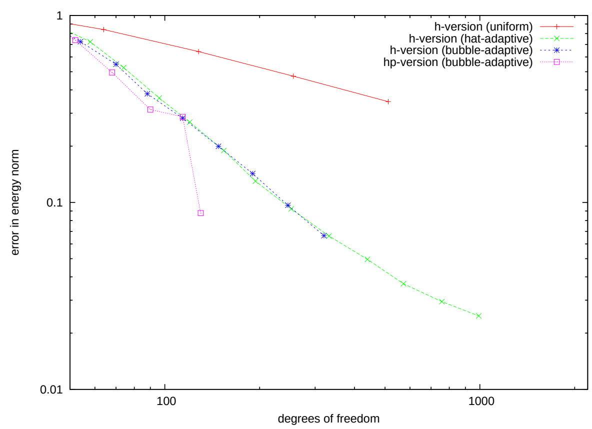


Figure 3.40: Convergence of the Lamé Contact Problem

$\lvert\sigma_k\rvert$	$J_S(\mathbf{u}_k)$	$\delta_{\mathbf{u}}$	Θ_h	$\eta_{u,h}$	$\eta_{\varphi,h}$	α
32	-34.3631	1.0280	0.2326	2.449	2.200	
64	-34.7135	0.8405	0.1534	1.772	1.522	0.291
128	-35.0095	0.6406	0.0908	1.282	1.084	0.392
256	-35.1958	0.4734	0.0493	0.920	0.768	0.436
512	-35.3002	0.3460	0.0257	0.656	0.543	0.452
1024	-35.3554	0.2541	0.0131	0.465	0.384	0.445

Table 3.6: Error $\delta_{\mathbf{u}} = \|\mathbf{u} - \mathbf{u}_k\|_S$ and hat-Haar-indicators for the uniform h-version (Lamé)

k	$\|\sigma_k\|$	$J_S(\mathbf{u}_k)$	$\delta_{\mathbf{u}}$	Θ_{hp}	$\eta_{u,hp}$	$\eta_{\varphi,hp}$	α
0	32	-34.3631	1.0246	2.3785	1.375	2.427	
1	44	-34.5922	0.9059	1.2402	1.018	1.704	0.386
2	56	-34.8731	0.7347	0.6692	0.757	1.235	0.869
3	72	-35.1141	0.5467	0.3990	0.572	0.890	1.176
4	92	-35.2792	0.3657	0.1803	0.382	0.613	1.640
5	116	-35.3390	0.2719	0.0940	0.271	0.436	1.278
6	148	-35.3733	0.1992	0.0521	0.196	0.313	1.277
7	184	-35.3938	0.1382	0.0275	0.137	0.220	1.677
8	238	-35.4033	0.0981	0.0108	0.096	0.153	1.333
9	316	-35.4083	0.0684	0.0052	0.067	0.108	1.268
10	410	-35.4104	0.0501	0.0026	0.047	0.076	1.196

Table 3.7: Error $\delta_{\mathbf{u}} = \|\mathbf{u} - \mathbf{u}_k\|_S$ and bubble-indicators for the adaptive h-version (Lamé)

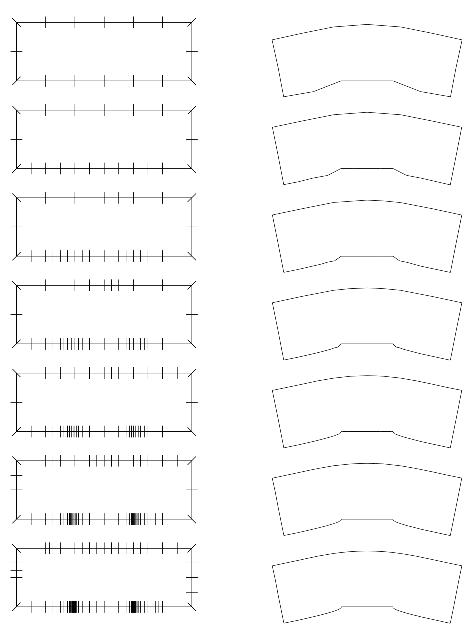


Figure 3.41: h-adaptive generated meshes for Lamé-BEM (bubble-indicator)

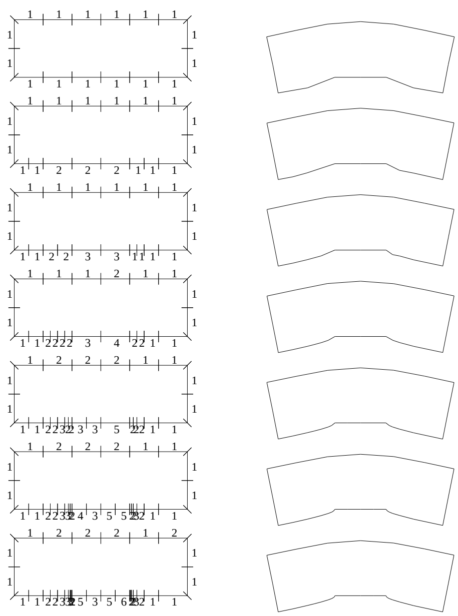


Figure 3.42: hp-adaptive generated meshes for Lamé-BEM (bubble-indicator)

Example 3.3. *This example solves the Hertz contact problem, i.e. an elastic body under constant load pressed against a rigid obstacle, using Finite elements.*

Find $\mathbf{u} \in K := \{\mathbf{v} \in H^1(\Omega) : \mathbf{v}_n \leq g|_{\Gamma_S}\}$ *such that*

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq \mathbf{l}(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K$$

which is equivalent to find $\mathbf{u}$ *such that*

$$J(\mathbf{u}) = \min_{\mathbf{v} \in K} J(\mathbf{v}) \text{ with } J(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - \mathbf{l}(\mathbf{v}), \quad a(\mathbf{u}, \mathbf{v}) = \int_\Omega \varepsilon(\mathbf{u}) : C : \varepsilon(\mathbf{v})\, dx$$

cont2/ex42h3in

```
! Lame, Signorini-problem, FEM, 2d Hertz contact problem
open(1) 'test.h' ; open(2) 'ex42h3in.dat'
geometry('Half-Circle',bmode=(/2,2/),dim=(/8.0,8.0/)); #ti
problem('Lame',nickname='FEMSIG')
R=25  ! rhs
#ep 2000.0 0.3
#pxbd 3 1 2 'ubd'
  0  2  8.  8. -8.  8.  -1                         ! Neumann boundary
  1  3 -8.  8. -5.656854250  2.343145750  0. 0. -3 ! Signorini
  1  3  0.  0.  5.656854250  2.343145750  8. 8. -3 ! SIgnorini

#pxg 1 1 2 'obsg'
0 2 -8. 0. 8. 0. 0. 1. 0

#cmode 1
J=2;H=2.
do I=1,10
 mesh('uniform',n=2,p=1,spline='obs',genspl='no')
 mesh('uniform',n=J,p=1,elements='triangles')
 compobstacle('uo','obs','y')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0)
 lft 16 R 0 R
 compdefect(obs='uo',to='u',mode='Sig')
 solve(eps=1.0d-8,mit='POLYAK'); T=SEC
 checksig 1.0e-4; #rno.
 computestress('u','sigma')
 open(1) 'ex42h3in'//I
 #taf. 'u'; #pnod. 'u'; #cx. 'u'
 #taf. 'sigma'; #pnod. 'sigma'; #cx. 'sigma'
 #obs.
 close(1)
 #ju.
 write(2) DOF,ITER,JU:12,T
 J=J*2; H=H/2
continue
end
```

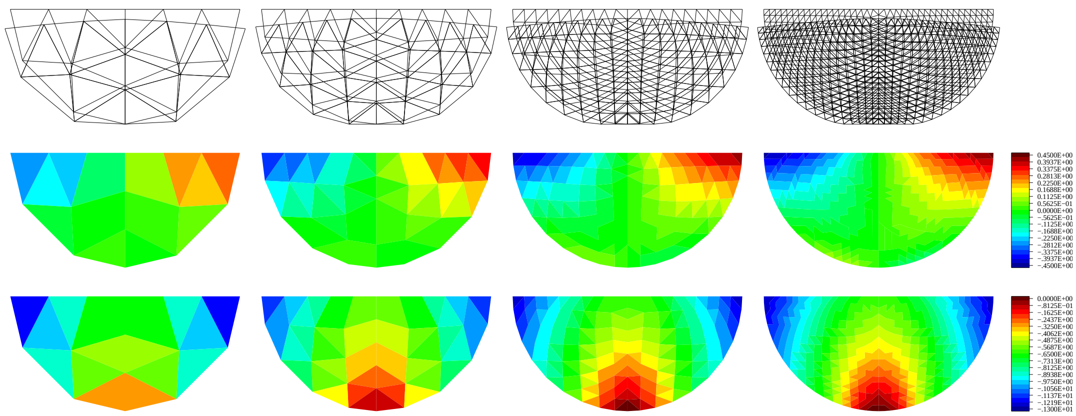


Figure 3.43: Deformed body (top), deformation in x (middle), deformation in y (bottom)

N	$J(u_N)$	$\|u - u_N\|_{H^1(\Omega)}$	α_N	Iter	τ
8	-1277.952	5.9331610		4	0.0000000
32	-1248.477	4.8731770	-0.141969	17	0.0000000
72	-1244.452	5.6388974	0.1799692	47	0.0000000
242	-1247.228	5.1230063	-0.079147	115	0.0000000
882	-1257.201	2.5099283	-0.551699	271	0.0100000
3528	-1259.489	1.3130666	-0.467353	614	0.1500000
13448	-1260.099	0.7089979	-0.460555	1228	1.1500000
53792	-1260.287	0.3559635	-0.497026	2624	13.550000
213858	-1260.335	0.1798722	-0.494557	6015	133.76000
852818	-1260.347	0.0912140	-0.490906	14234	1381.3900

Table 3.8: Convergence rate α_N of the h-version (FEM)

Example 3.4. *This example solves the Hertz contact problem, i.e. an elastic body under constant load pressed against a rigid obstacle, using Boundary elements.*

Find $\mathbf{u} \in K_\Gamma := \{\mathbf{v} \in H^{1/2}(\Gamma) : \mathbf{v}_n \le g|_{\Gamma_S}\}$ *such that*

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \ge \mathbf{l}(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K_\Gamma$$

which is equivalent to find $\mathbf{u}$ *such that*

$$J(\mathbf{u}) = \min_{\mathbf{v} \in K_\Gamma} J(\mathbf{v}) \text{ with } J(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - \mathbf{l}(\mathbf{v}), \quad a(\mathbf{u}, \mathbf{v}) = \langle S\mathbf{u}, \mathbf{v} \rangle$$

cont2/ex43h3in

```
! Lame, Signorini-problem, BEM, 2d Hertz contact problem
open(1) 'test.h' ; open(2) 'ex43hin.dat'
geometry('Half-Circle',bmode=(/2,2/),dim=(/8.0,8.0/),gm='ug')
geometry('Half-Circle',bmode=(/1,2/),dim=(/8.0,8.0/),gm='Dg'); #ti
#pxg. 'Dg'
problem('Lame',nickname='BEMSIGN')
R=25  ! rhs
#ep 2000.0 0.3
J=2;H=2.
#pxbd 3 1 2 'Dbd'
  0  2  8.  8. -8.  8.  -1                         ! Neumann boundary
  1  3 -8.  8. -5.656854250  2.343145750  0. 0. -3 ! Signorini
  1  3  0.  0.  5.656854250  2.343145750  8. 8. -3 ! SIgnorini

#pxg 1 1 2 'obsg'
0 2 -8. 0. 8. 0. 0. 1. 0

#cmode 1
do I=1,10
 mesh('uniform',n=2,p=1,spline='obs',genspl='no')
 mesh('uniform',n=J,p=1,spline='u')
 markrdn('D','Dbd')
 compobstacle('Do','obs','y')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0)
 compdefect(obs='Do',to='D',mode='y')
 lft 16 R 0 R
 solve(eps=1.0d-8,mdi='x=0',mit='POLYAK'); T=SEC
 checksig 1.0e-4
 #rno.
 eval('N=Matrix(SK)*D')
 eval('N=Matrix(SIV)*N')
 bem2fem(u0='D',t0='N',fem='u')
 open(1) 'ex43h3in'//I
 #taf. 'D'; #px. 'D'; #cx. 'D'; #hrc. 'D'; #lx. 'D'
 #taf. 'u'; #pnod. 'u'; #cx. 'u'
 #obs.
 close(1)
 #ju.
 write(2) DOF,ITER,JU,T
 J=J*2; H=H/2
continue
end
```

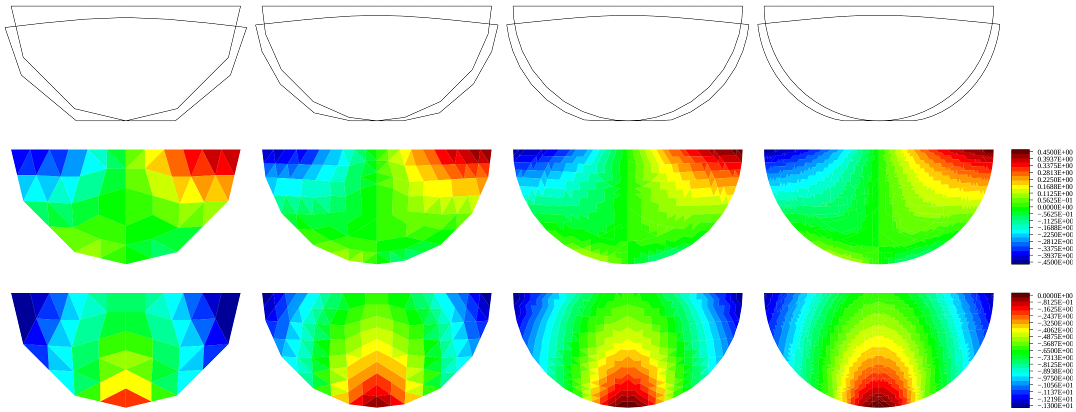


Figure 3.44: Deformed body (top), deformation in x (middle), deformation in y (bottom)

N	$J(u_N)$	$\|u-u_N\|_{H^{1/2}(\Gamma)}$	α_N	Iter	τ
8	-1814.467	33.290125		4	0.0000000
24	-1707.326	29.899003	-0.097792	14	0.0000000
40	-1342.997	12.856609	-1.652163	22	0.0000000
80	-1285.154	7.0431811	-0.868211	30	0.0000000
160	-1267.854	3.8738095	-0.862474	48	0.0000000
328	-1262.078	1.8586016	-1.023089	80	0.0100000
648	-1260.799	0.9467840	-0.990647	131	0.0700000
1304	-1260.461	0.4694678	-1.003104	292	0.7300000
2608	-1260.378	0.2332381	-1.009223	578	6.0100000
5216	-1260.357	0.1113553	-1.066633	1274	53.200000

Table 3.9: Convergence rate α_N of the h-version (BEM)

Example 3.5. *This example solves the Hertz contact problem, i.e. an elastic body under constant load pressed against a rigid obstacle, using Finite elements and a Penalty term.*

Find $\mathbf{u}^\epsilon \in H^1_D(\Omega)$ *such that*

$$a(\mathbf{u}^\epsilon, \mathbf{v}) - \langle p^\epsilon, \mathbf{v}_n \rangle_{\Gamma_S} = \mathbf{l}(\mathbf{v}) \quad \forall \mathbf{v} \in H^1_{D,0}(\Omega) \text{ with } p^\epsilon := -\frac{1}{\epsilon}(\mathbf{u}^\epsilon_n - g)^+$$

The Newton scheme for solving the non-linear penalty problem reads:

Let $\mathbf{u}^{(0)} \in H^1_D(\Omega)$*. For* $n = 1, 2, \ldots$ *until a stopping criterion is satisfied:*
Find $\boldsymbol{\delta} \in H^1_{D,0}(\Omega)$ *such that*

$$a(\boldsymbol{\delta}, \mathbf{v}) + \frac{1}{\epsilon} \int_{\Gamma_S} H(\mathbf{u}^{(n-1)}_n - g)\boldsymbol{\delta}_n \mathbf{v}_n \, ds = \mathbf{l}(\mathbf{v}) - a(\mathbf{u}^{(n-1)}, \mathbf{v}) - \frac{1}{\epsilon} \int_{\Gamma_S} (\mathbf{u}^{(n-1)}_n - g)^+ \mathbf{v}_n \, ds \quad \forall \mathbf{v} \in H^1_{D,0}(\Omega)$$

and $\mathbf{u}^{(n)} = \mathbf{u}^{(n-1)} + \boldsymbol{\delta}$.

cont2/ex44h3in

```
! Lame, Signorini-problem, FEM, 2d Hertz contact problem, Penalty
open(1) 'test.h' ; open(2) 'ex44h3in.dat'
defgeometry('Half-Circle','Half-Circle',2,(/2,2/),-1,g0=(/-1.0,-1.0/))
  1  4  -1.  1. -0.707106781187  0.292893218813  0.  0.  0. 1. 0
  1  4   0.  0.  0.707106781187  0.292893218813  1.  1. 0. 1. 0
enddefgeometry

defgeometry('Half-Circleu','Half-Circleu',2,(/2,2/),-1,g0=(/-10.0,-10.0/))
  1  4  -1. -1. -0.707106781187 -0.292893218813  0.  0.  0. -1. 0
  1  4   0.  0.  0.707106781187 -0.292893218813  1. -1.  0. -1. 0
enddefgeometry

geometry('Half-Circle',bmode=(/2,2/),dim=(/8.0,8.0/)); #ti
geometry('Half-Circleu',bmode=(/2,2/),dim=(/8.0,8.0/),gm='obsg')
problem('Lame',nickname='FEMSIGPEN')
eps=1.0d-3
R=25  ! rhs
EPS=1.0d-8
#ep 2000.0 0.3
#pxbd 3 1 2 'ubd'
  0  2  8.  8. -8.  8.  -2                         ! Dirichlet boundary
  1  3 -8.  8. -5.656854250  2.343145750  0. 0. -3 ! Signorini
  1  3  0.  0.  5.656854250  2.343145750  8. 8. -3 ! SIgnorini

#pxg 1 2 2 'obsg'
 0 4 -8. 0. 8. 0. 8. -1. -8. -1. 0

#cmode 1
J=2;H=2.
do I=1,20
 mesh('uniform',n=J,p=1,spline='obs',genspl='no')
 mesh('uniform',n=J,p=1,elements='triangles')
! compobstacle('uo','obs','y')
! compdefect(obs='uo',to='u',mode='Sig')
 clear('ux'); clear('u'); clear('obs')
 approx 0 R 'u_bd' 'u0'
 extend('u','u_bd','ux')
```

```
 NCNT=0; ITMAX=0
 do K=0,60
  matrix('analytic',ijn=6,sigma=0.17,mu=1.0)
  lft 16 R 0 R
!  compdefect(obs='uo',to='u',mode='Sig')
  solve(eps=EPS,mit='CG',cmode=-1); T=SEC; #rno.
  eval('ux=ux+u','no')
  norm('Nu','H1','u')
  write(0) 'Newton',Nu,NCNT
  NCNT=NCNT+1
  if (Nu<EPS*100); then
   exit
  fi
 continue
 computestress('ux','sigma')
 open(1) 'ex44h3in'//I
 #taf. 'ux'; #pnod. 'ux'; #cx. 'ux'
 #taf. 'sigma'; #pnod. 'sigma'; #cx. 'sigma'
 #ju. 'ux'
 write(2) DOF,ITER,JU:12,NCNT,T
 J=J*2; H=H/2; eps=eps/2
continue
end
```

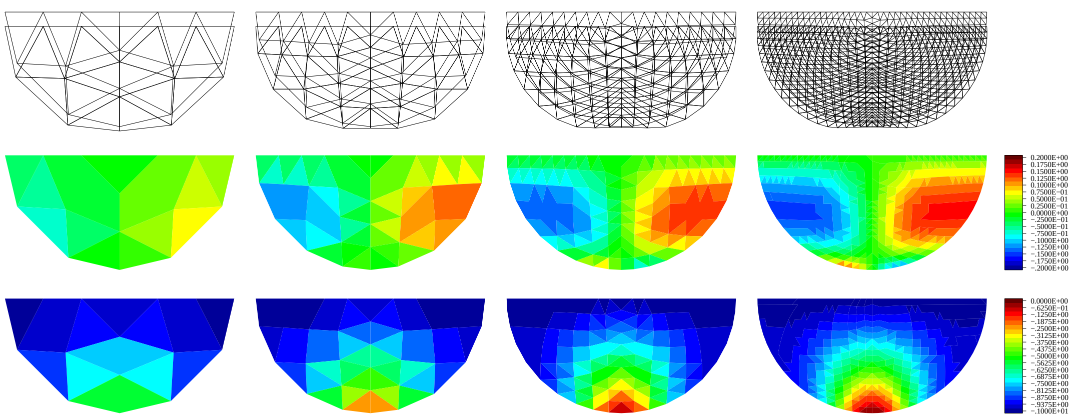


Figure 3.45: Deformed body (top), deformation in x (middle), deformation in y (bottom)

Example 3.6. *This example solves the Hertz contact problem, i.e. an elastic body under constant load pressed against a rigid obstacle, using Boundary elements and a Penalty term [9].*

Find $\mathbf{u}^\epsilon \in H_D^{1/2}(\Gamma)$ *such that*

$$\langle S\mathbf{u}^\epsilon, \mathbf{v}\rangle - \langle p^\epsilon, \mathbf{v}_n\rangle_{\Gamma_S} = \mathbf{l}(\mathbf{v}) \quad \forall \mathbf{v} \in H_0^{1/2}(\Gamma) \text{ with } p^\epsilon := -\frac{1}{\epsilon}(\mathbf{u}_n^\epsilon - g)^+$$

The Newton scheme for solving the non-linear penalty problem reads:

Let $\mathbf{u}^{(0)} \in H_D^{1/2}(\Gamma)$*. For* $n = 1, 2, \ldots$ *until a stopping criterion is satisfied:*
Find $\boldsymbol{\delta} \in H_{D,0}^{1/2}(\Omega)$ *such that*

$$\langle S\boldsymbol{\delta}, \mathbf{v}\rangle + \frac{1}{\epsilon}\int_{\Gamma_S} H(\mathbf{u}_n^{(n-1)} - g)\boldsymbol{\delta}_n \mathbf{v}_n \, ds = \mathbf{l}(\mathbf{v}) - \langle S\mathbf{u}^{(n-1)}, \mathbf{v}\rangle - \frac{1}{\epsilon}\int_{\Gamma_S} (\mathbf{u}_n^{(n-1)} - g)^+ \mathbf{v}_n \, ds \quad \forall \mathbf{v} \in H_{D,0}^{1/2}(\Gamma)$$

and $\mathbf{u}^{(n)} = \mathbf{u}^{(n-1)} + \boldsymbol{\delta}$*.*

3.2 Signorini Problems (Laplace)

Example 3.7. *[21] This example solves a Signorini problem for the Laplacian using BEM (see [21]).*

Example 3.8. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $\Gamma = \partial\Omega = \overline{\Gamma_d \cup \Gamma_n \cup \Gamma_s}$. Given: $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, $t_0 \in \tilde{H}^{-1/2}(\Gamma)$.*

Find $u \in H^1(\Omega)$ with

$$
\begin{aligned}
&-\Delta u = f \ \textit{in } \Omega \\
&\quad u = u_0 \ \textit{on } \Gamma_d, && \frac{\partial u}{\partial n} = t_0 \ \textit{on } \Gamma_n \\
&\quad u \le u_0, \quad \frac{\partial u}{\partial n} \le t_0 \ \textit{on } \Gamma_s, && 0 = (\frac{\partial u}{\partial n} - t_0)(u - u_0) \ \textit{on } \Gamma_s
\end{aligned}
$$

Define $\Phi : H^1(\Omega) \to \mathbb{R}$ and the subset of admissible functions C by

$$
\begin{aligned}
\Phi(u) &:= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega f \cdot u\,dx - \int_\Gamma t_0 \cdot u|_\Gamma\,ds \\
C &:= \{(u) \in H^1(\Omega) \,:\, u|_{\Gamma_d} = u_0, u \le u_0 \ \textit{on } \Gamma_s\}
\end{aligned}
$$

The minimization problem now reads:

Find $u \in C$ with

$$\Phi(u) = \min_{v \in C} \Phi(v).$$

Define $\tilde{\Phi} : H(\mathrm{div};\Omega) \to \mathbb{R}$ and the subset of admissible functions $\tilde{C}$ by

$$
\begin{aligned}
\tilde{\Phi}(q) &:= \frac{1}{2}\int_\Omega |q|^2\,dx - \int_\Gamma u_0(q \cdot \vec{n})\,ds \\
\tilde{C} &:= \{q \in H(\mathrm{div};\Omega) \,:\, q \cdot \vec{n} = t_0 \ \textit{on } \Gamma_n, q \cdot \vec{n} \le t_0 \ \textit{on } \Gamma_s, -\operatorname{div} q = f\}
\end{aligned}
$$

The minimization problem now reads:

Find $q \in \tilde{C}$ with

$$\tilde{\Phi}(q) = \min_{p \in \tilde{C}} \tilde{\Phi}(p).$$

Define $\mathcal{H} : H(\mathrm{div};\Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \to \mathbb{R}$ and the subset of admissible functions $\tilde{H}^{1/2}_+(\Gamma_s)$ by

$$
\begin{aligned}
\mathcal{H}(p, v, \mu) &:= \frac{1}{2}\int_\Omega |p|^2\,dx + \int_\Omega v \operatorname{div} p\,dx + \int_\Omega f \cdot v\,dx + \langle \mu, p \cdot \vec{n} \rangle_{\Gamma_s} - \langle u_0, p \cdot \vec{n} \rangle \\
\tilde{H}^{1/2}_+(\Gamma_s) &:= \{\mu \in \tilde{H}^{1/2}(\Gamma_s) \,:\, \mu \ge 0\}
\end{aligned}
$$

Find $(q, u, \lambda) \in H(\mathrm{div};\Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}_+(\Gamma_s)$ with

$$\mathcal{H}(q, v, \mu) \le \mathcal{H}(q, u, \lambda) \le \mathcal{H}(p, u, \lambda)$$

for all $(p, v, \mu) \in H(\mathrm{div};\Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}_+(\Gamma_s)$.

3.3 Obstacle Problems

Example 3.9. *We consider the obstacle problem*

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &\geq \psi \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma \end{aligned}$$

The variational formulation reads: Find $u \in K := \{v \in H_0^1(\Omega) : v \geq \psi\}$ *such that*

$$\int_\Omega \nabla u \nabla (v-u)\,dx \geq \int_\Omega f(v-u)\,dx$$

In this example we have $\Omega = [-1,1]^2$, $f = -1$ *and* $\psi(x) = \begin{cases} -0.2 \text{ if } \sqrt{x_1^2+x_2^2} \leq 0.5 \\ -10 \text{ else} \end{cases}$

cont2/ex48h3in

```
! Laplace- Obstacle-problem, FEM, 2d
open(1) 'test.h' ; open(2) 'ex48h3in.dat'
geometry('Square',bmode=(/2,2/)); #ti
problem('Laplace',nickname='FEMOBS')
#cmode 1; R=48  ! rhs
J=2;H=2.
do I=1,10
 mesh('uniform',n=J,p=1,elements='triangles')
 approx 0 R 'obs' 'u1'
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0); TM=SEC; WM=WSEC
 lft 16 R 0 R; TR=SEC;WR=WSEC
 compdefect(obs='obs',to='u',mode='Sig')
 solve(eps=1.0d-8,mit='POLYAK'); TS=SEC; WS=WSEC
 checksig 1.0e-4; #rno.
 open(1) 'ex48h3in'//I
 #taf. 'u'; #pnod. 'u'; #cx. 'u'; #cx. 'obs'; #obs.
 close(1)
 #ju. 'u'
 write(2) DOF,ITER,JU:12,TM,WM,TR,WR,TS,WS
 J=J*2; H=H/2
continue
end
```

The computations are done on a Core2 Duo T9500 with 2.6 GHz.

					Matrix		Rhs		Solver	
N	$J(u_N)$	$\|u-u_N\|_{H^1(\Omega)}$	α_N	Iter	τ_c	τ_w	τ_c	τ_w	τ_c	τ_w
1	-0.120000	0.5417578		1	0.0010	0.0185	0.0030	0.0024	0.0000	0.0003
9	-0.221667	0.3002801	-0.268568	5	0.0000	0.0002	0.0000	0.0004	0.0020	0.0002
49	-0.254668	0.1554526	-0.388515	14	0.0000	0.0002	0.0040	0.0028	0.0010	0.0016
225	-0.263679	0.0783829	-0.449218	39	0.0020	0.0025	0.0150	0.0148	0.0620	0.0624
961	-0.265978	0.0393153	-0.475243	176	0.0050	0.0054	0.0370	0.0268	0.0650	0.0519
3969	-0.266557	0.0196790	-0.487952	556	0.0180	0.0142	0.1500	0.1068	0.4979	0.3322
16129	-0.266702	0.0098384	-0.494443	1854	0.0420	0.0343	0.5879	0.3468	5.2992	3.2627
65025	-0.266739	0.0049202	-0.497041	6398	0.1750	0.1389	2.3456	1.4796	69.066	45.968
261121	-0.266748	0.0024605	-0.498472	22576	0.8039	0.5071	9.4246	5.2492	921.92	717.17

Table 3.10: Convergence rate α_N of the h-version (FEM, triangles)

Example 3.10. *In this example we consider the same obstacle problem as in the example before, but we apply a Monotone Multigrid algorithm as solver.*

cont2/ex48bh3in

```
! Laplace- Obstacle-problem, FEM, 2d
open(1) 'test.h' ; open(2) 'ex48bh3in.dat'
geometry('Square',bmode=(/2,2/)); #ti
problem('Laplace',nickname='FEMOBS')
R=48  ! rhs

#cmode 1
J=2;H=2.
mesh('uniform',n=J,p=1,elements='triangles')
do I=0,10
 approx 0 R 'obs' 'u1'
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0); TM=SEC; WM=WSEC
 lft 16 R 0 R ; TR=SEC; WR=WSEC
 compdefect(obs='obs',to='u',mode='Sig')
 mlevc(eps=1.0d-8,nu1=1,nu2=0,mtop=I,mnum=100,stp=0,hpmodus=0,omega=1.7,quiet=0)
 TS=SEC; WS=WSEC
 checksig 1.0e-4; #rno.
 open(1) 'ex48bh3in'//I
 #taf. 'u'; #pnod. 'u'; #cx. 'u';#cx. 'obs'
 #obs.
 close(1)
 #ju. 'u'
 write(2) DOF,ITER,JU:12,TM,WM,TR,WR,TS,WS
 refine('all')
 J=J*2; H=H/2
continue
end
```

The computations are done on a Core2 Duo T9500 with 2.6 GHz.

					Matrix		Rhs		Solver	
N	$J(u_N)$	$\|u-u_N\|_{H^1(\Omega)}$	α_N	Iter	τ_c	τ_w	τ_c	τ_w	τ_c	τ_w
1	-0.120000	0.5417578		2	0.0010	0.0004	0.0000	0.0002	0.0000	0.0002
9	-0.231852	0.2641927	-0.326840	53	0.0000	0.0002	0.0000	0.0004	0.0020	0.0008
49	-0.256635	0.1422409	-0.365372	59	0.0010	0.0003	0.0020	0.0018	0.0030	0.0059
225	-0.264093	0.0729107	-0.438428	53	0.0010	0.0007	0.0100	0.0047	0.0220	0.0204
961	-0.266076	0.0367269	-0.472304	51	0.0030	0.0021	0.0380	0.0215	0.0390	0.0287
3969	-0.266581	0.0183993	-0.487342	49	0.0100	0.0102	0.1480	0.0807	0.1280	0.0890
16129	-0.266708	0.0091960	-0.494649	47	0.0510	0.0319	0.5889	0.3214	0.4519	0.2991
65025	-0.266740	0.0045976	-0.497241	45	0.2050	0.1367	2.3406	1.2907	1.6068	1.2768
261121	-0.266748	0.0022987	-0.498625	42	0.8269	0.5477	9.3596	5.2488	5.9711	5.0387
1046529	-0.266750	0.0011498	-0.499023	40	3.3615	2.2172	37.621	21.011	23.159	21.445
4190209	-0.266751	0.0005745	-0.500193	40	13.378	9.0286	150.87	99.756	94.510	85.797

Table 3.11: Convergence rate α_N of the h-version (FEM, triangles)

Example 3.11. *Here we consider the obstacle problem with* $\Omega = [-1,1]^2$, $f = -1$ *and* $\psi(x) = \begin{cases} -0.2 \text{ if } \sqrt{x_1^2 + x_2^2} \leq 0.5 \\ -10 \text{ else} \end{cases}$ *using a penalty formulation, i.e. find* $u \in H_0^1(\Omega)$ *such that*

$$F_\epsilon(u) = 0$$

with $F_\epsilon(v) = -\Delta v - \epsilon^{-1}(v - \psi)_-^2 - f$.

Newton's method now reads: Let $u^{(0)} \in H_0^1(\Omega)$. *For* $n = 1, 2, \ldots$ *until a stopping criterion is satisfied:*
Find $\delta \in H_0^1(\Omega)$ *such that*

$$(F_\epsilon'(u^{(n-1)})\delta, w) = -(F_\epsilon(u^{(n-1)}), w) \quad \forall w \in H_0^1(\Omega)$$

and $u^{(n)} = u^{(n-1)} + \delta$. *We have* $(F_\epsilon'(u)v, w) = \int_\Omega \nabla v \nabla w \, dx + 2\epsilon^{-1} \int_\Omega (u - \psi)_- vw \, dx$.

Therefore the linearized system reads: Find $\delta \in H_0^1(\Omega)$ *such that*

$$\int_\Omega \nabla\delta\nabla w \, dx + 2\epsilon^{-1} \int_\Omega (u^{(n-1)} - \psi)_- \delta w \, dx = \int_\Omega f w \, dx - \int_\Omega \nabla u^{(n-1)} \nabla w \, dx + \epsilon^{-1} \int_\Omega (u^{(n-1)} - \psi)_-^2 w \, dx$$

4 Finite Element Methods (1D)

In this section we deal with one-dimensional finite elements on an interval.

4.1 Convergence

4.1.1 Laplace

Example 4.1. *Let $I = [0, 1]$, then we have*

$$-\partial_x^2 u(x) = f(x) \text{ in } I, \quad u(0) = u(1) = 0 \tag{7}$$

For $u(x) = x^\alpha(1-x)$ we obtain $\partial_x u(x) = \alpha x^{\alpha-1} - (\alpha+1)x^\alpha$, $\partial_x^2 u(x) = \alpha(\alpha-1)x^{\alpha-2} - (\alpha+1)\alpha x^{\alpha-1}$, i.e. $f(x) = (\alpha+1)\alpha x^{\alpha-1} - \alpha(\alpha-1)x^{\alpha-2}$.

We have $\|u\|_E^2 = \frac{\alpha}{4\alpha^2-1}$.

Let $\alpha = 0.7$.

fem1/ex1hin

```
open(1) 'test' ; open(2) 'ex1hin.dat'
geometry('Slit01') ; #ti
problem('Laplace',nickname='FEMHD')
EPS=1.0d-15
R=5; J=2
alpha=0.7
ENX=Sqrt(alpha/(4*alpha**2-1))
setalpha(alpha)
do I=1,14
 mesh('uniform',n=J,p=1) ; clear('u')
 matrix
 lft 0 R 0 R; #lx. 'u'
 solve(eps=EPS,mdi='x=1',mdc='diag',mit='CG'); T=SEC
 #rno.
 #hno. ENX
 #err. 32 R 'L2' 0 'u' ; E[0]=ERR
 #err. 32 R 'H10'0 'u' ; E[1]=ERR
 #no. 'L2'
 #no. 'H10'
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], COND, T, ITER
 J=J*2
continue
end
```

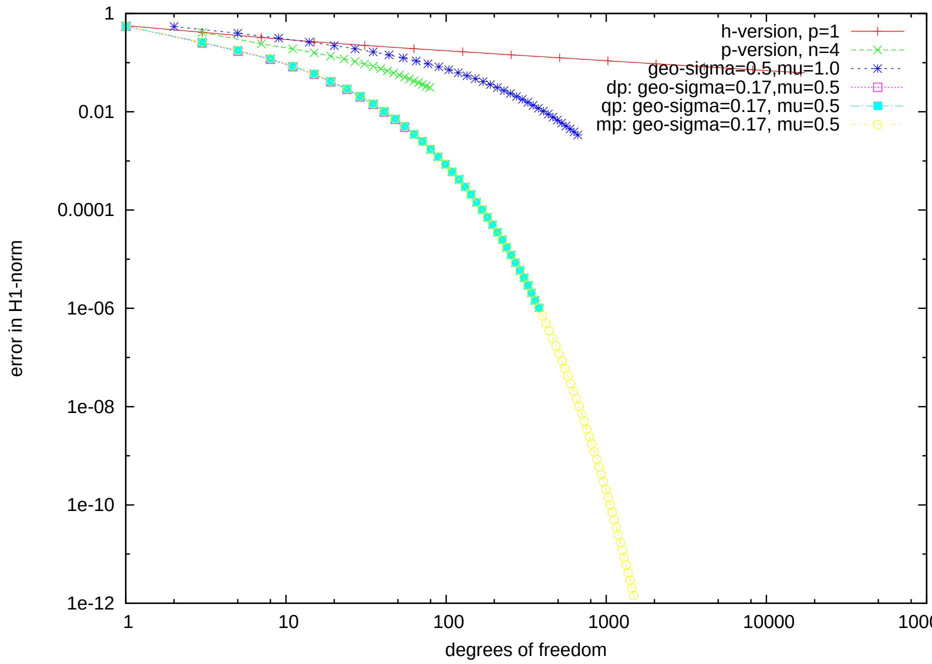


Figure 4.46: Homogenous Dirichlet problem, $\alpha = 0.7$.

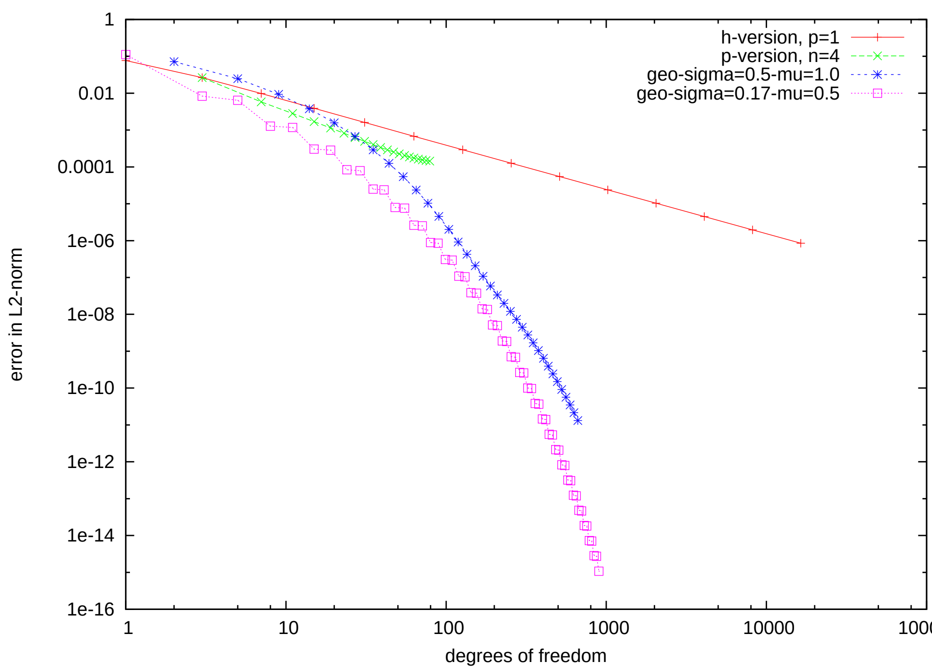


Figure 4.47: Homogenous Dirichlet problem, $\alpha = 0.7$.

4.1.2 Helmholtz

Example 4.2. *In this example we deal with the following Helmholtz problem. Let $I = [0,1]$, then we have*

$$-\partial_x^2 u(x) - k^2 u(x) = f(x) \text{ in } I, \quad u(0) = u(1) = 0$$

For $f \equiv 1$ we obtain the exact solution

$$u(x) = \frac{1}{k^2}\left(\frac{1-\cos(k)}{\sin(k)}\sin(k\,x) + \cos(k\,x) - 1\right)$$

In the following we document the convergence of h-version FEM (piecewise linear) for different values of the wave number.

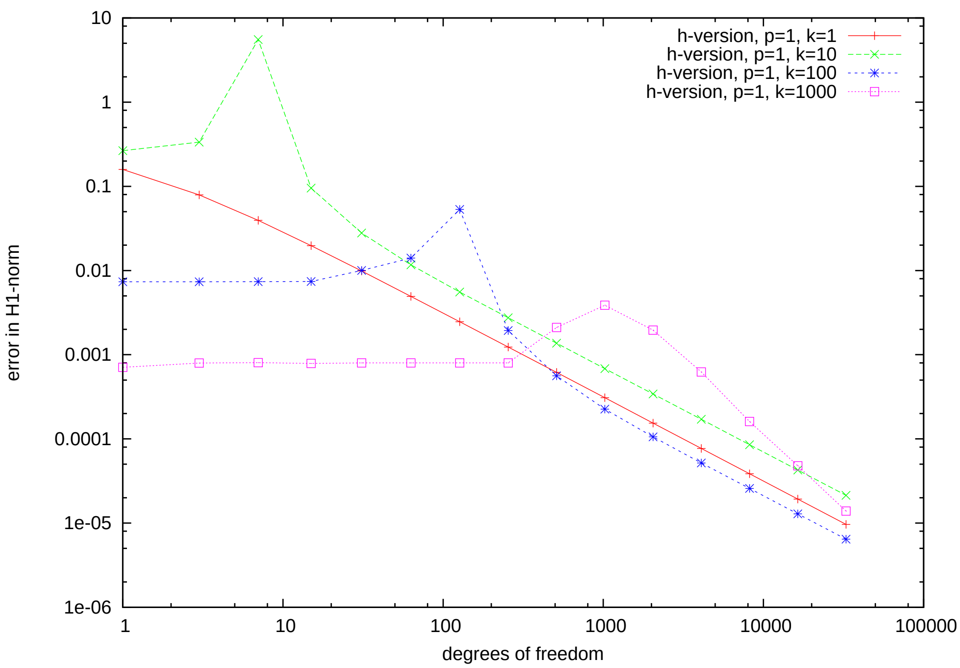


Figure 4.48: Homogenous Dirichlet Helmholtz problem.

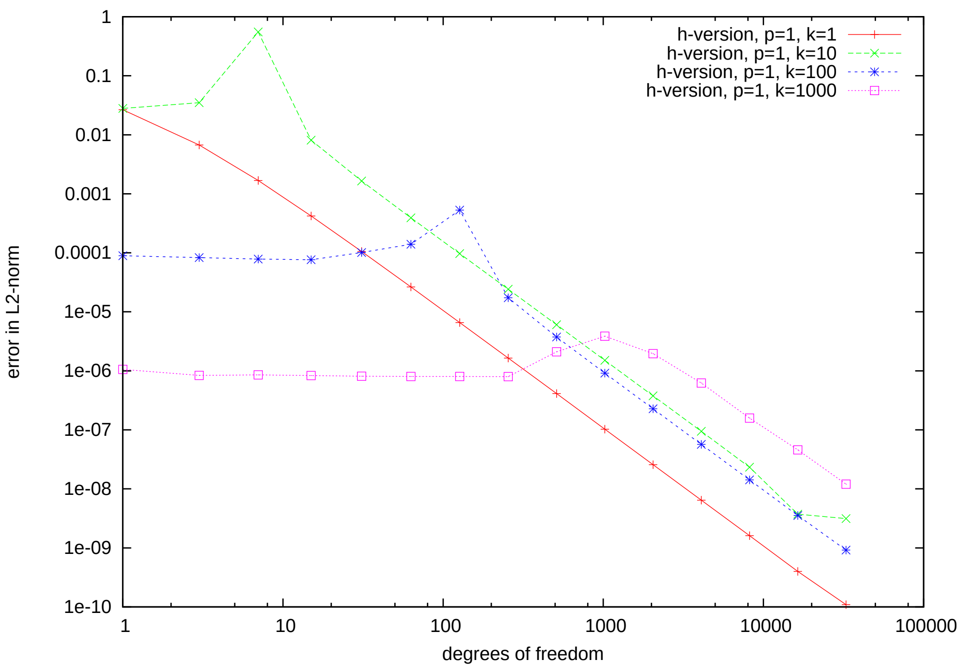


Figure 4.49: Homogenous Dirichlet Helmholtz problem.

4.1.3 Obstacle Problem

Example 4.3. *In this example we deal with the following obstacle problem.*

$$\begin{aligned} -\partial_x^2 u(x) &= f(x) \ in\ (-1,1), \quad u(-1)=u(1)=0 \\ u(x) &\geq \psi(x) \ in\ (-1,1) \end{aligned}$$

where we have $f(x) \equiv -2$, $\psi(x) = |x| - 1$. *The exact solution is given by*

$$u(x) = \begin{cases} -x-1 & for\ x \leq -\frac{1}{2} \\ x-1 & for\ x \geq \frac{1}{2} \\ x^2 - \frac{3}{4} & else \end{cases}$$

We solve this problem using the h-version, and using the p-version with Lagrange-Polynomials to the Gauss-Lobatto-nodes.

fem1/ex2pin

```
open(1) 'test'
geometry('Slit') ; #ti
problem('Laplace',nickname='FEMOBS')
#cmode 0
R=4
EPS=1.0d-10
do K=3,9,2
 open(2) 'ex2p'//K//'in.dat'
 do I=1,63
  mesh('uniform',n=K,p=I)
  matrix
  lft 64 R 0 R
  setdefect(spline='u')
  solve(eps=EPS,mdi='x=l/diag',mit='POLYAK',cmode=0); TS=SEC; WS=WSEC
  open(1) 'p'//K//'_'//I
  #taf. 'u'; #px. 'u'; #c. 'u'
  checksig 1.0d-6
  #rno. ; #hno.
  #no. 'H10'
  #err. 64 R 'L2' 0 'u' ; E[0]=ERR
  #err. 64 R 'H1' 0 'u' ; E[1]=ERR
  write(2) DOF,I,ENO,E[0], E[1]:12,OREST,IREST,ITER, TS, WS
 continue
 close(2)
continue
end
```

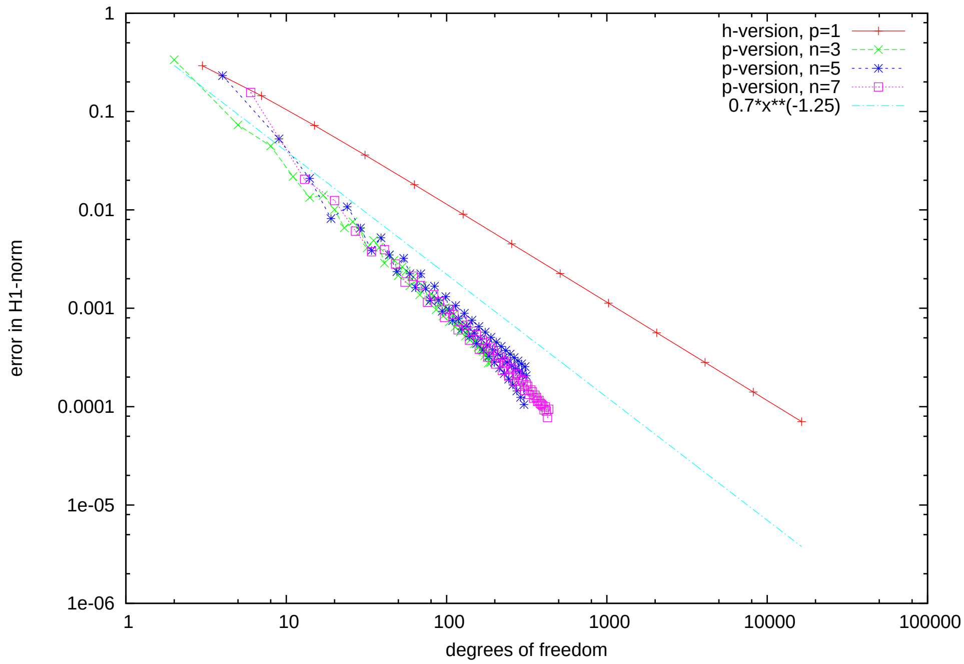


Figure 4.50: One-dimensional Obstacle problem.

N	$\|u - u_N\|_{H^1(I)}$	α_N	C_N
3	0.2922613		
7	0.1447879	0.83	0.7265859
15	0.0722251	0.91	0.8549023
31	0.0360914	0.96	0.9607385
63	0.0180431	0.98	1.0361547
127	0.0090212	0.99	1.0850643
255	0.0045106	0.99	1.1149037
511	0.0022553	1.00	1.1323925
1023	0.0011276	1.00	1.1423696
2047	0.0005638	1.00	1.1479579
4095	0.0002819	1.00	1.1510415
8191	0.0001410	1.00	1.1527239
16383	.7048E-04	1.00	1.1536442

Table 4.12: Convergence rate α_N of the h-version

N	p	$\|u - u_N\|_{H^1(I)}$	α_p
9	2	0.0527011	
24	5	0.0107588	1.734
39	8	0.0052140	1.541
54	11	0.0032206	1.513
69	14	0.0022412	1.503
84	17	0.0016753	1.499
99	20	0.0013137	1.496
114	23	0.0010663	1.493
129	26	0.0008882	1.490
144	29	0.0007552	1.486
159	32	0.0006527	1.481
174	35	0.0005719	1.475
189	38	0.0005069	1.468
204	41	0.0004537	1.458
219	44	0.0004097	1.446
234	47	0.0003728	1.431
249	50	0.0003415	1.414
264	53	0.0003149	1.393
279	56	0.0002921	1.368
294	59	0.0002724	1.338

Table 4.13: Convergence rate α_p of the h-version, $n = 5$

4.1.4 Heat-Equation

Example 4.4. *Let $\Omega \subset \mathbb{R}$. This example deals with the Heat equation*

$$\begin{aligned} \partial_t u(x,t) - \Delta u(x,t) &= f(x,t) \quad \text{in } \Omega \times (0,T] \\ u(x,0) &= u_0(x), \quad x \in \Omega \\ u(x,t) &= u_D(x,t), \quad x \in \Gamma = \partial\Omega \end{aligned}$$

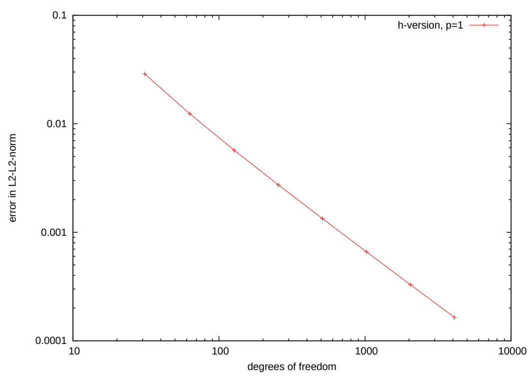
We choose Backward Euler discretization in time. $t_n = \Delta t \cdot n$. Let $H^1_{D,t_n} := \{v \in H^1(\Omega) : v|_\Gamma = u_D(x,t_n)\}$. Then for $n \geq 0$, $u^n \in H^1_{D,t_n}$ being known, we obtain $u^{n+1} \in H^1_{D,t_{n+1}}$ from

$$\frac{1}{\Delta t}\int_\Omega (u^{n+1} - u^n)v\,dx + \int_\Omega \nabla u^{n+1} \nabla v\,dx = \int_\Omega f \cdot v\,dx \quad \forall v \in H^1_0(\Omega).$$

Here we choose $\Omega = [-1,1]$ and $u(x,t) = -(1-x^2)(1-\exp(-t))$, $T = 10$.

fem1/ex34hin

```
! Heat-equation with homogeneous Dirichlet-conditions
open(1) 'test'; open(2) 'ex34hin.dat'
geometry('Slit'); #ti
EPS=1.0d-10
R=71
problem('Laplace',nickname='HEATHD')
DTS=0.1; TMAX=10.0
J=32
do I=1,8
 #time T1
 T=0; TL2=0; TH1=0
 DTS=TMAX/J
 TCNT=Iint(TMAX/DTS); DT=TMAX/TCNT
 mesh('uniform',n=J,p=1)
 matrix('analytic')
 #settime T
 approx 0 R 'u'
 #err. 16 R 'L2' 0 'u'; E[0]=ERR
 #err. 16 R 'H1' 0 'u'; E[1]=ERR
 do K=1,TCNT
  eval('uo=u'); AL2=E[0]**2; AH1=E[1]**2
  T=T+DT
  lft 16 R - R
  solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); #rno.
  #err. 16 R 'L2' 0 'u'; E[0]=ERR
  #err. 16 R 'H1' 0 'u'; E[1]=ERR
  #no. 'H1' 'u'
  TL2=TL2+(AL2+E[0]**2)/2*DT
  TH1=TH1+(AH1+E[1]**2)/2*DT
 continue
 #time T2
 TDIFF=T2-T1
 TL2=Sqrt(TL2); TH1=Sqrt(TH1)
 write(2) DOF,I,TL2,TH1,TDIFF
 J=J*2
continue
end
```



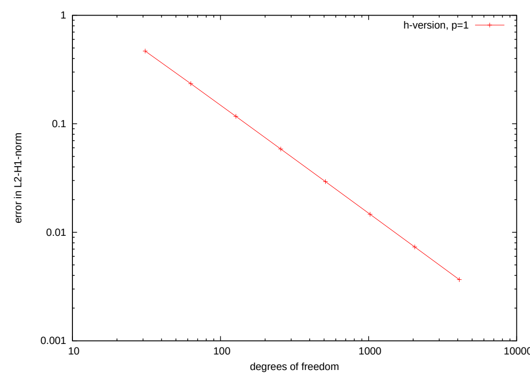


Figure 4.51: $\|u - u_n\|_{L^2([0,T],L^2(\Omega))}$ (left) and $\|u - u_n\|_{L^2([0,T],H^1(\Omega))}$ (right) — Heat equation.

4.1.5 Wave-Equation

Example 4.5. *Let $\Omega \subset \mathbb{R}$. This example deals with the Wave equation*

$$\begin{aligned} \partial_t^2 u(x,t) - c^2 \Delta u(x,t) &= f(x,t) \quad \text{in } \Omega \times (0,T] \\ u(x,0) &= u_0(x), \quad x \in \Omega \\ \partial_t u(x,0) &= \tilde{u}_0(x), \quad x \in \Omega \\ u(x,t) &= 0, \quad x \in \Gamma = \partial\Omega, t \in (0,T] \end{aligned}$$

We write the problem as a first order system, i.e. $u_1(x,t) = u(x,t)$, $u_2(x,t) = \partial_t u(x,t)$.

We choose Backward Euler discretization in time. $t_n = \Delta t \cdot n$. *Then for* $n \geq 0$, $(u_1^n, u_2^n) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *being known, we obtain* $(u_1^{n+1}, u_2^{n+1}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *from*

$$\begin{aligned} \frac{1}{\Delta t} \int_\Omega (u_2^{n+1} - u_2^n) v_1 \, dx + c^2 \int_\Omega \nabla u_1^{n+1} \nabla v_1 \, dx &= \int_\Omega f(x, t_{n+1}) \cdot v_1 \, dx \quad \forall v_1 \in H_0^1(\Omega), \\ \frac{1}{\Delta t} \int_\Omega (u_1^{n+1} - u_1^n) v_2 \, dx &= \int_\Omega u_2^{n+1} \cdot v_2 \, dx \quad \forall v_2 \in H_0^1(\Omega). \end{aligned}$$

Here we choose $\Omega = [0,1]$ *and* $u(x,t) = x(1-x)\cos(4x - ct)$ *with* $c = 2$ *and* $T = 3$.

fem1/ex35hin

```
! Wave-equation with homogeneous Dirichlet-conditions
open(1) 'test'; open(2) 'ex35hin.dat'
geometry('Slit01',gm='u'); #ti
EPS=1.0d-10
C=2
#setvelocity C
R=72
problem('Laplace',nickname='WAVEHD')
DTS=0.1; TMAX=3.0
J=32
do I=1,8
 #time T1
 T=0; TL2=0; TH1=0
 DTS=TMAX/J
 TCNT=Iint(TMAX/DTS); DT=TMAX/TCNT
 mesh('uniform',n=J,p=1,spline='u1',gm='u')
 matrix('analytic')
 #settime T
 approx 0 R 'u1' 'u'
 approx 0 R 'u2' 'ut'
 #err. 16 R 'L2' 0 'u1' 'u'; E[0]=ERR
 #err. 16 R 'H1' 0 'u1' 'u'; E[1]=ERR
 do K=1,TCNT
  eval('uo1=u1'); eval('uo2=u2'); AL2=E[0]**2; AH1=E[1]**2
  T=T+DT
  lft 16 R - R
  solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); #rno.
  #err. 16 R 'L2' 0 'u1' 'u'; E[0]=ERR
  #err. 16 R 'H1' 0 'u1' 'u'; E[1]=ERR
  #no. 'H10' 'u1'
  #no. 'L2'  'u2'
  TL2=TL2+(AL2+E[0]**2)/2*DT
```

```
  TH1=TH1+(AH1+E[1]**2)/2*DT
 continue
 #time T2
 TDIFF=T2-T1
 TL2=Sqrt(TL2); TH1=Sqrt(TH1)
 write(2) DOF,I,TL2,TH1,TDIFF
 J=J*2
continue
end
```

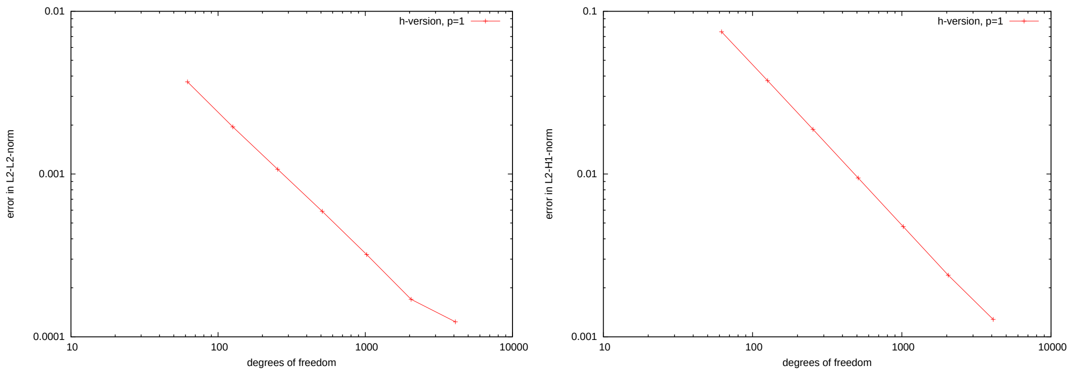


Figure 4.52: $\|u-u_1^n\|_{L^2([0,T],L^2(\Omega))}$ (left) and $\|u-u_1^n\|_{L^2([0,T],H^1(\Omega))}$ (right) — Wave equation.

5 Finite Element Methods (2D)

5.1 Convergence

5.1.1 Laplace

First we deal with the homogenous Dirichlet problem of the Laplacian on the L-Shape.

$$-\Delta u = f \text{ in } \Omega, \tag{8}$$
$$u = 0 \text{ on } \Gamma = \partial\Omega \tag{9}$$

Example 5.1. *This example solves the homogenous Dirichlet problem with*

$$f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$$

using the uniform h-version with rectangles. The exact energy norm is known to be

$$\|u\|_E = \pi\sqrt{\frac{3}{2}} = 3.84764949$$

fem2/ex1h4in

```
! FEM(2D)-problem on the L-Shape, h-version(4)
open(1) 'test' ; open(2) 'ex1h4in.dat'
geometry('L-Shape'); #ti
problem('Laplace',nickname='FEMHD')
EPS=1.0e-15
R=8; J=4
do I=1,8
 mesh('uniform',n=J,p=1,elements='rectangles')
 matrix; TM=SEC; WM=WSEC
 lft 8 R 0 R; TL=SEC; WL=WSEC
 solve(eps=EPS,mdi='x=1',mdc='diag',mit='CG'); TS=SEC; WS=WSEC
 #rno.
 #hno. 3.847649490
 open(1) 'ex1h4in'//I
 #taf. 'u'
 #px. 'u'
 #cx. 'u'
 #err. 8 R 'L2' 0 'u' ; E[0]=ERR
 #err. 8 R 'H10'0 'u' ; E[1]=ERR
 #no. 'L2'
 #no. 'H10'
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], COND, ITER, LMIN,LMAX,COND, TM, TL, TS, WM, WL, WS
 J=J*2
continue
end
```

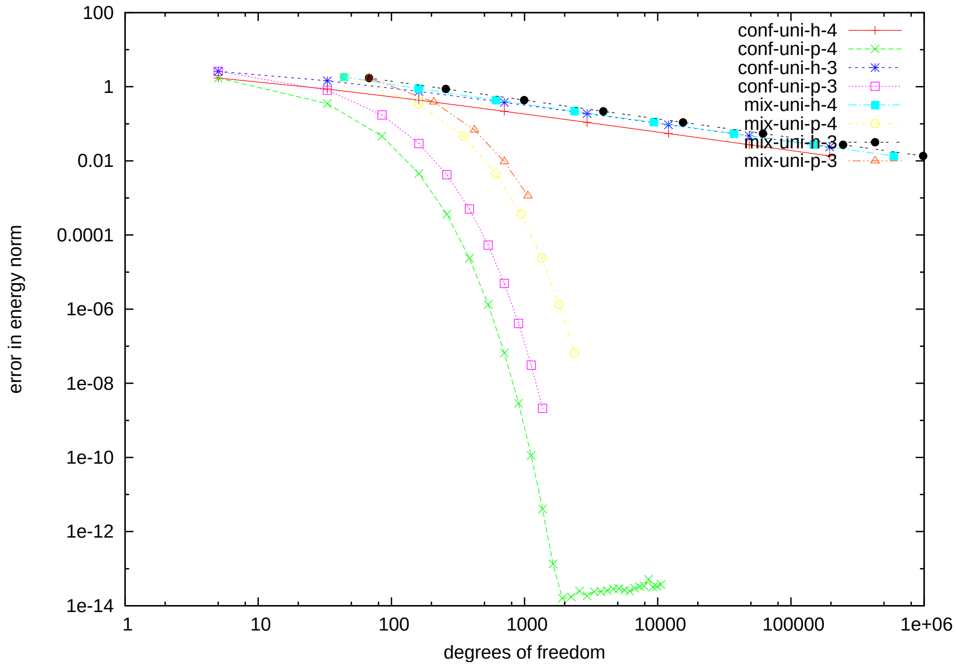


Figure 5.53: Homogenous Dirichlet problem, $f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y)$.

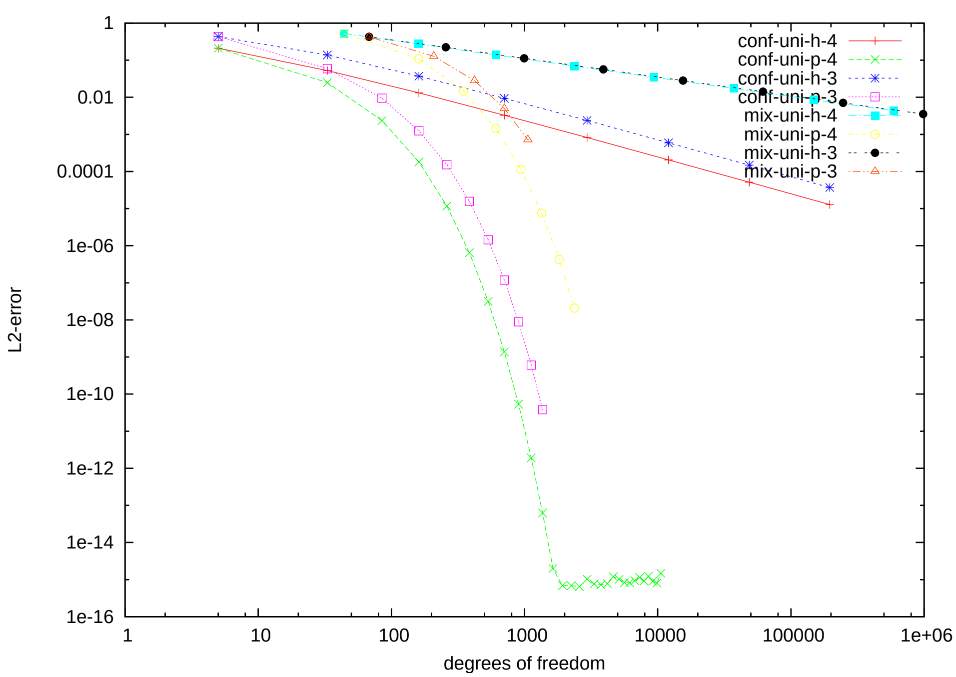


Figure 5.54: Homogenous Dirichlet problem, $f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y)$.

Example 5.2. *This example solves the homogenous Dirichlet problem with*

$$f(x,y) = 1$$

using the uniform h-version with rectangles. The exact energy norm is extrapolated to

$$\|u\|_E = 0.4626832638$$

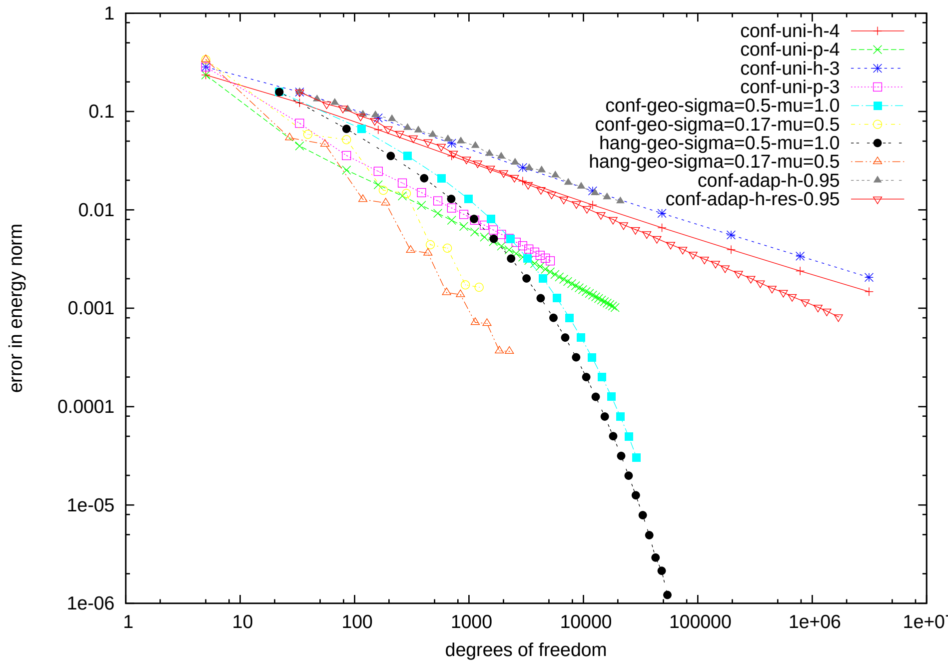


Figure 5.55: Homogenous Dirichlet problem, $f(x,y) = 1$.

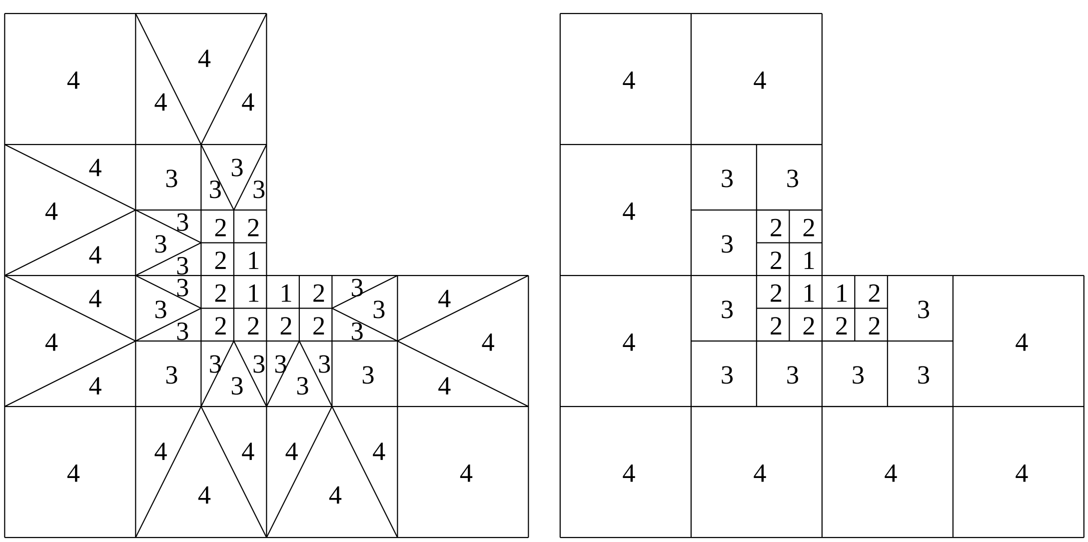


Figure 5.56: Conforming geometric mesh (left) and geometric mesh with hanging nodes (right)

Example 5.3. *Let* $\Omega = [-1,1]^2$, $\Gamma_D = \{-1\}\times[-1,1]$ *and* $\Gamma_N = \partial\Omega\backslash\Gamma_D$. *This example solves a Neumann problem for the Laplacian using a mixed formulation with Lagrange multipliers (see [1]).*

$$\begin{aligned} -\Delta u &= f = -(4+8x) \text{ in } \Omega \\ \frac{\partial u}{\partial n} &= \vec{n}\cdot\begin{pmatrix} 2x+3x^2+y^2 \\ 2y(1+x) \end{pmatrix} \text{ on } \Gamma_N \\ u &= 0 \text{ on } \Gamma_D \end{aligned}$$

with the exact solution

$$u(x,y) = (x^2+y^2)(1+x)$$

Find $(\sigma,u,\xi) \in H(\mathrm{div};\Omega)\times L^2(\Omega)\times \tilde{H}^{1/2}(\Gamma_N)$ *such that*

$$\begin{aligned} (\sigma,\tau) + (\operatorname{div}\tau, u) + \langle \tau\cdot n, \xi\rangle_{\Gamma_N} &= 0 \\ (\operatorname{div}\sigma, v) &= -(f,v) \\ \langle \sigma\cdot n, \lambda\rangle &= \langle g,\lambda\rangle_{\Gamma_N} \end{aligned}$$

for all $(\tau,v,\lambda) \in H(\mathrm{div};\Omega)\times L^2(\Omega)\times \tilde{H}^{1/2}(\Gamma_N)$. *We have* $\xi = -u|_{\Gamma_N}$ *and* $\sigma = \nabla u$.

fem2/ex4h3in

```
! mixed fem with Neumann-boundary (Lagrange-Multiplier)
open(1) 'test.h' ; open(2) 'ex4h3in.dat'
geometry('Square') ; #ti
#pxg 3 1 2 'Sg'
 0 2 -1.0 -1.0  1.0 -1.0  0.0 -1.0 0
 0 2  1.0 -1.0  1.0  1.0  1.0  0.0 0
 0 2  1.0  1.0 -1.0  1.0  0.0  1.0 0
problem('Laplace',igltyp=6); #pol 0 ! #pro 0 1 1 6 ; #pol 0
R=40   ! right hand side
G=1   ! solver
Q=8
EPS=1.0d-8
J=2;H=0.0625
do I=1,7
K=J
mesh('uniform',n=J,p=0,elements='triangles',spline='u',genspl='no')
mesh('uniform',n=K,p=1,spline='S')
#px. 'u' ; #px. 'N' ; #px. 'S'
matrix
lft 16 R 0 R
rlgs EPS -1 0 G 1 0 200 ; T=SEC
#cx. 'u' ; #cx. 'p' ; #cx. 'N' ; #cx. 'S'
#rno.
#err. Q R 'L2'   0 'u' ; E[0]=ERR ! FEM
#err. Q R 'L2'   0 'p' ; E[1]=ERR
#err. Q R 'L2'   0 'S' ; E[2]=ERR
#err. Q R 'L2'   0 'N' ; E[3]=ERR ! Neumann-Rand
#err. Q R 'Hdiv' 0 'p' ; E[4]=ERR
write(2) DOF,DOFU,E[0],E[1],E[2],E[3],E[4],ITER,T
J=J*2 ; H=H/2
continue
end
```

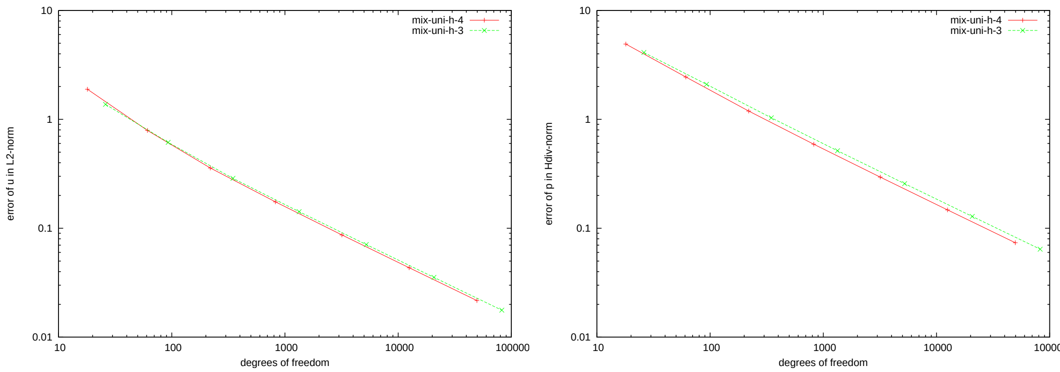


Figure 5.57: $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|p - p_n\|_{H(\mathrm{div};\Omega)}$ (right) — Mixed Neumann problem.

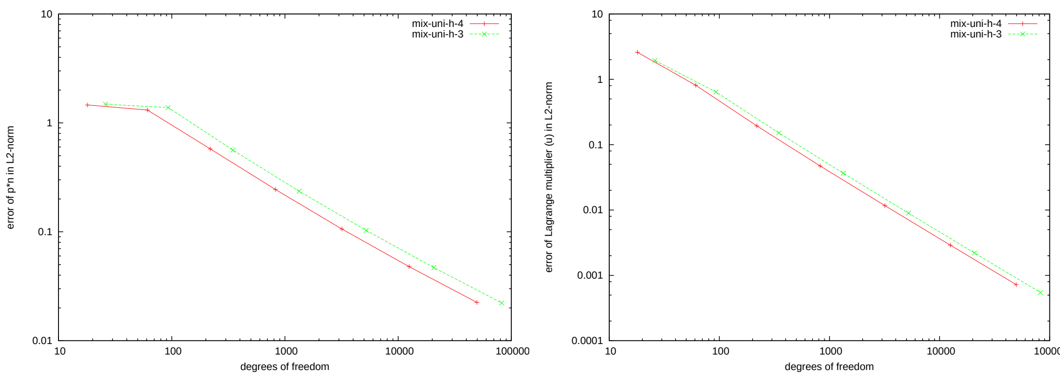


Figure 5.58: $\|p \cdot \vec{n} - p_n \cdot \vec{n}\|_{L^2(\Gamma)}$ (left) and $\|(-u) - \lambda_n\|_{L^2(\Gamma_N)}$ (right) — Mixed Neumann problem.

N	$E(u_N)_{L^2(\Omega)}$	α_N	$E(p_N)_{H(\mathrm{div};\Omega)}$	α_N	$E(\lambda_N)_{L^2(\Gamma_N)}$	α_N	$E(p_N \cdot \vec{n})_{L^2(\Gamma)}$	α_N
18	1.8886461		4.9170904		2.5893555		1.4605935	
61	0.7905197	0.714	2.4468487	0.572	0.8119938	0.950	1.3109844	0.089
219	0.3575252	0.621	1.1908586	0.563	0.1936145	1.122	0.5760745	0.643
823	0.1747434	0.541	0.5909142	0.529	0.0472703	1.065	0.2448177	0.646
3183	0.0868932	0.517	0.2948437	0.514	0.0116757	1.034	0.1060889	0.618
12511	0.0433876	0.507	0.1473416	0.507	0.0029002	1.017	0.0478885	0.581
49599	0.0216864	0.503	0.0736605	0.503	0.0007213	1.010	0.0224685	0.549

Table 5.14: Errors and convergence rates for Mixed Neumann Problem with rectangles

N	$E(u_N)_{L^2(\Omega)}$	α_N	$E(p_N)_{H(\mathrm{div};\Omega)}$	α_N	$E(\lambda_N)_{L^2(\Gamma_N)}$	α_N	$E(p_N \cdot \vec{n})_{L^2(\Gamma)}$	α_N
26	1.3718234		4.1088082		1.9355073		1.4829622	
93	0.6144346	0.630	2.0947182	0.529	0.6407308	0.867	1.3834982	0.054
347	0.2873128	0.577	1.0329807	0.537	0.1512122	1.097	0.5619718	0.684
1335	0.1420963	0.523	0.5144707	0.517	0.0365122	1.055	0.2359967	0.644
5231	0.0708753	0.509	0.2569530	0.508	0.0089348	1.031	0.1028940	0.608
20703	0.0354167	0.504	0.1284387	0.504	0.0022060	1.017	0.0469145	0.571
82367	0.0177058	0.502	0.0642144	0.502	0.0005462	1.011	0.0221975	0.542

Table 5.15: Errors and convergence rates for Mixed Neumann Problem with triangles

Example 5.4. *Let* $\Omega = [-1,1]^2 \backslash [0,1]^2$ *(L-Shape),* $\Gamma_D = \{0\} \times [0,1] \cup [0,1] \times \{0\}$ *and* $\Gamma_N = \partial\Omega \backslash \Gamma_D$. *This example solves a Neumann problem for the Laplacian using a mixed formulation with Lagrange multipliers (see [1]), for the adaptive version see [11].*

$$\begin{aligned} -\Delta u &= 0 \text{ in } \Omega \\ \frac{\partial u}{\partial n} &= \frac{\partial}{\partial n} r^{2/3} \sin(\frac{2}{3}(\varphi - \frac{\pi}{2})) \text{ on } \Gamma_N \\ u &= 0 \text{ on } \Gamma_D \end{aligned}$$

with the exact solution

$$u(x,y) = r^{2/3} \sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$$

fem2/ex5h3in

```
! mixed fem with Neumann-boundary (Lagrange-Multiplier)
open(1) 'test.h' ; open(2) 'ex5h3in.dat'
geometry('L-Shape') ; #ti
#pxg 4 1 2 'Sg'
 0 2  0.0  1.0 -1.0  1.0  0.0  1.0 0
 0 2 -1.0  1.0 -1.0 -1.0 -1.0  0.0 0
 0 2 -1.0 -1.0  1.0 -1.0  0.0 -1.0 0
 0 2  1.0 -1.0  1.0  0.0  1.0  0.0 0
problem('Laplace',nickname='SYMMIXFLAG')
R=1    ! right hand side
Q=8
EPS=1.0d-8
J=2;H=0.0625
do I=1,7
 K=J
 mesh('uniform',n=J,p=0,elements='triangles',spline='u',genspl='no')
 mesh('uniform',n=K,p=1,spline='S',gm='Sg')
 #px. 'u' ; #px. 'N' ; #px. 'S'
 matrix
 lft 16 R 0 R
 solve(eps=EPS,mdi='x=0',mit='GMRES',restrt=200); T=SEC; #rno.
 #cx. 'u' ; #cx. 'p' ; #cx. 'N' ; #cx. 'S'
 #err. Q R 'L2'   0 'u' ; E[0]=ERR ! FEM
 #err. Q R 'L2'   0 'p' ; E[1]=ERR
 #err. Q R 'L2'   0 'S' ; E[2]=ERR
 #err. Q R 'L2'   0 'N' ; E[3]=ERR ! Neumann-Rand
 #err. Q R 'Hdiv' 0 'p' ; E[4]=ERR
 write(2) DOF,DOFU,E[0],E[1],E[2],E[3],E[4],ITER,T
 J=J*2 ; H=H/2
continue
end
```

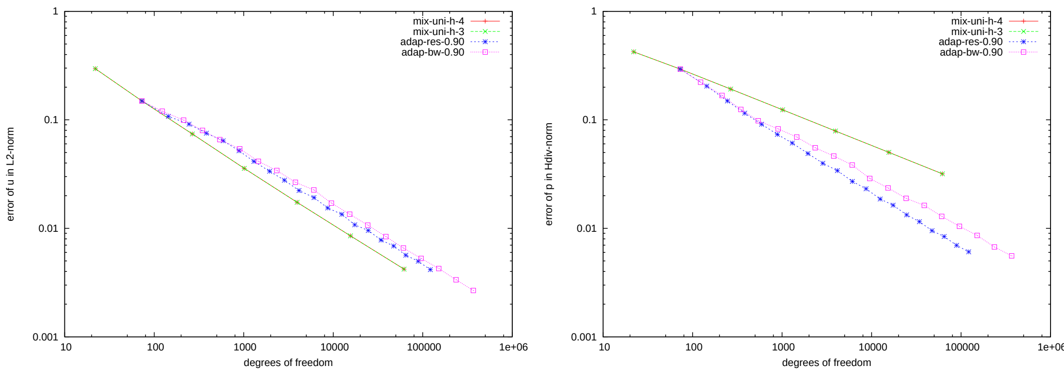


Figure 5.59: $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|p - p_n\|_{H(\mathrm{div};\Omega)}$ (right) — Mixed Neumann problem.

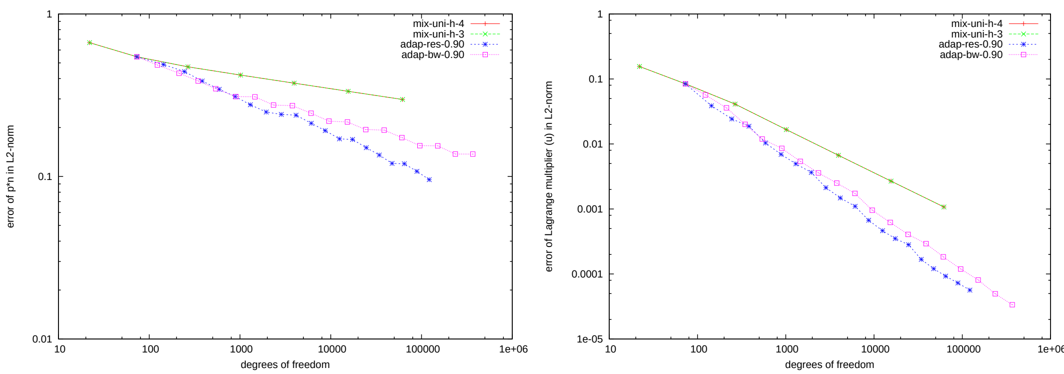


Figure 5.60: $\|p \cdot \vec{n} - p_n \cdot \vec{n}\|_{L^2(\Gamma)}$ (left) and $\|(-u) - \lambda_n\|_{L^2(\Gamma_N)}$ (right) — Mixed Neumann problem.

N	$E(u_N)_{L^2(\Omega)}$	α_N	$E(p_N)_{H(\mathrm{div};\Omega)}$	α_N	$E(\lambda_N)_{L^2(\Gamma_N)}$	α_N	$E(p_N \cdot \vec{n})_{L^2(\Gamma)}$	α_N
22	0.2961613		0.4239024		0.1555573		0.6657932	
73	0.1492915	0.571	0.2935250	0.306	0.0841846	0.512	0.5449981	0.167
267	0.0741690	0.539	0.1921723	0.327	0.0410493	0.554	0.4724582	0.110
1015	0.0357178	0.547	0.1236332	0.330	0.0166279	0.677	0.4206599	0.087
3951	0.0173767	0.530	0.0789576	0.330	0.0066917	0.670	0.3748203	0.085
15583	0.0085266	0.519	0.0501781	0.330	0.0026814	0.666	0.3339650	0.084
61887	0.0042102	0.512	0.0317869	0.331	0.0010709	0.666	0.2975455	0.084

Table 5.16: Errors and convergence rates for Mixed Neumann Problem with rectangles (uniform, $\tilde{h} = h$)

N	$E(u_N)_{L^2(\Omega)}$	α_N	$E(p_N)_{H(\mathrm{div};\Omega)}$	α_N	$E(\lambda_N)_{L^2(\Gamma_N)}$	α_N	$E(p_N \cdot \vec{n})_{L^2(\Gamma)}$	α_N
22	0.2961613		0.4239024		0.1555573		0.6657932	
73	0.1492915	0.571	0.2935250	0.306	0.0841846	0.512	0.5449981	0.167
267	0.0741690	0.539	0.1921723	0.327	0.0410493	0.554	0.4724582	0.110
1015	0.0357178	0.547	0.1236332	0.330	0.0166279	0.677	0.4206599	0.087
3951	0.0173767	0.530	0.0789576	0.330	0.0066917	0.670	0.3748203	0.085
15583	0.0085266	0.519	0.0501781	0.330	0.0026814	0.666	0.3339650	0.084
61887	0.0042102	0.512	0.0317869	0.331	0.0010709	0.666	0.2975455	0.084

Table 5.17: Errors and convergence rates for Mixed Neumann Problem with triangles (uniform, $\tilde{h} = h$)

N	$E(u_N)_{L^2(\Omega)}$	α_N	$E(p_N)_{H(\text{div};\Omega)}$	α_N	$E(\lambda_N)_{L^2(\Gamma_N)}$	α_N	$E(p_N \cdot \vec{n})_{L^2(\Gamma)}$	α_N
73	0.1492915		0.2935250		0.0841846		0.5449981	
144	0.1076699	0.481	0.2046182	0.531	0.0386234	1.147	0.4879773	0.163
245	0.0911765	0.313	0.1493435	0.593	0.0242453	0.876	0.4415598	0.188
383	0.0753215	0.428	0.1151962	0.581	0.0187438	0.576	0.3865611	0.298
590	0.0641205	0.373	0.0911027	0.543	0.0103587	1.372	0.3434908	0.273
886	0.0518077	0.524	0.0734545	0.530	0.0069220	0.991	0.3103103	0.250
1301	0.0414216	0.582	0.0610495	0.482	0.0049401	0.878	0.2760148	0.305
1946	0.0335636	0.522	0.0488985	0.551	0.0036427	0.757	0.2491436	0.254
2851	0.0277683	0.496	0.0397784	0.541	0.0021264	1.410	0.2408197	0.089
4151	0.0222971	0.584	0.0340512	0.414	0.0014778	0.969	0.2381418	0.030
6102	0.0191890	0.390	0.0270795	0.595	0.0010967	0.774	0.2124532	0.296
8686	0.0154091	0.621	0.0231472	0.444	0.0006693	1.399	0.1914706	0.295
12480	0.0134679	0.372	0.0186264	0.600	0.0004630	1.017	0.1703015	0.323
17400	0.0107778	0.670	0.0163413	0.394	0.0003499	0.843	0.1689324	0.024
24628	0.0095572	0.346	0.0132946	0.594	0.0002815	0.626	0.1503119	0.336
34244	0.0077964	0.618	0.0115125	0.437	0.0001674	1.577	0.1355012	0.315
47375	0.0068699	0.390	0.0095111	0.588	0.0001204	1.015	0.1202632	0.368
64671	0.0056507	0.628	0.0084022	0.398	.9258E-04	0.844	0.1196530	0.016
89058	0.0049705	0.401	0.0069661	0.586	.7261E-04	0.759	0.1074637	0.336

Table 5.18: Errors and convergence rates for Mixed Neumann Problem, adaptive version with residual error indicator, $\Theta = 0.90$

N	N_u	N_p	N_ξ	E_{tot}	η_{tot}	E_{tot}/η_{tot}
73	24	44	5	0.33990	1.47013	0.2312041
144	52	87	5	0.23442	1.05640	0.2219062
245	92	148	5	0.17665	0.85133	0.2074955
383	144	230	9	0.13891	0.69182	0.2007833
590	224	355	11	0.11189	0.56834	0.1968646
886	342	533	11	0.09015	0.46208	0.1951009
1301	502	782	17	0.07394	0.38325	0.1929326
1946	756	1169	21	0.05942	0.31008	0.1916326
2851	1116	1712	23	0.04856	0.25383	0.1913009
4151	1632	2492	27	0.04073	0.21082	0.1931903
6102	2405	3664	33	0.03321	0.17437	0.1904369
8686	3426	5213	47	0.02782	0.14571	0.1908984
12480	4940	7493	47	0.02299	0.12164	0.1890029
17400	6901	10445	54	0.01958	0.10262	0.1907810
24628	9784	14786	58	0.01638	0.08665	0.1889803
34244	13600	20551	93	0.01391	0.07328	0.1897506
47375	18844	28436	95	0.01173	0.06227	0.1884352
64671	25754	38814	103	0.01013	0.05334	0.1898296
89058	35490	53451	117	0.00856	0.04535	0.1887229

Table 5.19: Total Errors and Estimators for Mixed Neumann Problem, adaptive version with residual error indicator, $\Theta = 0.90$

N	η_{div}	η_{curl}	η_p	$\eta_{[p\cdot t]}$	η_{pt}	$\eta_{pt+\xi'}$	η_ξ	η_g
73	.39E-10	0.00000	0.97935	1.06107	0.15536	0.15302	0.11916	0.12054
144	.22E-08	0.00000	0.72020	0.74099	0.12875	0.09413	0.11465	0.09821
245	.74E-08	0.00000	0.61181	0.55851	0.09701	0.08396	0.11319	0.09617
383	.65E-07	0.00000	0.46529	0.49055	0.07666	0.07078	0.08739	0.05438
590	.58E-07	0.00000	0.38353	0.41284	0.05132	0.02818	0.04008	0.02105
886	.11E-06	0.00000	0.30733	0.33752	0.04765	0.02861	0.03998	0.02159
1301	.23E-06	0.00000	0.24828	0.28705	0.03237	0.02331	0.03078	0.01734
1946	.17E-06	0.00000	0.20116	0.23313	0.02314	0.01696	0.01943	0.01152
2851	.17E-06	0.00000	0.16788	0.18801	0.02392	0.00969	0.01412	0.00591
4151	.13E-06	0.00000	0.13395	0.16078	0.01853	0.00978	0.01292	0.00696
6102	.15E-06	0.00000	0.11489	0.12965	0.01239	0.00866	0.01151	0.00594
8686	.18E-06	0.00000	0.09267	0.11173	0.01046	0.00403	0.00511	0.00277
12480	.59E-06	0.00000	0.08101	0.09019	0.00756	0.00352	0.00499	0.00200
17400	.24E-06	0.00000	0.06476	0.07913	0.00618	0.00337	0.00453	0.00228
24628	.32E-06	0.00000	0.05726	0.06464	0.00412	0.00327	0.00436	0.00218
34244	.29E-06	0.00000	0.04683	0.05620	0.00355	0.00132	0.00180	0.00078
47375	.36E-06	0.00000	0.04127	0.04650	0.00241	0.00125	0.00176	0.00066
64671	.54E-06	0.00000	0.03393	0.04104	0.00215	0.00122	0.00168	0.00079
89058	.72E-06	0.00000	0.02983	0.03405	0.00166	0.00116	0.00154	0.00074

Table 5.20: Individual error indicators for Mixed Neumann Problem, adaptive version with residual error indicator, $\Theta = 0.90$

N	$E(u_N)_{L^2(\Omega)}$	α_N	$E(p_N)_{H(\text{div};\Omega)}$	α_N	$E(\lambda_N)_{L^2(\Gamma_N)}$	α_N	$E(p_N \cdot \vec{n})_{L^2(\Gamma)}$	α_N
22	0.2961613		0.4239024		0.1555573		0.6657932	
73	0.1492915	0.571	0.2935250	0.306	0.0841846	0.512	0.5449981	0.167
267	0.0741690	0.539	0.1921723	0.327	0.0410493	0.554	0.4724582	0.110
1015	0.0357178	0.547	0.1236332	0.330	0.0166279	0.677	0.4206599	0.087
3951	0.0173767	0.530	0.0789576	0.330	0.0066917	0.670	0.3748203	0.085

Table 5.21: Errors and convergence rates for Mixed Neumann Problem, uniform h-version with residual error indicator, $\tilde{h} = 2h$

N	N_u	N_p	N_ξ	E_{tot}	η_{tot}	E_{tot}/η_{tot}
22	6	13	3	0.54000	2.33470	0.2312938
73	24	44	5	0.33990	1.47013	0.2312041
267	96	160	11	0.21004	0.92375	0.2273767
1015	384	608	23	0.12976	0.56903	0.2280352
3951	1536	2368	47	0.08112	0.35396	0.2291895

Table 5.22: Total Errors and Estimators for Mixed Neumann Problem, uniform h-version with residual error indicator, $\tilde{h} = 2h$

N	η_{div}	η_{curl}	η_p	$\eta_{[p\cdot t]}$	η_{pt}	$\eta_{pt+\xi'}$	η_ξ	η_g
22	.22E-14	0.00000	1.99972	0.93082	0.36510	0.44715	0.32270	0.38492
73	.35E-11	0.00000	0.97935	1.06107	0.15536	0.15302	0.11916	0.12054
267	.18E-07	0.00000	0.48389	0.77925	0.08795	0.04600	0.04071	0.02029
1015	.45E-07	0.00000	0.24054	0.51223	0.05559	0.01511	0.01422	0.00594
3951	.77E-07	0.00000	0.11998	0.33107	0.03503	0.00522	0.00501	0.00191

Table 5.23: Individual error indicators for Mixed Neumann Problem, uniform h-version with residual error indicator, $\tilde{h} = 2h$

$$
\begin{aligned}
\eta_{\text{div}} &= \|\operatorname{div} p_h + f\|_{L^2(\Omega)} \\
\eta_{\text{curl}} &= \|h_T \operatorname{curl} p_h\|_{L^2(\Omega)} \\
\eta_p &= \|h_T p_h\|_{L^2(\Omega)} \\
\eta_{[p\cdot t]} &= \sum_e \|h_e^{1/2}[p_h \cdot t]\|_{L^2(e)} \\
\eta_{pt} &= \|h_e^{1/2} p_h \cdot t\|_{L^2(\Gamma_D)} \\
\eta_{pt+\xi'} &= \|h_e^{1/2}(p_h \cdot t + \xi'_{\tilde{h}}\|_{L^2(\Gamma_N)} \\
\eta_\xi &= \|h_e^{1/2}(\xi_{\tilde{h}} - \xi_h\|_{L^2(\Gamma_N)} \\
\eta_g &= (\log[1 + C_{\tilde{h}}(\Gamma_N)])^{1/2} \|h_e^{1/2}(g - p_h \cdot n)\|_{L^2(\Gamma_N)} \\
\eta_{tot} &= (\eta_{\text{div}}^2 + \eta_{\text{curl}}^2 + \eta_p^2 + \eta_{[p\cdot t]}^2 + \eta_{pt}^2 + \eta_{pt+\xi'}^2 + \eta_\xi^2 + \eta_g^2)^{1/2}
\end{aligned}
$$

N	$E(u_N)_{L^2(\Omega)}$	α_N	$E(p_N)_{H(\text{div};\Omega)}$	α_N	$E(\lambda_N)_{L^2(\Gamma_N)}$	α_N	$E(p_N \cdot \vec{n})_{L^2(\Gamma)}$	α_N
73	0.1492915		0.2935250		0.0841846		0.5449981	
123	0.1201638	0.416	0.2222761	0.533	0.0568563	0.752	0.4850009	0.224
213	0.0994555	0.344	0.1679056	0.511	0.0359113	0.837	0.4314508	0.213
346	0.0799517	0.450	0.1246915	0.613	0.0200509	1.201	0.3867902	0.225
541	0.0657315	0.438	0.0980190	0.538	0.0119055	1.166	0.3464830	0.246
902	0.0540457	0.383	0.0825650	0.336	0.0085675	0.644	0.3100845	0.217
1460	0.0414839	0.549	0.0693805	0.361	0.0054004	0.958	0.3088562	0.008
2342	0.0340957	0.415	0.0552306	0.483	0.0035889	0.865	0.2750783	0.245
3780	0.0265671	0.521	0.0464233	0.363	0.0024968	0.758	0.2726415	0.019
6069	0.0225974	0.342	0.0383405	0.404	0.0017354	0.768	0.2450870	0.225
9551	0.0170813	0.617	0.0288003	0.631	0.0009572	1.312	0.2188235	0.250
15268	0.0135201	0.498	0.0235659	0.428	0.0006202	0.925	0.2164942	0.023
24301	0.0107024	0.503	0.0189034	0.474	0.0004058	0.913	0.1942637	0.233
38779	0.0083945	0.520	0.0162618	0.322	0.0002928	0.698	0.1927600	0.017
60823	0.0065913	0.537	0.0128935	0.516	0.0001835	1.038	0.1733070	0.236
95912	0.0052815	0.486	0.0104380	0.464	0.0001191	0.949	0.1546438	0.250
151215	0.0042581	0.473	0.0086039	0.424	.8084E-04	0.851	0.1542212	0.006
235849	0.0033570	0.535	0.0067321	0.552	.4957E-04	1.100	0.1375036	0.258
367512	0.0026734	0.513	0.0055709	0.427	.3366E-04	0.873	0.1373286	0.003

Table 5.24: Errors and convergence rates for Mixed Neumann Problem, adaptive version with Bank-Weiser error indicator, $\Theta = 0.90$

N	N_u	N_p	N_ξ	E_{tot}	η_{tot}	E_{tot}/η_{tot}
73	24	44	5	0.33990	0.61905	0.5490683
123	42	74	7	0.25900	0.63823	0.4058000
213	76	128	9	0.19843	0.64377	0.3082274
346	128	208	10	0.14947	0.55592	0.2688772
541	202	324	15	0.11862	0.53154	0.2231563
902	343	541	18	0.09905	0.50631	0.1956337
1460	560	875	25	0.08102	0.48972	0.1654353
2342	908	1406	28	0.06501	0.43859	0.1482180
3780	1469	2268	43	0.05355	0.41013	0.1305578
6069	2372	3641	56	0.04454	0.40053	0.1111992
9551	3755	5734	62	0.03350	0.37545	0.0892228
15268	6009	9160	99	0.02718	0.36621	0.0742081
24301	9607	14587	107	0.02173	0.35730	0.0608078
38779	15350	23269	160	0.01830	0.34890	0.0524590
60823	24128	36495	200	0.01448	0.32930	0.0439777
95912	38129	57565	218	0.01170	0.33196	0.0352409
151215	60135	90728	352	0.00960	0.32904	0.0291764
235849	93920	141516	413	0.00752	0.32299	0.0232909
367512	146528	220542	442	0.00618	0.31983	0.0193207

Table 5.25: Total Errors and Estimators for Mixed Neumann Problem, adaptive version with Bank-Weiser error indicator, $\Theta = 0.90$

N	η_{div}	η_∇	η_u	$\eta_{\varphi+\xi}$	η_g
73	.42E-11	0.53482	0.13955	0.25136	0.12054
123	.17E-07	0.57639	0.11549	0.23317	0.08610
213	.29E-07	0.59323	0.09708	0.22365	0.05543
346	.50E-07	0.50898	0.08201	0.20424	0.03929
541	.63E-07	0.49672	0.06762	0.17523	0.02302
902	.77E-07	0.48326	0.05502	0.13983	0.01537
1460	.85E-07	0.47470	0.04146	0.11247	0.01079
2342	.13E-06	0.42861	0.03443	0.08609	0.00745
3780	.11E-06	0.39981	0.02667	0.08730	0.00540
6069	.14E-06	0.39416	0.02269	0.06732	0.00310
9551	.17E-06	0.37083	0.01710	0.05611	0.00217
15268	.16E-06	0.36326	0.01352	0.04433	0.00143
24301	.20E-06	0.35500	0.01069	0.03905	0.00088
38779	.20E-06	0.34713	0.00837	0.03408	0.00078
60823	.22E-06	0.32785	0.00658	0.03016	0.00041
95912	.28E-06	0.33088	0.00527	0.02634	0.00030
151215	.28E-06	0.32827	0.00425	0.02206	0.00022
235849	.31E-06	0.32232	0.00335	0.02066	0.00013
367512	.39E-06	0.31934	0.00267	0.01736	.91E-04

Table 5.26: Individual error indicators for Mixed Neumann Problem, adaptive version with Bank-Weiser error indicator, $\Theta = 0.90$

$$
\begin{aligned}
\eta_{\text{div}} &= \|\operatorname{div} p_h + f\|_{L^2(\Omega)} \\
\eta_\nabla &= \|p_h - \nabla\varphi_h\|_{[L^2(\Omega)]^2} \\
\eta_u &= \|u_h - \varphi_h\|_{L^2(\Omega)} \\
\eta_{\varphi+\xi} &= \|\varphi_h + \xi_{\tilde{h}}\|_{L^2(\Gamma_N)} \\
\eta_g &= (\log[1 + C_{\tilde{h}}(\Gamma_N)])^{1/2}\|h_e^{1/2}(g - p_h \cdot n)\|_{L^2(\Gamma_N)} \\
\eta_{tot} &= (\eta_{\text{div}}^2 + \eta_\nabla^2 + \eta_u^2 + \eta_{\varphi+\xi}^2 + \eta_g^2)^{1/2}
\end{aligned}
$$

N	N_u	N_p	N_ξ	E_{tot}	η_{tot}	E_{tot}/η_{tot}
73	24	44	5	0.33990	0.61905	0.5490683
267	96	160	11	0.21481	0.48319	0.4445715
1015	384	608	23	0.13172	0.33389	0.3944850
3951	1536	2368	47	0.08191	0.23361	0.3506354
15583	6144	9344	95	0.05129	0.16425	0.3122478

Table 5.27: Total Errors and Estimators for Mixed Neumann Problem, uniform h-version with Bank-Weiser error indicator, $\tilde{h} = 2h$

N	η_{div}	η_∇	η_u	$\eta_{\varphi+\xi}$	η_g
73	.35E-11	0.53482	0.13955	0.25136	0.12054
267	.24E-07	0.43617	0.08391	0.18914	0.02029
1015	.51E-07	0.30947	0.04101	0.11831	0.00594
3951	.82E-07	0.21912	0.02025	0.07841	0.00191
15583	.12E-06	0.15493	0.01005	0.05359	0.00065

Table 5.28: Individual error indicators for Mixed Neumann Problem, uniform h-version with Bank-Weiser error indicator, $\tilde{h} = 2h$

Example 5.5. *Let* $\Omega = [-1,1]^2 \backslash [0,1]^2$ *(L-Shape),* $\Gamma = \partial\Omega$*. This example solves a Dirichlet problem for the Laplacian.*

$$\begin{aligned} -\Delta u &= 0 \text{ in } \Omega \\ u &= r^{2/3} \sin(\frac{2}{3}(\varphi - \frac{\pi}{2})) \text{ on } \Gamma \end{aligned}$$

with the exact solution

$$u(x,y) = r^{2/3} \sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$$

We apply the residual error indicator on quadrilateral meshes with hanging nodes.

fem2/ex30h4a90in

```
! Dirichlet-FEM on the L-Shape, adaptive, hanging nodes, theta=0.90
open(1) 'test.h' ; open(2) 'ex30h4a90in.dat'
geometry('L-Shape') ; #ti
problem('Laplace',nickname='FEMNHD')
R=1    ! right hand side
Q=8
J=4;H=0.0625
mesh('uniform',n=J,p=1,elements='rectangles',mode='hanging')
do I=1,30
 approx 0 R 'u_bd' 'u0'
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
 lft 16 R 0 -1
 solve(eps=1.0d-10,mdi='x=0',mit='CG',abrflag=1,quiet=1);T=SEC; #rno.
 extend('u','u_bd','u_ex','Dirichlet')
 #err. Q R 'L2' 0 'u_ex' 'u' ; E[0]=ERR ! FEM
 #err. Q R 'H1' 0 'u_ex' 'u' ; E[1]=ERR
 resh 0.90 2 4 0
 write(2) DOF,ITER,E[0],E[1],ERREST,T
 refine(type='some',spline='u')
 J=J*2 ; H=H/2
continue
end
```

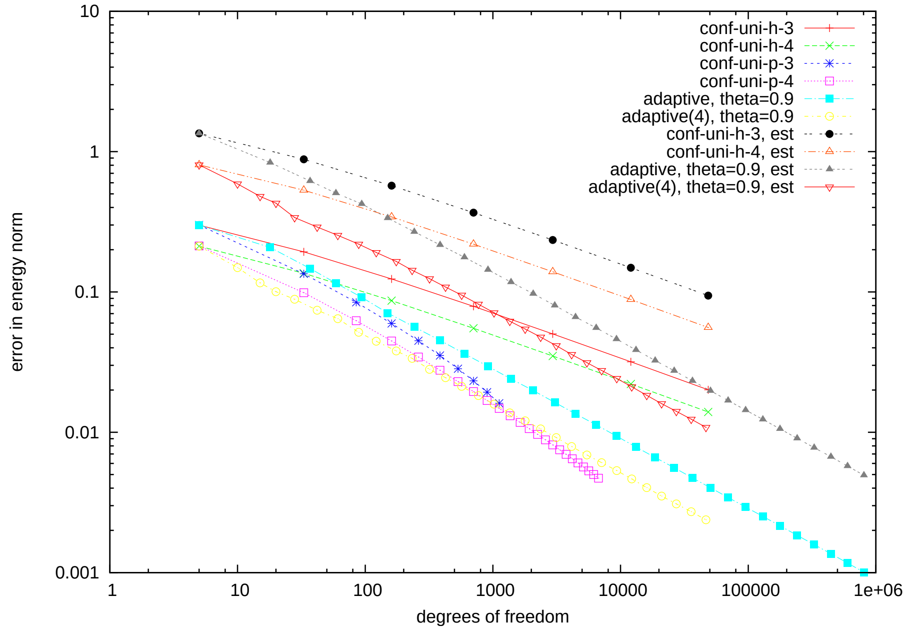


Figure 5.61: $\|u - u_h\|_{H^1(\Omega)}$

Example 5.6. *In this example we compare the finite element method with and without additional singularity functions at the tip. We compute the error by*

$$|u - u_N|^2_{H^1(\Omega)} = 2(J(u_N) - J(u)), \quad J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \int_{\Gamma_N} t_0\, v\, ds_x$$

and we use the extrapolated value $J(u) = -1.12034$.

N	$J(u_N)$	$\|u - u_N\|_{H^1(\Omega)}$	α
27	1.2292291	0.4666624	
111	1.1608195	0.2845251	-0.349994
447	1.1361365	0.1777319	-0.337786
1791	1.1267484	0.1131920	-0.325073
7167	1.1230128	0.0730833	-0.315484
28671	1.1214767	0.0476332	-0.308764
114687	1.1208303	0.0312447	-0.304173
458751	1.1205540	0.0205836	-0.301054
1835007	1.1204347	0.0136003	-0.298929
7340031	1.1203827	0.0090042	-0.297488

Table 5.29: Error for fem without singularity function

N	$J(u_N)$	$\lvert u-u_N\rvert_{H^1(\Omega)}$	α
28	1.2151759	0.4355082	
112	1.1582764	0.2754421	-0.330475
448	1.1351296	0.1719733	-0.339782
1792	1.1261330	0.1076182	-0.338131
7168	1.1226044	0.0672636	-0.339012
28672	1.1212035	0.0415040	-0.348288
114688	1.1206545	0.0249915	-0.365906
458752	1.1204479	0.0145370	-0.390855
1835008	1.1203752	0.0081290	-0.419293
7340032	1.1203517	0.0043684	-0.447976

Table 5.30: Error for fem with singularity function

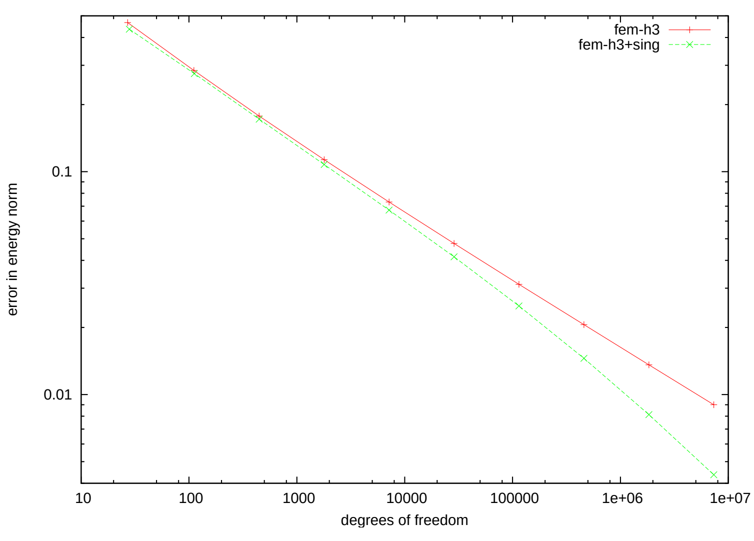


Figure 5.62: Energy error for fem without/with singularity function at tip.

Example 5.7. *Let $\Omega = [-1,1]^2 \backslash [0,1]^2$ (L-Shape), $\Gamma_D = \Gamma$ and $\Gamma_N = \partial\Omega \backslash \Gamma_D$. This example deals with a non-linear Laplace problem with mixed boundary conditions*

$$-\operatorname{div}(\varrho(|\nabla u|)\nabla u) = f \text{ in } \Omega$$
$$u = u_0 \text{ on } \Gamma_D$$
$$\varrho(|\nabla u|)\frac{\partial u}{\partial n} = t_0 \text{ on } \Gamma_N$$

The variational formulation is: Find $u \in H^1_D(\Omega) := \{v \in H^1(\Omega) : v|_{\Gamma_D} = u_0\}$ such that

$$\int_\Omega \varrho(|\nabla u|)\nabla u \nabla v \, dx = \int_\Omega f v \, dx + \int_{\Gamma_N} t_0 v \, ds \quad \forall v \in H^1_{D,0}(\Omega)$$

Let $\tilde{\varrho}(x) = \varrho(|x|)I + \varrho'(|x|)\frac{xx^t}{|x|}$. This gives rise to the following Newton scheme: Let $u^{(0)} \in H^1_D(\Omega)$. For $n = 1, 2, \dots$ until a stopping criterion is satisfied: Find $\delta \in H^1_{D,0}(\Omega)$ such that

$$\int_\Omega (\tilde{\varrho}(\nabla u^{(n-1)})\nabla\delta)\nabla v \, dx = \int_\Omega f v \, dx + \int_{\Gamma_N} t_0 v \, ds - \int_\Omega \varrho(|\nabla u^{(n-1)}|)\nabla u^{(n-1)} \nabla v \, dx \quad \forall v \in H^1_{D,0}(\Omega)$$

and $u^{(n)} = u^{(n-1)} + \delta$.

Here we choose $\varrho(t) = \frac{1}{6}(1 + \frac{5}{1+5t})$ and we stop if $\|\delta\|_{H^1(\Omega)} \le 10^{-8}$. We have $u_0 = r^{2/3}\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$.

fem2/ex32h4in

```
! Laplace, non-linear, L-Shape
open 'test.h' ; open(2) 'ex32h4in.dat'
geometry('L-Shape',styp=5,dim=(/0.25,0.25/)); #ti
problem('Laplace',nickname='FEMNHDNL')
R=51; NL=1 ! rhs, rho
EPS=1.0d-10
Q=16
J=4
do I=2,19
 mesh('uniform',n=J,p=1,elements='rectangles')
 approx 0 R 'u_bd' 'u0'; clear('u')
 extend('u','u_bd','u_ex')
 NCNT=0; ITMAX=0
 do K=0,200
  matrix('analytic',ijn=6,sigma=0.17,mu=1.0,nonlin=NL)
  lft 16 R 0 R NL
  solve(eps=EPS,mit='CG'); T=SEC; ITMAX=Max(ITER,ITMAX); #rno.
  eval('u_ex=u_ex+u')
  norm('NEWTON','H1','u')
  write(0) 'Newton', NEWTON, NCNT
  NCNT=NCNT+1
  if (NEWTON<EPS*100); then
   exit
  fi
 continue
 #err. Q R 'L2' 0 'u_ex' ; E[1]=ERR
 #err. Q R 'H1' 0 'u_ex' ; E[2]=ERR
 write(2) DOF,ITER,E[1],E[2],NCNT
 J=J*2
continue
end
```

DOF	$\|u - u_N\|_{L^2(\Omega)}$	α_{L^2}	$\|u - u_N\|_{H^1(\Omega)}$	α_{H^1}	It_{Newton}
5	0.0026734	—	0.0837548	—	5
33	0.0009865	-0.528302	0.0539445	-0.233133	6
161	0.0003697	-0.619265	0.0344836	-0.282335	6
705	0.0001404	-0.655607	0.0219286	-0.306537	6
2945	.5394E-04	-0.669123	0.0138970	-0.319039	6
12033	.2090E-04	-0.673601	0.0087877	-0.325617	6
48641	.8149E-05	-0.674287	0.0055491	-0.329118	6
195585	.3192E-05	-0.673538	0.0035010	-0.330994	6
784385	.1255E-05	-0.672121	0.0022075	-0.332052	6

Table 5.31: Non-Linear FEM, convergence rates and Newton steps

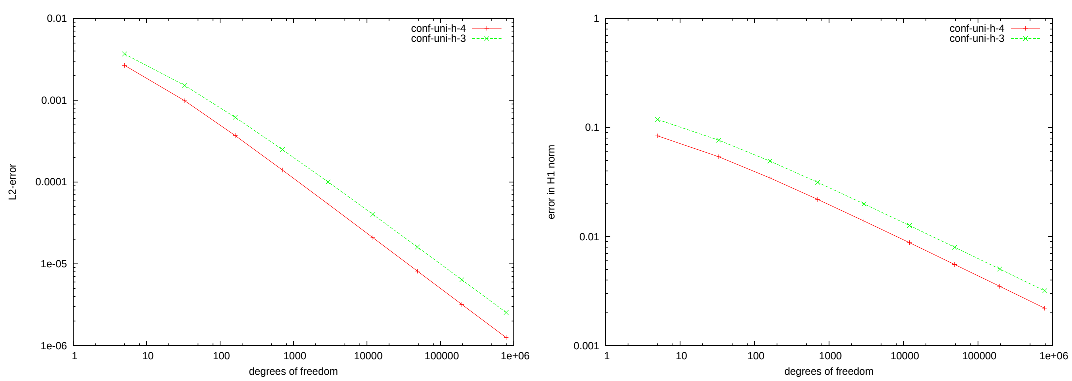


Figure 5.63: $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|u - u_n\|_{H^1(\Omega)}$ (right) — Non-linear.

Example 5.8. *Let* $\Omega = [-1,1]^2 \backslash [0,1]^2$ *(L-Shape),* $\Gamma_D = \Gamma$ *and* $\Gamma_N = \partial\Omega\backslash\Gamma_D$*. This example deals with a non-linear Laplace problem with mixed boundary conditions*

$$\begin{aligned} -\operatorname{div}(\varrho(|\nabla u|)\nabla u) &= f \ \textit{in } \Omega \\ u &= u_0 \ \textit{on } \Gamma_D \\ \varrho(|\nabla u|)\frac{\partial u}{\partial n} &= t_0 \ \textit{on } \Gamma_N \end{aligned}$$

Here we choose $\varrho(t) = (\varepsilon + t)^{p-2}$*. We have the exact solution* $u = r^{2/3}\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$ *and* $p = 3$*,* $\varepsilon = 0.00001$*.*

fem2/ex35h4in

```
! Laplace, non-linear, L-Shape, p-Laplacian p=3
open 'test.h' ; open(2) 'ex35h4in.dat'
geometry('L-Shape',styp=5,dim=(/0.25,0.25/)); #ti
problem('Laplace',nickname='FEMNHDNL')
R=51; NL=2 ! rhs, rho
P=3.0
setplap(p=P,eps=0.00001)
EPS=1.0d-10
Q=16
J=4
do I=2,19
 mesh('uniform',n=J,p=1,elements='rectangles')
 approx 0 R 'u_bd' 'u0'; clear('u')
 extend('u','u_bd','u_ex')
 NCNT=0; ITMAX=0
 do K=0,200
  matrix('analytic',ijn=6,sigma=0.17,mu=1.0,nonlin=NL)
  lft 16 R 0 R NL
  solve(eps=EPS,mit='CG'); T=SEC; ITMAX=Max(ITER,ITMAX); #rno.
  eval('u_ex=u_ex+u')
  norm('NEWTON','H1','u')
  write(0) 'Newton', NEWTON, NCNT
  NCNT=NCNT+1
  if (NEWTON<EPS*100); then
   exit
  fi
 continue
 #err. Q R 'L2' 0 'u_ex' 'u' P ; E[1]=ERR
 #err. Q R 'H1' 0 'u_ex' 'u' P ; E[2]=ERR
 write(2) DOF,ITER,E[1],E[2],NCNT
 J=J*2
continue
end
```

DOF	$\|u-u_N\|_{L^3(\Omega)}$	α_{L^3}	$\|u-u_N\|_{W^{1,p}(\Omega)}$	$\alpha_{W^{1,3}}$	It_{Newton}
5	0.0028079	—	0.0842060	—	6
33	0.0011047	-0.494345	0.0543268	-0.232238	21
161	0.0004425	-0.577254	0.0347406	-0.282106	22
705	0.0001806	-0.606825	0.0220947	-0.306455	22
2945	.7489E-04	-0.615713	0.0140031	-0.318997	23
12033	.3147E-04	-0.615956	0.0088552	-0.325584	23
48641	.1334E-04	-0.614447	0.0055919	-0.329096	23

Table 5.32: Non-Linear FEM (p-Laplacian), convergence rates and Newton steps

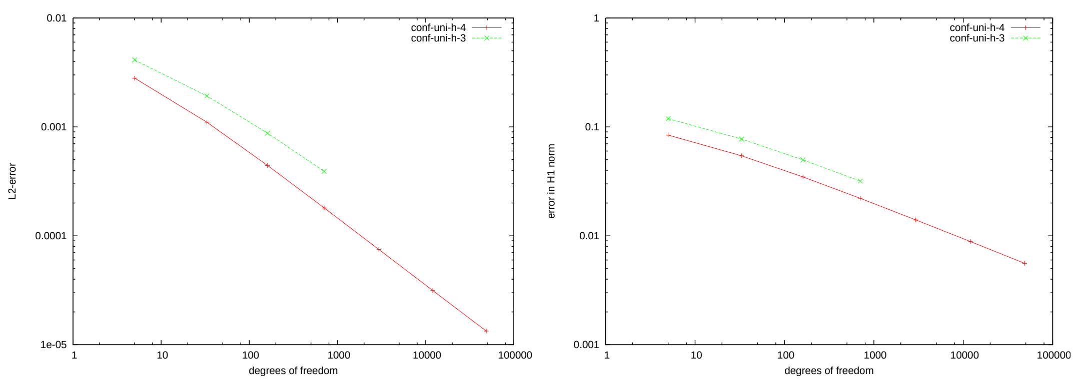


Figure 5.64: $\|u-u_n\|_{L^2(\Omega)}$ (left) and $\|u-u_n\|_{H^1(\Omega)}$ (right) — p-Laplacian.

Example 5.9. *Here we investigate the Discontinuous Galerkin method for the model problem*

$$\begin{aligned} -\Delta u &= f \ \ in\ \Omega, \\ u &= g \ \ on\ \Gamma. \end{aligned}$$

Find $u_h \in V_h$ such that

$$A_h(u_h, v) = F_h(v) \qquad \forall v \in V_h$$

where

$$\begin{aligned} A_h(u,v) &= \sum_{K\in\mathcal{T}_h} \int_K \nabla_h u \cdot \nabla_h v \, dx - \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \int_\kappa (\{\{\nabla_h v\}\} \cdot [[u]] + \{\{\nabla_h u\}\} \cdot [[v]]) ds \\ &+ \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \int_\kappa c[[u]] \cdot [[v]] ds, \\ F_h(v) &= \int_\Omega f v \, dx - \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \int_\kappa g \nabla_h v \cdot \mathbf{n} \, ds + \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \int_\kappa c g v \, ds \end{aligned}$$

with

$$\{\{v\}\} = \frac{1}{2}(v^+ + v^-), \quad \{\{\mathbf{q}\}\} = \frac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-), \quad [[v]] = v^+\mathbf{n}_{K^+} + v^-\mathbf{n}_{K^-}, \quad [[\mathbf{q}]] = \mathbf{q}^+ \cdot \mathbf{n}_{K^+} + \mathbf{q}^- \cdot \mathbf{n}_{K^-}.$$

On a boundary edge $\kappa \in \mathcal{E}_B(\mathcal{T}_h)$ we set $\{\{v\}\} = v$, $\{\{\mathbf{q}\}\} = \mathbf{q}$ and $[[v]] = v\mathbf{n}$.

Let h_K, k_K denote diameter and polynomial degree of element $K \in \mathcal{T}_h$.

We have $c = \gamma k^2 h^{-1}$ with

$$h(x) = \begin{cases} \min(h_K, h_{K'}), & x \in \kappa \in \mathcal{E}_I(\mathcal{T}_h), \kappa = \partial K \cap \partial K' \\ h_K, & x \in \kappa \in \mathcal{E}_B(\mathcal{T}_h), \kappa = \partial K \cap \Gamma, \end{cases}$$

$$k(x) = \begin{cases} \min(k_K, k_{K'}), & x \in \kappa \in \mathcal{E}_I(\mathcal{T}_h), \kappa = \partial K \cap \partial K' \\ k_K, & x \in \kappa \in \mathcal{E}_B(\mathcal{T}_h), \kappa = \partial K \cap \Gamma, \end{cases}$$

In our example we have the exact solution $u(r, \varphi) = r^{2/3} \sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$ and we choose $\gamma = 100$.

fem2/ex42h4in

```
! FEM(2D)-problem on the L-Shape, h-version(4)
open(1) 'test' ; open(2) 'ex42h4in.dat'
geometry('L-Shape'); #ti
problem('Laplace',nickname='DGFEMNHD')
#dg 100.
EPS=1.0e-15
R=1; J=4
do I=1,8
 mesh('uniform',n=J,p=1,elements='rectangles')
 matrix; TM=SEC; WM=WSEC
 lft 8 R 0 R ; TL=SEC; WL=WSEC
 solve(eps=EPS,mdi='x=1',mdc='no',mit='CG'); TS=SEC; WS=WSEC; #rno.
 #hno. 3.847649490
 #cx. 'u'
 #err. 8 R 'L2' 0 'u' ; E[0]=ERR
 #err. 8 R 'H10'0 'u' ; E[1]=ERR
```

```
 #no. 'L2'
 #no. 'H10'
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], COND, ITER, TM, TL, TS, WM, WL, WS
J=J*2
continue
end
```

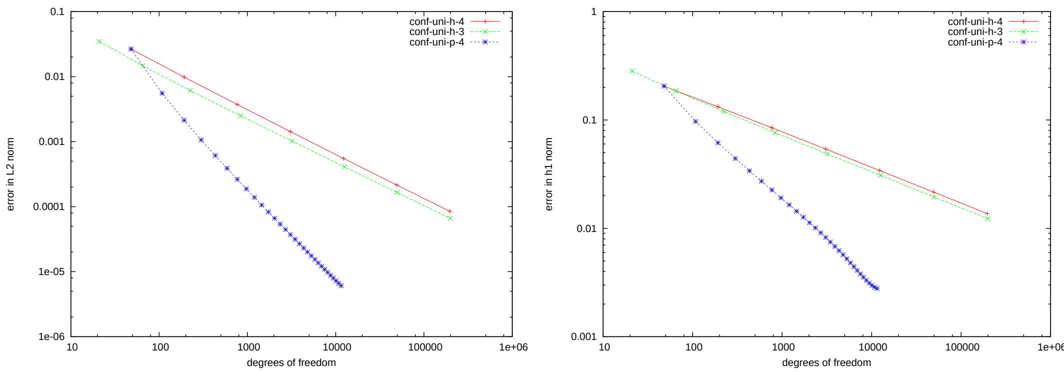


Figure 5.65: Laplace-2d DG: $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|u - u_n\|_{H^1(\Omega)}$ (right).

Example 5.10. *Here we investigate solvers for the Discontinuous Galerkin method for the model problem*

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= g \text{ on } \Gamma. \end{aligned}$$

We have $\Omega = [-1,1]^2$ *and* $u(x,y) = \sin(\pi x)\sin(\pi y)$, $\gamma = 100$ *and* $p = 1$.

DOF	$\Vert u-u_N\Vert_{L^2(\Omega)}$	$\Vert u-u_N\Vert_{H^1(\Omega)}$	$\lambda_{\min}$	$\lambda_{\max}$	cond	#it	τ_{iter}
64	0.23998	1.99018	0.26425	189.98	718.92602	24	0.0002
256	0.06059	1.00269	0.07419	197.18	2657.8488	82	0.0007
1024	0.01519	0.50303	0.01909	199.25	.1044E+05	189	0.0042
4096	0.00380	0.25175	0.00481	199.81	.4156E+05	397	0.0310
16384	0.00095	0.12590	0.00120	199.95	.1661E+06	817	0.2207
65536	0.00024	0.06296	0.00030	199.99	.6641E+06	1599	1.9530
262144	.59E-04	0.03148	.75E-04	200.00	.2656E+07	3097	18.067
1048576	.15E-04	0.01574	.19E-04	200.00	.1062E+08	6120	151.28

Table 5.33: Condition number and iteration number, unpreconditioned DG system, $p = 1$

DOF	$\Vert u-u_N\Vert_{L^2(\Omega)}$	$\Vert u-u_N\Vert_{H^1(\Omega)}$	$\lambda_{\min}$	$\lambda_{\max}$	cond	#it	τ_{iter}
144	0.02858	0.40389	0.04987	850.60	.1706E+05	183	0.0013
576	0.00383	0.10194	0.03982	880.56	.2211E+05	640	0.0142
2304	0.00049	0.02552	0.01848	889.09	.4812E+05	1047	0.0738
9216	.61E-04	0.00638	0.00478	891.35	.1864E+06	1529	0.3657
36864	.76E-05	0.00160	0.00120	891.94	.7417E+06	2344	2.2325
147456	.95E-06	0.00040	0.00030	892.08	.2963E+07	4374	19.967
589824	.13E-06	.10E-03	.75E-04	892.12	.1185E+08	8576	160.31
2359296	.84E-07	.25E-04	.19E-04	892.13	.4739E+08	16896	1342.0

Table 5.34: Condition number and iteration number, unpreconditioned DG system, $p = 2$

DOF	$\Vert u-u_N\Vert_{L^2(\Omega)}$	$\Vert u-u_N\Vert_{H^1(\Omega)}$	$\lambda_{\min}$	$\lambda_{\max}$	cond	#it	τ_{iter}
256	0.00271	0.05337	0				
1024	0.00018	0.00675	0				
4096	.11E-04	0.00085	0.01088	1998.7	.1837E+06	2754	0.4650
16384	.70E-06	0.00011	0.00478	2003.8	.4191E+06	3280	1.9381
65536	.44E-07	.13E-04	0.00120	2005.1	.1667E+07	4352	13.105
262144	.19E-07	.17E-05	0.00030	2005.4	.6661E+07	7174	89.363
1048576	.53E-07	.12E-05	.75E-04	2005.5	.2664E+08	13872	744.60
4194304	.10E-06	.24E-05	.19E-04	2005.5	.1065E+09	27174	5436.1

Table 5.35: Condition number and iteration number, unpreconditioned DG system, $p = 3$

DOF	$\|u-u_N\|_{L^2(\Omega)}$	$\|u-u_N\|_{H^1(\Omega)}$	$\lambda_{\min}$	$\lambda_{\max}$	cond	#it	τ_{iter}
400	0.00131	0.03332	0				
1600	.14E-04	0.00071	0				
6400	.21E-06	.21E-04	0.00248	3552.2	.1429E+07	8386	3.0488710
25600	.75E-08	.13E-05	0.00246	3561.1	.1445E+07	9284	14.181178
102400	.76E-08	.34E-06	0.00120	3563.5	.2967E+07	10607	68.252031
409600	.23E-07	.59E-06	0.00030	3564.0	.1184E+08	13475	357.93308
1638400	.35E-07	.14E-05	.75E-04	3564.2	.4734E+08	20580	2240.5793
6553600	.11E-06	.26E-05	.19E-04	3564.2	.1893E+09	38800	.1698E+05

Table 5.36: Condition number and iteration number, unpreconditioned DG system, $p = 4$

DOF	$\lambda_{\min}$	$\lambda_{\max}$	cond	#it	τ_{iter}
64	0.01179	1.4287	121.14	21	0.0012
256	0.01038	1.7476	168.36	51	0.0036
1024	0.01040	1.9925	191.56	71	0.0118
4096	0.01067	2.1854	204.89	84	0.0482
16384	0.01058	2.3389	221.16	102	0.2091
65536	0.01068	2.4613	230.52	105	0.9620
262144	0.01071	2.5588	238.82	113	4.2395
1048576	0.01077	2.6365	244.74	122	18.602
4194304	0.01080	2.6985	249.79	128	79.975
16777216	0.01075	2.7480	255.54	138	355.41

Table 5.37: Condition number and iteration number, MG-preconditioned DG system

DOF	$\lambda_{\min}$	$\lambda_{\max}$	cond	#it	τ_{iter}
64	0.75901	476.63	627.96	60	0.0007
256	0.72310	1078.3	1491.3	140	0.0024
1024	0.78465	2271.5	2894.9	227	0.0107
4096	0.82306	4651.4	5651.3	347	0.0560
16384	0.83408	9407.9	.1E+05	491	0.2703
65536	0.83675	.2E+05	.2E+05	687	1.6837
262144	0.84188	.4E+05	.5E+05	959	10.392
1048576	0.84845	.8E+05	.9E+05	1346	66.417

Table 5.38: Condition number and iteration number, BPX-preconditioned DG system

Example 5.11. *Here we investigate the double well problem as described in [8].*

Let $\Omega = (0,1)^2$, $F_1 = (-1,0), F_2 = (1,0)$ *and* $f_0(x) = -\frac{3}{128}(x-0.5)^5 - \frac{1}{3}(x-0.5)^3$, $f(x,y) = f_0(x)$, $g(x,y) = 0$, $\alpha = 1$. *Let*

$$I^{**}(v) := \int_\Omega W^{**}(\nabla v(x))\,dx + \alpha \int_\Omega |f(x) - v(x)|^2\,dx - \int_\Omega g(x)\cdot v(x)\,dx \qquad (v \in W^{1,p}(\Omega)).$$

We obtain

$$DI^{**}(u;v) = \int_\Omega DW^{**}(\nabla u(x))\nabla v\,dx + 2\alpha \int_\Omega (u-f)v\,dx - \int_\Omega g(x)\cdot v(x)\,dx$$

$$D^2I^{**}(u;v,w) = \int_\Omega (D^2W^{**}(\nabla u(x))\nabla v)\nabla w\,dx + 2\alpha \int_\Omega vw\,dx$$

Then we have

$$u(x,y) = f_1(x) := \begin{cases} f_0(x) & \text{for } 0 \le x \le 1/2, \\ \frac{1}{24}(x-0.5)^3 + x - 0.5 & \text{for } 1/2 \le x \le 1. \end{cases}$$

We abbreviate the stress $\sigma := DW^{**}(\nabla u)$ *and the indicator for microstructure* $\xi := Q(\nabla u)$.

We have the error estimate

$$\|\sigma - \sigma_h\|_{4/3} + \alpha\|u - u_h\|_2 \le c \cdot \left(\sum_{T\in\mathcal{T}_h} \eta_h(T)\right)^{3/8}$$

with

$$\eta_h(T) := h_T^{p'} \cdot \int_T |g + 2\alpha(f - u_h) + \operatorname{div}\sigma_h|^{p'}\,dx + \frac{1}{2}\sum_{E\subset\partial T\setminus\Gamma} h_E \cdot \int_E |[\sigma_h \cdot n_E]|^{p'}\,ds.$$

Here we have $p = 4, p' = 4/3, \frac{1}{p} + \frac{1}{p'} = 1$.

The minimal energy is $E := \min I^{**} = \frac{1409}{30000}$ *and microstructure is present in* $\Omega_m = (0,0.5)\times(0,1)$.

fem2/ex66h3in

```
! Laplace, double-well, Square
open 'test.h' ; open(2) 'ex66h3in.dat'; #ti
#pxg 1 2 2 'ug'
0 4 0. 0. 1. 0. 1. 1. 0. 1. 0

problem('Laplace',nickname='FEMDWNL')
R=65; NL=0 ! rhs, rho
#initdw -1. 0. 1. 0.
#initdw.
EPS=1.0d-10; Q=16; J=4
do I=2,10
 mesh('uniform',n=J,p=1,elements='triangles')
 approx 0 R 'u_bd' 'u0'; clear('u')
 extend('u','u_bd','u_ex')
 NCNT=0; ITMAX=0
 do K=0,200
  matrix('analytic',ijn=6,sigma=0.17,mu=1.0,nonlin=NL)
  lft 16 R 0 R NL
```

```
  solve(eps=EPS,mit='CG'); T=SEC; ITMAX=Max(ITER,ITMAX); #rno.
  norm('NEWTON','H1','u')
  eval('u_ex=u_ex+u')
  write(0) 'Newton', NEWTON, NCNT
  NCNT=NCNT+1
  if (NEWTON<EPS*100); then
   exit
  fi
 continue
 open(1) 'ex66h3u'//I
 #taf. 'u_ex' ; #px. 'u_ex' ; #cx. 'u_ex'
 dwenergy('u_ex',16)
 #err. Q R 'L2' 1 'u_ex' 'u' ; E[1]=ERR
 #err. Q R 'H1' 0 'u_ex' 'u' ; E[2]=ERR
 EDW=DWEN-1409./30000
 write(2) DOF,ITER,E[1],E[2],DWEN,EDW,NCNT
 J=J*2
continue
end
```

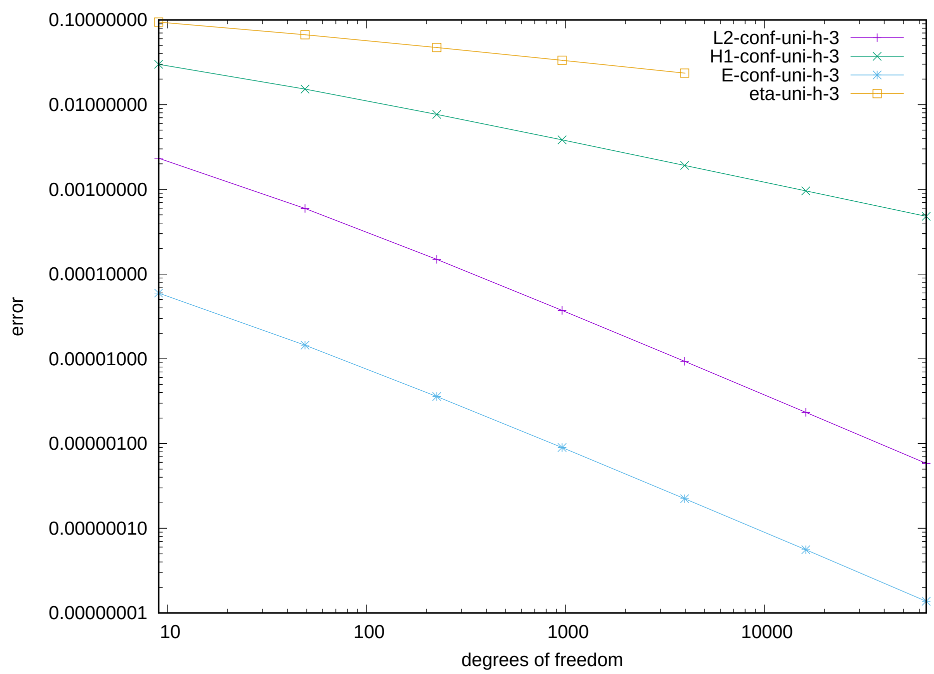


Figure 5.66: Error in different norms

Example 5.12. *In this example we investigate the homogenous Dirichlet problem of the Laplacian on the L-Shape* $[-1,1]^2 \backslash [0,1]^2$.

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma = \partial\Omega \end{aligned}$$

We are expecially interested in the point-wise convergence, using a postprocessing scheme

$$\begin{aligned}
G(x,y) &:= -\frac{1}{2\pi}\log|x-y| \\
\psi(x,\varrho) &= \begin{cases} 1 & \text{for } |x| \le \varrho/2 \\ 0 & \text{for } |x| \ge \varrho \end{cases} \\
&= \frac{1}{2}\left(1-\chi(4\frac{x}{\varrho}-3)\right) \\
\chi(t) &= \begin{cases} -1 & \text{for } t \le -1 \\ 2.4609375t - 3.28125t^3 + 2.953125t^5 - 1.40625t^7 + 0.2734375t^9 & \text{for} -1 < t < 1 \\ +1 & \text{for } t >= 1 \end{cases} \\
H(x) &:= G(x,x_0)\psi(x-x_0) \\
\Psi &:= -\int_{\omega(x_0,\varrho)} f(x)H(x)\,dx \\
\tilde{u}_N(x_0) &:= \int_{\omega^*} u_N(x)\Delta H(x)\,dx - \Psi \\
|u(x_0) - \tilde{u}_N(x_0)| &\le C\,N^{\alpha}
\end{aligned}$$

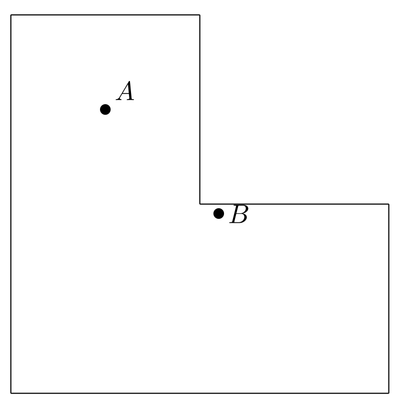


Figure 5.67: L-Shape with points of interest $A = (-0.5, 0.5), B = (0.1, -0.05)$

In our example we have $f \equiv 1$ *and a ll computations are done using uniform meshes with triangles and piecewise linear polynomials. The singular integral* Ψ *is computed using a composite quadrature scheme with geometrical refinement towards the singular point.*

All smooth integrals are done using a tensor product rule with a 24 point 1d-Gauss-Quadrature formula and Duffy transformation.

In Table 5.39 we present the point-evaluation $u_N(A)$*, the point-wise error* $\delta u(A) = u(A) - u_N(A)$*, the convergence rate* α_A*, the post-processed value* $\tilde{u}_N(A)$*, the error* $|u - \tilde{u}_N(A)|$ *and the rate* α*.*

Table 5.40 gives the corresponding values in point B, nearer to the boundary of the L-Shape. $u(A)$ and $u(B)$ have been determined by using Aitken extrapolation.

DOF	$u_N(A)$	$\delta u(A)$	α_A	$\tilde{u}_N(A)$	$\|u - \tilde{u}_N(A)\|$	α
5	0.0913462	0.0110160		-0.057080	0.1594419	
33	0.0986836	0.0036786	-0.581	0.1024786	0.0001164	-3.827
161	0.1011860	0.0011761	-0.719	0.0980565	0.0043057	2.2781
705	0.1019774	0.0003848	-0.757	0.1009705	0.0013916	-0.765
2945	0.1022308	0.0001313	-0.752	0.1021084	0.0002538	-1.190
12033	0.1023154	.4672E-04	-0.734	0.1022482	0.0001139	-0.569
48641	0.1023450	.1719E-04	-0.716	0.1023298	.3235E-04	-0.901
195585	0.1023557	.6485E-05	-0.700	0.1023543	.7841E-05	-1.019
784385	0.1023597	.2491E-05	-0.689	0.1023594	.2802E-05	-0.741
3141633	0.1023612	.9691E-06	-0.680	0.1023611	.1047E-05	-0.710

Table 5.39: Pointwise error and postprocessed pointwise error in $A = (-0.5, 0.5)$, $\varrho = 0.025$

DOF	$u_N(B)$	$\delta u(B)$	α_B	$\tilde{u}_N(B)$	$\|u - \tilde{u}_N(B)\|$	α
5	0.0091346	0.0208761		0.0424652	0.0124544	
33	0.0179303	0.0120804	-0.290	0.0137162	0.0162945	0.1424
161	0.0245250	0.0054857	-0.498	0.0253744	0.0046364	-0.793
705	0.0284157	0.0015950	-0.836	0.0284872	0.0015235	-0.754
2945	0.0294341	0.0005766	-0.712	0.0298624	0.0001483	-1.630
12033	0.0298247	0.0001860	-0.804	0.0297881	0.0002226	0.2886
48641	0.0299358	.7491E-04	-0.651	0.0299569	.5382E-04	-1.016
195585	0.0299834	.2730E-04	-0.725	0.0299826	.2817E-04	-0.465
784385	0.0299999	.1084E-04	-0.665	0.0299999	.1078E-04	-0.691
3141633	0.0300066	.4110E-05	-0.699	0.0300066	.4161E-05	-0.686

Table 5.40: Pointwise error and postprocessed pointwise error in $B = (0.1, -0.05)$, $\varrho = 0.025$

Example 5.13. *Here we investigate the same problem as in Example 5.1 using the multi-precision version of maiprogs.*

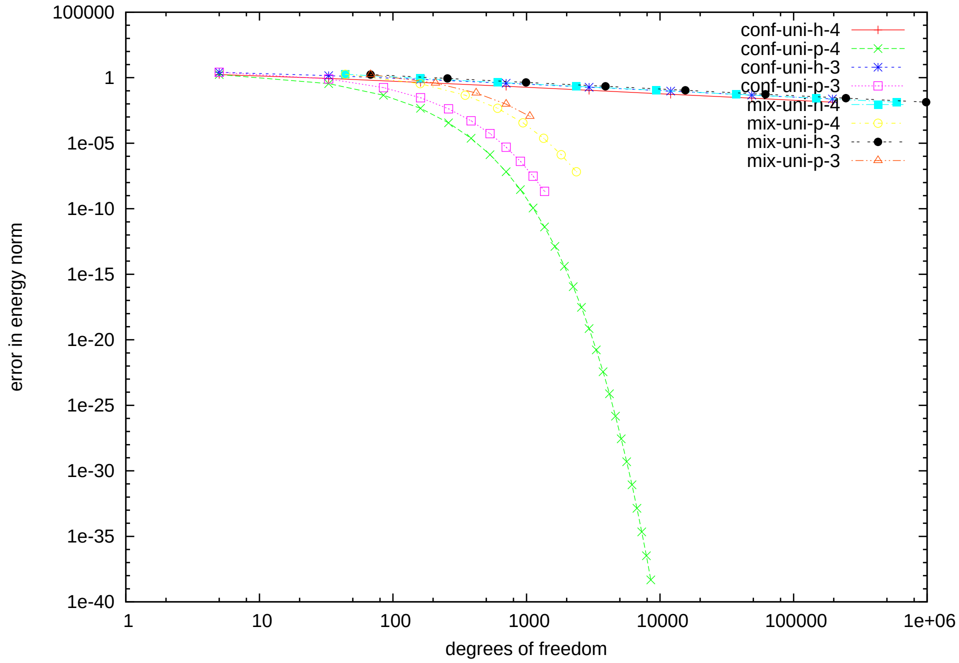


Figure 5.68: Homogenous Dirichlet problem, $f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y)$.

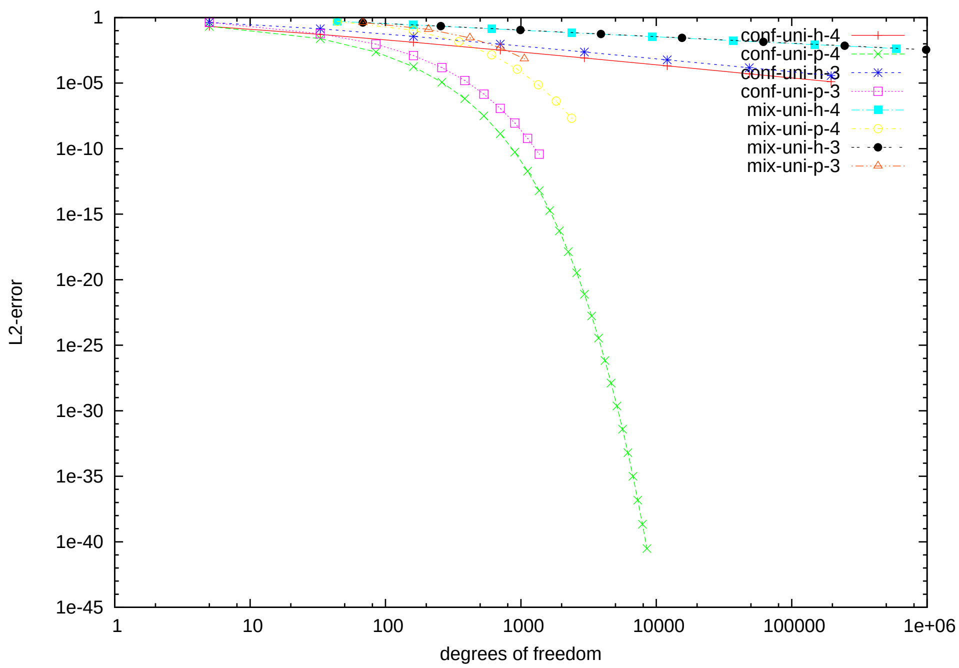


Figure 5.69: Homogenous Dirichlet problem, $f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y)$.

Example 5.14. *Here we rewrite the PDE from Example 5.1 as a L^2-Least-Squares Problem with homogenuous Dirichlet data. We first obtain the first order system*

$$\begin{aligned} -\operatorname{div} p &= f(x) \;\; in \; \Omega \\ p &= \nabla u \;\; in \; \Omega \\ u(x) &= 0, \quad x \in \Gamma \end{aligned}$$

then we obtain the Least-Squares minimization problem

$$(u,p) = \underset{(v,q)\in H_0^1(\Omega)\times H(\operatorname{div};\Omega)}{\text{minarg}} \mathcal{F}(v,q;f)$$

with

$$\mathcal{F}(v,q;f) := \| - \operatorname{div} q - f\|^2_{L^2(\Omega)} + \|q - \nabla v\|^2_{L^2(\Omega)}$$

The variational formulation now reads: Find $(u,p) \in H_0^1(\Omega) \times H(\operatorname{div};\Omega)$ s.t.

$$\begin{aligned} (\nabla u, \nabla v) - (p, \nabla v) &= 0 \\ -(\nabla u, q) + (p,q) + (\operatorname{div} p, \operatorname{div} q) &= -(f, \operatorname{div} q) \end{aligned}$$

for all $(v,q) \in H_0^1(\Omega) \times H(\operatorname{div};\Omega)$.

fem2/ex100h3in

```
! Laplace, Least-Squares-L2-formulation with homogenuous Dirichlet data
open(1) 'test'; open(2) 'ex100h3in.dat'
geometry('L-Shape'); #ti
problem('Laplace',nickname='LS2HD')
EPS=1.0e-15
R=8; J=4
do I=1,10
 mesh('uniform',n=J,p=1,elements='triangles')
 matrix; TM=SEC; WM=WSEC
 show('matrix')
 lft 8 R 0 R; TL=SEC; WL=WSEC
 solve(eps=EPS,mdi='x=1',mdc='diag',mit='CG'); TS=SEC; WS=WSEC
 #rno.
! open(1) 'ex100h3in'//I
 #hno. 3.847649490
! #taf. 'u'; #px. 'u'; #cx. 'u'
! #taf. 'p'; #px. 'p'; #cx. 'p'
 #err. 8 R 'L2'   0 'u' ; E[0]=ERR
 #err. 8 R 'H10'  0 'u' ; E[1]=ERR
 #err. 8 R 'L2'   0 'p' 'p' ; E[2]=ERR
 #err. 8 R 'HDIV' 0 'p' 'p' ; E[3]=ERR
 #no. 'L2' 'u'
 #no. 'H10' 'u'
 write(2) DOF,DOFU,DOFP,I,ENO,ENOERR,E[0],E[1],E[2],E[3],COND,ITER,TM,TL,TS,WM,WL,WS
 J=J*2
continue
end
```

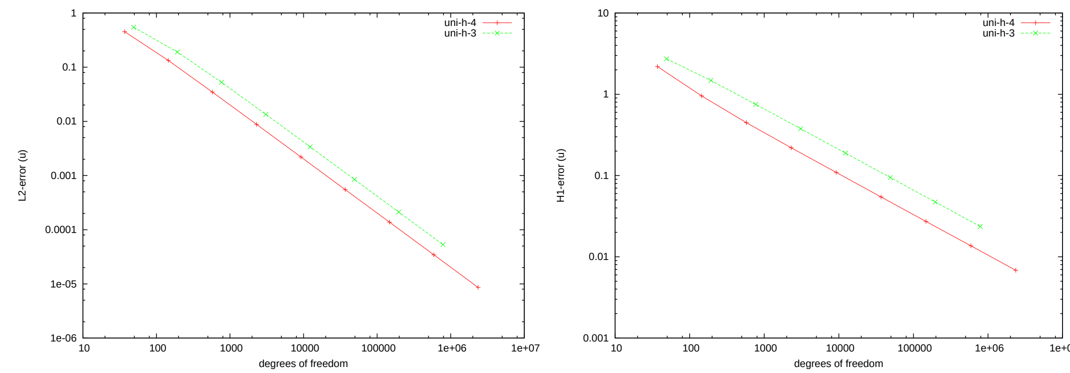


Figure 5.70: Laplace (L2-Least Squares): $\|u - u_n\|_{L^2(\Omega)}$ (left) and $|u - u_n|_{H^1(\Omega)}$ (right).

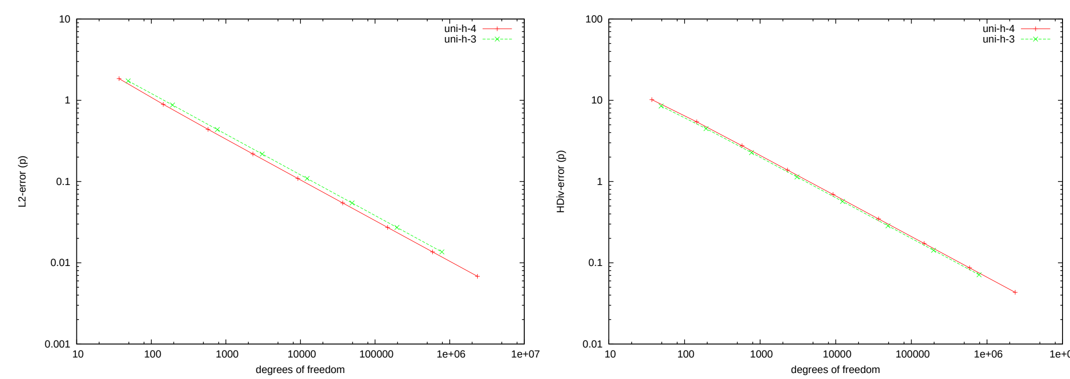


Figure 5.71: Laplace (L2-Least Squares): $\|p - p_n\|_{L^2(\Omega)}$ (left) and $|p - p_n|_{H(\mathrm{div};\Omega)}$ (right).

Example 5.15. *Here we rewrite a non-linear PDE as L^2-Least-Squares Problem with homogenuous Dirichlet data. We first obtain the first order system*

$$
\begin{aligned}
-\operatorname{div} p &= f(x) \ \textit{in } \Omega \\
p &= \varrho(|\nabla u|)\nabla u \ \textit{in } \Omega \\
u(x) &= 0, \quad x \in \Gamma
\end{aligned}
$$

then we obtain the Least-Squares minimization problem

$$(u, p) = \underset{(v,q)\in H_0^1(\Omega)\times H(\operatorname{div};\Omega)}{\text{minarg}} \mathcal{F}(v, q; f)$$

with

$$\mathcal{F}(v, q; f) := \| - \operatorname{div} q - f\|^2_{L^2(\Omega)} + \|q - \varrho(|\nabla v|)\nabla v\|^2_{L^2(\Omega)}$$

Example 5.16. *Similar to Example 5.14 we can investigate a L^2-Least-Squares Problem with (possible) mixed boundary conditions. We first obtain the first order system*

$$\begin{aligned} -\operatorname{div} p &= f(x) \text{ in } \Omega \\ p &= \nabla u \text{ in } \Omega \\ u(x) &= u_D(x), \quad x \in \Gamma_D \\ p(x) \cdot n &= p_N(x), \quad x \in \Gamma_N \end{aligned}$$

Introducing the spaces

$$\begin{aligned} H^1_D(\Omega) &:= \{u \in H^1(\Omega) \,:\, u|_{\Gamma_D} = u_D\} \\ H_N(\operatorname{div};\Omega) &:= \{p \in H(\operatorname{div};\Omega) \,:\, p \cdot n|_{\Gamma_N} = u_N\} \end{aligned}$$

we obtain the Least-Squares minimization problem

$$(u,p) = \underset{(v,q) \in H^1_D(\Omega) \times H_N(\operatorname{div};\Omega)}{\text{minarg}} \mathcal{F}(v,q;f)$$

with

$$\mathcal{F}(v,q;f) := \| -\operatorname{div} q - f\|^2_{L^2(\Omega)} + \|q - \nabla v\|^2_{L^2(\Omega)}$$

The variational formulation now reads: Find $(u,p) \in H^1_D(\Omega) \times H_N(\operatorname{div};\Omega)$ s.t.

$$\begin{aligned} (\nabla u, \nabla v) - (p, \nabla v) &= 0 \\ -(\nabla u, q) + (p, q) + (\operatorname{div} p, \operatorname{div} q) &= -(f, \operatorname{div} q) \end{aligned}$$

for all $(v,q) \in H^1_{D,0}(\Omega) \times H_{N,0}(\operatorname{div};\Omega)$.

Example 5.17. *Here we rewrite the PDE from Example 5.1 as a H^{-1}-Least-Squares Problem with (possible) mixed boundary conditions. We first obtain the first order system*

$$
\begin{aligned}
-\operatorname{div} p &= f(x) \ \ in\ \Omega \\
p &= \nabla u \ \ in\ \Omega \\
u(x) &= u_D(x), \quad x \in \Gamma_D \\
p(x)\cdot n &= p_N(x), \quad x \in \Gamma_N
\end{aligned}
$$

Introducing the spaces

$$
\begin{aligned}
H^1_D(\Omega) &:= \{u \in H^1(\Omega) \,:\, u|_{\Gamma_D} = u_D\} \\
H_N(\operatorname{div};\Omega) &:= \{p \in H(\operatorname{div};\Omega) \,:\, p\cdot n|_{\Gamma_N} = u_N\}
\end{aligned}
$$

we obtain the Least-Squares minimization problem

$$
(u,p) = \underset{(v,q)\in H^1_D(\Omega)\times H_N(\operatorname{div};\Omega)}{\text{minarg}} \mathcal{F}(v,q;f)
$$

with

$$
\mathcal{F}(v,q;f) := \| -\operatorname{div} q - f\|^2_{H^{-1}(\Omega)} + \|q - \nabla v\|^2_{L^2(\Omega)}
$$

fem2/ex110rth4in

```
! Least Squares, FEMHD, MG, RT, 110
open(1) 'test.h' ; open(2) 'ex110mgrth4in.dat'
open(3) 'ex110bprth4in.dat'; open(4) 'ex110ivrth4in.dat'
geometry('L-Shape')) ; #ti
problem('Laplace',nickname='LSFEMHD')
! #taf 'C0' 2 2 1 3 'p'
#taf 'Hdiv' 1 2 1 10 'p'
R= 8   ! right hand side
EPS=1.0d-8
J=8;H=0.0625
do I=1,8
 mesh('uniform',n=J,p=1,elements='quadrilaterals')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
 lft 16 R 0 R
 defprec(mode='ID',spline='p',name='Pp')

 defprec(mode='MG',spline='u',name='Pu',mat='I+A',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)

 lsqsolve(eps=EPS,mdi='x=1',mdc='scp',scp='Pp:0.5*Pu',mit='CG',mnum=1000); T=SEC

 #err. 16 R 'L2' 0 'u' ; E[1]=ERR ! FEM
 #err. 16 R 'H1' 0 'u' ; E[2]=ERR
 #err. 16 R 'L2' 0 'p' ; E[3]=ERR
 write(2) DOF,DOFP,DOFU,E[1],E[2],E[3],ITER,COND,T

 defprec(mode='BPX',spline='u',name='Pu',mat='I+A',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)

 lsqsolve(eps=EPS,mdi='x=1',mdc='scp',scp='Pp:0.5*Pu',mit='CG',mnum=1000); T=SEC
 #err. 16 R 'L2' 0 'u' ; E[1]=ERR ! FEM
 #err. 16 R 'H1' 0 'u' ; E[2]=ERR
```

```
 #err. 16 R 'L2' 0 'p' ; E[3]=ERR
 write(3) DOF,DOFP,DOFU,E[1],E[2],E[3],ITER,COND,T

 defprec(mode='INVCG',spline='u',name='Pu',mat='I+A')

 lsqsolve(eps=EPS,mdi='x=1',mdc='scp',scp='Pp:0.5*Pu',mit='CG',mnum=1000); T=SEC

 #err. 16 R 'L2' 0 'u' ; E[1]=ERR ! FEM
 #err. 16 R 'H1' 0 'u' ; E[2]=ERR
 #err. 16 R 'L2' 0 'p' ; E[3]=ERR
 write(4) DOF,DOFP,DOFU,E[1],E[2],E[3],ITER,COND,T
 J=J*2 ; H=H/2
continue
end
```

DOF	$\|u-u_N\|_{H^1(\Omega)}$	α_{H^1,L^2}	$\|u-u_N\|_{L^2(\Omega)}$	α_{u,L^2}	$\|\vartheta-\vartheta_N\|_{L^2(\Omega)}$	α_{ϑ,L^2}	κ
145	0.9898028	—	0.1454175	—	0.9078811	—	14.870170
577	0.4528237	0.4934	0.0379781	0.8471	0.4410636	0.456	19.879457
2305	0.2202422	0.4881	0.0095993	0.9313	0.2187156	0.475	22.758384
9217	0.1093162	0.4900	0.0024064	0.9677	0.1091235	0.486	24.826784
36865	0.0545564	0.4938	0.0006020	0.9844	0.0545322	0.493	26.447964
147457	0.0272654	0.4966	0.0001505	0.9925	0.0272624	0.496	27.736891
589825	0.0136311	0.4982	.3764E-04	0.9960	0.0136307	0.498	28.769259
2359297	0.0068154	0.4991	.9417E-05	0.9976	0.0068153	0.499	29.615695

Table 5.41: Linear H^{-1}-Least-Squares FEM (MG), convergence rates and condition numbers

5.1.2 Lamé

Example 5.18. *Let* $\Omega = [-1,1]^2$, $\Gamma_D = \{-1\} \times [-1,1]$ *and* $\Gamma_N = \partial\Omega \backslash \Gamma_D$. *This example solves a mixed boundary value problem for the Lamé equation using the primal formulation with* $E = 2000, \nu = 0.3$.

$$
\begin{aligned}
-\Delta^* \mathbf{u} = \mathbf{f} = -\Delta^* \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix} (x+1) &= \begin{pmatrix} -6\lambda - 8\mu \\ -2\lambda - 8\mu \end{pmatrix} \\
\mathcal{T}\mathbf{u} = \mathcal{T} \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix} (x+1) & \text{ on } \Gamma_N \\
\mathbf{u} = 0 & \text{ on } \Gamma_D
\end{aligned}
$$

fem2/ex82h4in

```
! Lame, FEM, mixed Dirichlet and Neumann boundary conditions
open(1) 'test.h' ; open(2) 'ex82h4in.dat'; #ti
geometry('Square')
#pxbd 4 1 2 'ubd'
 0 2 -1. -1.  1. -1. -5
 0 2  1. -1.  1.  1. -5
 0 2  1.  1. -1.  1. -5
 0 2 -1.  1. -1. -1. -1
problem('Lame',nickname='FEMNHD')
#ep 2000.0 0.3
R=11   ! right hand side
EPS=1.0d-8; J=1
do I=1,8
 mesh('uniform',n=J,p=1,elements='quadrilaterals')
 approx 0 R 'u_bd'
 matrix
 lft 16 R 0 R 0
 solve(eps=EPS,mdi='x=1',mdc='no',mit='CG'); T=SEC; #rno.
 extend('u','u_bd','u_ex','Dirichlet')
 #err. 24 R 'L2' 0 'u_bd' ; E[1]=ERR
 #err. 24 R 'L2' 0 'u_ex' ; E[3]=ERR ! FEM
 #err. 24 R 'H1' 0 'u_ex' ; E[4]=ERR
 write(2) DOF,ITER,E[1],0.0,E[3],E[4],T
 J=J*2
continue
end
```

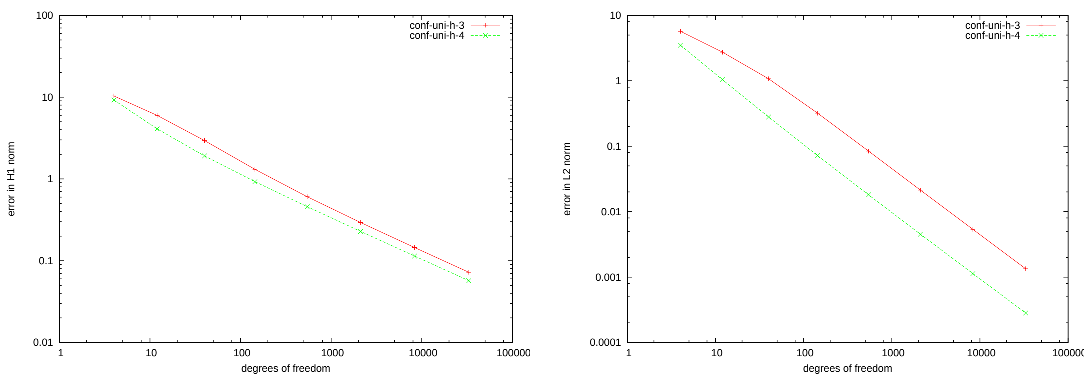


Figure 5.72: Error in $H^1(\Omega)$-norm (left) and in $L^2(\Omega)$-norm (right)

Example 5.19. *Here we investigate the Discontinuous Galerkin method for nearly incompressible linear elasticity, cf. [16]:*

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= \frac{2(1+\nu)}{E}\mathbf{f} \ \textit{in}\ \Omega,\\
\operatorname{div}\mathbf{u} + (1-2\nu)p &= 0 \ \textit{in}\ \Omega,\\
\mathbf{u} &= \mathbf{g} \ \textit{on}\ \Gamma
\end{aligned}
$$

Find $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ *such that*

$$
\begin{aligned}
A_h(\mathbf{u}_h, \mathbf{v}) + B_h(\mathbf{v}, p_h) &= F_h(\mathbf{v}),\\
-B_h(\mathbf{u}_h, q) + C_h(p_h, q) &= G_h(q) \qquad \forall (\mathbf{v}, q) \in V_h \times Q_h
\end{aligned}
$$

where

$$
\begin{aligned}
A_h(\mathbf{u},\mathbf{v}) =& \sum_{K\in\mathcal{T}_h}\int_K \nabla_h\mathbf{u} : \nabla_h\mathbf{v}\,dx - \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)}\int_\kappa (\{\{\nabla_h\mathbf{v}\}\} : [[\mathbf{u}_h]] + \{\{\nabla_h\mathbf{u}\}\} : [[\mathbf{v}]])ds\\
&+ \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)}\int_\kappa c[[\mathbf{u}]]\cdot[[\mathbf{v}]]ds,\\
B_h(\mathbf{v},q) =& -\int_\Omega q\nabla_h\cdot\mathbf{v}\,dx + \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)}\int_\kappa \{\{q\}\}[[\mathbf{v}]]\,ds,\\
C_h(p,q) =& (1-2\nu)\int_\Omega pq\,dx,\\
F_h(v) =& \frac{2(1+\nu)}{E}\int_\Omega \mathbf{f}\mathbf{v}\,dx - \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)}\int_\kappa \mathbf{g}\nabla_h\mathbf{v}\cdot\mathbf{n}\,ds + \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)}\int_\kappa c\mathbf{g}\mathbf{v}\,ds,\\
G_h(q) =& -\sum_{\kappa\in\mathcal{E}_\mathcal{B}(\mathcal{T}_h)}\int_\kappa q\mathbf{g}\mathbf{n}\,ds
\end{aligned}
$$

with

$$
\{\{q\}\} = \frac{1}{2}(q^+ + q^-), \qquad \{\{\mathbf{v}\}\} = \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-), \qquad \{\{\underline{\tau}\}\} = \frac{1}{2}(\underline{\tau}^+ + \underline{\tau}^-).
$$

Jumps are given by

$$
[[q]] = q^+\mathbf{n}_{K^+} + q^-\mathbf{n}_{K^-}, \quad [[\mathbf{v}]] = \mathbf{v}^+\cdot\mathbf{n}_{K^+} + \mathbf{v}^-\cdot\mathbf{n}_{K^-}, \quad [[\underline{\tau}]] = \underline{\tau}^+\mathbf{n}_{K^+} + \underline{\tau}^-\cdot\mathbf{n}_{K^-}.
$$

On a boundary edge $\kappa \in \mathcal{E}_B(\mathcal{T}_h)$ *we set* $\{\{q\}\} = q$, $\{\{\mathbf{v}\}\} = \mathbf{v}$ *and* $[[q]] = q\mathbf{n}$, $[[\mathbf{v}]] = \mathbf{v}\cdot\mathbf{n}$.

Let h_K, k_K *denote diameter and polynomial degree of element* $K \in \mathcal{T}_h$.

We have $c = \gamma k^2 h^{-1}$ *with*

$$
\begin{aligned}
h(x) &= \begin{cases} \min(h_K, h_{K'}), & x\in\kappa\in\mathcal{E}_I(\mathcal{T}_h), \kappa = \partial K\cap\partial K'\\ h_K, & x\in\kappa\in\mathcal{E}_B(\mathcal{T}_h), \kappa = \partial K\cap\Gamma,\end{cases}\\
k(x) &= \begin{cases} \min(k_K, k_{K'}), & x\in\kappa\in\mathcal{E}_I(\mathcal{T}_h), \kappa = \partial K\cap\partial K'\\ k_K, & x\in\kappa\in\mathcal{E}_B(\mathcal{T}_h), \kappa = \partial K\cap\Gamma,\end{cases}
\end{aligned}
$$

fem2/ex43h3in

```
! FEM(2D)-problem on the L-Shape, h-version(3), Lame, Discontinuous Galerkin
open(1) 'test' ; open(2) 'ex43h3in.dat'
```

```
geometry('L-Shape'); #ti
problem('Lame',nickname='DGFEMNHD')
#dg 100
E=1; nu=0.4
#ep E nu
EN=2*(1+nu)/E; IN=1-2*nu
EPS=1.0e-15
R=3; J=4
do I=1,8
 mesh('uniform',n=J,p=1,elements='triangles')
 matrix
 show('matrix')
 lft 8 R 0 R
 solve(eps=EPS,mdi='x=1',mdc='no',mit='CG'); T=SEC; #rno.
 #hno. 3.847649490
 #cx. 'u'
 #err. 8 R 'L2' 0 'u' ; E[0]=ERR
 #err. 8 R 'H10'0 'u' ; E[1]=ERR
 #no. 'L2'
 #no. 'H10'
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], COND, T, ITER
J=J*2
continue
end
```

Example 5.20. *2d-Stein-Benchmark, elastic case, plain-strain. E-module* $E = 206900.0$, $\nu = 0.29$. $\mathcal{T} = (0, 450.0)^T$.

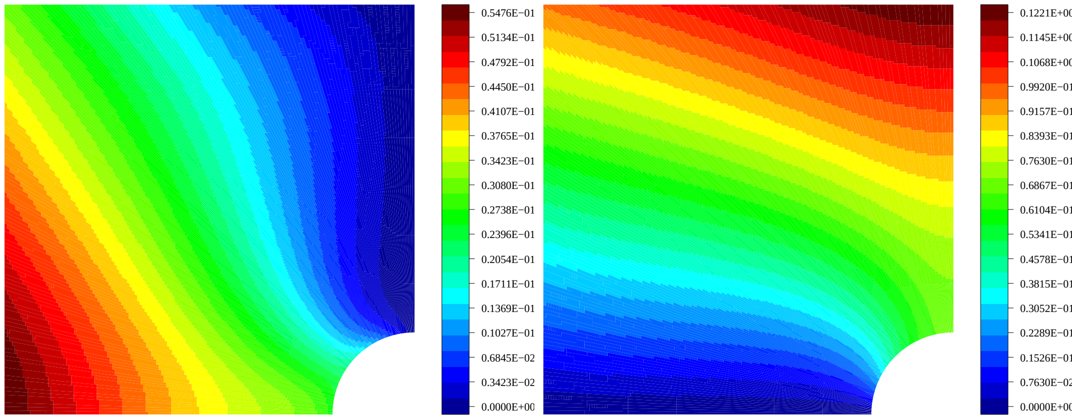


Figure 5.73: Deformation x and y-components.

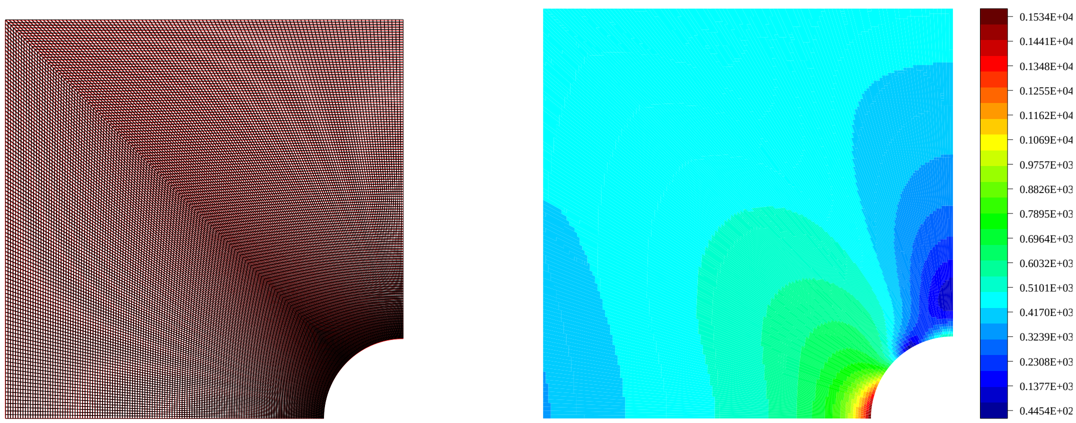


Figure 5.74: Deformation total and stress.

Example 5.21. *2d-Stein-Benchmark, elasto-plastic case, plain-strain. E-module* $E = 206900.0$*,* $\nu = 0.29$*.* $\mathcal{T} = (0, 450.0)^T$*.*

Figure 5.75: Deformation x and y-components.

Figure 5.76: Deformation total and stress.

5.1.3 Helmholtz

Example 5.22. *In this example we investigate the Helmholtz equation for different wave numbers, strongly and weakly imposed Dirichlet conditions and enrichment of the discrete function space by global ray-functions.*

$$-\Delta u - k^2 u = f \text{ in } \Omega$$
$$u = g \text{ on } \Gamma$$

Additionally to the standard fe method with strong boundary conditions we also use a saddle point formulation.

Find $(u, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ *such that*

$$(\nabla u, \nabla v)_\Omega + (u, \lambda)_\Gamma = (f, u)_\Omega$$
$$(v, \mu)_\Gamma = (g, \mu)_\Gamma$$

for all $(v, \mu) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$. *We note that there holds* $\lambda = -\frac{\partial u}{\partial n}$.

For the computations we split $u = u^p + u^e$, $\lambda = \lambda^p + \lambda^e$, *where* u^p, λ^p *are the standard polynomial test- and trial-functions, whereas* u^e, λ^e *are ray functions in different directions and their normal derivatives.*

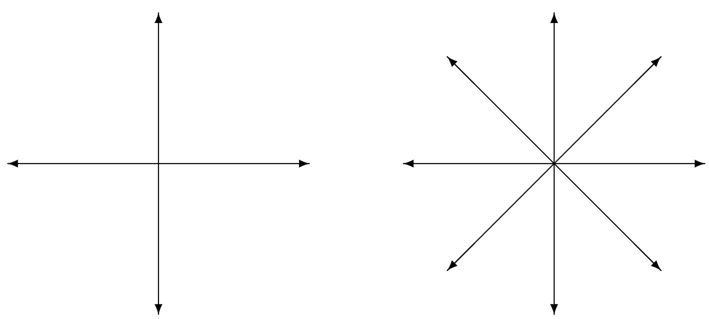


Figure 5.77: 4 rays and 8 rays

The exact solution in this example is $u(x, y) = e^{ik(\cos(0.1), \sin(0.1))(x,y)}$ *with* $k = 5$.

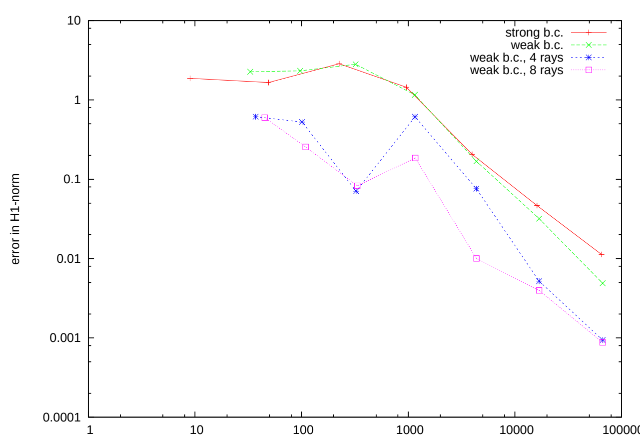


Figure 5.78: Helmholtz equation with rays

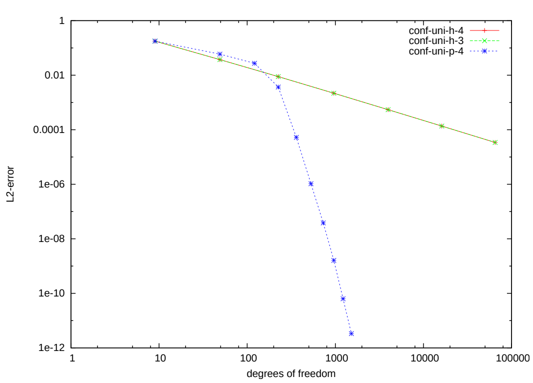
Example 5.23. *This example solves the homogenous Dirichlet problem of the Helmholtz equation on the square* $\Omega = [-1, 1]^2$.

$$-\Delta u - k^2 u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma = \partial\Omega$$

We use $u = \sin(\pi x)\sin(\pi y)$, *i.e.* $f = (2\pi^2 - k^2)u$ *and wavenumber* $k = 10$.

fem2/ex72h4in

```
open(1) 'test.h' ; open(2) 'ex72h4in.dat' ! Helmholtz-FEM problem
geometry('Square') ; #ti
problem('Helmholtz',nickname='FEMHD')
R=8    ! right hand side
#kw 10.0
EPS=1.0d-8
J=4;H=0.0625; ITER=0
do I=1,7
 mesh('uniform',n=J,p=1,elements='quadrilaterals')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
 lft 16 R 0 R
 solve(eps=EPS,mdi='x=0',mdc='no',mit='GMRES',quiet=0); T=SEC; #rno.
 #hno.
 #cx. 'u'
 #err. 24 R 'L2' 0 'u' ; E[3]=ERR ! FEM
 #err. 24 R 'H1' 0 'u' ; E[4]=ERR
 #no. 'L2' 'u'
 #no. 'H1' 'u'
 write(2) DOF,ITER,E[3],E[4],T
 J=J*2 ; H=H/2
continue
end
```



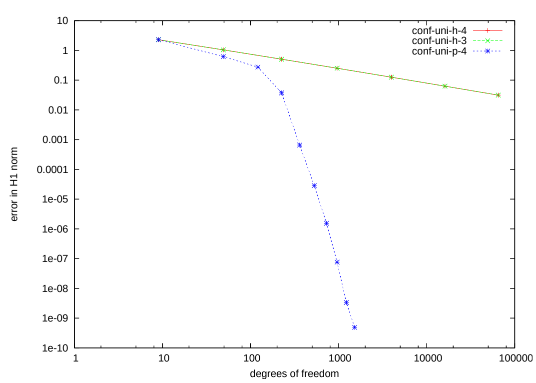


Figure 5.79: $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|u - u_n\|_{H^1(\Omega)}$ (right).

Example 5.24. *Let* $\Omega = [-1,1]^2 \backslash [0,1]^2$ *(L-Shape),* $\Gamma = \partial\Omega$*. This example solves an inhomogenous Dirichlet problem for the Helmholtz equation.*

$$\begin{aligned} -\Delta u - k^2 u &= 0 \text{ in } \Omega \\ u &= \tilde{J}_{2/3}(kr)\sin(\frac{2}{3}(\varphi - \frac{\pi}{2})) \text{ on } \Gamma \end{aligned}$$

with the exact solution

$$u(x,y) = \tilde{J}_{2/3}(kr)\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$$

$\tilde{J}_{2/3}(x) = \Gamma(2/3+1)J_{2/3}(x)$ *is a rescaled Bessel function.*

fem2/ex40h4in

```
! Dirichlet-FEM on the L-Shape, uniform mesh, Helmholtz
open(1) 'test.h'
geometry('L-Shape') ; #ti
problem('Helmholtz',nickname='FEMNHD')
R=1    ! right hand side
do K=1,10
 KW=Real(K)/2.0; #kw KW
 J=4;H=0.0625; Q=8
 open(2) 'ex40h4k'//KW:3//'in.dat'
 do I=1,9
  mesh('uniform',n=J,p=1,elements='rectangles')
  approx 0 R 'u_bd' 'u0'
  matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
  lft 16 R 0 R
  solve(eps=1.0d-10,mdi='x=0',mit='CG',abrflag=1,quiet=1);T=SEC; #rno.
  extend('u','u_bd','u_ex')
  #err. Q R 'L2' 0 'u_ex' 'u' ; E[0]=ERR ! FEM
  #err. Q R 'H1' 0 'u_ex' 'u' ; E[1]=ERR
  norm('NO','H1','u_ex')
  write(2) DOF,ITER,E[0],E[1],T,NO
  J=J*2 ; H=H/2
 continue
 close(2)
continue
end
```

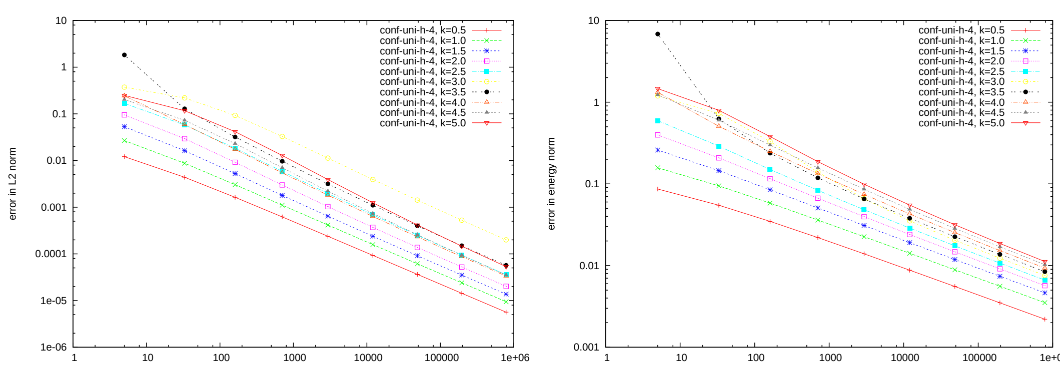


Figure 5.80: Helmholtz (2d-FEM): $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|u - u_n\|_{H^1(\Omega)}$ (right).

Example 5.25. *This example solves the homogenous Dirichlet problem of the Helmholtz equation on the square* $\Omega = [-1, 1]^2$ *for different wave numbers* k*.*

$$\begin{aligned} -\Delta u - k^2 u &= f \ \textit{ in } \Omega \\ u &= 0 \ \textit{ on } \Gamma = \partial\Omega \end{aligned}$$

We use $f \equiv 1$ *and wavenumbers* $k = 5$ *and* $k = 10$*.*

fem2/ex120h4in

```
! FEM(2D)-problem on the Square, h-version(4)
open(1) 'test' ; open(2) 'ex120h4in.dat'
geometry('Square01'); #ti
problem('Helmholtz',nickname='FEMHD')
EPS=1.0e-15
R=23; KA[0:3]=(/5,10,20,40/)
do K=0,3
 #kwc KA[K]
 J=4
 do I=1,8
  mesh('uniform',n=J,p=1,elements='rectangles')
  matrix; TM=SEC; WM=WSEC
  lft 8 R 0 R; TL=SEC; WL=WSEC
  solve(eps=EPS,mdi='x=1',mdc='no',mit='CG'); TS=SEC; WS=WSEC
!  solve(eps=EPS,mdi='unchanged',mdc='no',mit='CG'); TS=SEC; WS=WSEC
  #rno.; #hno.
  open(1) 'ex120h4_'//KA[K]//'_'//I
  #taf. 'u'; #px. 'u'; #cx. 'u'
  write(2) DOF, I, KA[K], ENO, TM, TL, TS, WM, WL, WS
  J=J*2
 continue
continue
end
```

5.1.4 Biharmonic

Example 5.26. *In this example we consider the clamped plate, i.e.*

$$\Delta^2 u = f \text{ in } \Omega, \quad u = 0, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

using a mixed formulation:
Find $(p, u) \in H^1(\Omega) \times H_0^1(\Omega)$ *such that*

$$\begin{aligned} a(p,q) + b(q,u) &= 0 & \forall q \in H^1(\Omega) \\ b(p,v) &= -\int_\Omega f v \, dx & \forall v \in H_0^1(\Omega) \end{aligned}$$

with $a(p,q) := \int_\Omega p \cdot q \, dx; b(q,u) := \int_\Omega \nabla u \cdot \nabla q \, dx$. *In our example we have* $u(x,y) = (\sin(\pi x)\sin(\pi y))^2$ *and* $\Omega = [-1,1]^2$.

fem2/ex22h4in

```
! FEM(2D)-problem on the Square, Clamped Plate, h-version(4)
open(1) 'test' ; open(2) 'ex22h4in.dat'
geometry('Square') ; #gm. ; #ti
problem('Bilaplace',nickname='MIX') ; #pro.
R=1; J=4
do I=1,8
mesh('uniform',n=J,p=1,elements='rectangles')
matrix
lft 8 R 0 R
!defprec(mode='INVCG',spline='u',name='Pu',mat='I+GG')
defprec(mode='MG',spline='u',name='Pu',mat='I+GG',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)
!defprec(mode='INVCG',spline='p',name='Pp',mat='I+GG')
defprec(mode='MG',spline='p',name='Pp',mat='I+GG',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)
solve(eps=1.0d-10,mti='x=0',mdc='p.Pp.p:u.Pu.u',mit='GMRES',restart=300); T=SEC
!rlgs 1.0d-10 1 1 0 1 1; T=SEC
#rno.
#hno. 27.9
#err. 8 R 'L2' 0 'u' ; E[0]=ERR
#err. 8 R 'H1' 0 'u' ; E[1]=ERR
#no. 'L2'
#no. 'H10'
write(2) DOF, I, ENO, ENOERR, E[0], E[1], ITER, COND, T
J=J*2
continue
end
```

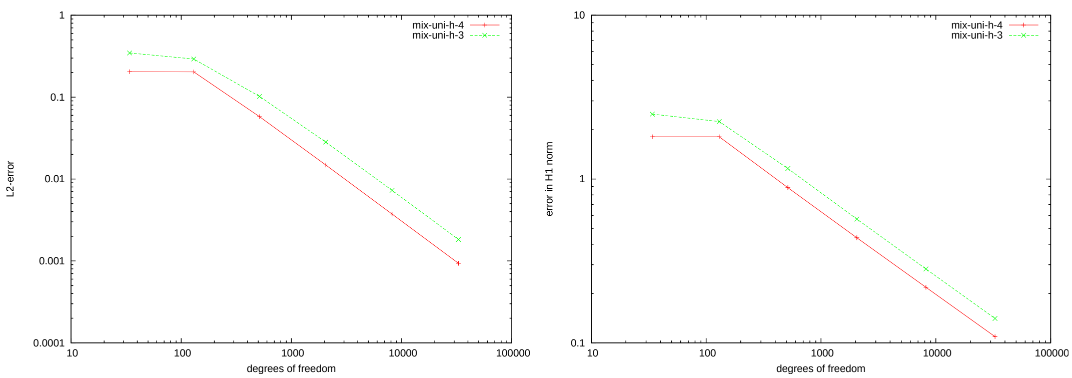


Figure 5.81: $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|u - u_n\|_{H^1(\Omega)}$ (right) — Clamped plate.

5.1.5 Stokes

Example 5.27. *Here we investigate the Stokes problem*

$$\begin{aligned} -\nu\Delta\vec{u} + \nabla p &= f \text{ in } \Omega \\ \operatorname{div}\vec{u} &= 0 \text{ in } \Omega \\ \vec{u} &= u_0 \text{ on } \partial\Omega \end{aligned}$$

The variational formulation reads: Find $(\vec{u}, p, \xi) \in H^1_D(\Omega)^d X \times L^2(\Omega) \times \mathbb{R}$ *such that*

$$\begin{aligned} \nu\int_\Omega \nabla\vec{u}\nabla\vec{v}\,dx \quad -\int_\Omega \operatorname{div}\vec{v}p\,dx \qquad\qquad &= \int_\Omega f\vec{v} \\ -\int_\Omega \operatorname{div}\vec{u}q\,dx \qquad\qquad\qquad +\xi\int_\Omega q \quad &= 0 \\ \chi\int_\Omega p\,dx \qquad\qquad\qquad\qquad\qquad &= 0 \end{aligned}$$

for all $(\vec{v}, q, \chi) \in H^1_0(\Omega)^d \times L^2(\Omega) \times \mathbb{R}$.

In this example we choose $\Omega = [-1,1]^2$, $\nu = 1$ *and* $\vec{u} = \frac{1}{\nu}(-\log r + (x_0 - \bar{x}_0)^2/r^2, (x_0 - \bar{x}_0)(x_1 - \bar{x}_1)/r^2)$ *and* $p = 2(x_0 - \bar{x}_0)/r^2$ *with* $r = |x - \bar{x}|$ *and* $\bar{x} = (0, 1.5)$, *such that* $f \equiv 0$.

fem2/ex60h4in

```
! Stokes on Square
open(1) 'test.h'; open(2) 'ex60h4in.dat'; #ti

problem('Stokes',nickname='FEMNHD')
geometry('Square')
NU=1.; #stokes NU
R=10
J=2
do I=0,8
 mesh('uniform',n=J,p=2,elements='rectangles',spline='u',gm='ug',genspl='no')
 mesh('global',n=1,spline='xi',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix
 lft 16 R 0 R
 solve(eps=1.e-10,mdc='no',mit='MINRES',quiet=0,restart=400); T=SEC
 #rno.
 extend('u','u_bd','u_ex')
 #taf. 'u_ex'; #px. 'u_ex'; #cx. 'u_ex'
 #taf. 'p'; #px. 'p'; #cx. 'p'
 #taf. 'xi'; #px. 'xi'; #cx. 'xi'
 #err. 16 R 'L2' 0 'u_ex' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u_ex' 'u'; E[2]=ERR
 #err. 16 R 'L2' 0 'p' 'p'; E[3]=ERR

 write(2) DOF,E[1],E[2],E[3],T,ITER
 J=J*2
continue
end
```

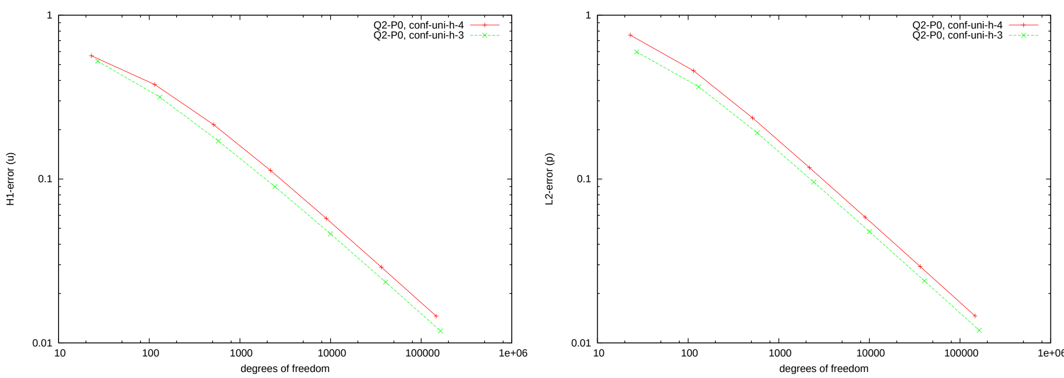


Figure 5.82: Stokes (2d-FEM): $\|u - u_n\|_{H^1(\Omega)}$ (left) and $\|p - p_n\|_{L^2(\Omega)}$ (right).

Example 5.28. *Here we investigate the nonlinear Stokes problem*

$$\begin{aligned} -\operatorname{div}(\psi(|\nabla u|)\nabla \vec{u}) + \nabla p &= f \ \textit{ in } \Omega \\ \operatorname{div} \vec{u} &= 0 \ \textit{ in } \Omega \\ \vec{u} &= u_0 \ \textit{ on } \partial\Omega \end{aligned}$$

using a dual-dual mixed formulation [6].

Find $(t,(\sigma,p),(u,\xi)) \in X_1 \times M_1 \times M$ *such that*

$$\begin{array}{llll} \int_\Omega \psi(|t|)t : s & -\int_\Omega \sigma : s - \int_\Omega p \operatorname{tr}(s) & & = 0 \\ -\int_\Omega \tau : t - \int_\Omega q \operatorname{tr}(t) & & -\int_\Omega u \cdot \operatorname{div}\tau + \xi \int_\Omega \operatorname{tr}(\tau) & = -\langle \tau n, u_0 \rangle \\ & -\int_\Omega v \cdot \operatorname{div}\sigma + \eta \int_\Omega \operatorname{tr}(\sigma) & & = \int_\Omega f \cdot v \end{array}$$

for all $(s,(\tau,q),(v,\eta)) \in X_1 \times M_1 \times M$, *with* $X_1 = L^2(\Omega)^{2\times 2}$, $M_1 = H(\operatorname{div};\Omega) \times L^2(\Omega)$, $M = L^2(\Omega)^2 \times \mathbb{R}$.

We have $t = \nabla u$ *and the stress-like tensor* $\sigma = \psi(|t|)t - pI$.

In this example we choose $\Omega = [-1,1]^2$, $\nu = 1$, $\psi(|t|) = \nu$ *and* $\vec{u} = \frac{1}{\nu}(-\log r + (x_0 - \bar{x}_0)^2/r^2, (x_0-\bar{x}_0)(x_1-\bar{x}_1)/r^2)$ *and* $p = 2(x_0-\bar{x}_0)/r^2$ *with* $r = |x - \bar{x}|$ *and* $\bar{x} = (0, 1.5)$, *such that* $f \equiv 0$.

fem2/ex61h4in

```
! Stokes on Square (dual-dual-mixed)
open(1) 'test.h'; open(2) 'ex61h4in.dat'; #ti

problem('Stokes',nickname='DDMIX')
geometry('Square')
NU=1.; #stokes NU
R=10
J=2
do I=0,8
 mesh('uniform',n=J,p=0,elements='rectangles',spline='u',gm='ug',genspl='no')
 mesh('global',n=1,spline='xi',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix
 lft 16 R 0 R
 solve(eps=1.e-10,mdc='diag',mit='GMRES',quiet=0,restart=400); T=SEC
 #rno.
 #taf. 'u'; #px. 'u'; #cx. 'u'
 #taf. 'p'; #px. 'p'; #cx. 'p'
 #taf. 'xi'; #px. 'xi'; #cx. 'xi'
 #err. 16 R 'L2' 0 'u' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u' 'u'; E[2]=ERR
 #err. 16 R 'L2' 0 'p' 'p'; E[3]=ERR
 #err. 16 R 'L2' 0 't' 'tgrad'; E[4]=ERR
 #err. 16 R 'L2' 0 'sigma' 'tsigma' ; E[5]=ERR
 #err. 16 R 'Hdiv' 0 'sigma' 'tsigma' ; E[6]=ERR

 write(2) DOF,E[1],E[2],E[3],E[4],E[5],E[6],T,ITER
 J=J*2
continue
```

```
end
```

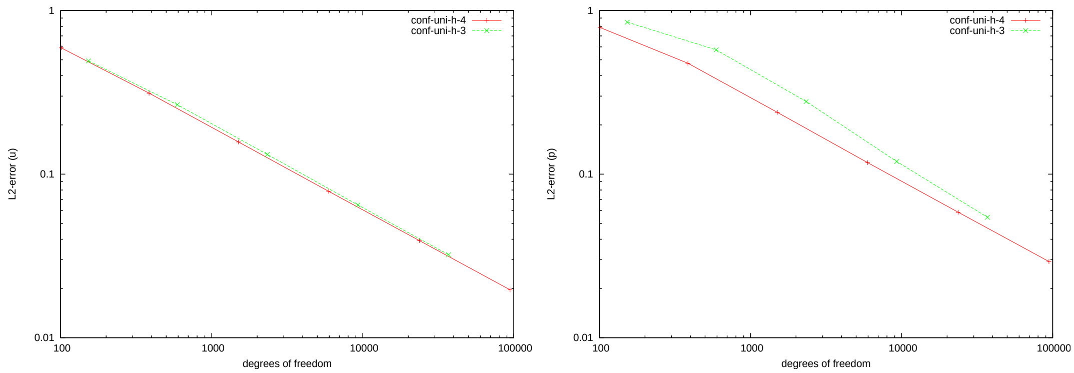


Figure 5.83: Stokes (2d-FEM): $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|p - p_n\|_{L^2(\Omega)}$ (right).

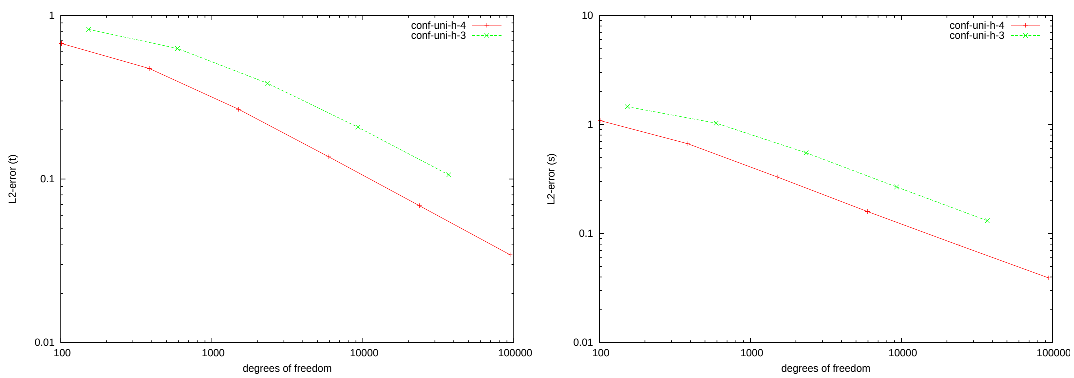


Figure 5.84: Stokes (2d-FEM): $\|t - t_n\|_{L^2(\Omega)}$ (left) and $\|\sigma - \sigma_n\|_{L^2(\Omega)}$ (right).

Example 5.29. *Here we investigate the linear Stokes problem*

$$
\begin{aligned}
-\nu \operatorname{div}(\nabla \vec{u}) + \nabla p &= f \ \text{ in } \Omega \\
\operatorname{div} \vec{u} &= 0 \ \text{ in } \Omega \\
\vec{u} &= u_0 \ \text{ on } \partial\Omega
\end{aligned}
$$

using a dual-dual mixed formulation.

Find $(t, (\sigma, p), (u, \gamma, \xi)) \in X_1 \times M_1 \times M$ *such that*

$$
\begin{array}{llll}
2\nu \int_\Omega t : s & -\int_\Omega \sigma : s - \int_\Omega p \ \mathrm{tr}(s) & & = 0 \\
-\int_\Omega \tau : t - \int_\Omega q \,\mathrm{tr}(t) & & -\int_\Omega u \cdot \operatorname{div}\tau - \int_\Omega \gamma : \tau + \xi \int_\Omega \mathrm{tr}(\tau) & = -\langle \tau n, u_0 \rangle_\Gamma \\
& -\int_\Omega v \cdot \operatorname{div}\sigma + \eta \int_\Omega \mathrm{tr}(\sigma)\, dx - \int_\Omega \sigma : \delta & & = \int_\Omega f \cdot v,
\end{array}
$$

for all $(s, (\tau, q), (v, \delta, \eta)) \in X_1 \times M_1 \times M$ *with* $X_1 = [L^2(\Omega)]^{2\times 2}$, $M_1 = H(\mathrm{div};\Omega) \times L^2(\Omega)$, $M = [L^2(\Omega)]^2 \times H_0 \times \mathbb{R}$.

fem2/ex62h3in

```
! Stokes on Square (dual-dual-mixed, stress tensor)
open(1) 'test.h'; open(2) 'ex62h3in.dat'; #ti

problem('Stokes',nickname='DDMIX2')
geometry('Square')
NU=4.; #stokes NU
R=10
setstokesx( (/ 0.0,1.5 /) )
J=2
do I=0,8
 mesh('uniform',n=J,p=0,elements='triangles',spline='u',gm='ug',genspl='no')
 mesh('global',n=1,spline='xi',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix
 lft 16 R 0 R
 solve(eps=1.e-10,mdc='diag',mdi='x=1',mit='MINRES',quiet=0,restart=400); T=SEC
 #rno.
 show('matrix')
 #taf. 'u'; #px. 'u'; #cx. 'u'
 #taf. 'p'; #px. 'p'; #cx. 'p'
 #taf. 'xi'; #px. 'xi'; #cx. 'xi'
 #err. 16 R 'L2' 0 'u' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u' 'u'; E[2]=ERR
 #err. 16 R 'L2' 0 'p' 'p'; E[3]=ERR
 #err. 16 R 'L2' 0 't' 'tstrain'; E[4]=ERR
 #err. 16 R 'L2' 0 'sigma' 'sigma' ; E[5]=ERR
 #err. 16 R 'Hdiv' 0 'sigma' 'sigma' ; E[6]=ERR

 write(2) DOF,E[1],E[2],E[3],E[4],E[5],E[6],T,ITER,RNORM
 J=J*2
continue
end
```

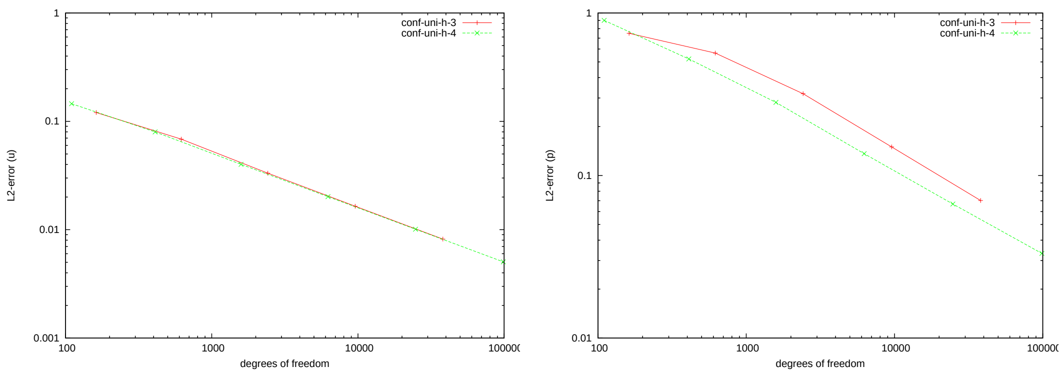


Figure 5.85: Stokes (2d-FEM): $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|p - p_n\|_{L^2(\Omega)}$ (right).

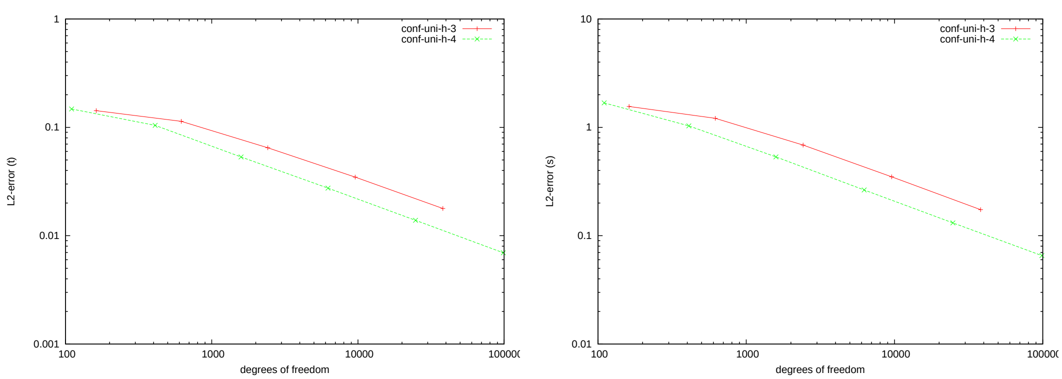
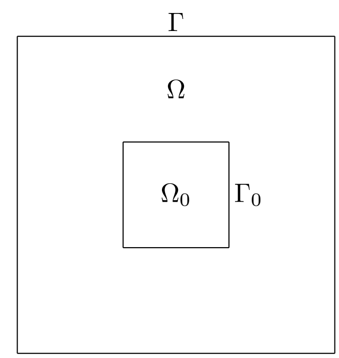


Figure 5.86: Stokes (2d-FEM): $\|t - t_n\|_{L^2(\Omega)}$ (left) and $\|\sigma - \sigma_n\|_{L^2(\Omega)}$ (right).

Example 5.30.

Here we use the formulation of the previous example 5.29 on a domain with a square hole and the singularity inside.
The exact solution of our model problem is given by $\nu = 4$ *and* $\vec{u} = \frac{1}{\nu}(-\log r + (x_0 - \bar{x}_0)^2/r^2, (x_0 - \bar{x}_0)(x_1 - \bar{x}_1)/r^2)$ *and* $p = 2(x_0 - \bar{x}_0)/r^2$ *with* $r = |x - \bar{x}|$ *and* $\bar{x} = (0, 0.5)$*, such that* $f \equiv 0$*.*



Square domain with hole

fem2/ex63h3in

```
! Stokes on Square with Hole (dual-dual-mixed, stress tensor)
open(1) 'test.h'; open(2) 'ex63h3in.dat'; #ti

problem('Stokes',nickname='DDMIX2')
geometry('Square with Hole')
NU=4.; #stokes NU
R=11
J=2
do I=0,8
 mesh('uniform',n=J,p=0,elements='triangles',spline='u',gm='ug',genspl='no')
 mesh('global',n=1,spline='xi',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix
 lft 16 R 0 R
 solve(eps=1.e-10,mdc='diag',mdi='x=1',mit='MINRES',quiet=0,restart=400); T=SEC
 #rno.
 show('matrix')
 #taf. 'u'; #px. 'u'; #cx. 'u'
 #taf. 'p'; #px. 'p'; #cx. 'p'
 #taf. 'xi'; #px. 'xi'; #cx. 'xi'
 #err. 16 R 'L2' 0 'u' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u' 'u'; E[2]=ERR
 #err. 16 R 'L2' 0 'p' 'p'; E[3]=ERR
 #err. 16 R 'L2' 0 't' 'tstrain'; E[4]=ERR
 #err. 16 R 'L2' 0 'sigma' 'sigma' ; E[5]=ERR
 #err. 16 R 'Hdiv' 0 'sigma' 'sigma' ; E[6]=ERR

 write(2) DOF,E[1],E[2],E[3],E[4],E[5],E[6],T,ITER
 J=J*2
continue
end
```

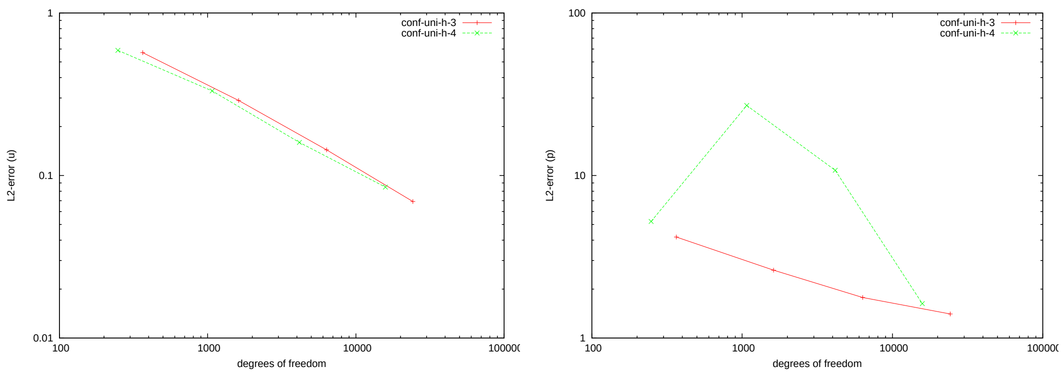


Figure 5.87: Stokes (2d-FEM): $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|p - p_n\|_{L^2(\Omega)}$ (right).

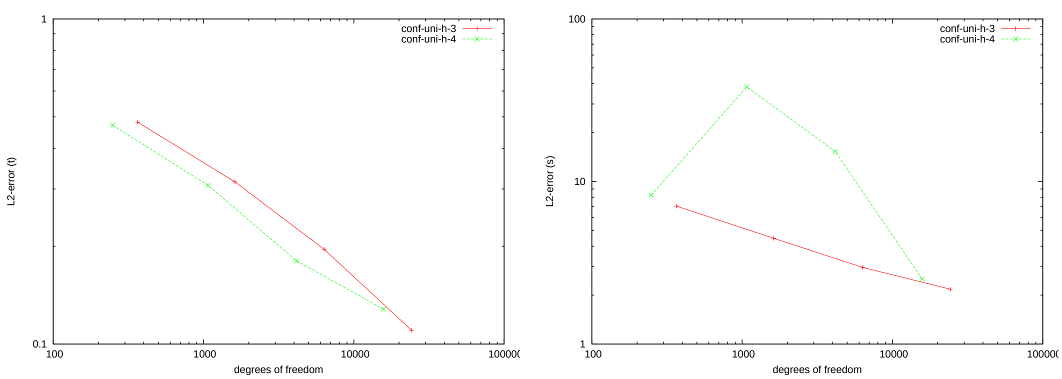


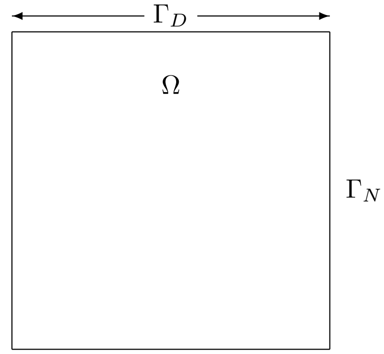
Figure 5.88: Stokes (2d-FEM): $\|t - t_n\|_{L^2(\Omega)}$ (left) and $\|\sigma - \sigma_n\|_{L^2(\Omega)}$ (right).

Example 5.31. *Here we investigate the linear Stokes problem with mixed boundary data*

Let $f \in [L^2(\Omega)]^2,\ t_0 \in [H^{-1/2}(\Gamma_N)]^2$, $u_0 \in [H^{1/2}(\Gamma)]^2$

$$\begin{aligned} -\nu \operatorname{div}(\nabla u) + \nabla p &= f \ \textit{in}\ \Omega \\ \operatorname{div} u &= 0 \ \textit{in}\ \Omega \\ u &= u_0 \ \textit{on}\ \Gamma_D \\ \sigma \cdot n &= t_0 \ \textit{on}\ \Gamma_N \end{aligned}$$

using a dual-dual mixed formulation.



Square domain

Find $(t,(\sigma,p),(u,\gamma,\psi,\xi)) \in X_1 \times M_1 \times M$ *such that*

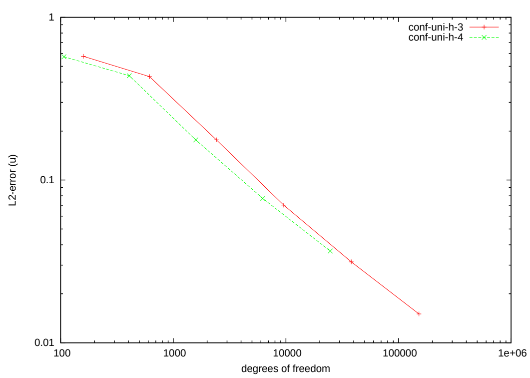
$$\begin{aligned}
&2\nu \int_\Omega t : s - \int_\Omega \sigma : s - \int_\Omega p \operatorname{tr}(s) = 0 \\
&-\int_\Omega \tau : t - \int_\Omega q \operatorname{tr}(t) - \int_\Omega u \cdot \operatorname{div} \tau - \int_\Omega \gamma : \tau + \xi \int_\Omega \operatorname{tr}(\tau) + \langle \tau n, \psi \rangle_{\Gamma_N} = -\langle \tau n, u_0 \rangle_\Gamma \\
&-\int_\Omega v \cdot \operatorname{div} \sigma + \eta \int_\Omega \operatorname{tr}(\sigma) - \int_\Omega \sigma : \delta + \langle \sigma n, \zeta \rangle_{\Gamma_N} = \int_\Omega f \cdot v + \langle t_0, \zeta \rangle_{\Gamma_N},
\end{aligned}$$

for all $(s,(\tau,q),(v,\delta,\zeta,\eta)) \in X_1 \times M_1 \times M$ *with* $X_1 = [L^2(\Omega)]^{2\times 2}$, $M_1 = H(\operatorname{div};\Omega) \times L^2(\Omega)$, $M = [L^2(\Omega)]^2 \times H_0 \times \tilde{H}^{1/2}(\Gamma_N) \times \mathbb{R}$.

fem2/ex64h3in

```
! Stokes on Square (dual-dual-mixed, stress tensor, Neumann, linear)
open(1) 'test.h'; open(2) 'ex64h3in.dat'; #ti

problem('Stokes',nickname='DDMIX2NEU')
geometry('Square',dim=(/3.,3./))
#pxbd 4 1 2 'ubd'
 0 2 -3. -3.  3. -3. -2
 0 2  3. -3.  3.  3. -2
 0 2  3.  3. -3.  3. -1
 0 2 -3.  3. -3. -3. -2
#pxg 3 1 2 'psig'
 0 2 -3  3 -3 -3 -1  0 0
 0 2 -3 -3  3 -3  0 -1 0
 0 2  3 -3  3  3  1  0 0

NU=4.; #stokes NU
R=10
setstokesx( (/0.0,4.0/) )
J=2
do I=0,8
 mesh('uniform',n=J,p=0,elements='triangles',spline='u',gm='ug',genspl='no')
 mesh('global',n=1,spline='xi',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix
```

```
 lft 16 R 0 R
 #rci. 'psi'
 defprec(mode='INVCG',spline='u',name='Pu',mat='Mu')
 defprec(mode='INVCG',spline='t',name='Pt',mat='Mt')
 defprec(mode='INVCG',spline='sigma',name='Psigma',mat='HD')
 defprec(mode='INVCG',spline='p',name='Pp',mat='Mp')
 defprec(mode='INVCG',spline='psi',name='Ppsi',mat='W')
 defprec(mode='ID',spline='xi',name='Pxi',mat='I')
 defprec(mode='INVCG',spline='g',name='Pg',mat='Mg')

 solve(eps=1.e-10,mdc='diag',mdi='x=1',mit='MINRES',quiet=0,restart=400); T=SEC
! solve(eps=1.e-10,mdc='u.Pu.u:t.Pt.t:sigma.Psigma.sigma:p.Pp.p:psi.Ppsi.psi:xi.Pxi.xi:g.Pg.g',mdi=
 #rno.
 show('matrix')
 #taf. 'u'; #px. 'u'; #cx. 'u'
 #taf. 'p'; #px. 'p'; #cx. 'p'
 #taf. 'xi'; #px. 'xi'; #cx. 'xi'
 #err. 16 R 'L2' 0 'u' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u' 'u'; E[2]=ERR
 #err. 16 R 'L2' 0 'p' 'p'; E[3]=ERR
 #err. 16 R 'L2' 0 't' 'tstrain'; E[4]=ERR
 #err. 16 R 'L2' 0 'sigma' 'sigma' ; E[5]=ERR
 #err. 16 R 'Hdiv' 0 'sigma' 'sigma' ; E[6]=ERR
 #no. 'L2' 'psi'; E[7]=NORM

 write(2) DOF,E[1],E[2],E[3],E[4],E[5],E[6],E[7],T,ITER,RNORM
 J=J*2
continue
end
```



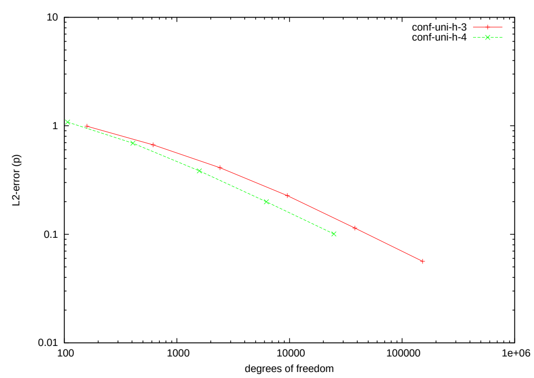
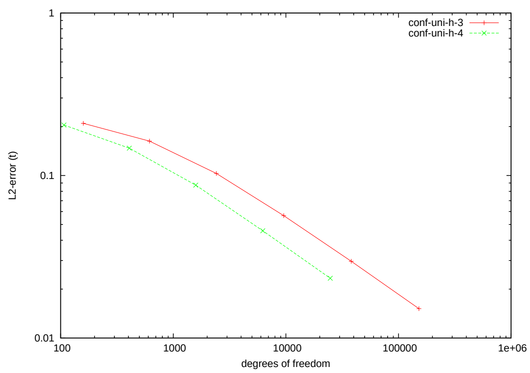


Figure 5.89: Stokes (2d-FEM, mixed b.c.): $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|p - p_n\|_{L^2(\Omega)}$ (right).



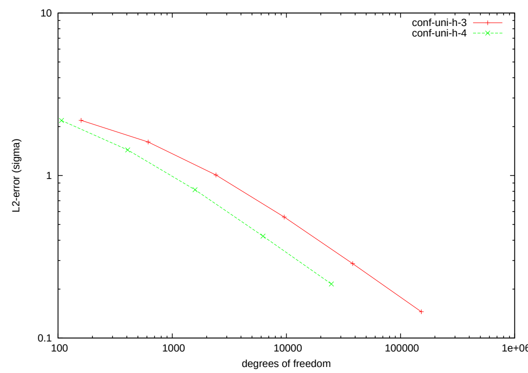


Figure 5.90: Stokes (2d-FEM, mixed b.c.): $\|t - t_n\|_{L^2(\Omega)}$ (left) and $\|\sigma - \sigma_n\|_{L^2(\Omega)}$ (right).

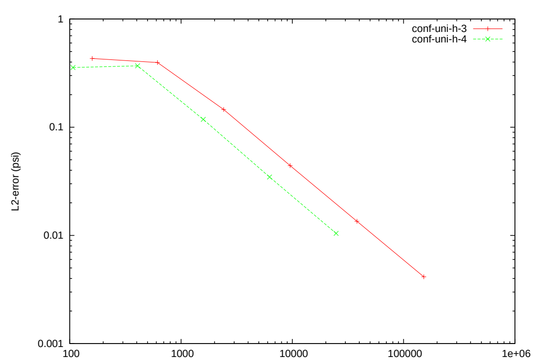


Figure 5.91: Stokes (2d-FEM, mixed b.c.): $\|\psi - \psi_n\|_{L^2(\Gamma_N)}$.

5.1.6 Transport

Example 5.32. *Here we investigate the Transport problem*

$$\beta \cdot \nabla u + c \cdot u = f \;\; in \; \Omega$$
$$u = u_0 \;\; on \; \Gamma_{inflow}$$

The variational formulation, using the Galerkin method, is:
Find $u_h \in V^h := \{u^h \in P_k \,|\, u^h = u_0 \;\; on \; \Gamma_{inflow}\}$, *such that*

$$(u_\beta^h + cu^h, v^h) = (f, v^h), \quad \forall v^h \in V_0^h = \{v^h \in P_k \,|\, v^h = 0 \;\; on \; \Gamma_{inflow}\}$$

In this example we use $\beta = (1, \tan 35°)$, $c = 1$, $u(x, y) = \exp(x)\sin(x)\sin(y)$, *see [4].*

fem2/ex99h3in

```
! Convection on Square with triangles
open(1) 'test.h'; open(2) 'ex99h3in.dat'
problem('Convection',nickname='PGCSBD'); #ti
#pxg 1 2 2 'ug'
0 4  0. 0.  1. 0.  1. 1.  0. 1.  0
#pxbd 2 1 2 'ubd'
0 2 0. 0. 1. 0. -2
0 2 0. 0. 0. 1. -2

R=0; J=2
do I=0,8
 mesh('uniform',n=J,p=1,elements='triangles',spline='u',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix
 lft 16 R 0 R
 solve(eps=1.e-10,mdc='no',mit='CGNR'); #rno.
 extend('u','u_bd','u_ex','Dirichlet')
 open(1) 'ex99h3_'//I
 #taf. 'u_ex'; #px. 'u_ex'; #cx. 'u_ex'
 close(1)
 #err. 16 R 'L2' 0 'u_ex' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u_ex' 'u'; E[2]=ERR
 write(2) DOF,E[1],E[2]
 J=J*2
continue
end
```

Example 5.33. *Similarly to Example 5.32 we investigate here the Transport Problem*

$$\beta \cdot \nabla u + c \cdot u = f \;\; in \; \Omega$$
$$u = u_0 \;\; on \; \Gamma_{inflow}$$

The variational formulation, using the SUPG method, reads:
Find $u_h \in V^h := \{u^h \in P_k \,|\, u^h = u_0 \;\; on \; \Gamma_{inflow}\}$, *such that*

$$(u_\beta^h + cu^h, v^h) + \sum_\Delta (u_\beta^h + cu^h, \frac{\xi_\Delta h_\Delta}{2|\beta_\Delta|} v_\beta^h)_\Delta = (f, v^h) + \sum_\Delta (f, \frac{\xi_\Delta h_\Delta}{2|\beta_\Delta|} v_\beta^h)_\Delta, \quad \forall v^h \in V_0^h$$

with $V_0^h = \{v^h \in P_k \,|\, v^h = 0 \;\; on \; \Gamma_{inflow}\}$. *In this example we use* $\beta = (1, \tan 35°)$, $c = 1$, $u(x, y) = \exp(x)\sin(x)\sin(y)$, *see [4].*

fem2/ex97h3in

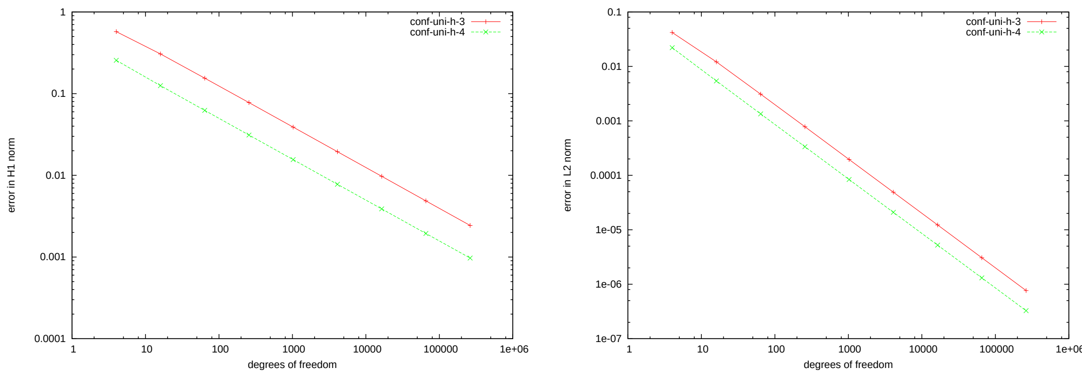


Figure 5.92: Error in $H^1(\Omega)$-norm (left) and in $L^2(\Omega)$-norm (right)

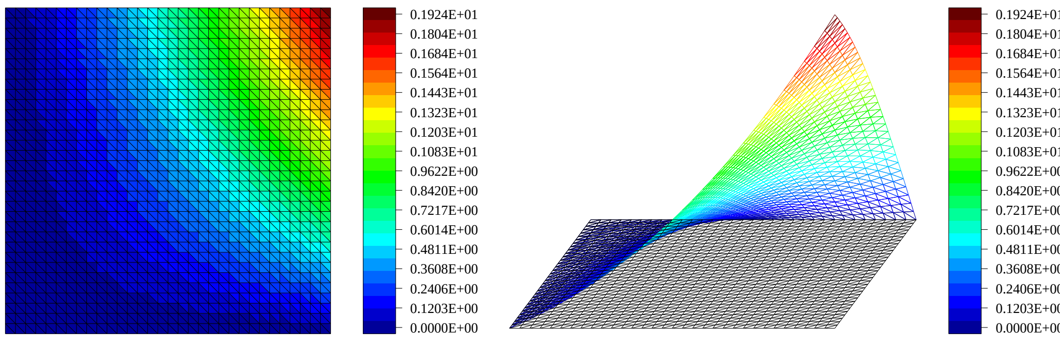


Figure 5.93: Plot of the Galerkin solution $u(x, y) = \exp(x)\sin(x)\sin(y)$

```
! Convection on Square with triangles, SUPG
open(1) 'test.h'; open(2) 'ex97h3in.dat'
problem('Convection',nickname='SUPGCSBD'); #ti
#pxg 1 2 2 'ug'
0 4  0. 0.  1. 0.  1. 1.  0. 1.  0
#pxbd 2 1 2 'ubd'
0 2 0. 0. 1. 0. -2
0 2 0. 0. 0. 1. -2

R=0; J=2
do I=0,8
 mesh('uniform',n=J,p=1,elements='triangles',spline='u',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix
 lft 16 R 0 R
 solve(eps=1.e-10,mdc='no',mit='CGNR'); #rno.
 extend('u','u_bd','u_ex','Dirichlet')
 open(1) 'ex97h3_'//I
 #lx. 'u'
 #taf. 'u_ex'; #px. 'u_ex'; #cx. 'u_ex'
 close(1)
 #err. 16 R 'L2' 0 'u_ex' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u_ex' 'u'; E[2]=ERR
 write(2) DOF,E[1],E[2]
 J=J*2
```

```
continue
end
```

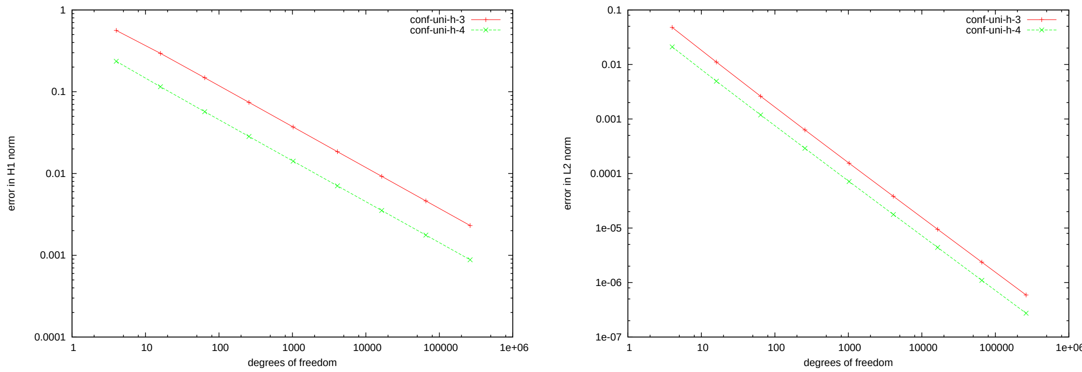


Figure 5.94: Error in $H^1(\Omega)$-norm (left) and in $L^2(\Omega)$-norm (right)

Example 5.34. *Similarly to Example 5.32 we investigate here the Transport Problem*

$$\beta \cdot \nabla u + c \cdot u = f \ \text{ in } \Omega$$
$$u = u_0 \ \text{ on } \Gamma_{inflow}$$

The variational formulation, using the Discontinous Galerkin method, reads: Find $u_h \in V^h := \{u_h \in L^2(\Omega) \, : \, u_h|_\tau \in P_k(\tau), \tau \in \mathcal{T}_h\}$, *such that*

$$\sum_{\tau \in \mathcal{T}_h} (u^h_\beta + u^h, v)_\tau - \sum_{\tau \in \mathcal{T}_h} \int_{\partial\tau_-} [u^h] v_+ n \cdot \beta \, ds = (f, v) \quad \forall v \in V^h$$

and $u^h_- = u_0$ *on* Γ_{inflow}.

$$a_h^{upw}(v_h, w_h) = \int_\Omega \{\mu v_h w_h + (\beta \cdot \nabla_h v_h) w_h\} + \int_\Gamma (\beta \cdot n)^\ominus v_h w_h$$
$$- \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) [\![v_h]\!] \{\{w_h\}\} + \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\gamma}{2} |\beta \cdot n_F| [\![v_h]\!] [\![w_h]\!]$$
$$a_h^{upw}(u_h, w_h) = \int_\Omega f w_h + \int_\Gamma (\beta \cdot n)^\ominus g w_h$$

Here we use

$$\partial\tau_- = \{x \in \partial\tau \, : \, n(x) \cdot \beta < 0\}$$
$$\partial\tau_+ = \{x \in \partial\tau \, : \, n(x) \cdot \beta \geq 0\}$$
$$v_-(x) = \lim_{s \to 0^-} v(x + s\beta)$$
$$v_+(x) = \lim_{s \to 0^+} v(x + s\beta)$$
$$[v] = v_+ - v_-$$

with $n(x)$ *being the outward unit normal to* $\partial\tau$.

We can define two graph norms

$$|v|_\beta^2 = \|v\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_\tau \int_{\partial\tau_-} [v]^2 |n \cdot \beta| \, ds + \frac{1}{2} \int_{\Gamma_+} v_-^2 n \cdot \beta \, ds$$
$$\|v\|_\beta^2 = |v|_\beta^2 + h \sum_\tau \|v_\beta\|_{L^2(\tau)}^2$$

Example 5.35. *Here we investigate Galerkin and SUPG method for a transport problem with a jump in the boundary data. We have*

$$r(s) = r_0 + s, \qquad \varphi(s) = \varphi_0 + s/k$$

and

$$\beta(x) = \frac{\partial x}{\partial s} = \partial_s(e_r(\varphi(s))r(s)) = e_\varphi r\frac{\partial \varphi}{\partial s} + e_r\frac{\partial r}{\partial s} = e_\varphi r/k + e_r$$

Finally we have

$$\begin{aligned} c = \operatorname{div}\beta &= (\operatorname{div} e_\varphi) r\frac{\partial \varphi}{\partial s} + e_\varphi \nabla r\frac{\partial \varphi}{\partial s} + e_\varphi r\nabla\frac{\partial \varphi}{\partial s} + \operatorname{div}(e_r)\frac{\partial r}{\partial s} + e_r\nabla\frac{\partial r}{\partial s} \\ &= e_\varphi r\nabla\frac{\partial \varphi}{\partial s} + \frac{1}{r}\frac{\partial r}{\partial s} + e_r\nabla\frac{\partial r}{\partial s} \\ &= \frac{1}{r} \end{aligned}$$

We prescribe $u(r_0,\varphi) = 1$ *for* $\varphi \in [-\pi/2, \pi/2]$ *and* $u(r_0,\varphi) = 2$ *for* $\varphi \in [-\pi, \pi/2] \cup [\pi/2, \pi]$. *We have* $r_0 = 1, r_1 = 2, k = 1$.

DOF	$\|u-u_h\|_{L^2(\Omega)}$	α	It	τ_{mat}	τ_{rhs}	τ_{solve}
16	0.6752845	—	11	0.0100000	0.0100000	0.0000000
64	0.5114712	-0.200421	57	0.0100000	0.0200000	0.0000000
256	0.3751443	-0.223604	294	0.0000000	0.0200000	0.0100000
1024	0.2754346	-0.222868	936	0.0300000	0.0700000	0.1200000
4096	0.2024423	-0.222100	2219	0.1400000	0.2300000	1.0300000
16384	0.1503251	-0.214712	4519	0.5000000	0.9500000	8.2700000
65536	0.1123843	-0.209823	8721	1.9800000	3.7900000	74.310000
262144	0.0842619	-0.207744	16459	7.9500000	15.150000	604.54000
1048576	0.0634182	-0.204992	30790	31.820000	60.440000	5397.2200

Table 5.42: $L^2(\Omega)$-norm (Galerkin)

DOF	$\|u-u_h\|_{L^2(\Omega)}$	α	It	τ_{mat}	τ_{rhs}	τ_{solve}
16	0.6481758	—	7	0.0000000	0.0000000	0.0000000
64	0.4694561	-0.232697	21	0.0000000	0.0200000	0.0000000
256	0.3464436	-0.219185	72	0.0100000	0.0400000	0.0000000
1024	0.2573060	-0.214568	153	0.0400000	0.1700000	0.0100000
4096	0.1930780	-0.207151	305	0.1700000	0.6400000	0.1300000
16384	0.1456847	-0.203167	587	0.6600000	2.6000000	1.0500000
65536	0.1105604	-0.199007	1108	2.7300000	10.310000	9.2600000
262144	0.0842005	-0.196467	2092	10.570000	40.940000	74.990000
1048576	0.0643251	-0.194224	3863	43.070000	166.02000	677.17000

Table 5.43: $L^2(\Omega)$-norm (SUPG)

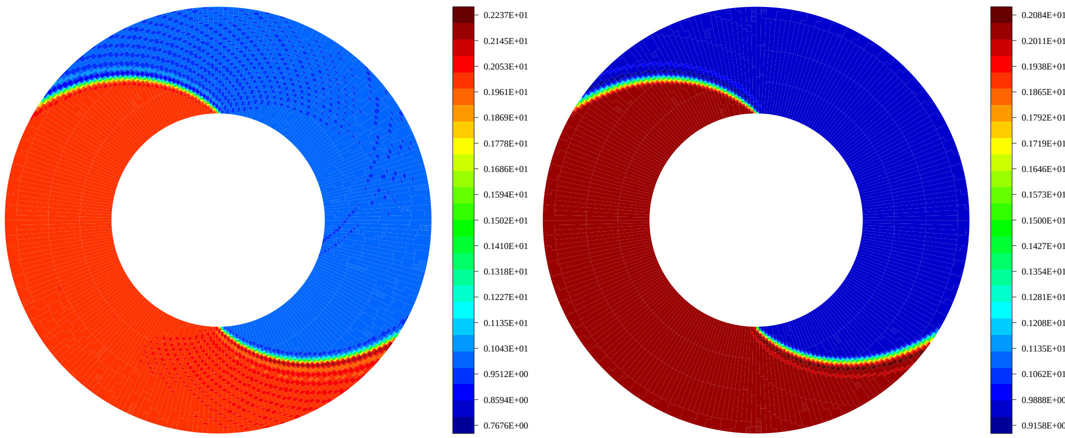


Figure 5.95: Galerkin (left) and SUPG (right)

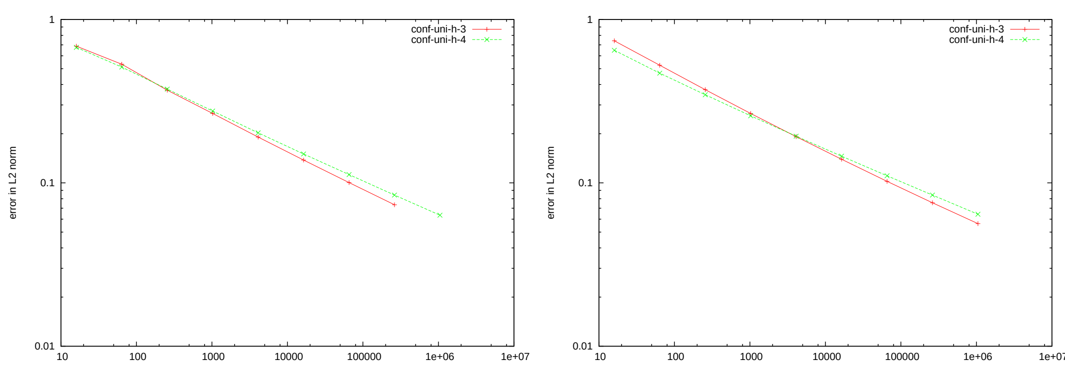


Figure 5.96: Galerkin (left) and SUPG (right)

Example 5.36. *Here we investigate the transport problem from Example 5.35 using a weak boundary condition. Using* $\langle u, v\rangle_- = \int_{\Gamma_-} uv\beta \cdot n\, ds$, *we obtain the variational Galerkin formulation:*
Find $u_h \in V^h := \{u^h \in P_k\}$, *such that*

$$(u^h_\beta + cu^h, v^h) - \langle u^h, v^h\rangle_- = (f, v^h) - \langle u_0, v^h\rangle_-, \quad \forall v^h \in V^h$$

DOF	$\|u-u_h\|_{L^2(\Omega)}$	α	It	τ_{mat}	τ_{rhs}	τ_{solve}
24	0.5305024	—	11	0.0000000	0.0000000	0.0000000
80	0.4170681	-0.199818	40	0.0000000	0.0000000	0.0000000
288	0.3150784	-0.218925	206	0.0100000	0.0100000	0.0100000
1088	0.2394548	-0.206493	665	0.0300000	0.0600000	0.0900000
4224	0.1822904	-0.201088	1568	0.1300000	0.2400000	0.7500000
16640	0.1389060	-0.198248	3153	0.5000000	0.9400000	5.9600000
66048	0.1060655	-0.195667	6135	2.0000000	3.7900000	58.420000
263168	0.0809211	-0.195732	11609	8.3700000	15.330000	465.18000

Table 5.44: $L^2(\Omega)$-norm (Galerkin)

DOF	$\|u-u_h\|_{L^2(\Omega)}$	α	It	τ_{mat}	τ_{rhs}	τ_{solve}
24	0.5722828	—	10	0.0040000	0.0040010	0.0000000
80	0.4299614	-0.237495	27	0.0040000	0.0080000	0.0000000
288	0.3238759	-0.221194	78	0.0080010	0.0280020	0.0080000
1088	0.2444501	-0.211678	159	0.0400030	0.1200070	0.0200010
4224	0.1854443	-0.203663	310	0.1480090	0.4960310	0.1440090
16640	0.1412698	-0.198452	599	0.6160370	1.9721240	1.1040690
66048	0.1079813	-0.194922	1116	2.3441460	7.9804980	10.088631
263168	0.0827254	-0.192730	2100	9.4605910	31.733983	81.205074

Table 5.45: $L^2(\Omega)$-norm (SUPG)

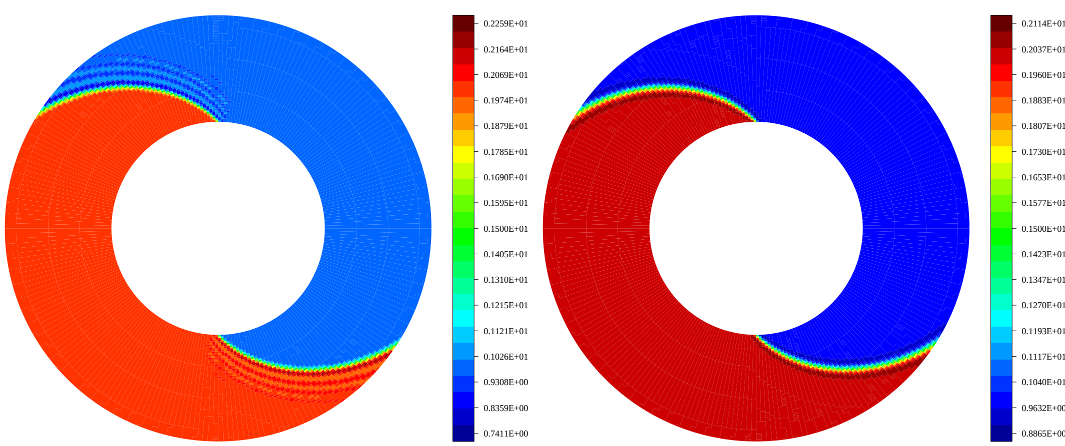


Figure 5.97: Galerkin (left) and SUPG (right)

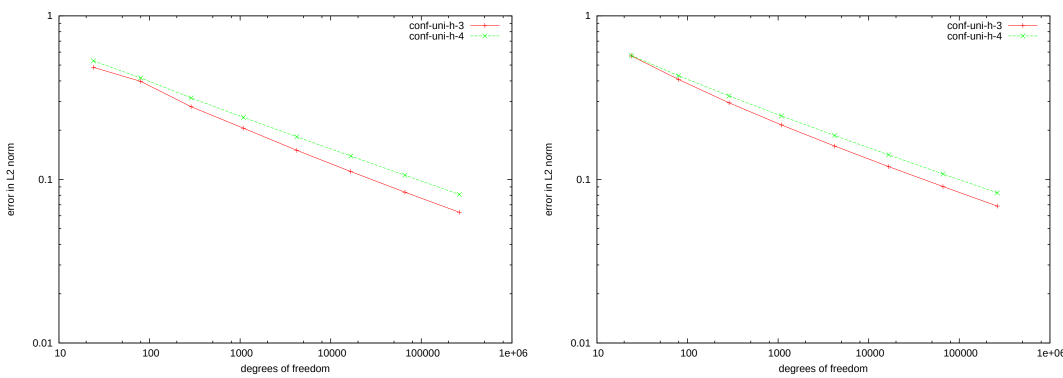


Figure 5.98: Galerkin (left) and SUPG (right)

Example 5.37. *Here we investigate the coupled Poisson-Transport problem*

$$\begin{aligned}
-\Delta u &= \rho \ \textit{in}\ \Omega \\
\operatorname{div}(E \cdot \rho) &= 0 \ \textit{in}\ \Omega \\
E &= -\nabla u \ \textit{in}\ \Omega \\
\rho &= \rho_0 \ \textit{on}\ \Gamma_{\textit{inflow}} \\
u &= u_0 \ \textit{on}\ \Gamma_{\textit{inflow}} \\
u &= u_1 \ \textit{on}\ \Gamma_{\textit{outflow}}
\end{aligned}$$

We use the Galerkin method for both the Poisson-equation and the transport equation. We have $u_0 = 4, u_1 = 1, \rho_0 = 2$. *The annular domain has the radii* $r_0 = 1, r_1 = 2$.

The solution algorithm is based on the fixed-point iteration

$$\begin{aligned}
\rho^0 &= 0 \ \textit{in}\ \Omega \\
-\Delta u^n &= \rho^n \ \textit{in}\ \Omega \\
u^n &= u_0 \ \textit{on}\ \Gamma_{\textit{inflow}} \\
u^n &= u_1 \ \textit{on}\ \Gamma_{\textit{outflow}} \\
(-\nabla u^n)\nabla \rho^{n+1} + \rho^n \rho^{n+1} &= 0 \ \textit{in}\ \Omega \\
\rho^{n+1} &= \rho_0 \ \textit{on}\ \Gamma_{\textit{inflow}}
\end{aligned}$$

fem2/ex93h4in

```
! Elliptic-Convection on Annular Domain with quadrilaterals, Gal-Gal
open(1) 'test.h'; open(2) 'ex93h4in.dat'; open(3) 'ex93h4int.dat'; #ti

geometry('Annular')
problem('Laplace',nickname='FEMrhoNHD',pnum=1)
problem('Convection',nickname='PGCuSBD',pnum=2)
#pxbd 4 1 2 'ubd'
1 3  0.0  1.0  0.707107  0.707107 1.0 0.0  -2
1 3  0.0 -1.0  0.707107 -0.707107 1.0 0.0  -2
1 3 -1.0 0.0 -0.707107 -0.707107  0.0 -1.0 -2
1 3 -1.0 0.0 -0.707107  0.707107  0.0  1.0 -2

setehc(u0=4.0,u1=1.0,rho0=2.0,r0=1.0,r1=2.0,epsilon0=1.0)

R=80; EPS=1.0d-12; EEPS=1.0d-8
J=2
do I=0,8
 TM1=0;TL1=0;TS1=0;WM1=0;WL1=0;WS1=0; TM2=0;TL2=0;TS2=0;WM2=0;WL2=0;WS2=0
 #time T1 W1
 switch_problem(pnum=1)
 mesh('graded',n=J,p=2,beta=1.0,elements='rectangles',spline='u',gm='ug')
 switch_problem(pnum=2)
 genspl

 switch_problem(pnum=1)
 approx 0 R 'u_bd' 'u0'  ! Laplace boundary condition
 switch_problem(pnum=2)
 approx 0 R 'rho_bd' 'u0' ! Transport inflow
```

```
 clear('rho_ex'); clear('u_ex')
 NCNT=0 ; ITP=0; ITC=0
 do K=0,3000
  switch_problem(pnum=1)  ! Start Laplace
  matrix; TM1=TM1+SEC; WM1=WM1+WSEC
  lft 16 R 0 R; TL1=TL1+SEC; WL1=WL1+WSEC
  solve(eps=EPS,mdc='diag',mit='CG',quiet=0); TS1=TS1+SEC; WS1=WS1+WSEC
  #rno.; ITP=Max(ITER,ITP)
  eval('uold=u_ex')
  extend('u','u_bd','u_ex','Dirichlet')

  switch_problem(pnum=2) ! Switch to transport
  matrix; TM2=TM2+SEC; WM2=WM2+WSEC
  lft 16 R 0 R; TL2=TL2+SEC; WL2=WL2+WSEC
  solve(eps=EPS,mit='CGNE',quiet=0); TS2=TS2+SEC; WS2=WS2+WSEC
  #rno.; ITC=Max(ITC,ITER)
  eval('rhoold=rho_ex')
  extend('rho','rho_bd','rho_ex','Dirichlet')
! check for convergence
  norm('NU','H1','uold','u_ex')
  norm('NR','L2','rhoold','rho_ex')
  FIX=Sqrt(NU*NU+NR*NR)
  write(0) 'FIX Iterations',FIX,NCNT
  NCNT=NCNT+1
  if (FIX<=EEPS); then
   exit
  fi
 continue

 open(1) 'ex93h4_'//I
 #taf. 'u_ex'; #pnod. 'u_ex'; #cx. 'u_ex'
 #taf. 'rho_ex'; #pnod. 'rho_ex'; #cx. 'rho_ex'; #cx. 'rho_bd'
 close(1)
 switch_problem(pnum=1)
 #err. 16 R 'L2' 0 'u_ex' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u_ex' 'u'; E[2]=ERR
 switch_problem(pnum=2)
 #err. 16 R 'L2' 0 'rho_ex' 'u'; E[3]=ERR
 #err. 16 R 'H1' 0 'rho_ex' 'u'; E[4]=ERR

 write(2) DOF,E[1],E[2],E[3],E[4],NCNT,FIX,NU,NR,ITP,ITC
 #time T2 W2
 write(3) DOF,TM1,TL1,TS1,TM2,TL2,TS2,T2-T1,WM1,WL1,WS1,WM2,WL2,WS2,W2-W1
 J=J*2
continue
end
```

DOF	$\|u-u_h\|_{H^1(\Omega)}$	α	$\|\rho-\rho_h\|_{L^2(\Omega)}$	α	It_{FP}
16	1.4431768	—	0.0948688	—	13
64	0.4698718	-0.809454	0.0247641	-0.968842	13
256	0.1585214	-0.783795	0.0062938	-0.988123	13
1024	0.0547148	-0.767337	0.0015855	-0.994497	13
4096	0.0191126	-0.758704	0.0003978	-0.997411	13
16384	0.0067169	-0.754328	.9963E-04	-0.998696	13
65536	0.0023676	-0.752185	.2493E-04	-0.999349	13
262144	0.0008358	-0.751098	.6234E-05	-0.999826	13
1048576	0.0002953	-0.750488	.1559E-05	-0.999769	13

Table 5.46: Convergence of the coupled Poisson-Transport problem (quadrilaterals)

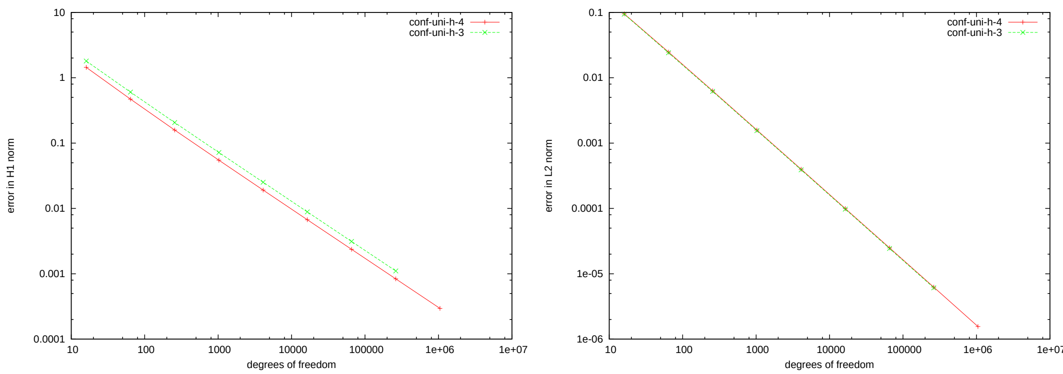


Figure 5.99: Poisson-problem (left) and Transport-problem (right)

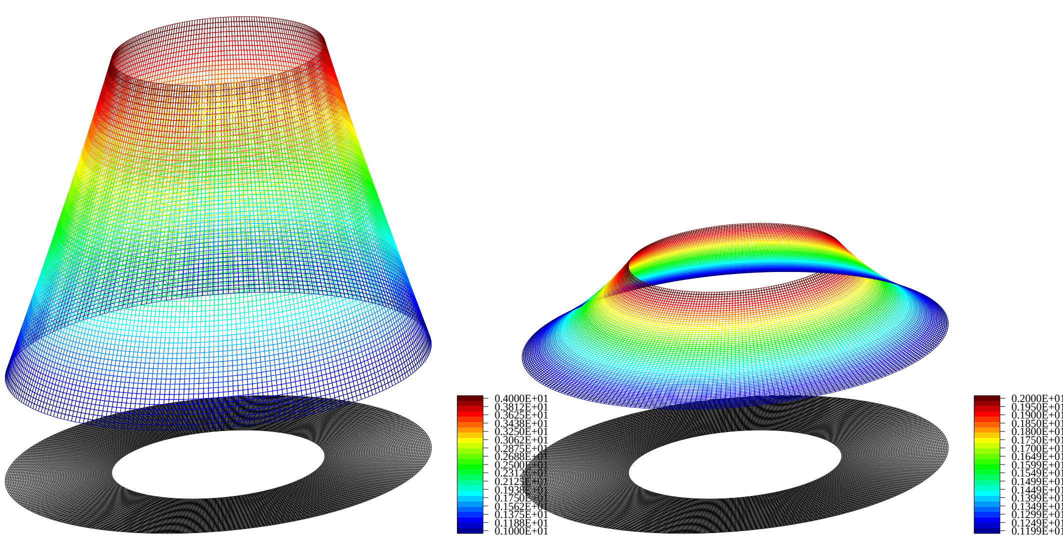


Figure 5.100: Poisson (left) and Transport (right)

5.1.7 Heat-Equation

Example 5.38. *Let $\Omega \subset \mathbb{R}^2$. This example deals with the Heat equation*

$$\begin{aligned} \partial_t u(x,t) - \Delta u(x,t) &= f(x,t) \quad \text{in } \Omega \times (0,T] \\ u(x,0) &= u_0(x), \quad x \in \Omega \\ u(x,t) &= u_D(x,t), \quad (x,t) \in \Gamma_D \times (0,T] \\ \frac{\partial u(x,t)}{\partial n} &= u_N(x,t), \quad (x,t) \in \Gamma_N \times (0,T] \end{aligned}$$

We choose Backward Euler discretization in time. $t_n = \Delta t \cdot n$. Let $H^1_{D,t_n} := \{v \in H^1(\Omega) : v|_{\Gamma_D} = u_D(x,t_n)\}$. Then for $n \geq 0$, $u^n \in H^1_{D,t_n}$ being known, we obtain $u^{n+1} \in H^1_{D,t_{n+1}}$ from

$$\frac{1}{\Delta t}\int_\Omega (u^{n+1} - u^n)v\,dx + \int_\Omega \nabla u^{n+1}\nabla v\,dx = \int_\Omega f \cdot v\,dx \quad \forall v \in H^1_0(\Omega).$$

Here we choose $\Omega = [-1,1]^2$, $\Gamma_D = \partial\Omega$ and $u(x,t) = -(1-x_1^2)(1-x_2^2)(1-\exp(-t))$, $T = 10$.

fem2/ex34h4in

```
! Heat-equation with homogeneous Dirichlet-conditions
open(1) 'test'; open(2) 'ex34h4in.dat'
geometry('Square'); #ti
EPS=1.0d-10
R=71
problem('Laplace',nickname='HEATHD')
DTS=0.1; TMAX=1.0
J=16
do I=1,5
 #time T1
 T=0; TL2=0; TH1=0
 mesh('uniform',n=J,p=1,elements='rectangles')
 DTS=TMAX/DOF*3; TCNT=Iint(TMAX/DTS); DT=TMAX/TCNT
 matrix('analytic')
 #settime T
 approx 0 R 'u'
 #err. 4 R 'L2' 0 'u'; E[0]=ERR
 #err. 4 R 'H1' 0 'u'; E[1]=ERR
 do K=1,TCNT
  eval('uo=u'); AL2=E[0]**2; AH1=E[1]**2
  T=T+DT
  lft 16 R - R
  solve(eps=EPS,mti='x=0',mdc='no',mit='CG'); #rno.
  #err. 2 R 'L2' 0 'u'; E[0]=ERR
  #err. 2 R 'H1' 0 'u'; E[1]=ERR
  #no. 'H1' 'u'
  TL2=TL2+(AL2+E[0]**2)/2*DT
  TH1=TH1+(AH1+E[1]**2)/2*DT
 continue
 #time T2
 TDIFF=T2-T1
 TL2=Sqrt(TL2); TH1=Sqrt(TH1)
 write(2) DOF,I,TL2,TH1,TDIFF
 J=J*2
continue
end
```

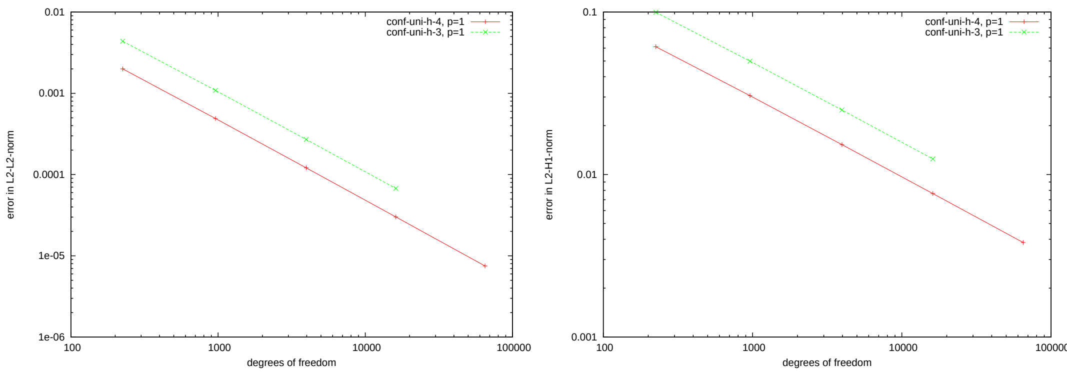


Figure 5.101: $\|u-u_n\|_{L^2([0,T],L^2(\Omega))}$ (left) and $\|u-u_n\|_{L^2([0,T],H^1(\Omega))}$ (right) — Heat equation.

Example 5.39. *Here we rewrite the PDE from example 5.38 as a mixed Problem. After Backward-Euler discretization in time we first obtain the first order system*

$$\begin{aligned} \frac{u_i(x)-u_{i-1}(x)}{\Delta t} - \operatorname{div} p_i &= f(x,t_i) \ \text{ in } \Omega \\ p_i &= \nabla u_i \ \text{ in } \Omega \\ u_i(x) &= u_D(x,t_i), \quad x \in \Gamma_D \\ p \cdot n &= u_N(x,t_i), \quad x \in \Gamma_N \end{aligned}$$

Example 5.40. *Here we rewrite the PDE from example 5.38 as a Least-Squares Problem. After Backward-Euler discretization in time we first obtain the first order system*

$$\begin{aligned}
\frac{u_i(x)-u_{i-1}(x)}{\Delta t} - \operatorname{div} p_i &= f(x,t_i) \ \textit{in } \Omega \\
p_i &= \nabla u_i \ \textit{in } \Omega \\
u_i(x) &= u_D(x,t_i), \quad x \in \Gamma_D \\
p \cdot n &= u_N(x,t_i), \quad x \in \Gamma_N \\
u_0(x) &= u(x,0), \quad x \in \Omega
\end{aligned}$$

Introducing the spaces

$$\begin{aligned}
H^1_{D,t_i} &:= \{u \in H^1(\Omega) \,:\, u|_{\Gamma_D} = u_D(\cdot,t_i)\} \\
H_{N,t_i}(\operatorname{div};\Omega) &:= \{p \in H(\operatorname{div};\Omega) \,:\, p\cdot n|_{\Gamma_N} = u_N(\cdot,t_i)\}
\end{aligned}$$

we obtain the Least-Squares minimization problem

$$(u_i,p_i) = \operatorname*{minarg}_{(v,q)\in H^1_{D,t_i}\times H_{N,t_i}(\operatorname{div};\Omega)} \mathcal{F}(v,q;u_{i-1},f)$$

with

$$\mathcal{F}(v,q;u_{i-1},f) := \|\frac{v(x)-u_{i-1}(x)}{\Delta t} - \operatorname{div} q - f(x,t_i)\|^2_{L^2(\Omega)} + \|q-\nabla v\|^2_{L^2(\Omega)}$$

The variational formulation now reads: Find $(u_i,p_i) \in H^1_{D,t_i} \times H_{N,t_i}(\operatorname{div};\Omega)$ *s.t.*

$$\begin{aligned}
\left(\frac{1}{\Delta t}u_i, \frac{1}{\Delta t}v\right) + (\nabla u_i, \nabla v) - \left(\operatorname{div} p_i, \frac{1}{\Delta t}v\right) - (p_i,\nabla v) &= \left(\frac{1}{\Delta t}u_{i-1}+f, \frac{1}{\Delta t}v\right) \\
-\left(\frac{1}{\Delta t}u_i, \operatorname{div} q\right) - (\nabla u_i, q) + (p_i,q) + (\operatorname{div} p_i, \operatorname{div} q) &= -\left(\frac{1}{\Delta t}u_{i-1}+f, \operatorname{div} q\right)
\end{aligned}$$

for all $(v,q) \in H^1_{D,t_i,0} \times H_{N,t_i,0}(\operatorname{div};\Omega)$.

fem2/ex68h3in

```
! Heat-equation with homogeneous Dirichlet-conditions, Least Squares
geometry('Square'); #ti
EPS=1.0d-10
R=71; Q=4
problem('Laplace',nickname='HEATLS2HD')
DTS=0.1; TMAX=1.0
J=2
open(2)'ex68h3in.dat'
do I=1,6
 #time T1
 T=0; TUL2=0; TUH1=0; TPL2=0; TPHD=0
 mesh('uniform',n=J,p=1,elements='triangles')
 DTS=TMAX/DOF*3; TCNT=Iint(TMAX/DTS); DT=TMAX/TCNT
 matrix('analytic')
 show('matrix')
 #settime T
 approx 0 R 'u'
 approx 1 R 'p'
 #err. Q R 'L2' 0 'u';   E[0]=ERR
 #err. Q R 'H1' 0 'u';   E[1]=ERR
```

```
 #err. Q R 'L2' 0 'p';   E[2]=ERR
 #err. Q R 'Hdiv' 0 'p'; E[3]=ERR
 do K=1,TCNT
  eval('uo=u')
  UL2=E[0]**2; UH1=E[1]**2; PL2=E[2]**2; PHD=E[3]**2
  T=T+DT
  #settime T
  lft 8 R - R
  solve(eps=EPS,mdi='x=0',mdc='no',mit='CG'); #rno.
  #err. Q R 'L2' 0 'u';   E[0]=ERR
  #err. Q R 'H1' 0 'u';   E[1]=ERR
  #err. Q R 'L2' 0 'p';   E[2]=ERR
  #err. Q R 'Hdiv' 0 'p'; E[3]=ERR
  TUL2=TUL2+(UL2+E[0]**2)/2*DT
  TUH1=TUH1+(UH1+E[1]**2)/2*DT
  TPL2=TPL2+(PL2+E[2]**2)/2*DT
  TPHD=TPHD+(PHD+E[3]**2)/2*DT
 continue
 #time T2; TDIFF=T2-T1
 TUL2=Sqrt(TUL2); TUH1=Sqrt(TUH1); TPL2=Sqrt(TPL2); TPHD=Sqrt(TPHD)
 write(2) DOF, DOFU, DOFP, TUL2, TUH1, TPL2, TPHD, TDIFF
 J=J*2
continue
end
```

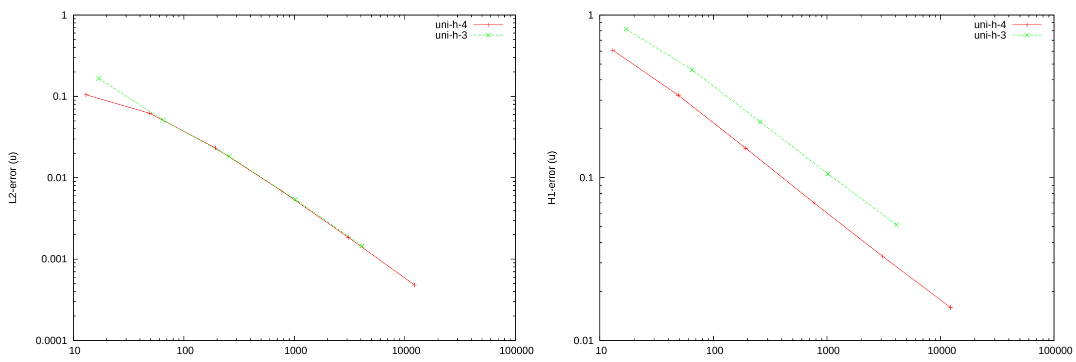


Figure 5.102: Heat equation (L2-Least Squares): $\|u - u_n\|_{L^2((0,T),L^2(\Omega))}$ (left) and $|u - u_n|_{L^2((0,T),H^1(\Omega))}$ (right).

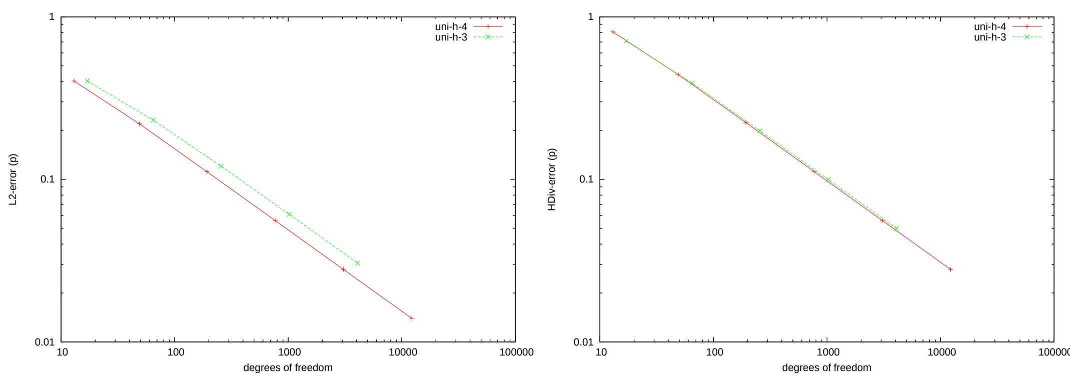


Figure 5.103: Heat equation (L2-Least Squares): $\|p - p_n\|_{L^2((0,T),L^2(\Omega))}$ (left) and $|p - p_n|_{L^2((0,T),H(\mathrm{div};\Omega))}$ (right).

5.2 Solvers

Here we investigate the performance of different solvers and preconditioners for h- and p-version for the 2d-FEM.

Example 5.41. *Using the configuration of example 5.2 for the Poisson-equation with the uniform h-version with rectangles we apply the multigrid-algorithm with V-cycle and one pre- and one post-smoothing step (damped Jacobi $\omega = 0.5$). The iteration stops if the last relative change of the iterate is less than 10^{-10}.*

fem2/ex3mcgin

```
! h-version, multigrid, Laplace
open(1) 'test.mcg' ; open(2) 'ex3mcgin.dat' ; open(3) 'ex3mcg.tex'
geometry('L-Shape'); #ti
problem('Laplace',nickname='FEMHD')
EPS=1.0d-10
R=15; J=4
do I=0,9
mesh('uniform',n=J,p=1,elements='rectangles')
matrix
lft 4 R 0 R
defprec(mode='MG',spline='u',name='Pu',mat='A',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)

solve(eps=EPS,mdi='x=1',mdc='u.Pu.u',mit='CG'); T=SEC
#rno.
#hno. 0.46268307
write(2) DOF,I,LMIN,LMAX,COND,T,ITER
write(3) DOF//'&'//LMIN:7//'&'//LMAX:6//'&'//COND:6//'&'//ITER//'&'//T:6//'\\'
J=J*2
continue
end
```

	Multigrid				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
5	1.00000	1.0000	1.0000	2	0.0000
33	0.59788	1.0000	1.6726	11	0.0000
161	0.57101	1.0000	1.7513	13	0.0100
705	0.54788	1.0000	1.8252	14	0.0100
2945	0.53050	1.0000	1.8849	14	0.0300
12033	0.51721	0.9999	1.9332	14	0.1600
48641	0.50682	0.9997	1.9726	14	0.9500
195585	0.49855	0.9996	2.0049	15	8.6500
784385	0.49189	0.9990	2.0309	15	108.86

Table 5.47: Conjugate Gradients with multigrid-preconditioner

	BPX				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
5	1.89907	3.1009	1.6329	4	0.0000
33	1.82446	5.2617	2.8840	14	0.0000
161	1.74262	6.6642	3.8242	20	0.0000
705	1.67990	7.7159	4.5931	23	0.0100
2945	1.63197	8.5485	5.2381	25	0.0200
12033	1.59420	9.2337	5.7921	26	0.1100
48641	1.56385	9.8082	6.2718	27	0.8400
195585	1.53911	10.295	6.6888	28	7.8300
784385	1.51871	10.710	7.0521	28	108.85
3141633	1.50173	11.066	7.3690	29	1727.5

Table 5.48: Conjugate Gradients with BPX-preconditioner

	Hierarchical				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
5	1.89907	3.1009	1.6329	4	0.0000
33	0.88088	4.1143	4.6707	15	0.0000
161	0.58889	4.9776	8.4526	29	0.0100
705	0.42315	5.7729	13.643	37	0.0100
2945	0.31965	6.4166	20.074	46	0.0700
12033	0.24972	6.9366	27.777	54	0.3800
48641	0.20035	7.3623	36.748	60	2.6300
195585	0.16421	7.7139	46.977	69	25.800
784385	0.13700	8.0071	58.444	75	366.25

Table 5.49: Conjugate Gradients with Hierarchical-preconditioner

	CG				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
5	1.89907	3.1009	1.6329	4	0.0000
33	0.57272	3.6887	6.4406	14	0.0000
161	0.14914	3.9170	26.265	30	0.0000
705	0.03760	3.9790	105.82	61	0.0000
2945	0.00941	3.9947	424.33	121	0.0200
12033	0.00235	3.9987	1698.8	240	0.1900
48641	0.00059	3.9997	6797.3	472	2.1500
195585	0.00015	3.9999	.3E+05	927	20.440
784385	.37E-04	4.0000	.1E+06	1791	167.32
3141633	.92E-05	4.0000	.4E+06	3572	1358.5

Table 5.50: Conjugate Gradients without preconditioner

Example 5.42. *For the Lamé-equation with $E = 20000$ and $\sigma = 0.3$ on a square domain with the uniform h-version with rectangles we apply the multigrid-algorithm with V-cycle and one pre- and one post-smoothing step (damped Jacobi $\omega = 0.5$). The iteration stops if the last relative change of the iterate is less than 10^{-10}.*

fem2/ex13mcgin

```
! h-version, multigrid, Lame
open(1) 'test.mcg' ; open(2) 'ex13mcgin.dat' ; open(3) 'ex13mcg.tex'
geometry('Square'); #ti
problem('Lame',nickname='FEMHD')
#ep 20000.0 0.3
EPS=1.0d-10; R=14; J=4
do I=0,8
mesh('uniform',n=J,p=1,elements='rectangles'); #g.
matrix
lft 4 R 0 R
defprec(mode='MG',spline='u',name='Pu',mat='A',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)

solve(eps=EPS,mdi='x=1',mdc='u.Pu.u',mit='CG'); T=SEC
#rno.
#hno. 701.66
write(2) DOF,I,LMIN,LMAX,COND,T,ITER
write(3) DOF//'&'//LMIN:7//'&'//LMAX:6//'&'//COND:6//'&'//ITER//'&'//T:6//'\\'
J=J*2
continue
end
```

	Multigrid				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
18	1.00000	1.0000	1.0000	2	0.0000
98	0.36913	1.0000	2.7091	15	0.0000
450	0.31760	0.9998	3.1479	18	0.0200
1922	0.30934	0.9999	3.2322	19	0.0800
7938	0.30737	0.9999	3.2531	20	0.3600
32258	0.30673	0.9999	3.2599	20	1.6900
130050	0.30609	0.9999	3.2668	20	8.5200
522242	0.30565	0.9999	3.2714	20	58.050
2093058	0.30538	0.9999	3.2743	20	720.92

Table 5.51: Conjugate Gradients with multigrid-preconditioner (Lamé)

	BPX				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
18	.17E+05	.9E+05	4.9118	6	0.0000
98	.18E+05	.1E+06	7.7567	21	0.0000
450	.16E+05	.2E+06	10.720	31	0.0100
1922	.16E+05	.2E+06	12.872	36	0.0300
7938	.16E+05	.2E+06	14.457	39	0.1600
32258	.15E+05	.2E+06	15.793	41	0.8700
130050	.15E+05	.3E+06	16.898	42	5.4900
522242	.15E+05	.3E+06	17.819	44	47.580
2093058	.15E+05	.3E+06	18.594	45	687.35

Table 5.52: Conjugate Gradients with BPX-preconditioner (Lamé)

	Hierarchical				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
18	.17E+05	.9E+05	4.9118	6	0.0000
98	9849.98	.1E+06	11.609	25	0.0000
450	6865.71	.1E+06	19.392	37	0.0000
1922	4944.14	.1E+06	29.889	48	0.0300
7938	3750.27	.2E+06	42.538	57	0.2200
32258	2939.73	.2E+06	57.538	66	1.2400
130050	2364.58	.2E+06	74.904	74	7.6000
522242	1941.98	.2E+06	94.643	83	61.110
2093058	1622.60	.2E+06	116.75	91	756.33

Table 5.53: Conjugate Gradients with Hierarchical-preconditioner (Lamé)

	CG				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
18	.17E+05	.9E+05	4.9118	6	0.0000
98	4798.25	.1E+06	21.143	24	0.0000
450	1228.61	.1E+06	86.342	50	0.0000
1922	308.976	.1E+06	347.23	101	0.0500
7938	77.3577	.1E+06	1390.8	200	0.4700
32258	19.3465	.1E+06	5565.2	396	4.5400
130050	4.83707	.1E+06	.2E+05	781	39.050
522242	1.20930	.1E+06	.9E+05	1541	346.38
2093058	0.30233	.1E+06	.4E+06	2982	2792.6

Table 5.54: Conjugate Gradients without preconditioner (Lamé)

6 FEM-BEM coupling (2D)

6.1 Convergence

6.1.1 Laplace

Example 6.1. ***Symmetric coupling***

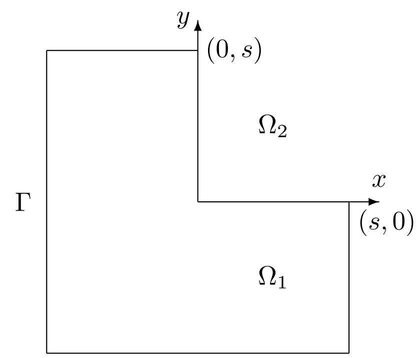


Figure 6.104: Transmission problem on *L*-shaped boundary

Let $\Omega_1 \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ, and $\Omega_2 := \mathbb{R}^2 \setminus \overline{\Omega}_1$. We consider the model transmission problem of finding $u_1 \in H^1(\Omega_1)$, $u_2 \in H^1_{loc}(\overline{\Omega}_2)$ such that

$$\begin{aligned} \Delta u_1 &= f \quad \text{in } \Omega_1 && (10)\\ \Delta u_2 &= 0 \quad \text{in } \Omega_2 && (11)\\ u_1 &= u_2 + u_0 \quad \text{on } \Gamma && (12)\\ \frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} + t_0 \quad \text{on } \Gamma && (13)\\ u_2(x) &= A \log|x| + o(1), \quad |x| \to \infty && (14) \end{aligned}$$

The symmetric coupling formulation now reads: Find $(u, \sigma) \in H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)$, such that:

$$\begin{aligned} & 2(\nabla u, \nabla v)_{L^2(\Omega)} + \langle Wu, v\rangle + \langle (K' - I)\sigma, v\rangle \\ &\quad = \; 2(f, v)_{L^2(\Omega)} + \langle (I + K')t_0, v\rangle + \langle Wu_0, v\rangle \quad \forall v \in H^1(\Omega) \\ & \langle (K - I)u, \psi\rangle - \langle V\sigma, \psi\rangle \\ &\quad = \; -\langle Vt_0, \psi\rangle + \langle (K - I)u_0, \psi\rangle \quad \forall \psi \in \tilde{H}^{-1/2}(\Gamma) \end{aligned}$$

We prescribe the transmission conditions (12) and (13) by

$$\begin{aligned} u_0(r, \phi) &= r^{2/3} \sin[\frac{2}{3}(2\pi - \varphi)] - \log \bar{r} && (15)\\ t_0(r, \phi) &= \frac{\partial f}{\partial n}, && (16) \end{aligned}$$

where $\bar{r} = |\frac{1}{s}(x, y) - (-0.5, -0.5)|$. Here $0 < s < 1$ is a scaling factor so that the single layer potential is positive definite. Then the exact solution of (10)–(14) is known to be

$$\begin{aligned} u_1(r, \varphi) &= (r/s)^{2/3} \sin[\frac{2}{3}(2\pi - \varphi)] \quad \text{in } \Omega_1 \\ u_2(\bar{r}, \varphi) &= \log \bar{r} \quad \text{in } \Omega_2. \end{aligned}$$

coup2/ex1h4in

```
open(1) 'test.h' ; open(2) 'ex1h4in.dat' ;
geometry('L-Shape',5,dim=(/0.25,0.25/)) ; #ti
problem('Laplace',nickname='SYMC')
R=0    ! right hand side
Q=8
J=4;H=0.0625
do I=1,10
mesh('uniform',n=J,p=1,elements='quadrilaterals')
matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
lft 16 R 0 -1

defprec(mode='MG',spline='u',name='Pu',mat='I+A',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)
!defprec(mode='BPX',spline='u',name='Pu',mtop=I,hpmodus=0,mdc=0)
!defprec(mode='ID',spline='u',name='Pu')
!defprec(mode='INVCG',spline='u',name='Pu',mat='A',mit=0)

defprec(mode='MG',spline='N',name='PN',mat='V',mtop=I,hpmodus=0,stp=2,mdc=2, &
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=2)
!defprec(mode='BPX',spline='N',name='PN',mtop=I,hpmodus=0,stp=2,mdc=2)
!defprec(mode='INV',spline='N',name='PN',mat='V',mit=0)

solve(eps=1.0d-10,mdi='x=0',mit='HMCR',mdc='u.Pu.u:N.PN.N',abrflag=1,quiet=1); T=SEC
#cx. 'u' ; #cx. 'N'
#rno.
#err. Q R 'L2' 0 'N' ; E[1]=ERR ! Neumann-Rand
! #err. Q R 'E'  0 'N' ; E[2]=ERR
#err. Q R 'L2' 0 'u' ; E[3]=ERR ! FEM
#err. Q R 'H1' 0 'u' ; E[4]=ERR
#no. 'L2' 'u'
#no. 'H1' 'u'
write(2) DOF,ITER,E[1],0.0,E[3],E[4],T
J=J*2 ; H=H/2
continue
end
```

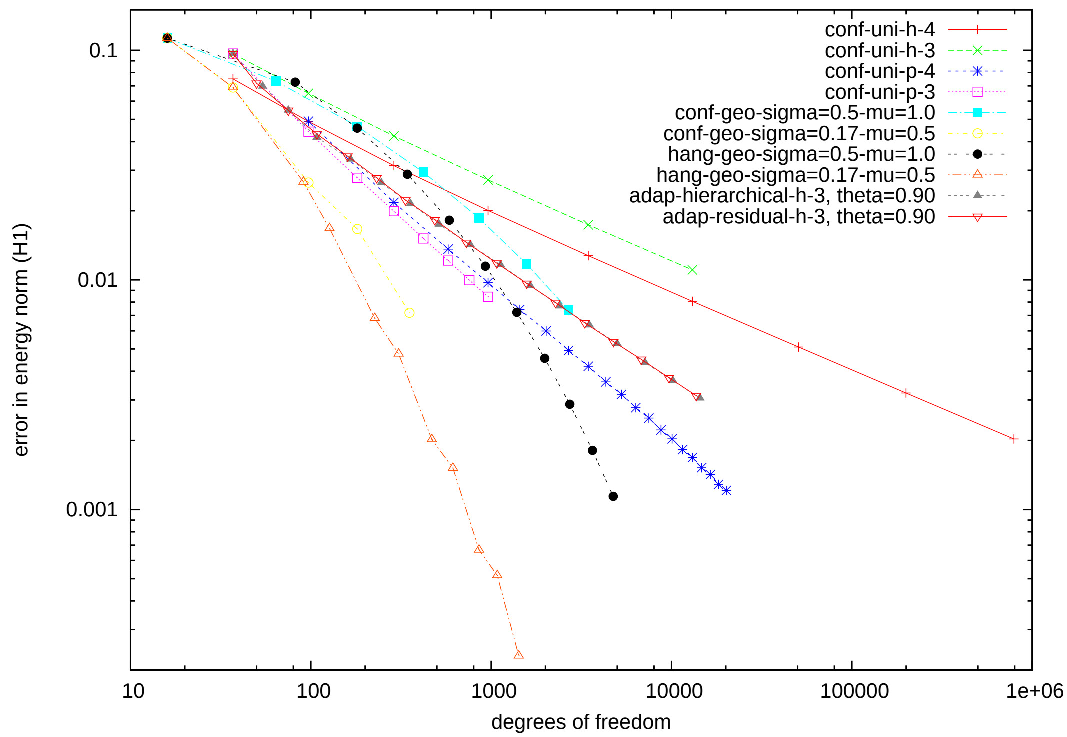


Figure 6.105: FEM-BEM Coupling on the L-Shape

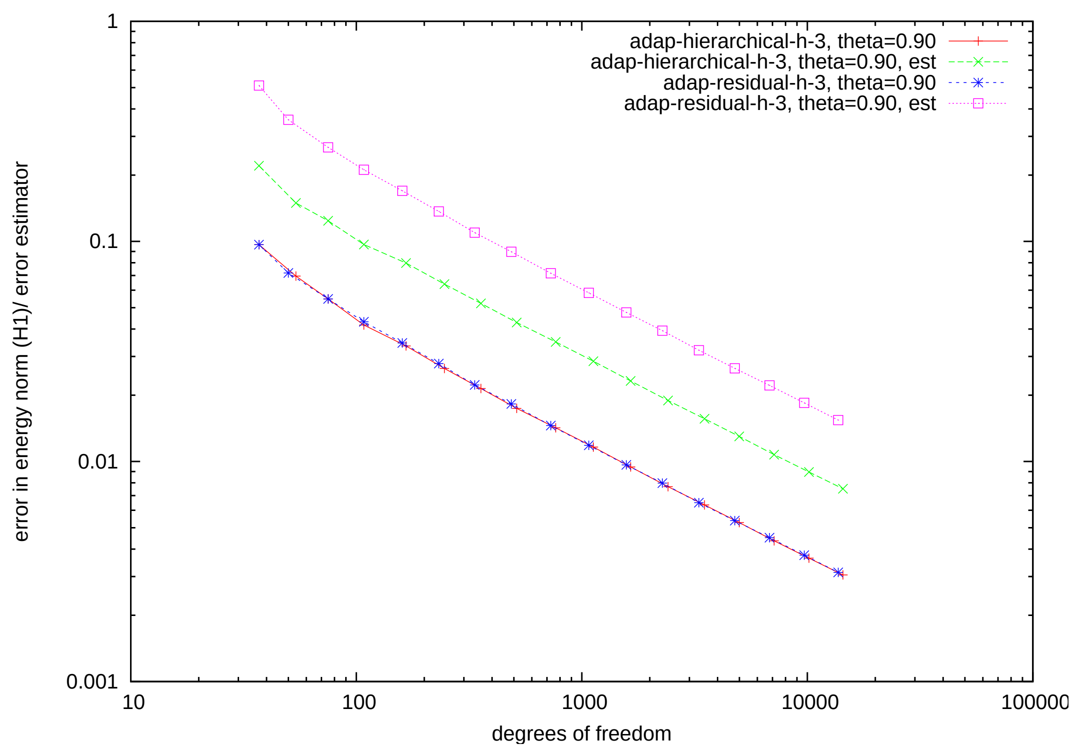


Figure 6.106: FEM-BEM Coupling on the L-Shape, Error estimators

Example 6.2. ***Symmetric coupling with Schur-complement***

coup2/ex2h4in

```
open(1) 'test' ; open(2) 'ex2h4in.dat'
geometry('L-Shape',styp=5,dim=(/0.25,0.25/)) ; #ti
problem('Laplace',nickname='SYMCS')
R=0    ! rhs
J=4;H=0.0625
do I=1,9
 mesh('uniform',n=J,p=1,elements='rectangles')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
 lft 16 R 0 -1
 defprec(mode='MG',spline='u',name='Pu',mat='I+A',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)
 solve(eps=1.0d-10,mdi='x=0',mit='CG',mdc='u.Pu.u',abrflag=1,quiet=1); T=SEC
 #rno.
 #err. 16 R 'L2' 0 'u' ; E[1]=ERR
 #err. 16 R 'H1' 0 'u' ; E[2]=ERR
 write(2) DOF,ITER,E[1],E[2],T,LMIN,LMAX,COND
 J=J*2; H=H/2
continue
end
```

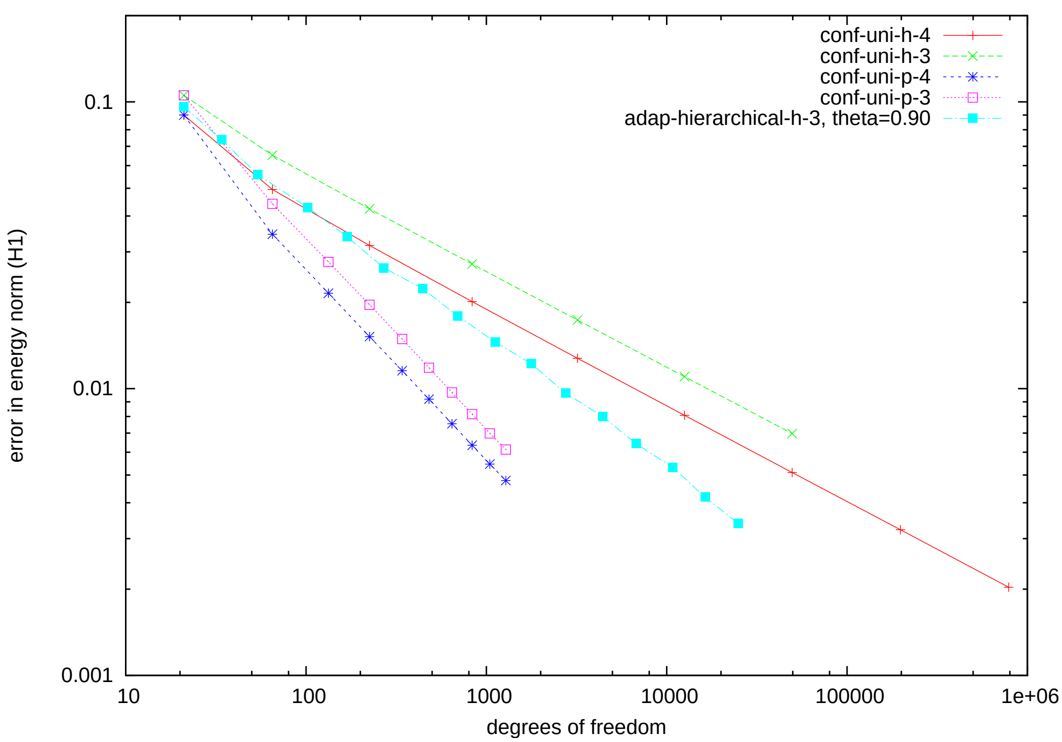


Figure 6.107: FEM-BEM Coupling with Schur-complement on the L-Shape

Example 6.3. ***Symmetric mixed FEM-BEM coupling***

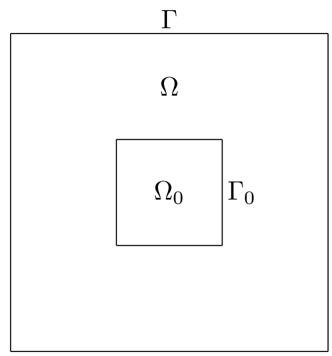


Figure 6.108: Transmission problem on square with hole

$$u(x,y) = \bar{u}(x,y)(1-\chi(x)\chi(y)), \qquad \bar{u}(x,y) = \frac{-2xy}{(x^2+y^2)^2}$$

$$\chi(x) = \begin{cases} \frac{1}{2}(1+\cos\frac{\pi}{2}(|x|-1)) & 1 \le |x| \le 3 \\ 1 & |x| \le 1 \\ 0 & |x| \ge 3 \end{cases}$$

coup2/ex3h3in

```
open(1) 'testc' ; open(2) 'ex3h3in.dat' ; #ti
load('mix3geom.dat') ! load geometry description
problem('Laplace',nickname='SYMMIXC')
#setc 4 6 0.17 ; #setc 6 - 1.0
R=30  ! rhs
EPS=1.0d-8
J=2;H=0.0625
do I=1,7
 mesh('uniform',n=J,p=0,elements='triangles')
 matrix
 lft 16 R 3 R
 solve(eps=EPS,mdi='x=0',mdc='no',mit='GMRES',abrflag=1,quiet=0); T=SEC; #rno.
 open(1) 'mixu'//I
 #taf. 'u' -1 ; #px. 'u' ; #rci. 'u' ; #cx. 'u'
 open(1) 'mixp'//I
 #taf. 'p' -1 ; #px. 'p' ; #rci. 'p' ; #cx. 'p'
 open(1) 'mixd'//I
 #taf. 'D' -1 ; #px. 'D' ; #rci. 'D' ; #cx. 'D'
 open(1) 'mixn'//I
 #taf. 'N' -1 ; #px. 'N' ; #rci. 'N' ; #cx. 'N'
 #err. 16 R 'L2'   0 'u' ; E[0]=ERR
 #err. 16 R 'L2'   0 'p' ; E[1]=ERR
 #err. 16 R 'L2'   0 'D' ; E[2]=ERR
 #err. 16 R 'L2'   0 'N' ; E[3]=ERR
 #err. 16 R 'Hdiv' 0 'p' ; E[4]=ERR
 write(2) DOF, DOFU, E[0], E[1], E[2], E[3], E[4], ITER, T
 J=J*2 ; H=H/2
continue
end
```

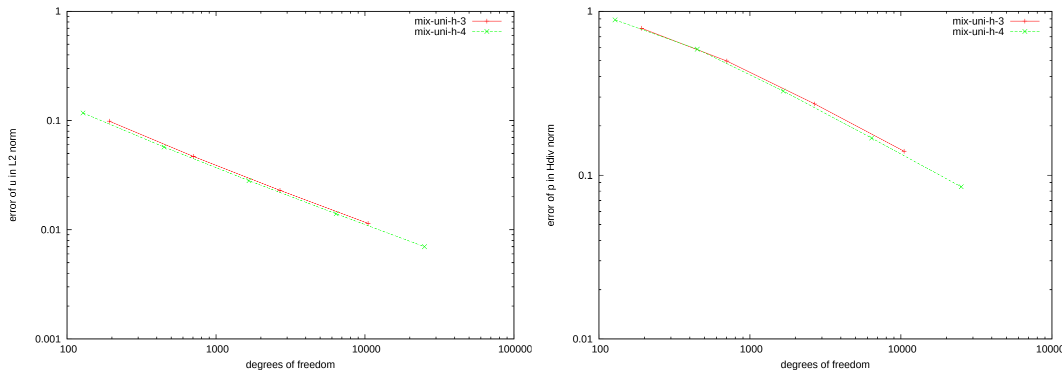


Figure 6.109: Error of u in L2-norm (left) and Error of p in Hdiv-norm (right)

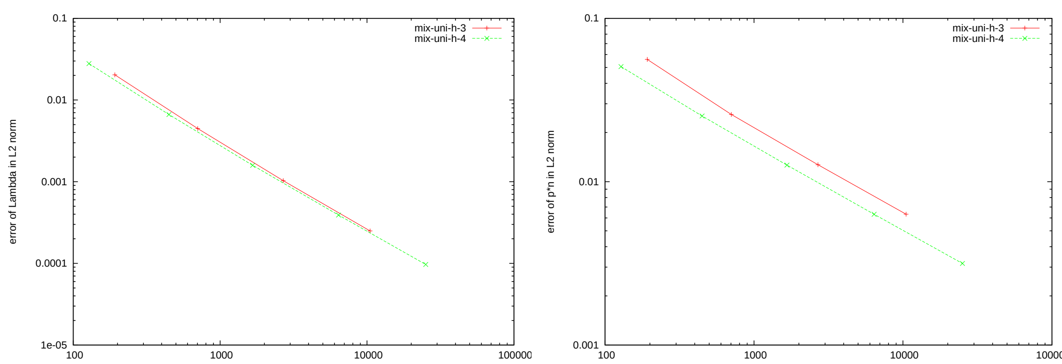


Figure 6.110: Error of λ in L2-norm (left) and Error of $p \cdot \vec{n}$ in L2-norm (right)

Example 6.4. ***Symmetric mixed FEM-BEM coupling with Lagrange Multiplier*** *In this example we use the geometry and data from Example 6.3. The variational formulation uses a Lagrange Multiplier to eliminate the kernel of W*

Find $(p, u, \varphi, \xi) \in H(\mathrm{div}; \Omega) \times L^2(\Omega) \times H^{1/2}(\Gamma) \times \mathbb{R}$, such that

$$\int_\Omega p \cdot q - \int_\Omega u \,\mathrm{div}\, q + \langle V p \cdot n, q \cdot n \rangle - \langle (K + I)\varphi, q \cdot n \rangle + \int_\Gamma \xi \psi = \langle u_0 + V t_0 - (K + I) u_0, q \cdot n \rangle$$

$$- \int_\Omega v \,\mathrm{div}\, p = - \int_\Omega f v$$

$$\langle (K' + I)\psi, p \cdot n \rangle - \langle W \varphi, \psi \rangle = \langle -(K' + I) t_0 - W u_0, \psi \rangle$$

$$\int_\Gamma \chi \varphi = 0$$

for all $(q, v, \psi, \chi) \in H(\mathrm{div}; \Omega) \times L^2(\Omega) \times H^{1/2}(\Gamma) \times \mathbb{R}$.

coup2/ex37h3in

```
open(1) 'testc' ; open(2) 'ex37h3in.dat' ; #ti
load('mix3geom.dat') ! load geometry description
problem('Laplace',nickname='SYMMIXCLG')
#setc 4 6 0.17 ; #setc 6 - 1.0
R=30  ! rhs
EPS=1.0d-8
J=2;H=0.0625
do I=1,7
 mesh('uniform',n=J,p=0,elements='triangles',genspl='no')
 mesh('global',n=1,spline='xi',gm='ubd')
 matrix
 lft 16 R 3 R
 solve(eps=EPS,mdi='x=0',mdc='no',mit='GMRES',abrflag=1,quiet=0); T=SEC; #rno.
 open(1) 'mix37u'//I
 #taf. 'u' -1 ; #px. 'u' ; #rci. 'u' ; #cx. 'u'
 open(1) 'mix37p'//I
 #taf. 'p' -1 ; #px. 'p' ; #rci. 'p' ; #cx. 'p'
 open(1) 'mix37d'//I
 #taf. 'D' -1 ; #px. 'D' ; #rci. 'D' ; #cx. 'D'; #taf. 'xi' -1; #cx. 'xi'
 open(1) 'mix37n'//I
 #taf. 'N' -1 ; #px. 'N' ; #rci. 'N' ; #cx. 'N'
 #err. 16 R 'L2'   0 'u' ; E[0]=ERR
 #err. 16 R 'L2'   0 'p' ; E[1]=ERR
 #err. 16 R 'L2'   0 'D' ; E[2]=ERR
 #err. 16 R 'L2'   0 'N' ; E[3]=ERR
 #err. 16 R 'Hdiv' 0 'p' ; E[4]=ERR
 write(2) DOF, DOFU, E[0], E[1], E[2], E[3], E[4], ITER, T
 J=J*2 ; H=H/2
continue
end
```

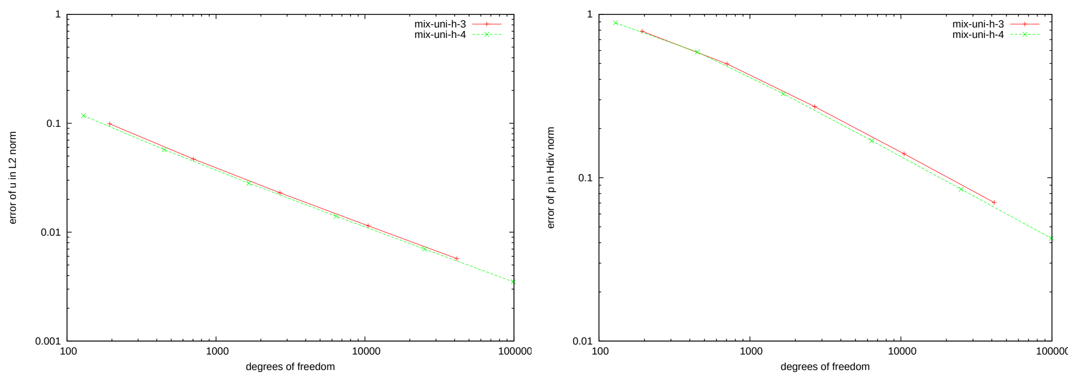


Figure 6.111: Error of u in L2-norm (left) and Error of p in Hdiv-norm (right)

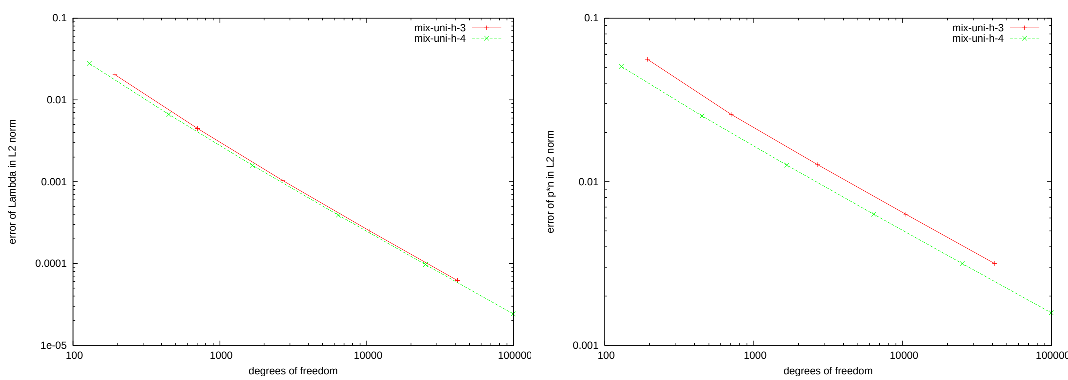


Figure 6.112: Error of λ in L2-norm (left) and Error of $p \cdot \vec{n}$ in L2-norm (right)

Example 6.5. ***Least Squares FEM-BEM coupling[20]***

coup2/ex62rth4in

```
! Least Squares, MG, RT, 62
open(1) 'test.h' ; open(2) 'ex62mgrth4in.dat'
open(3) 'ex62bprth4in.dat'; open(4) 'ex62ivrth4in.dat'
geometry('L-Shape',5,dim=(/0.25,0.25/)) ; #ti
problem('Laplace',nickname='LSC')
 #taf 'C0' 2 2 1 3 'p'
#taf 'Hdiv' 1 2 1 10 'p'
R= 0   ! right hand side
G= 0   ! solver
B=1100 ! max iter
S=0    ! smoother
C=1    ! decomposition
EPS=1.0d-8
J=8;H=0.0625
do I=1,8
mesh('uniform',n=J,p=1,elements='quadrilaterals')
matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
lft 16 R 0 R ; #l. 'u'
defprec(mode='ID',spline='p',name='Pp')

defprec(mode='MG',spline='u',name='Pu',mat='I+A',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)
defprec(mode='MG',spline='N',name='PN',mat='V',mtop=I,hpmodus=0,stp=2,mdc=2, &
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=2)

lsqsolve(eps=EPS,mdi='x=1',mdc='scp',scp='Pp:0.5*Pu:PN',mit='CG',mnum=1000); T=SEC

#err. 24 R 'L2' 0 'N' ; E[1]=ERR ! Neumann-Rand
#err. 24 R 'L2' 0 'D' ; E[2]=ERR
#err. 16 R 'L2' 0 'u' ; E[3]=ERR ! FEM
#err. 16 R 'H1' 0 'u' ; E[4]=ERR
#err. 16 R 'L2' 0 'p' ; E[5]=ERR
#err. 16 R 'H1' 0 'D' ; E[6]=ERR
write(2) DOF,DOFP,DOFU,DOFN,E[1],E[2],E[3],E[4],E[5],E[6],ITER,COND,T

defprec(mode='BPX',spline='u',name='Pu',mat='I+A',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)
defprec(mode='BPX',spline='N',name='PN',mat='V',mtop=I,hpmodus=0,stp=2,mdc=2, &
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=2)

lsqsolve(eps=EPS,mdi='x=1',mdc='scp',scp='Pp:0.5*Pu:PN',mit='CG',mnum=1000); T=SEC
#err. 24 R 'L2' 0 'N' ; E[1]=ERR ! Neumann-Rand
#err. 24 R 'L2' 0 'D' ; E[2]=ERR
#err. 16 R 'L2' 0 'u' ; E[3]=ERR ! FEM
#err. 16 R 'H1' 0 'u' ; E[4]=ERR
#err. 16 R 'L2' 0 'p' ; E[5]=ERR
#err. 16 R 'H1' 0 'D' ; E[6]=ERR
write(3) DOF,DOFP,DOFU,DOFN,E[1],E[2],E[3],E[4],E[5],E[6],ITER,COND,T


defprec(mode='INVCG',spline='u',name='Pu',mat='I+A')
```

```
defprec(mode='INV',spline='N',name='PN',mat='V')

lsqsolve(eps=EPS,mdi='x=1',mdc='scp',scp='Pp:0.5*Pu:PN',mit='CG',mnum=1000); T=SEC

#err. 24 R 'L2' 0 'N' ; E[1]=ERR ! Neumann-Rand
#err. 24 R 'L2' 0 'D' ; E[2]=ERR
#err. 16 R 'L2' 0 'u' ; E[3]=ERR ! FEM
#err. 16 R 'H1' 0 'u' ; E[4]=ERR
#err. 16 R 'L2' 0 'p' ; E[5]=ERR
#err. 16 R 'H1' 0 'D' ; E[6]=ERR
write(4) DOF,DOFP,DOFU,DOFN,E[1],E[2],E[3],E[4],E[5],E[6],ITER,COND,T
J=J*2 ; H=H/2
continue
end
```

DOF	$\|\vartheta - \vartheta_N\|_{L^2(\Omega)}$	α_{ϑ,L^2}	$\|u - u_N\|_{L^2(\Omega)}$	α_{u,L^2}	$\|\sigma - \sigma_N\|_{L^2(\Gamma)}$	α_{σ,L^2}	κ
209	0.0552177	—	0.0007822	—	0.4571975	—	106.26266
705	0.0353794	0.6422	0.0002619	1.5785	0.4085217	0.162	119.69710
2561	0.0225073	0.6525	.9471E-04	1.4674	0.3650370	0.162	134.27148
9729	0.0142646	0.6580	.4507E-04	1.0713	0.3261994	0.162	142.09749
37889	0.0090200	0.6612	.2743E-04	0.7164	0.2914880	0.162	145.70241
149505	0.0056957	0.6633	.1781E-04	0.6231	0.2604425	0.162	149.06988
593921	0.0035934	0.6645	.1154E-04	0.6260	0.2326494	0.163	150.92103
2367489	0.0022659	0.6653	.7394E-05	0.6422	0.2077351	0.163	152.12219

Table 6.55: Linear Least-Squares FEM-BEM coupling, convergence rates and condition numbers

Example 6.6. *Let $\Omega_1 = [-1,1]^2 \backslash [0,1]^2$ (L-Shape), $\Omega_2 = \mathbb{R}^2 \backslash \Omega_1$. This example deals with a non-linear Laplace FEM-BEM coupling problem.*

$$\begin{aligned} -\operatorname{div}(\varrho(|\nabla u_1|)\nabla u_1) &= f \ \textit{in } \Omega_1 \\ \Delta u_2 &= 0 \ \textit{in } \Omega_2 \\ u_1 &= u_2 + u_0 \ \textit{on } \Gamma \\ \varrho(|\nabla u_1|)\frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} + t_0 \ \textit{on } \Gamma \\ u_2(x) &= A\log|x| + o(1), \quad |x| \to \infty \end{aligned}$$

The (non-linear) variational formulation is: Find $(u, \sigma) \in H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)$, such that:

$$\begin{aligned} & 2(\varrho(|\nabla u|)\nabla u, \nabla v)_{L^2(\Omega)} + \langle Wu, v\rangle + \langle (K' - I)\sigma, v\rangle \\ & \quad = \quad 2(f, v)_{L^2(\Omega)} + \langle (I + K')t_0, v\rangle + \langle Wu_0, v\rangle \quad \forall v \in H^1(\Omega) \\ & \langle (K - I)u, \psi\rangle - \langle V\sigma, \psi\rangle \\ & \quad = \quad -\langle Vt_0, \psi\rangle + \langle (K - I)u_0, \psi\rangle \quad \forall \psi \in \tilde{H}^{-1/2}(\Gamma) \end{aligned}$$

Let $\tilde{\varrho}(x) = \varrho(|x|)I + \varrho'(|x|)\frac{xx^t}{|x|}$. This gives rise to the following Newton scheme: Let $(u^{(0)}, \sigma^{(0)}) \in H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)$. For $n = 1, 2, \ldots$ until a stopping criterion is satisfied: Find $(\delta, \theta) \in H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)$ such that

$$\begin{aligned} & 2\int_\Omega (\tilde{\varrho}(\nabla u^{(n-1)})\nabla\delta)\nabla v \, dx + \langle W\delta, v\rangle + \langle (K' - I)\vartheta, v\rangle \\ & = 2\int_\Omega fv \, dx + \langle (I + K')t_0, v\rangle + \langle Wu_0, v\rangle \\ & \quad - 2\int_\Omega \varrho(|\nabla u^{(n-1)}|)\nabla u^{(n-1)}\nabla v \, dx - \langle Wu^{(n-1)}, v\rangle - \langle (K' - I)\sigma^{(n-1)}, v\rangle \quad \forall v \in H^1(\Omega) \\ & \langle (K - I)\delta, \psi\rangle - \langle V\vartheta, \psi\rangle \\ & = -\langle Vt_0, \psi\rangle + \langle (K - I)u_0, \psi\rangle - \langle (K - I)u^{(n-1)}, \psi\rangle + \langle V\sigma^{(n-1)}, \psi\rangle \quad \forall \psi \in \tilde{H}^{-1/2}(\Gamma) \end{aligned}$$

and $u^{(n)} = u^{(n-1)} + \delta$, $\sigma^{(n)} = \sigma^{(n-1)} + \theta$.

Here we choose $\varrho(t) = \frac{1}{6}(1 + \frac{5}{1+5t})$ and we stop if $\|\delta\|_{H^1(\Omega)} + \|\theta\|_{H^{-1/2}(\Gamma)} \le 10^{-8}$. We have $u_0 = r^{2/3}\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$, $t_0 = \frac{\partial u_0}{\partial n}$, $f = 0$.

coup2/ex34h4in

```
! Laplace, non-linear, L-Shape, FEM-BEM coupling
open 'test.h' ; open(2) 'ex34h4in.dat'
geometry('L-Shape',styp=5,dim=(/0.25,0.25/)); #ti
problem('Laplace',nickname='SYMCNL')
R=50; NL=1 ! rhs, rho
EPS=1.0d-10; Q=16
J=4
do I=2,19
 mesh('uniform',n=J,p=1,elements='rectangles')
 approx 0 R 'dt' 't0' ; approx 0 R 'du' 'u0'
 clear('ux'); clear('Dx'); clear('Nx')
 NCNT=0; ITMAX=0
 do K=0,200
  matrix('analytic',ijn=6,sigma=0.17,mu=1.0,nonlin=NL)
  lft 16 R 0 R NL
defprec(mode='MG',spline='u',name='Pu',mat='I+ANL',mtop=I,hpmodus=0,stp=2,mdc=0,&
```

DOF	$\|u-u_N\|_{L^2(\Omega)}$	α_{u,L^2}	$\|u-u_N\|_{H^1(\Omega)}$	α_{u,H^1}	$\|\sigma-\sigma_N\|_{L^2(\Gamma)}$	α_{σ,L^2}	It_{Newton}
37	0.0093172	—	0.1544821	—	0.5472259	—	5
97	0.0022429	-1.477592	0.0586558	-1.004771	0.1378859	-1.430219	5
289	0.0005663	-1.260769	0.0334225	-0.515205	0.0784760	-0.516282	5
961	0.0001474	-1.120200	0.0208308	-0.393490	0.0599076	-0.224701	5
3457	.3894E-04	-1.039796	0.0131585	-0.358828	0.0515423	-0.117484	6
13057	.1126E-04	-0.933662	0.0083200	-0.344945	0.0455267	-0.093387	6
50689	.4914E-05	-0.611308	0.0052564	-0.338559	0.0404447	-0.087264	6

Table 6.56: Non-Linear FEM, convergence rates and Newton steps

```
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)
!defprec(mode='BPX',spline='u',name='Pu',mtop=I,hpmodus=0,mdc=0)
!defprec(mode='ID',spline='u',name='Pu')
!defprec(mode='INVCG',spline='u',name='Pu',mat='ANL',mit=0)

defprec(mode='MG',spline='N',name='PN',mat='xV',mtop=I,hpmodus=0,stp=2,mdc=2, &
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=2)
!defprec(mode='BPX',spline='N',name='PN',mtop=I,hpmodus=0,stp=2,mdc=2)
!defprec(mode='INV',spline='N',name='PN',mat='xV',mit=0)

solve(eps=EPS,mdi='x=0',mit='HMCR',mdc='u.Pu.u:N.PN.N',abrflag=1,quiet=1);T=SEC
  ITMAX=Max(ITER,ITMAX); #rno.
  eval('Dx=Dx+D','no'); eval('ux=ux+u','no'); eval('Nx=Nx+N','no')
  norm('Nu','H1','u'); norm('NN','L2','N')
  NEWTON=Sqrt(Nu*Nu+NN*NN)
  write(0) 'Newton', NEWTON, NCNT, Nu,NN
  NCNT=NCNT+1
  exit(NEWTON<EPS*100)
 continue
 #err. Q R 'L2' 0 'ux' 'u' ; E[1]=ERR
 #err. Q R 'H1' 0 'ux' 'u' ; E[2]=ERR
 #err. Q R 'L2' 0 'Nx' 't0' ; E[4]=ERR
 write(2) DOF,ITER,E[1],E[2],E[4],NCNT
 J=J*2
continue
end
```

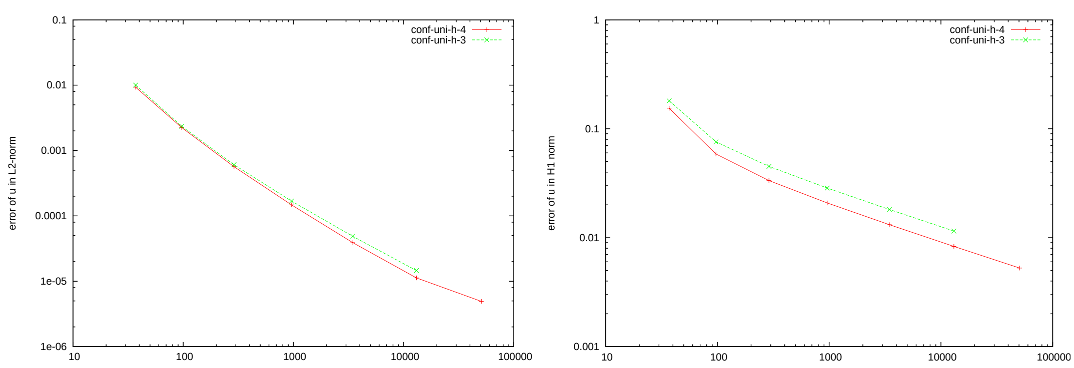


Figure 6.113: $\|u-u_n\|_{L^2(\Omega)}$ (left) and $\|u-u_n\|_{H^1(\Omega)}$ (right) — Non-linear.

Example 6.7. *Let $\Omega_1 = [-1,1]^2 \backslash [0,1]^2$ (L-Shape), $\Omega_2 = \mathbb{R}^2 \backslash \Omega_1$. This example deals with a non-linear Laplace FEM-BEM coupling problem.*

$$\begin{aligned}
-\operatorname{div}(\varrho(|\nabla u_1|)\nabla u_1) &= f \ \text{in } \Omega_1 \\
\Delta u_2 &= 0 \ \text{in } \Omega_2 \\
u_1 &= u_2 + u_0 \ \text{on } \Gamma \\
\varrho(|\nabla u_1|)\frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} + t_0 \ \text{on } \Gamma \\
u_2(x) &= A \log|x| + o(1), \quad |x| \to \infty
\end{aligned}$$

The (non-linear) variational formulation is: Find $u \in H^1(\Omega)$, such that:

$$2(\varrho(|\nabla u|)\nabla u, \nabla v)_{L^2(\Omega)} + \langle Su, v\rangle = 2(f,v)_{L^2(\Omega)} + \langle t_0, v\rangle + \langle Su_0, v\rangle \quad \forall v \in H^1(\Omega)$$

Let $\tilde{\varrho}(x) = \varrho(|x|)I + \varrho'(|x|)\frac{xx^t}{|x|}$. This gives rise to the following Newton scheme: Let $u^{(0)} \in H^1(\Omega)$. For $n = 1, 2, \ldots$ until a stopping criterion is satisfied: Find $\delta \in H^1(\Omega)$ such that

$$\begin{aligned}
&2\int_\Omega (\tilde{\varrho}(\nabla u^{(n-1)})\nabla\delta)\nabla v\,dx + \langle \tilde{S}\delta, v\rangle \\
&= 2\int_\Omega fv\,dx + \langle t_0, v\rangle + \langle \tilde{S}u_0, v\rangle - 2\int_\Omega \varrho(|\nabla u^{(n-1)}|)\nabla u^{(n-1)}\nabla v\,dx - \langle \tilde{S}u^{(n-1)}, v\rangle \quad \forall v \in H^1(\Omega)
\end{aligned}$$

and $u^{(n)} = u^{(n-1)} + \delta$.

Here we choose $\varrho(t) = \frac{1}{6}(1 + \frac{5}{1+5t})$ and we stop if $\|\delta\|_{H^1(\Omega)} \leq 10^{-8}$. We have $u_0 = r^{2/3}\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$, $t_0 = \frac{\partial u_0}{\partial n}$, $f = 0$.

DOF	$\|u - u_N\|_{H^1(\Omega)}$	α_{u,H^1}	It_{Newton}
21	0.1544821	—	5
65	0.0586558	-0.857086	5
225	0.0334225	-0.452969	5
833	0.0208308	-0.361208	5
3201	0.0131585	-0.341234	5
12545	0.0083200	-0.335617	6
49665	0.0052564	-0.333738	6
197633	0.0033181	-0.333104	6
788481	0.0020932	-0.332948	6

Table 6.57: Non-Linear FEM, convergence rates and Newton steps (Example 6.7, uniform-h4)

DOF	$\|u-u_N\|_{H^1(\Omega)}$	α_{u,H^1}	η_N	It_{Newton}	η_N/E_N	κ_N
21	0.1806867	—	0.0966305	5	0.5347959	—
35	0.0926990	-1.306526	0.0460938	5	0.4972416	0.4770109
69	0.0693000	-0.428595	0.0343828	5	0.4961443	0.7459311
128	0.0533507	-0.423285	0.0292706	5	0.5486451	0.8513152
230	0.0338001	-0.778814	0.0178149	5	0.5270665	0.6086278
378	0.0250610	-0.602138	0.0133225	5	0.5316029	0.7478291
659	0.0177304	-0.622551	0.0093986	5	0.5300839	0.7054682
1119	0.0128197	-0.612498	0.0066856	5	0.5215099	0.7113400
1901	0.0096018	-0.545401	0.0050660	5	0.5276094	0.7577480
3197	0.0072038	-0.552758	0.0038316	5	0.5318859	0.7563364
5284	0.0054660	-0.549409	0.0028958	5	0.5297841	0.7557678
8739	0.0042231	-0.512763	0.0022552	5	0.5340153	0.7787831
14201	0.0032684	-0.527827	0.0017359	5	0.5311161	0.7697322
22828	0.0025457	-0.526454	0.0013542	5	0.5319558	0.7801141
36387	0.0020151	-0.501341	0.0010730	5	0.5324798	0.7923497
57728	0.0015958	-0.505478	0.0008488	5	0.5318962	0.7910531

Table 6.58: Non-Linear FEM, convergence rates and Newton steps (Example 6.7, adaptive)

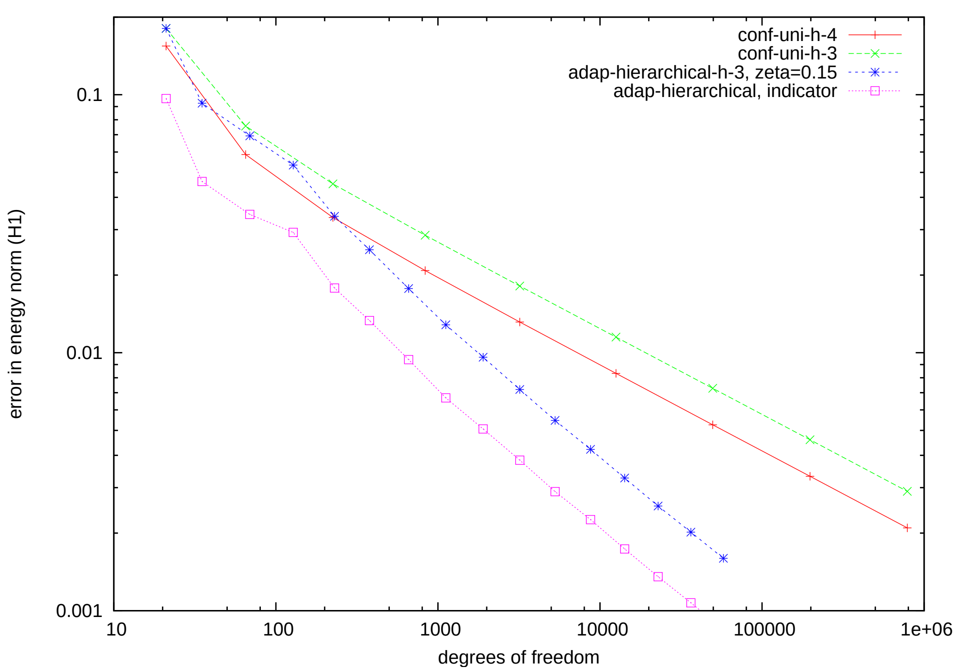


Figure 6.114: $\|u-u_n\|_{H^1(\Omega)}$ — Non-linear.(Example 6.7)

Example 6.8. *(cf. [10]) Let* $\Omega_1 = [-1,1]^2 \backslash [0,1]^2$ *(L-Shape),* $\Omega_2 = \mathbb{R}^2 \backslash \Omega_1$*. This example deals with a non-linear Laplace FEM-BEM coupling problem.*

$$\begin{aligned}
-\operatorname{div}(\varrho(|\nabla u_1|)\nabla u_1) &= f \ \textit{in } \Omega_1 \\
\Delta u_2 &= 0 \ \textit{in } \Omega_2 \\
u_1 &= u_2 + u_0 \ \textit{on } \Gamma \\
\varrho(|\nabla u_1|)\frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} + t_0 \ \textit{on } \Gamma \\
u_2(x) &= A \log|x| + o(1), \quad |x| \to \infty
\end{aligned}$$

Using boundary elements we can rewrite this in the form

$$\begin{aligned}
-\operatorname{div}\boldsymbol{\sigma} &= f \ \textit{in } \Omega_1 \\
\varrho(|\boldsymbol{\vartheta}|)\boldsymbol{\vartheta} &= \boldsymbol{\sigma} \ \textit{in } \Omega_1 \\
u_2 &= (I+K)\phi - V\frac{\partial u_2}{\partial n} \\
W\phi + (I+K')\frac{\partial u_2}{\partial n} &= 0
\end{aligned}$$

The (non-linear) dual-dual variational formulation is: Find $(\boldsymbol{\vartheta}, \boldsymbol{\sigma}, \phi, u) \in [L^2(\Omega)]^2 \times H(\operatorname{div};\Omega) \times H^{1/2}(\Gamma) \times L^2(\Omega) =: X$*, such that:*

$$\begin{aligned}
2(\varrho(|\boldsymbol{\vartheta}|)\boldsymbol{\vartheta}, \boldsymbol{\xi}) - 2(\boldsymbol{\sigma}, \boldsymbol{\xi}) &= 0 \\
-2(\boldsymbol{\vartheta}, \boldsymbol{\tau}) - \langle V\boldsymbol{\sigma}\cdot n, \boldsymbol{\tau}\cdot n\rangle + \langle (I+K)\phi, \boldsymbol{\tau}\cdot n\rangle - 2(u, \operatorname{div}\boldsymbol{\tau}) &= \langle (I+K)t_0 - Vt_0 - 2u_0, \boldsymbol{\tau}\cdot n\rangle \\
\langle \tilde{W}\phi, \psi\rangle + \langle (I+K')\boldsymbol{\sigma}\cdot n, \psi\rangle &= \langle (I+K')t_0 + Wt_0, \psi\rangle \\
-2(\operatorname{div}\boldsymbol{\sigma}, v) &= 2(f, v)
\end{aligned}$$

for all $(\boldsymbol{\xi}, \boldsymbol{\tau}, \psi, v) \in X$*.*

Let $\tilde{\varrho}(x) = \varrho(|x|)I + \varrho'(|x|)\frac{xx^t}{|x|}$*. This gives rise to the following Newton scheme: Let* $(\boldsymbol{\vartheta}^{(0)}, \boldsymbol{\sigma}^{(0)}, \phi^{(0)}, u^{(0)}) \in X$*. For* $n = 1, 2, \ldots$ *until a stopping criterion is satisfied: Find* $(\boldsymbol{\delta}_\theta, \boldsymbol{\delta}_\sigma, \delta_\phi, \delta_u) \in X$ *such that*

$$\begin{aligned}
2(\tilde{\varrho}(|\boldsymbol{\vartheta}^{(n-1)}|)\boldsymbol{\delta}_\theta, \boldsymbol{\xi}) - 2(\boldsymbol{\delta}_\sigma, \boldsymbol{\xi}) &= 0 - \{2(\varrho(|\boldsymbol{\vartheta}^{(n-1)}|)\boldsymbol{\vartheta}^{(n-1)}, \boldsymbol{\xi}) - 2(\boldsymbol{\sigma}^{(n-1)}, \boldsymbol{\xi})\} \\
-2(\boldsymbol{\delta}_\theta, \boldsymbol{\tau}) - \langle V\boldsymbol{\delta}_\sigma\cdot n, \boldsymbol{\tau}\cdot n\rangle + \langle (I+K)\delta_\phi, \boldsymbol{\tau}\cdot n\rangle - 2(\delta_u, \operatorname{div}\boldsymbol{\tau}) &= \langle (I+K)t_0 - Vt_0 - 2u_0, \boldsymbol{\tau}\cdot n\rangle \\
&\quad -\{-2(\boldsymbol{\vartheta}^{(n-1)}, \boldsymbol{\tau}) - \langle V\boldsymbol{\sigma}^{(n-1)}\cdot n, \boldsymbol{\tau}\cdot n\rangle + \langle (I+K)\phi^{(n-1)}, \boldsymbol{\tau}\cdot n\rangle - 2(u^{(n-1)}, \operatorname{div}\boldsymbol{\tau})\} \\
\langle \tilde{W}\delta_\phi, \psi\rangle + \langle (I+K')\boldsymbol{\delta}_\sigma\cdot n, \psi\rangle &= \langle (I+K')t_0 + Wt_0, \psi\rangle \\
&\quad -\{\langle \tilde{W}\phi^{(n-1)}, \psi\rangle + \langle (I+K')\boldsymbol{\sigma}^{(n-1)}\cdot n, \psi\rangle\} \\
-2(\operatorname{div}\boldsymbol{\delta}_\sigma, v) &= 2(f, v) - \{-2(\operatorname{div}\boldsymbol{\sigma}^{(n-1)}, v)\}
\end{aligned}$$

for all $(\boldsymbol{\xi}, \boldsymbol{\tau}, \psi, v) \in X$*. And compute* $\boldsymbol{\vartheta}^{(n)} = \boldsymbol{\vartheta}^{(n-1)} + \boldsymbol{\delta}_\theta$*,* $\boldsymbol{\sigma}^{(n)} = \boldsymbol{\sigma}^{(n-1)} + \boldsymbol{\delta}_\sigma$*,* $\phi^{(n)} = \phi^{(n-1)} + \delta_\phi$*,* $u^{(n)} = u^{(n-1)} + \delta_u$*.*

Here we choose $\varrho(t) = \frac{1}{6}(1 + \frac{5}{1+5t})$ *and we stop if* $\|\boldsymbol{\delta}_\theta\|_{L^2(\Omega)} + \|\boldsymbol{\delta}_\sigma\|_{H(\operatorname{div};\Omega)} + \|\delta_\phi\|_{H^{-1/2}(\Gamma)} + \|\delta_u\|_{L^2(\Omega)} \le 10^{-8}$*. We have* $u_0 = r^{2/3}\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$*,* $t_0 = \frac{\partial u_0}{\partial n}$*,* $f = 0$*.*

Example 6.9. *(cf. [10]) Let $\Omega_1 = [-1,1]^2\backslash[0,1]^2$ (L-Shape), $\Omega_2 = \mathbb{R}^2\backslash\Omega_1$. This example deals with a non-linear Laplace FEM-BEM coupling problem.*

$$\begin{aligned} -\operatorname{div}(\varrho(|\nabla u_1|)\nabla u_1) &= f \ \text{ in } \Omega_1 \\ \Delta u_2 &= 0 \ \text{ in } \Omega_2 \\ u_1 &= u_2 + u_0 \ \text{ on } \Gamma \\ \varrho(|\nabla u_1|)\frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} + t_0 \ \text{ on } \Gamma \\ u_2(x) &= A\log|x| + o(1), \quad |x| \to \infty \end{aligned}$$

The (non-linear) dual-dual variational formulation with the inverse Poincare-Steklov operator is: Find $(\boldsymbol{\vartheta}, \boldsymbol{\sigma}, u) \in [L^2(\Omega)]^2 \times H(\operatorname{div};\Omega) \times L^2(\Omega) =: X$, such that:

$$\begin{aligned} 2(\varrho(|\boldsymbol{\vartheta}|)\boldsymbol{\vartheta}, \boldsymbol{\xi}) - 2(\boldsymbol{\sigma}, \boldsymbol{\xi}) &= 0 \\ -2(\boldsymbol{\vartheta}, \boldsymbol{\tau}) - \langle R\boldsymbol{\sigma}\cdot n, \boldsymbol{\tau}\cdot n\rangle - 2(u, \operatorname{div}\boldsymbol{\tau}) &= \langle -Rt_0 - 2u_0, \boldsymbol{\tau}\cdot n\rangle \\ -2(\operatorname{div}\boldsymbol{\sigma}, v) &= 2(f, v) \end{aligned}$$

for all $(\boldsymbol{\xi}, \boldsymbol{\tau}, \psi, v) \in X$.

Let $\tilde{\varrho}(x) = \varrho(|x|)I + \varrho'(|x|)\frac{xx^t}{|x|}$. This gives rise to the following Newton scheme: Let $(\boldsymbol{\vartheta}^{(0)}, \boldsymbol{\sigma}^{(0)}, u^{(0)}) \in X$. For $n = 1, 2, \ldots$ until a stopping criterion is satisfied: Find $(\boldsymbol{\delta}_\theta, \boldsymbol{\delta}_\sigma, \delta_u) \in X$ such that

$$\begin{aligned} 2(\tilde{\varrho}(|\boldsymbol{\vartheta}^{(n-1)}|)\boldsymbol{\delta}_\theta, \boldsymbol{\xi}) - 2(\boldsymbol{\delta}_\sigma, \boldsymbol{\xi}) &= 0 - \{2(\varrho(|\boldsymbol{\vartheta}^{(n-1)}|)\boldsymbol{\vartheta}^{(n-1)}, \boldsymbol{\xi}) - 2(\boldsymbol{\sigma}^{(n-1)}, \boldsymbol{\xi})\} \\ -2(\boldsymbol{\delta}_\theta, \boldsymbol{\tau}) - \langle \tilde{R}\boldsymbol{\delta}_\sigma\cdot n, \boldsymbol{\tau}\cdot n\rangle - 2(\delta_u, \operatorname{div}\boldsymbol{\tau}) &= \langle -\tilde{R}t_0 - 2u_0, \boldsymbol{\tau}\cdot n\rangle \\ &\quad -\{-2(\boldsymbol{\vartheta}^{(n-1)}, \boldsymbol{\tau}) - \langle \tilde{R}\boldsymbol{\sigma}^{(n-1)}\cdot n, \boldsymbol{\tau}\cdot n\rangle - 2(u^{(n-1)}, \operatorname{div}\boldsymbol{\tau})\} \\ -2(\operatorname{div}\boldsymbol{\delta}_\sigma, v) &= 2(f, v) - \{-2(\operatorname{div}\boldsymbol{\sigma}^{(n-1)}, v)\} \end{aligned}$$

for all $(\boldsymbol{\xi}, \boldsymbol{\tau}, v) \in X$. And compute $\boldsymbol{\vartheta}^{(n)} = \boldsymbol{\vartheta}^{(n-1)} + \boldsymbol{\delta}_\theta$, $\boldsymbol{\sigma}^{(n)} = \boldsymbol{\sigma}^{(n-1)} + \boldsymbol{\delta}_\sigma$, $u^{(n)} = u^{(n-1)} + \delta_u$.

Here we choose $\varrho(t) = \frac{1}{6}(1 + \frac{5}{1+5t})$ and we stop if $\|\boldsymbol{\delta}_\theta\|_{L^2(\Omega)} + \|\boldsymbol{\delta}_\sigma\|_{H(\operatorname{div};\Omega)} + \|\delta_\phi\|_{H^{-1/2}(\Gamma)} + \|\delta_u\|_{L^2(\Omega)} \leq 10^{-8}$. We have $u_0 = r^{2/3}\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$, $t_0 = \frac{\partial u_0}{\partial n}$, $f = 0$.

Example 6.10. *Let $\Omega = [-1,1]^2 \backslash [0,1]^2$ (L-Shape), $\Omega_c = \mathbb{R}^2 \backslash \Omega$. This example deals with a non-linear Laplace FEM-BEM coupling problem, modeling Coulomb friction, see [22].*

$$
\begin{aligned}
-\operatorname{div}(\varrho(|\nabla u_1|)\cdot \nabla u_1) &= f \ \text{in } \Omega,\\
-\Delta u &= 0 \ \text{in } \Omega_c,\\
u(x) &= a + o(1), \quad |x| \to \infty \ \text{(radiation condition)}
\end{aligned}
$$

transmission conditions on Γ_t

$$u_1|_{\Gamma_t} - u_2|_{\Gamma_t} = u_0|_{\Gamma_t} \ \text{and} \ \varrho(|\nabla u_1|)\frac{\partial u_1}{\partial n}|_{\Gamma_t} - \frac{\partial u_2}{\partial n}|_{\Gamma_t} = t_0|_{\Gamma_t},$$

and Coulomb friction conditions with given friction force $g > 0$ on Γ_s,

$$
\begin{aligned}
\varrho(|\nabla u_1|)\frac{\partial u_1}{\partial n}|_{\Gamma_s} - \frac{\partial u_2}{\partial n}|_{\Gamma_s} = t_0|_{\Gamma_s}, \quad |\varrho(|\nabla u_1|)\frac{\partial u_1}{\partial n}| \le g \ \text{on } \Gamma_s,\\
-\varrho(|\nabla u_1|)\frac{\partial u_1}{\partial n}(u_0 + u_2 - u_1) + g|u_0 + u_2 - u_1| = 0 \ \text{on } \Gamma_s.
\end{aligned}
$$

Using

$$
\begin{aligned}
G(u) &= \int_\Omega g(|\nabla u|)\,dx, \quad g(t) = \int_0^t s\cdot\varrho(s)\,ds\\
\lambda(u,v) &= \int_\Omega f\cdot u\,dx + \langle t_0 + Su_0, u|_\Gamma + v\rangle\\
J(u,v) &:= G(u) + \frac{1}{2}\langle S(u|_\Gamma + v), u|_\Gamma + v\rangle - \lambda(u,v)\\
j(v) &= \int_{\Gamma_s} g\cdot|v|\,ds
\end{aligned}
$$

the minimization problem reads: Find $(u,v) \in H^1(\Omega)\times\tilde{H}^{1/2}(\Gamma_s)$ such that

$$J(u,v) + j(v) = \min J(\tilde{u},\tilde{v}) + j(\tilde{v})$$

Using

$$\mathcal{A}(u,v;r,s) = DG(u,r) + \langle S(u|_\Gamma + v), r|_\Gamma + s\rangle$$

the variational formulation reads: Find $(\hat{u},\hat{v}) \in H^1(\Omega)\times\tilde{H}^{1/2}(\Gamma_s)$ such that

$$\mathcal{A}(\hat{u},\hat{v}; u-\hat{u}, v-\hat{v}) + j(v) - j(\hat{v}) \ge \lambda(u-\hat{u}, v-\hat{v}) \quad \forall (u,v)\in H^1(\Omega)\times\tilde{H}^{1/2}(\Gamma_s)$$

In this example we choose $\varrho \equiv 1$, $g = 0.5$, $f = 0$ and $u_0 = r^{2/3}\sin\frac{2}{3}(\varphi - \frac{\pi}{2})$, $t_0 = \frac{\partial u_0}{\partial n}$.

coup2/ex19h4itin

```
! FEM-BEM coupling with Coulomb friction
open 'testc' ; open(2) 'ex19h4itcgin.dat' ;
! open(3) 'ex19h4itmcgin.dat' ; open(4) 'ex19h4itbpxin.dat'
geometry('L-Shape',styp=5,dim=(/0.25,0.25/)); #ti
problem('Laplace',nickname='SYMCCF')
#pxg 2 1 2 'Sg'
 0 2 -0.25 -0.25  0.25 -0.25   0.0 -1.0 0
 0 2 -0.25 -0.25 -0.25  0.25  -1.0  0.0 0

R=1    ! rhs
EPS=1.0d-8; EPSF=1.0d-10
```

```
GG=1.0 ! friction coefficient
RHO=25. ! damping factor for Uzawa

J=4;H=0.0625
do I=1,10
 mesh('uniform',n=J,p=1,elements='rectangles')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0) ; T[0]=SEC
 lft 16 R 3 -1
! now use an uzawa solver for the friction part

 eval('RS=Rhs(S)','no') ! save right hand side for S
 UzIT=0; clear('F'); #time T0
 do
  eval('Rhs(S)=RS-GG*Matrix(G,trans)*F','no') ! factor g=GG
  solve(eps=EPSF,mdi='x=0',mdc='no',mit='CG'); #rno.
  eval('FOLD=F','no')
  RG=RHO*GG
  eval('F=F+RG*S','no')
  frictionpro('F')
  norm('NormF','Euklid','F')
  norm('Delta','Euklid','F','FOLD')
  UzIT=UzIT+1
  ! check for convergence
  exit(Delta<=EPS*NormF) ! exit Uzawa loop
 continue
 #time T1 ; T=T1-T0
 eval('Rhs(S)=RS','no')
 Gu('u','GU'); #ju.
 xmy('D','S' ,'D','DSD'); xmy('S','SS','D','VSD'); xmy('S','SSS','S','VSV')
 Suv=DSD+2*VSD+VSV
 xy('Rhs(u)','u','uu'); xy('Rhs(S)','S','SS')
 xmy('F','G','S','COR')
 JC=GU+0.5*Suv/2-uu/2-SS/2+GG*COR/2
 JC3=JU+GG*COR
 write(2) DOF,I,T,UzIT,JU:12,NormF,Delta,JC:12,COR,JC3:12
 open(1) 'uni19u'//I; #taf. 'u'; #px. 'u' ; #cx. 'u'; close(1)
 open(1) 'uni19s'//I; #taf. 'S'; #px. 'S' ; #cx. 'S'; close(1)
 open(1) 'uni19f'//I; #taf. 'F'; #px. 'F' ; #cx. 'F'; close(1)
 J=J*2; H=H/2
continue
end
```

DOF	J	δJ	α_J	It_{Uzawa}	$\tau(s)$
28	-0.410449	0.0051273	—	2	0.0100000
80	-0.412887	0.0026888	-0.614835	2	0.0000000
256	-0.414170	0.0014058	-0.557564	2	0.0100000
896	-0.414848	0.0007278	-0.525506	2	0.0300000
3328	-0.415201	0.0003750	-0.505269	2	0.2000000
12800	-0.415382	0.0001935	-0.491126	2	1.8200000
50176	-0.415476	0.0001004	-0.480567	2	16.500000
198656	-0.415524	.5234E-04	-0.473250	2	262.12000
790528	-0.415549	.2731E-04	-0.470930	2	1572.5100

Table 6.59: FEM-BEM with friction, convergence rates and Uzawa steps

Example 6.11. *Let* $\Omega = [-\frac{1}{4}, \frac{1}{4}]^2 \backslash [0, \frac{1}{4}]^2$ *(L-Shape),* $\Omega_c = \mathbb{R}^2 \backslash \Omega$*. This example deals with a non-linear p-Laplacian FEM-BEM coupling problem, see [14].*

In this example we choose $\varrho(t) = (\varepsilon + t)^{p-2}$*, with* $p = 3$ *and* $\varepsilon = 0.00001$*,* $u_0 = r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2})$*,* $t_0 = \frac{\partial u_0}{\partial n}$ *and* $f = -\operatorname{div}(\varrho(|\nabla u_0|)\nabla u_0)$*.*

coup2/ex35h4in

```
! Laplace, non-linear, L-Shape, FEM-BEM coupling with Schur, p-Laplacian p=3
open 'test.h' ; open(2) 'ex35h4in.dat'
geometry('L-Shape',styp=5,dim=(/0.25,0.25/)); #ti
problem('Laplace',nickname='SYMCSNL')
R=50; NL=2 ! rhs, rho
P=3
setplap(p=P,eps=0.00001)
EPS=1.0d-10; Q=16
J=4
do I=2,19
 mesh('uniform',n=J,p=1,elements='rectangles')
 approx 0 R 'du' 'u0'
 clear('ux'); clear('Dx')
 NCNT=0; ITMAX=0; #time T0
defprec(mode='MG',spline='u',name='Pu',mat='I+GG',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0,keep='yes')
 do K=0,200
  if (K==0); then
   matrix('analytic',ijn=6,sigma=0.17,mu=1.0,nonlin=NL)
  else
   matrix('analytic',ijn=6,sigma=0.17,mu=1.0,nonlin=NL,arg='u.ANL(ux).u')
  fi
  lft 16 R 0 R NL

  solve(eps=EPS,mdi='x=0',mit='CG',mdc='u.Pu.u',abrflag=1,quiet=1);T=SEC
  ITMAX=Max(ITER,ITMAX); #rno.
  eval('Dx=Dx+D'); eval('ux=ux+u')
  norm('Nu','H1','u',pexp=P)
  NEWTON=Nu
  write(0) 'Newton', NEWTON, NCNT
  NCNT=NCNT+1
  exit(NEWTON<EPS*100)
 continue
 #time T1
 T[1]=T1-T0
 #err. Q R 'L2' 0 'ux' 'u' P ; E[1]=ERR
 #err. Q R 'H1' 0 'ux' 'u' P ; E[2]=ERR
 #err. Q R 'Quasi' 0 'ux' 'u' P ; E[3]=ERR
 write(2) DOF,ITER,E[1],E[2],E[3],NCNT,ITMAX,T[1]
 J=J*2
 destroyprec('Pu')
continue
end
```

DOF	$\|u-u_h\|_{W^{1,3}(\Omega)}$	α	$\|u-u_h\|_{\text{Quasi}}$	α	It_{Newton}	$\tau(s)$
21	0.1711499	—	0.1293512	—	22	0.22401
65	0.1308635	-0.238	0.0860870	-0.360	22	0.42403
225	0.1039326	-0.186	0.0612225	-0.274	23	1.66810
833	0.0826578	-0.175	0.0438478	-0.255	23	6.80443
3201	0.0657091	-0.170	0.0314280	-0.247	23	27.2897
12545	0.0522196	-0.168	0.0225589	-0.243	24	120.800
49665	0.0414910	-0.167	0.0162319	-0.239	24	560.127
197633	0.0329617	-0.167	0.0117169	-0.236	24	2678.24

Table 6.60: p-Laplacian (h-4), FEM-BEM, convergence rates

DOF	$\|u-u_h\|_{W^{1,3}(\Omega)}$	α	$\|u-u_h\|_{\text{Quasi}}$	α	η	δ_u/η	δ_q/η	It_{New}	$\tau(s)$
21	0.1945908	—	0.1510064	—	1.0265069	0.1895660	0.1471070	22	0.62004
65	0.1535874	-0.209	0.1081632	-0.295	0.6897276	0.2226783	0.1568202	22	2.21214
225	0.1219287	-0.186	0.0774765	-0.269	0.5157851	0.2363944	0.1502108	22	8.61654
833	0.0969249	-0.175	0.0555005	-0.255	0.3942211	0.2458643	0.1407852	23	35.9982
3201	0.0770270	-0.171	0.0396882	-0.249	0.3040212	0.2533606	0.1305442	23	144.185
12545	0.0611994	-0.168	0.0283778	-0.246	0.2357796	0.2595619	0.1203573	24	608.702
49665	0.0486160	-0.167	0.0203130	-0.243	0.1836408	0.2647342	0.1106127	24	2529.72
197633	0.0386151	-0.167	0.0145686	-0.241	0.1435315	0.2690357	0.1015011	24	.11E+05

Table 6.61: p-Laplacian (h-3), FEM-BEM, convergence rates, estimator η, reliability δ_u/η and δ_q/η

DOF	$\|u-u_h\|_{W^{1,3}(\Omega)}$	α	$\|u-u_h\|_{\text{Quasi}}$	α	η	δ_u/η	δ_q/η	It_{New}	$\tau(s)$
21	0.1945908	—	0.1510064	—	1.0265069	0.1895660	0.1471070	22	0.19601
32	0.1602214	-0.461	0.1205155	-0.535	0.8037363	0.1993457	0.1499441	22	0.33202
54	0.1275298	-0.436	0.0918131	-0.520	0.6027735	0.2115717	0.1523177	22	0.64804
93	0.1019990	-0.411	0.0699054	-0.501	0.4422717	0.2306252	0.1580599	22	1.13207
152	0.0821754	-0.440	0.0540462	-0.524	0.3248199	0.2529876	0.1663882	23	2.00013
249	0.0679251	-0.386	0.0449420	-0.374	0.2458318	0.2763072	0.1828161	23	3.35221
400	0.0558447	-0.413	0.0369614	-0.412	0.1902594	0.2935187	0.1942685	23	5.70036
625	0.0439784	-0.535	0.0277857	-0.639	0.1478669	0.2974188	0.1879102	24	9.89662
986	0.0352491	-0.485	0.0217361	-0.539	0.1157059	0.3046439	0.1878565	24	17.4491
1528	0.0279287	-0.531	0.0167409	-0.596	0.0907169	0.3078666	0.1845400	25	31.1619
2322	0.0222760	-0.540	0.0129489	-0.614	0.0713898	0.3120334	0.1813831	25	53.9794
3620	0.0177640	-0.510	0.0102552	-0.525	0.0562611	0.3157421	0.1822787	25	106.743
5544	0.0142059	-0.524	0.0080233	-0.576	0.0444089	0.3198886	0.1806687	25	205.341
8449	0.0112965	-0.544	0.0063426	-0.558	0.0351042	0.3217991	0.1806792	26	422.390
12810	0.0090396	-0.536	0.0050706	-0.538	0.0277777	0.3254265	0.1825421	26	1060.07
19222	0.0072288	-0.551	0.0040370	-0.562	0.0219954	0.3286505	0.1835384	26	2400.17
29006	0.0057984	-0.536	0.0032478	-0.529	0.0174348	0.3325762	0.1862826	27	5459.89
43593	0.0046615	-0.536	0.0026230	-0.524	0.0138332	0.3369792	0.1896163	27	.13E+05

Table 6.62: p-Laplacian (adaptive), FEM-BEM, convergence rates, estimator η, reliability δ_u/η and δ_q/η

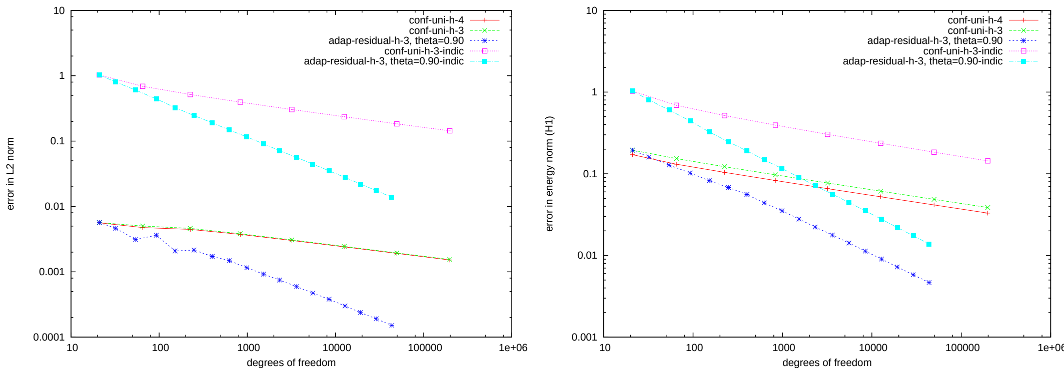


Figure 6.115: $\|u - u_n\|_{L^3(\Omega)}$ (left) and $\|u - u_n\|_{W^{1,3}(\Omega)}$ (right) — p-Laplacian.

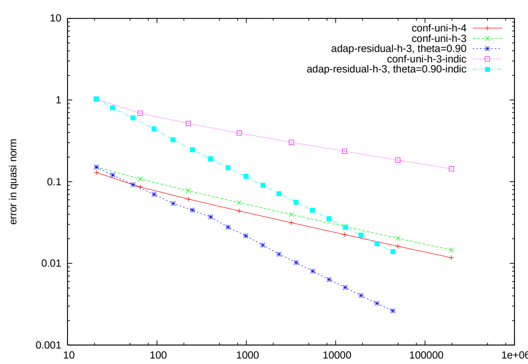


Figure 6.116: $\|u - u_n\|_{\text{Quasi}}$ — p-Laplacian.

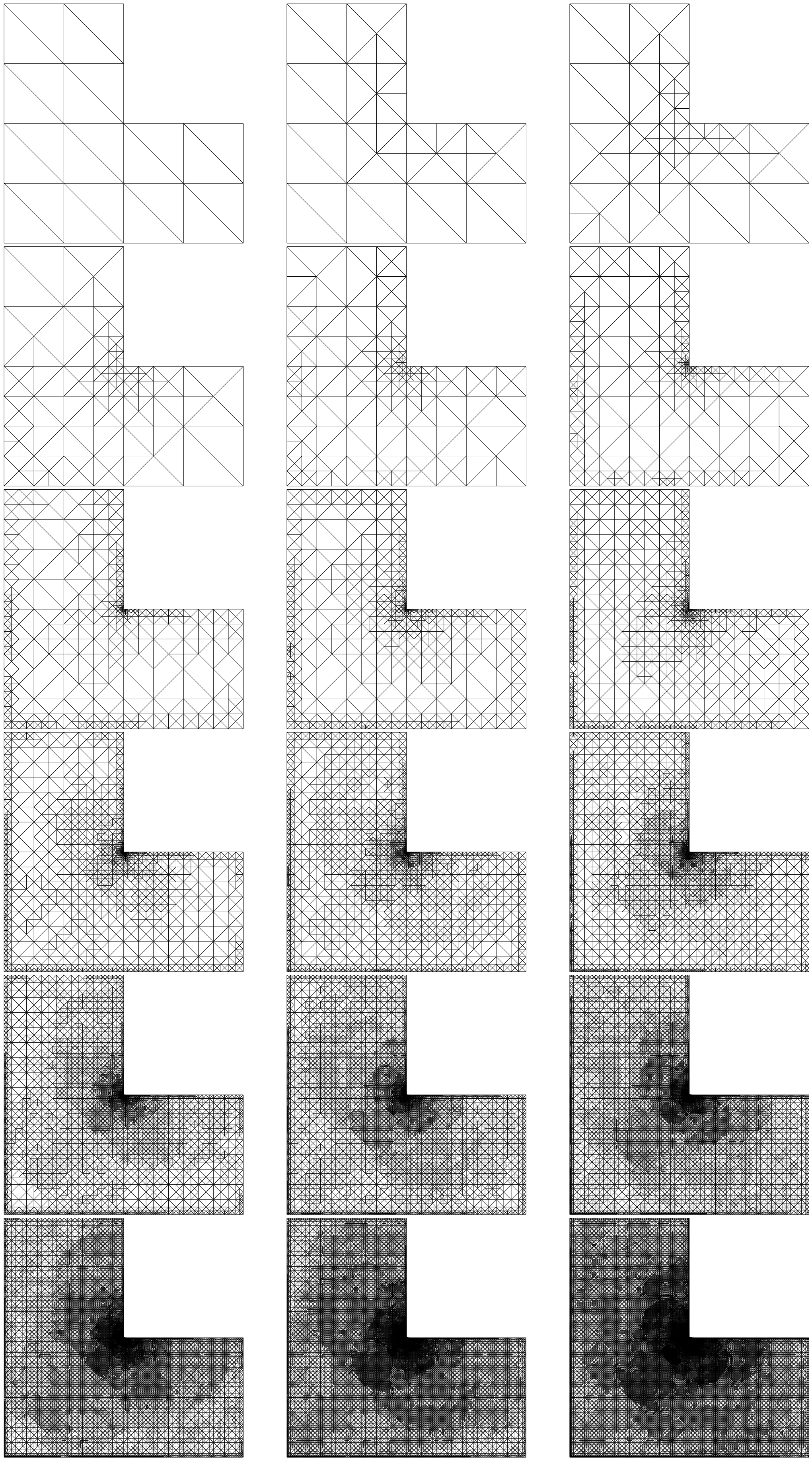


Figure 6.117: Meshes

Example 6.12. *Let* $\Omega = [-\frac{1}{4}, \frac{1}{4}]^2 \backslash [0, \frac{1}{4}]^2$ *(L-Shape),* $\Omega_c = \mathbb{R}^2 \backslash \Omega$*. This example deals with a non-linear p-Laplacian FEM-BEM coupling problem, modeling Coulomb friction, see [14], cf. [22].*

In this example we choose $\varrho(t) = (\varepsilon + t)^{p-2}$*, with* $p = 3$ *and* $\varepsilon = 0.00001$*,* $g = 0.5$*,* $f = 0$ *and* $u_0 = r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2})$*,* $t_0 = \frac{\partial u_0}{\partial n}$*.*

coup2/ex20h4itin

```
! FEM-BEM coupling with Coulomb friction, non-linear, L-Shape, p-Laplacian p=3
open 'testc' ; open(2) 'ex20h4itcgin.dat' ;
! open(3) 'ex20h4itmcgin.dat' ; open(4) 'ex20h4itbpxin.dat'
geometry('L-Shape',styp=5,dim=(/0.25,0.25/)); #ti
problem('Laplace',nickname='SYMCCFNL')
#pxg 2 1 2 'Sg'
 0 2 -0.25 -0.25  0.25 -0.25   0.0 -1.0 0
 0 2 -0.25 -0.25 -0.25  0.25  -1.0  0.0 0

R=51; NL=2 ! rhs, rho
setplap(p=3.0,eps=0.00001)
EPS=1.0d-8; EPSF=1.0d-10
GG=1.0 ! friction coefficient
RHO=25. ! damping factor for Uzawa

J=4;H=0.0625
do I=1,10
 mesh('uniform',n=J,p=1,elements='rectangles')
 approx 0 R 'du' 'u0'
 clear('ux'); clear('Dx'); clear('Sx'); clear('F')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0,nonlin=NL)
! now use an uzawa solver for the friction part
 UzIT=0; UNCNT=0; ITMAX=0; #time T0
 do
! create and solve non-linear system
  NCNT=0
  do K=0,200
   matrix('analytic',ijn=6,sigma=0.17,mu=1.0,nonlin=NL,arg='u.ANL(ux).u')
   lft 16 R 0 R NL
   eval('Rhs(S)=Rhs(S)-GG*Matrix(G,trans)*F') ! factor g=GG
   solve(eps=EPSF,mdi='x=0',mit='CG',abrflag=1,quiet=1)
   ITMAX=Max(ITER,ITMAX); #rno.
   eval('Dx=Dx+D'); eval('ux=ux+u'); eval('Sx=Sx+S')
   norm('Nu','H1','u'); norm('NS','L2','S')
   NEWTON=Sqrt(Nu*Nu+NS*NS)
   write(0) 'Newton', NEWTON, NCNT,Nu,NS
   NCNT=NCNT+1
   exit(NEWTON<EPS*100)
  continue
  UNCNT=Max(UNCNT,NCNT)

  eval('FOLD=F')
  RG=RHO*GG
  eval('F=F+RG*Sx')
  frictionpro('F')
  norm('NormF','Euklid','F')
```

DOF	J	δJ	α_J	It_{Uzawa}	$\tau(s)$
28	-0.511609	0.0172493	—	2	0.1900000
80	-0.517938	0.0109198	-0.435498	2	0.6400000
256	-0.521857	0.0070009	-0.382185	2	2.4400000
896	-0.524293	0.0045655	-0.341256	2	11.050000
3328	-0.525841	0.0030167	-0.315787	2	61.850000
12800	-0.526865	0.0019933	-0.307602	2	437.55000
50176	-0.527571	0.0012871	-0.320187	2	4218.3600

Table 6.63: p-Laplacian, FEM-BEM with friction, convergence rates and Uzawa steps

```
   norm('Delta','Euklid','F','FOLD')
   UzIT=UzIT+1
   ! check for convergence
   exit(Delta<=EPS*NormF) ! exit Uzawa loop
  continue
  #time T1 ; T[1]=T1-T0
  eval('u=ux'); eval('S=Sx')
  clear('ux'); clear('Dx'); clear('Sx')
  lft 16 R 0 R NL
  Gu('u','GU');
  xmy('D','S' ,'D','DSD'); xmy('S','SS','D','VSD');
  xmy('S','SSS','S','VSV')
  Suv=DSD+2*VSD+VSV
  xy('Rhs(u)','u','uu'); xy('Rhs(S)','S','SS')
  xmy('F','G','S','COR')
  JC=GU+0.5*Suv/2-uu/2-SS/2+GG*COR/2
  write(2) DOF,I,T[1],UzIT,UNCNT,ITMAX,JC:12,NormF,Delta,COR
  open(1) 'uni20h4u'//I; #taf. 'u'; #px. 'u' ; #cx. 'u'; close(1)
  open(1) 'uni20h4s'//I; #taf. 'S'; #px. 'S' ; #cx. 'S'; close(1)
  open(1) 'uni20h4f'//I; #taf. 'F'; #px. 'F' ; #cx. 'F'; close(1)
  J=J*2; H=H/2
continue
end
```

Example 6.13. *We solve the double-well problem (see [13])*

Find $(\hat{u}, \hat{v}) \in \mathcal{A} := \{(u,v) \in X \,:\, v \geq 0$ *and* $\langle S(u|_{\partial\Omega} + v - u_0), 1\rangle = 0$ *if* $n = 2\}$ *such that*

$$\int_\Omega DW^{**}(\nabla\hat{u})\nabla(u - \hat{u}) + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u - \hat{u})|_{\partial\Omega} + v - \hat{v}\rangle \geq \lambda(u - \hat{u}, v - \hat{v})$$

for all $(u,v) \in \mathcal{A}$. (u,v) *is a minimizer of*

$$J(u,v) = \int_\Omega W^{**}(\nabla u) + \frac{1}{2}\langle S(u|_\Gamma + v), u|_\Gamma + v\rangle - \lambda(u,v).$$

We have

$$X := W^{1,4}(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)$$
$$\lambda(u,v) = \langle t_0 + Su_0, u|_{\partial\Omega} + v\rangle + \int_\Omega fu\,dx$$

Let $A = \frac{1}{2}(F_2 - F_1)$ *and* $B = \frac{1}{2}(F_1 + F_2)$. *Let* $\mathbb{P} = \mathbb{I} - A_0 \cdot A_0^t$ *be the orthogonal projection onto the subspace of vectors orthogonal to* A, $A_0 = A \cdot |A|^{-1}$ *and* $Q(F) = \max\{0, |F - B|^2 - |A|^2\}$.

$$\begin{aligned} W^{**}(F) &= (\max\{0, |F-B|^2 - |A|^2\})^2 + 4|A|^2|F-B|^2 - 4(A(F-B))^2 \\ DW^{**}(F) &= 4[\max\{0, |F-B|^2 - |A|^2\} \cdot \mathbb{I} + 2|A|^2\mathbb{P}](F - B) \\ D^2W^{**}(F)(G,H) &= \ldots + 8(|A|^2G^t \cdot H - (A^t \cdot G)(A^t \cdot H)) \end{aligned}$$

We abbreviate the stress $\sigma := DW^{**}(\nabla\hat{u})$ *and the indicator for microstructure* $\xi := Q(\nabla\hat{u})$.

We use the a-posteriori error estimator

$$\begin{aligned} &\|\sigma - \sigma_h\|^2_{L^{\frac{4}{3}}(\Omega)} + \|(\hat{u} - \hat{u}_h)|_\Gamma + \hat{v} - \hat{v}_h\|^2_{H^{1/2}(\Gamma)} + \|\mathbb{P}\nabla\hat{u} - \mathbb{P}\nabla\hat{u}_h\|^2_{L^2(\Omega)} \\ &+ \|(\xi + \xi_h)^{1/2}A\nabla(\hat{u} - \hat{u}_h)\|^2_{L^2(\Omega)} + \|\xi - \xi_h\|^2_{L^2(\Omega)} \\ &\qquad \lesssim \eta_\Omega^2 + \eta_C^2 + \eta_S^2 + \operatorname{dist}_{\tilde{H}^{-1/2}(\Gamma)}(V^{-1}(I - K)(\hat{u}_h|_\Gamma + \hat{v}_h - u_0), W_h^{-\frac{1}{2}}(\Gamma))^2 \end{aligned}$$

with

$$\begin{aligned} \eta_\Omega^2 &= \sum_K h_K\|f\|_{L^{4/3}(K)} + \sum_{E \cap \Gamma = \emptyset} h_E\|[\nu_E \cdot \sigma_h]\|_{L^2(E)}, \\ \eta_C^2 &= \sum_{E \subset \Gamma_s} \|(\nu_E \cdot \sigma_h)_+\|_{L^2(E)} + \sum_{E \subset \Gamma_s} \int_E (\nu_E \cdot \sigma_h)_- \hat{v}_g, \\ \eta_S^2 &= \sum_{E \subset \Gamma} h_E^{1/2}\|S_h(\hat{u}_h|_\Gamma + \hat{v}_h - u_0) + (v_\Gamma \cdot \sigma_h) - t_0\|_{L^2(E)} \end{aligned}$$

Let $\Omega = (0,1)^2$, $F_1 = (-1,0)$, $F_2 = (1,0)$ *and*

$$f_0(x) = -\frac{3}{128}(x - 0.5)^5 - \frac{1}{3}(x - 0.5)^3, \qquad \bar{f}(x,y) = f_0(x).$$

We define

$$\bar{u}(x,y) = \begin{cases} f_0(x) & \text{for } 0 \leq x \leq 1/2, \\ \frac{1}{24}(x - 0.5)^3 + x - 0.5 & \text{for } 1/2 \leq x \leq 1. \end{cases}$$

We set $\Gamma_s = \emptyset$, $f \equiv 0$, $u_0 := \bar{u}|_{\partial\Omega}$, $t_0 := \frac{\partial}{\partial n}\bar{u}$ *and* $\alpha = 1$.

DOF	$J(u_h)$	$J(u_h) - J(u)$	γ	It_{New}	$\tau(s)$
25	-1.39482	0.22812	—	8	0.2899559
81	-1.41004	0.14695	-0.374078	10	0.1779729
289	-1.41646	0.09359	-0.354741	12	0.7238900
1089	-1.41924	0.05662	-0.378793	12	3.2235100
4225	-1.42023	0.03489	-0.357151	14	19.724001
16641	-1.42060	0.02173	-0.345502	19	209.54514
66049	-1.42075	0.01336	-0.352715	29	3209.6921

Table 6.64: Experimental convergence rates and Newton iteration numbers (uniform mesh)

Example 6.14. ***Annular coupling*** *Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ_0. Let Ω be an annular domain bounded by Γ_0 and the circular boundary Γ, whose interior region contains Ω_0. Then, given $f \in L^2(\Omega)$ we consider the exterior transmission problem: Find $u_1 \in H^1(\Omega)$, $u_2 \in H^1_{loc}(\mathbb{R}^2 \backslash (\Omega_0 \cup \Omega))$ such that*

$$
\begin{aligned}
u_1 &= 0 \textit{ on } \Gamma_0 \\
\Delta u_1 &= f \textit{ in } \Omega \\
u_1 &= u_2 \textit{ on } \Gamma \\
\frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} \textit{ on } \Gamma \\
-\Delta u_2 &= 0 \textit{ in } \mathbb{R}^2 \backslash (\Omega_0 \cup \Omega) \\
u_2(x) &= \mathcal{O}(1) \textit{ as } \|x\| \to \infty
\end{aligned}
$$

coup2/ex38a00in

```
open(1) 'test.h' ; open(2) 'ex38a00in.dat' ;
geometry('AnnularSquare',bmode=(/2,2/)) ; #ti
problem('Laplace',nickname='SYMCW')

#pxbd 4 1 2 'ubd'
0 2 0.5 0.5 0.5 -0.5 -2
0 2 0.5 -0.5 -0.5 -0.5 -2
0 2 -0.5 -0.5 -0.5 0.5 -2
0 2 -0.5 0.5 0.5 0.5 -2

R=32   ! right hand side
Q=8
J=4;H=0.0625
mesh('uniform',n=J,p=1,elements='triangles')
do I=1,10
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
 lft 16 R 0 R

 solve(eps=1.0d-10,mdi='x=0',mit='CG',abrflag=1,quiet=1); T=SEC
 #rno.
 open(1) 'ex38a00u'//I
 #taf. 'u'
 #pnod. 'u'
 #cx. 'u'
 #err. Q R 'L2' 0 'u' ; E[1]=ERR ! FEM
 #err. Q R 'H1' 0 'u' ; E[2]=ERR
 adap(theta=0.0,admode=2,gq=4,tlocfs=0,mode='h',spline='u')
 #no. 'L2' 'u'
 #no. 'H1' 'u'
 close(1)
 write(2) DOF,ITER,E[1],E[2],T,ERREST,LAPUF,JUMP,TUW
 refine('u')
continue
end
```

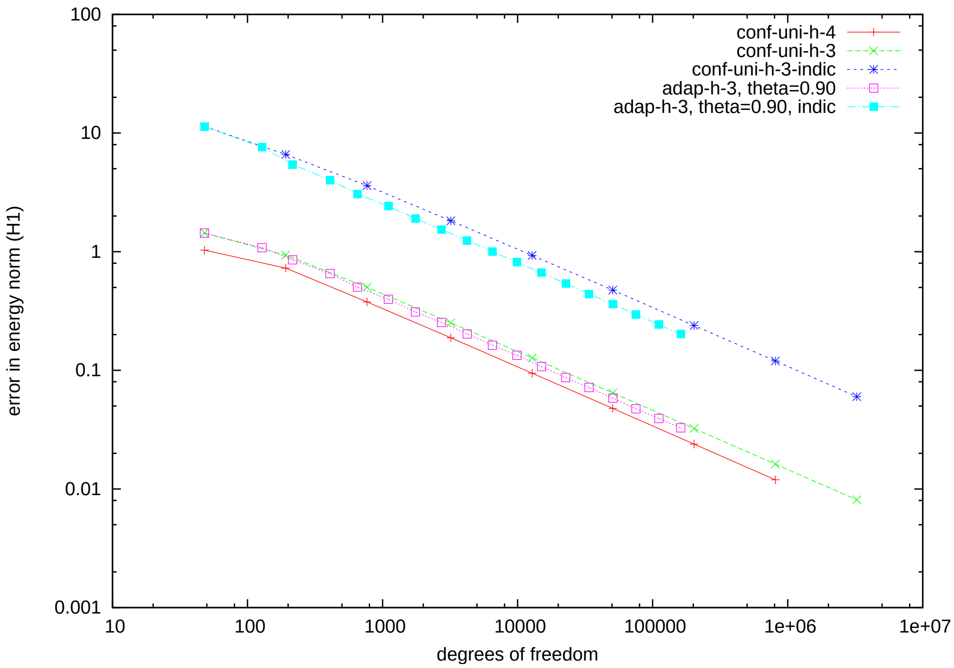


Figure 6.118: $\|u - u_n\|_{H^1(\Omega)}$

6.1.2 Stokes

Example 6.15. *Here we investigate the linear Stokes problem (see [2])*

$$\begin{aligned} -\nu\,\mathrm{div}(\nabla\vec{u}) + \nabla p &= f \ \ in\ \Omega \\ -\nu\,\mathrm{div}(\nabla\vec{u}) + \nabla p &= 0 \ \ in\ \mathbb{R}^2\backslash(\Omega\cup\Omega_0) \\ \mathrm{div}\,\vec{u} &= 0 \ \ in\ \mathbb{R}^2\backslash(\Omega\cup\Omega_0) \\ \vec{u} &= u_0 \ \ on\ \Gamma_0 \\ \vec{u} &= \Sigma\log|x| + \mathcal{O}(1) \ \ as\ |x|\to\infty \\ p &= \mathcal{O}(|x|^{-1}) \ \ as\ |x|\to\infty \end{aligned}$$

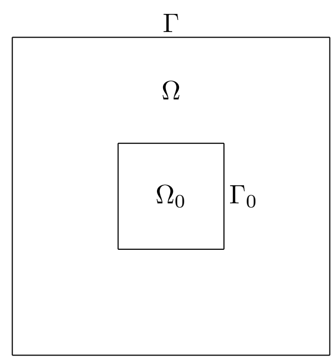
using a dual-dual mixed formulation with FEM-BEM coupling.

Find $((t,\phi),(\sigma,p),(u,\gamma,\xi,\alpha)) \in X_1\times M_1\times M$ *such that*

$$\begin{aligned} &2\nu\int_\Omega t:s + \langle\tilde{W}\phi,\mu\rangle - \int_\Omega \sigma:s - \int_\Omega p\,\mathrm{tr}(s) + \langle\Big(\frac{I}{2}+K'\Big)(\sigma\,n),\mu\rangle = 0 \\ &-\int_\Omega t:\tau - \int_\Omega q\,\mathrm{tr}(t) + \langle\tau n,\Big(\frac{I}{2}+K\Big)\,\phi\rangle - \langle\tau n,\tilde{V}(\sigma\,n)\rangle - \int_\Omega u\,\mathrm{div}\,\tau - \int_\Omega \gamma:\tau + \xi\int_\Omega \mathrm{tr}(\tau) + \langle\tau n,\alpha\rangle = \langle\tau n,u_0\rangle_{\Gamma_0} \\ &-\int_\Omega v\,\mathrm{div}\,\sigma + \eta\int_\Omega \mathrm{tr}(\sigma) - \int_\Omega \sigma:\delta + \langle\sigma n,\beta\rangle = \beta\Sigma + \int_\Omega fv, \end{aligned}$$

for all $((s,\mu),(\tau,q),(v,\delta,\eta,\beta)) \in X_1\times M_1\times M$ *with* $X_1 = [L^2(\Omega)]^{2\times 2}\times\big[H^{1/2}(\Gamma)\big]^2$, $M_1 = H(\mathrm{div};\Omega)\times L^2(\Omega)$, $M = [L^2(\Omega)]^2\times H_0\times\mathbb{R}\times\mathbb{R}^2$.

The exact solution of our model problem is given by $\nu = 4$ *and* $\vec{u} = \frac{1}{\nu}(-\log r + (x_0-\bar{x}_0)^2/r^2, (x_0-\bar{x}_0)(x_1-\bar{x}_1)/r^2)$ *and* $p = 2(x_0-\bar{x}_0)/r^2$ *with* $r = |x-\bar{x}|$ *and* $\bar{x} = (0, 0.5)$, *such that* $f\equiv 0$.



Square domain with hole

We introduce the following notation for the components of the error estimator.

$$\eta_t^2 = \sum_T \|\boldsymbol{\sigma}_h - (2\nu\mathbf{t}_h - p_h\mathbf{I})\|^2_{[L^2(T)]^{2\times 2}} \qquad \eta_{tr}^2 = \sum_T \|\mathrm{tr}(\mathbf{t}_h)\|^2_{L^2(T)}$$

$$\eta_{t\gamma\varphi}^2 = \sum_T \|\mathbf{t}_h + \boldsymbol{\gamma}_h - \nabla\boldsymbol{\varphi}_h\|^2_{[L^2(T)]^{2\times 2}} \qquad \eta_{\sigma\sigma}^2 = \sum_T \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^T\|^2_{[L^2(T)]^{2\times 2}}$$

$$\eta_{u-\varphi}^2 = \sum_T \|u_h - \varphi_h\|^2_{[L^2(T)]^2} \qquad \eta_{g-\varphi}^2 = \sum_{e\subset\Gamma_0} \|g - \varphi_h\|^2_{H^{1/2}(e)}$$

$$\eta_V^2 = \sum_{e\subset\Gamma} \|\boldsymbol{\varphi}_h + \tilde{\mathbf{V}}(\boldsymbol{\sigma}_h\mathbf{n}) - (\frac{1}{2}\mathbf{I}+\mathbf{K})\phi_h - \alpha_h\|^2_{H^{1/2}(e)} \qquad \eta_\xi^2 = \sum_T h_T^2|\xi_h|^2$$

$$\eta_W^2 = \sum_{e\subset\Gamma} \|\tilde{\mathbf{W}}\phi_h + (\frac{1}{2}\mathbf{I}+\mathbf{K}')(\boldsymbol{\sigma}_h\mathbf{n})\|^2_{L^2(e)} \qquad \eta_f^2 = \sum_T \|\mathrm{div}\,\boldsymbol{\sigma}_h - \mathbf{f}\|^2_{[L^2(T)]^2}$$

DOF	δ_t	α_t	δ_σ	α_σ	δ_u	α_u	δ_p	α_p	η	α_η
1275	0.29794		2.94644		0.25484		1.22479		8.97907	
4915	0.18076	0.370	1.76079	0.382	0.12844	0.508	0.71036	0.404	6.45700	0.244
19299	0.09992	0.433	0.96970	0.436	0.06420	0.507	0.38817	0.442	4.50208	0.264

Table 6.65: Uniform mesh, errors and convergence rates

We use the following abbrevations for the different errors $\delta_t = \|\mathbf{t} - \mathbf{t}_h\|_{L^2(\Omega)}$, $\delta_\sigma = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathrm{div};\Omega)}$, $\delta_u = \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$, $\delta_p = \|p - p_h\|_{L^2(\Omega)}$.

DOF	η_t	η_{tr}	$\eta_{\sigma\sigma}$	$\eta_{t\gamma\varphi}$	$\eta_{u-\varphi}$	$\eta_{g-\varphi}$	η_ξ	η_f	η_V	η_W
1275	.6E-10	.3E-10	1.9053	0.8485	0.2545	0.2259	0.0170	.1E-10	0.3616	8.7193
4915	.3E-09	.1E-09	1.3500	0.5685	0.1313	0.0881	0.0013	.2E-10	0.3011	6.2795
19299	.4E-08	.2E-08	0.7950	0.5272	0.0654	0.0311	.6E-04	.2E-09	0.2839	4.3901

Table 6.66: Uniform mesh, error indicators

coup2/ex54h3in

```
! Stokes on Square with Hole (dual-dual-mixed, fem-bem, stress tensor)
open(1) 'test.h'; open(2) 'ex54h3in.dat'; #ti

problem('Stokes',nickname='DDMIX2C')
geometry('Square with Hole')
#pxg 4 1 2 'Ng'
0 2 -3. -3.  3. -3.  0. -1. 0
0 2  3. -3.  3.  3.  1.  0. 0
0 2  3.  3. -3.  3.  0.  1. 0
0 2 -3.  3. -3. -3. -1.  0. 0

#pxbd 4 1 2 'ubd'
 0 2 -1. -1.  1. -1. -2
 0 2  1. -1.  1.  1. -2
 0 2  1.  1. -1.  1. -2
 0 2 -1.  1. -1. -1. -2

NU=4.; #stokes NU
R=11
J=2
do I=0,8
 mesh('uniform',n=J,p=0,elements='triangles',spline='u',gm='ug',genspl='no')
 mesh('global',n=1,spline='alpha',gm='Ng',genspl='no')
 mesh('global',n=1,spline='xi',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix
 lft 16 R 0 R
 load('ex54auxin')

 solve(eps=1.e-10,mdc='no',mdi='x=1',mit='MINRES',quiet=0,restart=400); T=SEC
 #rno.
 show('matrix')
 #taf. 'u'; #px. 'u'; #cx. 'u'
 #taf. 'p'; #px. 'p'; #cx. 'p'
 #taf. 'xi'; #px. 'xi'; #cx. 'xi'
 #taf. 'alpha'; #px. 'alpha'; #cx. 'alpha'; #lx. 'alpha'
 #err. 16 R 'L2' 0 'u' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u' 'u'; E[2]=ERR
 #err. 16 R 'L2' 0 'p' 'p'; E[3]=ERR
 #err. 16 R 'L2' 0 't' 'tstrain'; E[4]=ERR
 #err. 16 R 'L2' 0 'sigma' 'sigma' ; E[5]=ERR
 #err. 16 R 'Hdiv' 0 'sigma' 'sigma' ; E[6]=ERR
 #err. 16 R 'L2' 1 'D' 'u0'; E[7]=ERR
 #err. 16 R 'L2' 0 'N' 't0'; E[8]=ERR

 write(2) DOF,E[1],E[2],E[3],E[4],E[5],E[6],E[7],E[8],T,ITER
 J=J*2
continue
end
```

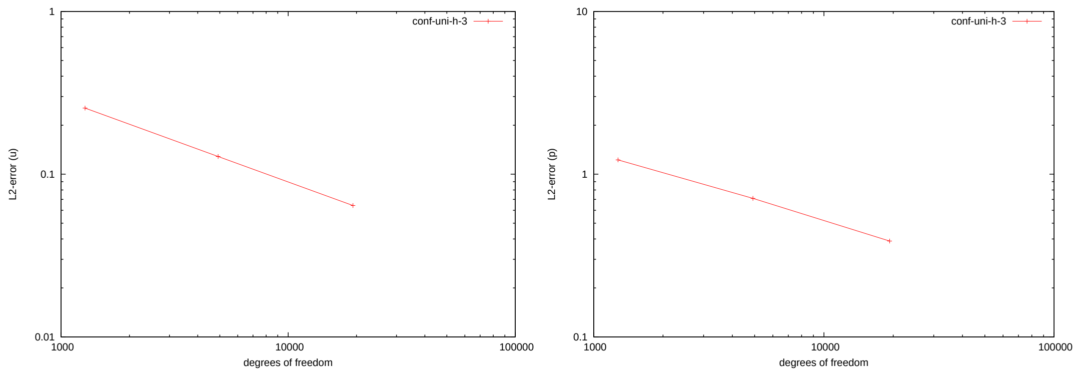


Figure 6.119: Stokes (2d-FEM-BEM): $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|p - p_n\|_{L^2(\Omega)}$ (right).

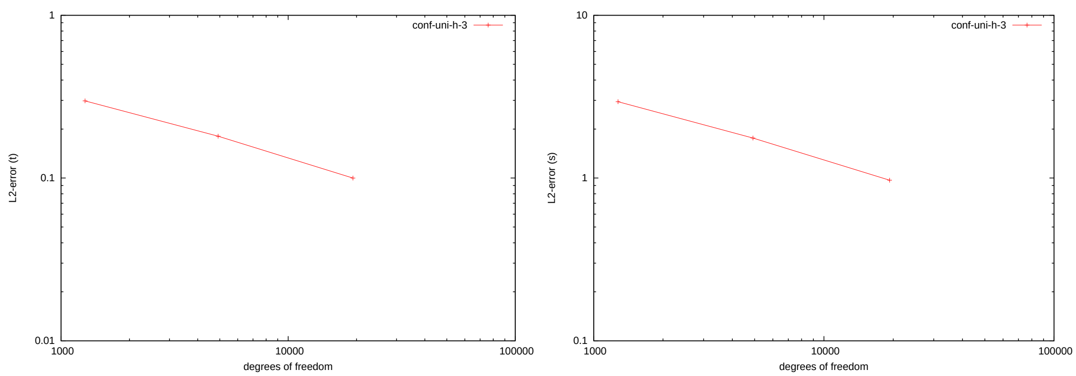


Figure 6.120: Stokes (2d-FEM-BEM): $\|t - t_n\|_{L^2(\Omega)}$ (left) and $\|\sigma - \sigma_n\|_{L^2(\Omega)}$ (right).

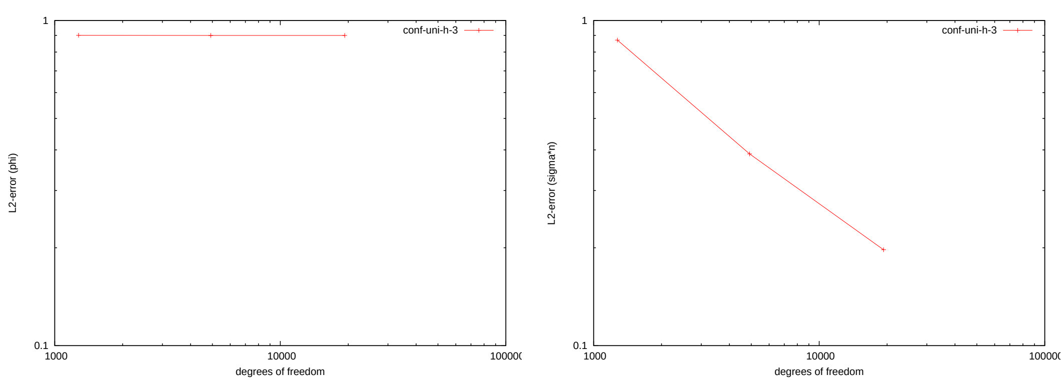


Figure 6.121: Stokes (2d-FEM-BEM): $\|\phi - \phi_n\|_{L^2(\Gamma)}$ (left) and $\|\sigma \cdot \vec{n} - \sigma_n \cdot \vec{n}\|_{L^2(\Gamma)}$ (right).

Example 6.16. *Here we investigate the previous example with the exact solution with* $\bar{x} = (0, 0.9)$*. The adaptive version is computed with* $\vartheta = 0.90$ *using a blue-green refinement strategy.*

DOF	δ_t	α_t	δ_σ	α_σ	δ_u	α_u	δ_p	α_p	η	α_η
1275	0.47205		5.22012		0.30461		2.54835		9.60597	
4915	0.34589	0.230	3.64023	0.267	0.15868	0.483	1.67247	0.312	7.31227	0.202
19299	0.25059	0.236	2.47140	0.283	0.07942	0.506	1.02198	0.360	5.20241	0.249

Table 6.67: Uniform mesh, errors and convergence rates

DOF	η_t	η_{tr}	$\eta_{\sigma\sigma}$	$\eta_{t\gamma\varphi}$	$\eta_{u-\varphi}$	$\eta_{g-\varphi}$	η_ξ	η_f	η_V	η_W
1275	.4E-10	.5E-10	2.5384	1.1158	0.3045	0.2672	0.0095	.2E-10	0.3330	9.1821
4915	.3E-09	.2E-09	2.1219	1.0145	0.1652	0.2689	0.0055	.6E-10	0.2928	6.9103
19299	.2E-07	.2E-07	1.7553	0.8426	0.0815	0.1942	0.0015	.1E-08	0.2745	4.8119

Table 6.68: Uniform mesh, error indicators

DOF	δ_t	α_t	δ_σ	α_σ	δ_u	α_u	δ_p	α_p	η	α_η
1969	0.47205		5.22012		0.30461		2.54835		9.60597	
3096	0.47564	-0.02	5.24581	-0.01	0.28608	0.139	2.55338	-0.00	7.97700	0.411
4888	0.44543	0.144	4.74346	0.220	0.25021	0.293	2.21382	0.312	7.14214	0.242
8125	0.39247	0.249	4.12920	0.273	0.22391	0.219	1.89634	0.305	5.90371	0.375
13010	0.34880	0.251	3.57556	0.306	0.19065	0.342	1.58084	0.387	5.19833	0.270
21222	0.27424	0.491	2.87082	0.449	0.15976	0.361	1.30923	0.385	4.24730	0.413

Table 6.69: Adaptive version, $\vartheta = 0.9$, errors and convergence rates

DOF	η_t	η_{tr}	$\eta_{\sigma\sigma}$	$\eta_{t\gamma\varphi}$	$\eta_{u-\varphi}$	$\eta_{g-\varphi}$	η_ξ	η_f	η_V	η_W
1969	.8E-14	.3E-14	2.5384	1.1158	0.3045	0.2672	0.0095	.1E-13	0.3330	9.1821
3096	.9E-14	.8E-14	2.5311	1.1448	0.2951	0.2672	0.0145	.8E-14	0.3005	7.4610
4888	.9E-14	.2E-13	2.5607	1.1767	0.2338	0.2779	0.0138	.5E-14	0.2907	6.5461
8125	.1E-12	.3E-13	1.9425	1.0319	0.1957	0.2470	0.0023	.6E-13	0.2845	5.4622
13010	.6E-12	.1E-12	2.0909	0.8329	0.1718	0.2299	0.0021	.6E-13	0.2728	4.6691
21222	.1E-11	.3E-12	1.3139	0.8335	0.1437	0.1907	0.0006	.1E-12	0.2855	3.9344

Table 6.70: Adaptive version, $\vartheta = 0.9$, error indicators

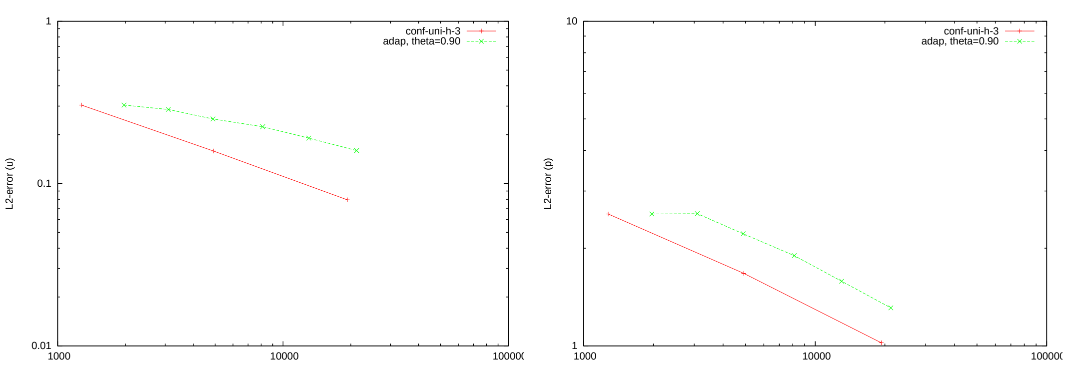


Figure 6.122: Stokes (2d-FEM-BEM): $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|p - p_n\|_{L^2(\Omega)}$ (right).

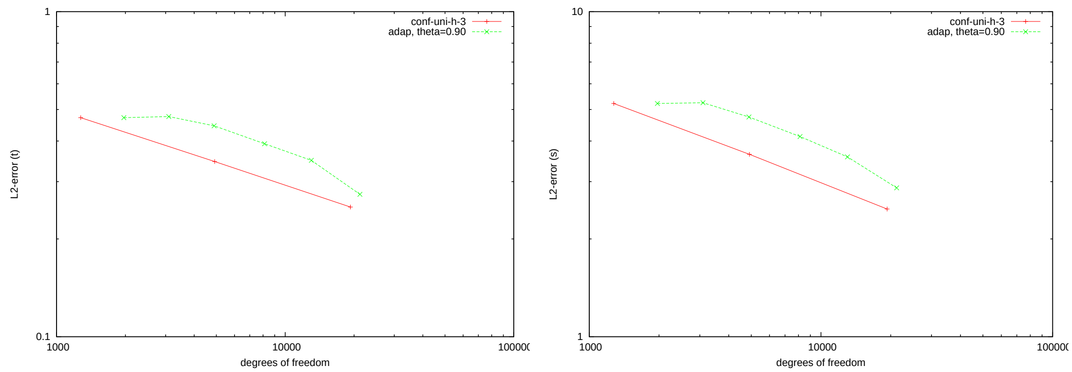


Figure 6.123: Stokes (2d-FEM-BEM): $\|t - t_n\|_{L^2(\Omega)}$ (left) and $\|\sigma - \sigma_n\|_{L^2(\Omega)}$ (right).

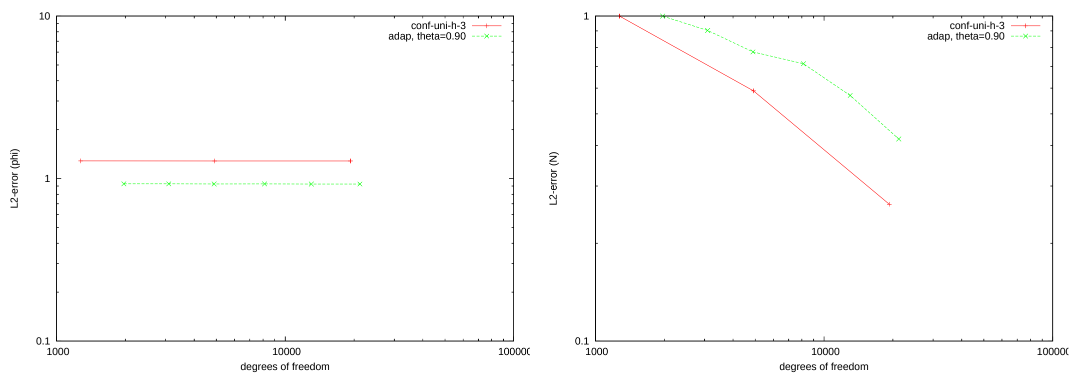


Figure 6.124: Stokes (2d-FEM-BEM): $\|\phi - \phi_n\|_{L^2(\Gamma)}$ (left) and $\|\sigma \cdot \vec{n} - \sigma_n \cdot \vec{n}\|_{L^2(\Gamma)}$ (right).

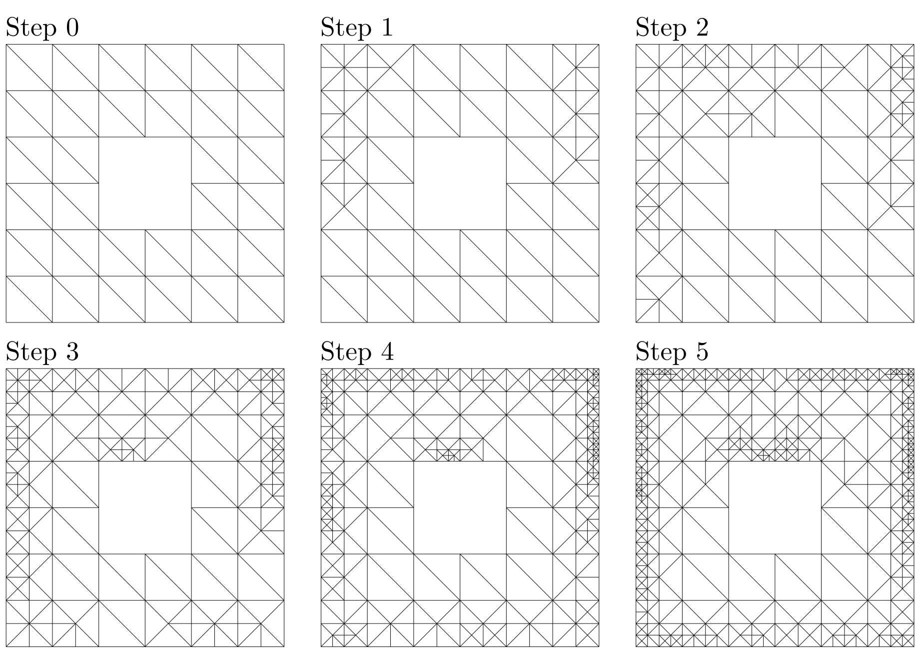


Figure 6.125: Adaptively refined meshes

Example 6.17. *Here we investigate the coupling between a linear Stokes flow with a quasi-Newtonian flow and mixed boundary conditions (see [3]).*

$$
\begin{aligned}
-\operatorname{div}(\sigma_0(\vec{u}_0,p_0)) &= 0 \ \textit{in}\ \Omega_0\\
-\operatorname{div}(\sigma(\vec{u},p)) &= 0 \ \textit{in}\ \Omega\\
\operatorname{div}\vec{u}_0 &= 0 \ \textit{in}\ \Omega_0\\
\operatorname{div}\vec{u} &= 0 \ \textit{in}\ \Omega\\
\vec{u} &= \vec{u}_0 \ \textit{on}\ \Gamma\\
\sigma_0(\vec{u}_0,p_0)\cdot\vec{n} &= \sigma(\vec{u},p)\cdot\vec{n} \ \textit{on}\ \Gamma\\
\vec{u} &= 0 \ \textit{on}\ \Gamma_D\\
\sigma(\vec{u},p)\cdot\vec{n} &= g \ \textit{on}\ \Gamma_N
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_0(\vec{u}_0,p_0) &= -p_0 I + 2\nu e(\vec{u}_0)\\
\sigma(\vec{u},p) &= -pI + k(|e(\vec{u})|)e(\vec{u})
\end{aligned}
$$

using a dual-dual mixed formulation with FEM-BEM coupling.

Find $((t,\phi),(p,\sigma),(u,\gamma,\psi,\xi,\alpha)) \in X_1 \times M_1 \times M$ *such that*

$$
\begin{aligned}
&\int_\Omega \kappa(t):s + \langle \tilde{W}\phi,\mu\rangle \quad -\int_\Omega \sigma:s - \int_\Omega p\,\mathrm{tr}(s) + \langle \Big(\frac{I}{2}+K'\Big)(\sigma\, n),\mu\rangle &&= 0\\
&-\int_\Omega t:\tau - \int_\Omega q\,\mathrm{tr}(t) + \langle \tau n, \Big(\frac{I}{2}+K\Big)\phi\rangle \quad -\langle \tau n, \tilde{V}(\sigma\, n)\rangle \quad -\int_\Omega u \operatorname{div}\tau + \xi\int_\Omega \mathrm{tr}(\tau) - \int_\Omega \gamma:\tau\\
&\qquad + \langle \tau n,\alpha\rangle - \langle \tau n,\psi\rangle_{\Gamma_N} &&= 0\\
&-\int_\Omega v \operatorname{div}\sigma + \eta\int_\Omega \mathrm{tr}(\sigma) - \int_\Omega \sigma:\delta\\
&\qquad + \langle \sigma n,\beta\rangle + \langle \sigma n,\nu\rangle_{\Gamma_N} &&= \int_\Omega f v + \langle g,\nu\rangle_{\Gamma_N},
\end{aligned}
$$

for all $((s,\mu),(q,\tau),(v,\delta,\nu,\eta,\beta)) \in X_1 \times M_1 \times M$ *with* $X_1 = [L^2(\Omega)]^{2\times 2} \times \left[H^{1/2}(\Gamma)\right]^2$, $M_1 = L^2(\Omega) \times H(\mathrm{div};\Omega)$, $M = [L^2(\Omega)]^2 \times H_0 \times [\tilde{H}^{1/2}(\Gamma_N)]^2 \times \mathbb{R} \times \mathbb{R}^2$.

coup2/ex56h3in

```
! Stokes on Square with Hole (dual-dual-mixed, fem-bem, stress tensor, quasi-Newton, inner problem)
open(1) 'test.h'; open(2) 'ex56h3in.dat'; #ti

problem('Stokes',nickname='DMIXQNLINm')
geometry('Square with Hole')
#pxg 4 1 2 'Ng'
0 2 -1. -1.  1. -1.  0.  1. 0
0 2  1. -1.  1.  1. -1.  0. 0
0 2  1.  1. -1.  1.  0. -1. 0
0 2 -1.  1. -1. -1.  1.  0. 0

#pxbd 4 1 2 'ubd'
```

```
 0 2 -3. -3.  3. -3. -3
 0 2  3. -3.  3.  3. -3
 0 2  3.  3. -3.  3. -2
 0 2 -3.  3. -3. -3. -3

#pxg 3 1 2 'psig'
 0 2 -3  3 -3 -3 -1  0 0
 0 2 -3 -3  3 -3  0 -1 0
 0 2  3 -3  3  3  1  0 0

NU=4.; #stokes NU
R=20
setstokesx( (/0.0,4.0/) )

J=2
do I=0,8
 mesh('uniform',n=J,p=0,elements='triangles',spline='u',gm='ug',genspl='no')
 mesh('global',n=1,spline='alpha',gm='Ng',genspl='no')
 mesh('global',n=1,spline='xi',gm='ug')
! approx 0 R 'u_bd' 'u0'
 matrix
 setstokesu('D',10)
 setstokesp('N',10)
 lft 16 R 0 R
! load('ex56auxin')

 solve(eps=1.e-10,mdc='no',mdi='x=1',mit='MINRES',quiet=0,restart=400); T=SEC

 #rno.
 #taf. 'u'; #px. 'u'; #cx. 'u'
 #taf. 'p'; #px. 'p'; #cx. 'p'
 #taf. 'xi'; #px. 'xi'; #cx. 'xi'
 #taf. 'alpha'; #px. 'alpha'; #cx. 'alpha'; #lx. 'alpha'
 #err. 16 R 'L2' 0 'u' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u' 'u'; E[2]=ERR
 #err. 16 R 'L2' 0 'p' 'p'; E[3]=ERR
 #err. 16 R 'L2' 0 't' 'tstrain'; E[4]=ERR
 #err. 16 R 'L2' 0 'sigma' 'sigma' ; E[5]=ERR
 #err. 16 R 'Hdiv' 0 'sigma' 'sigma' ; E[6]=ERR
 #err. 16 R 'L2' 0 'D' 'u0'; E[7]=ERR
 #err. 16 R 'L2' 0 'N' 't0'; E[8]=ERR
 #no. 'L2' 'psi'; E[9]=NORM
 stokesphi('D')
 stokestau('N')


 write(2) DOF,E[1],E[2],E[3],E[4],E[5],E[6],E[7],E[8],E[9],T,ITER,RNORM
 J=J*2
continue
end
```

Example 6.18. *In this example we study a finite domain.*

DOF	δ_t	α_t	δ_σ	α_σ	δ_u	α_u	δ_p	α_p	δ_ϕ	α_ϕ
1251	0.12917		1.24181		0.10325		0.48696		0.00713	
4861	0.07582	0.393	0.72515	0.396	0.05394	0.478	0.28102	0.405	0.00191	0.969
19185	0.04078	0.452	0.38810	0.455	0.02735	0.495	0.14867	0.464	0.00090	0.552
76249	0.02088	0.485	0.19823	0.487	0.01338	0.518	0.07549	0.491	0.00025	0.914

Table 6.71: Uniform mesh, errors and convergence rates

We use the following abbrevations for the different errors $\delta_t = \|\mathbf{t} - \mathbf{t}_h\|_{L^2(\Omega)}$, $\delta_\sigma = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathrm{div};\Omega)}$, $\delta_u = \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$, $\delta_p = \|p - p_h\|_{L^2(\Omega)}$, $\delta_\phi = \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{L^2(\Gamma)}$.

6.2 Solvers

Example 6.19. *The current example illustrates the action of various preconditioners for the symmetric coupling problem of Example 6.1 using the three block structure of the corresponding matrix.*

	unpreconditioned			
N	λ_1	λ_2	λ_3	λ_4
21 + 16	-0.281397	-0.003829019	0.1689593	6.72458
65 + 32	-0.121623	-0.000999241	0.0630029	7.52636
225 + 64	-0.051534	-0.000255293	0.0214018	7.85703
833 + 128	-0.021990	-.643860E-04	0.0067335	7.96104

Table 6.72: Extreme eigenvalues of the unpreconditioned matrix

	Multigrid			
N	λ_1	λ_2	λ_3	λ_4
21 + 16	-1.512749	-0.998895141	0.2066764	5.25391
65 + 32	-1.391804	-0.996934960	0.1060927	5.20907
225 + 64	-1.355113	-0.994387678	0.0540560	5.22073
833 + 128	-1.346358	-0.993351037	0.0272909	5.22454

Table 6.73: Extreme eigenvalues of the Multigrid preconditioner (3 block)

	BPX			
N	λ_1	λ_2	λ_3	λ_4
21 + 16	-1.322343	-0.464617902	0.2451262	6.73274
65 + 32	-1.538297	-0.505876002	0.1478522	10.6177
225 + 64	-1.765061	-0.476756998	0.0831051	13.3739
833 + 128	-1.907576	-0.448575078	0.0445902	15.4471

Table 6.74: Extreme eigenvalues of the BPX preconditioner (3 block)

	Hierarchical			
N	λ_1	λ_2	λ_3	λ_4
21 + 16	-6.539824	-0.215031591	0.3966674	6.72956
65 + 32	-6.305538	-0.245673936	0.1973150	8.81948
225 + 64	-6.190308	-0.182472969	0.0980939	10.4027
833 + 128	-6.134569	-0.149238689	0.0488286	11.9389

Table 6.75: Extreme eigenvalues of the Hierarchical preconditioner (3 block)

Example 6.20. *The current example illustrates the action of various preconditioners for the symmetric coupling problem of Example 6.1 using the two block structure of the corresponding matrix by merging the Galerkin matrix of the hyper singular operator into the fem matrix.*

	Multigrid + 1 Mass-matrix			
N	λ_1	λ_2	λ_3	λ_4
21 + 16	-7.665445	-0.998889207	1.6177304	6.70327
65 + 32	-7.672777	-0.996805664	1.3520912	6.71657
225 + 64	-7.676337	-0.994296562	1.2574645	6.72103
833 + 128	-7.677938	-0.993298672	1.2291103	6.72224

Table 6.76: Extreme eigenvalues of the Multigrid preconditioner (2 block)

	BPX			
N	λ_1	λ_2	λ_3	λ_4
21 + 16	-1.442482	-0.470800305	0.5771855	9.04521
65 + 32	-1.789122	-0.510989128	0.6471639	11.6533
225 + 64	-2.038636	-0.519039068	0.6753368	13.8683
833 + 128	-2.225165	-0.520506117	0.6870637	15.8023

Table 6.77: Extreme eigenvalues of the BPX preconditioner (2 block)

	Hierarchical			
N	λ_1	λ_2	λ_3	λ_4
21 + 16	-6.539824	-0.215031591	0.3966674	6.72956
65 + 32	-6.625417	-0.245702503	0.3791439	9.30444
225 + 64	-6.672382	-0.182927313	0.3690236	10.9913
833 + 128	-6.697843	-0.149998967	0.3546350	12.3795

Table 6.78: Extreme eigenvalues of the Hierarchical preconditioner (2 block)

Example 6.21. *Here we investigate the condition number of the preconditioned matrix from the FEM-BEM coupling with Signorini interface described in [7].*

$$\mathcal{A}_h := \begin{pmatrix} A & B & 0 \\ B^T & C + S_{\Gamma\Gamma} & S_{S\Gamma} \\ 0 & S_{S\Gamma}^T & S_{SS} \end{pmatrix}$$

We are using Multigrid and BPX as preconditioners for the $\begin{pmatrix} A & B \\ B^T & C + S_{\Gamma\Gamma} \end{pmatrix}$*-block and the* S_{SS}*-block.*

	Unpreconditioned		
N	$\lambda_{\min}$	$\lambda_{\max}$	κ
13 + 8	0.3501285	6.6385243	18.960251
49 + 16	0.1239584	7.5062141	60.554323
201 + 24	0.0375316	7.8541909	209.26861
801 + 32	0.0103780	7.9606813	767.07492
3161 + 40	0.0027327	7.9898222	2923.7884

Table 6.79: Extreme eigenvalues of the unpreconditioned matrix

	Multigrid		
N	$\lambda_{\min}$	$\lambda_{\max}$	κ
13 + 8	1.3790842	1.9948052	1.4464710
49 + 16	1.1586240	1.9963555	1.7230400
201 + 24	1.0922783	1.9977578	1.8289825
801 + 32	1.0533323	1.9987166	1.8975177
3161 + 40	1.0260201	1.9992925	1.9485901

Table 6.80: Extreme eigenvalues of the Multigrid preconditioner

	BPX		
N	$\lambda_{\min}$	$\lambda_{\max}$	κ
13 + 8	1.0175671	7.7724317	7.6382495
49 + 16	1.0878180	11.049247	10.157256
201 + 24	1.1095415	13.713844	12.359919
801 + 32	1.1151510	15.745539	14.119647
3161 + 40	1.1161782	17.357227	15.550588

Table 6.81: Extreme eigenvalues of the BPX preconditioner

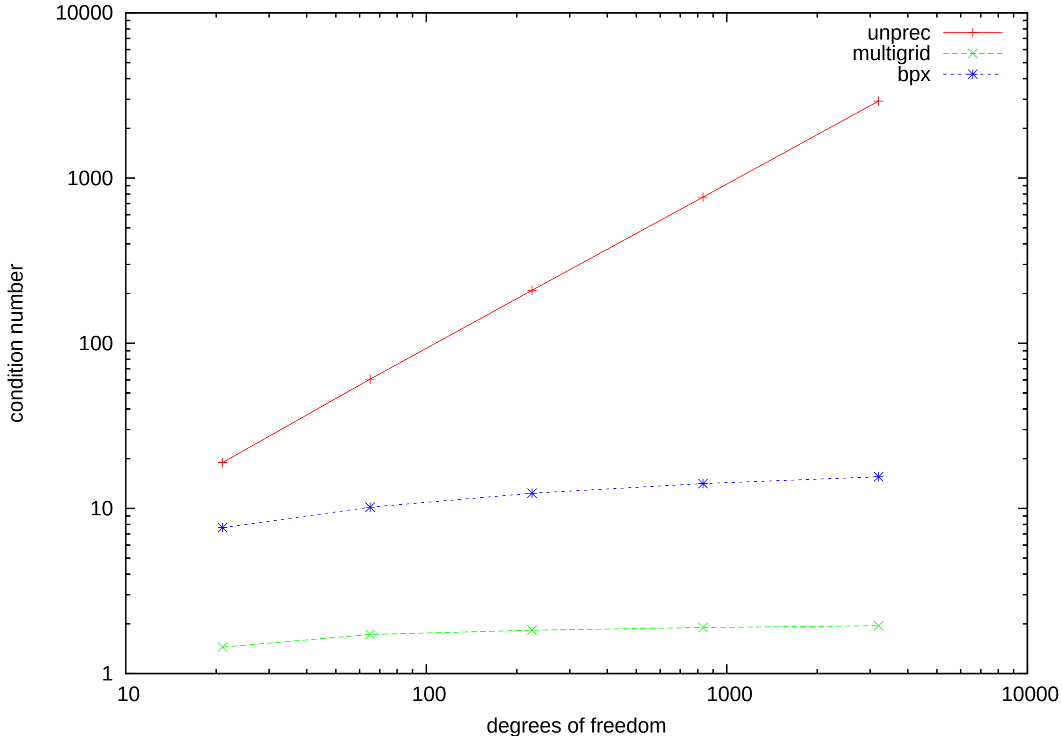


Figure 6.126: Preconditioned Variational Inequality

	Unpreconditioned			
N	time	iter	inner restarts	outer restarts
13 + 8	0.0000000	13	1	1
49 + 16	0.0000000	37	1	1
201 + 24	0.0000000	68	1	1
801 + 32	0.0000000	130	1	1
3161 + 40	0.0100000	263	1	1
12497 + 48	0.0000000	2467	1	1
49609 + 56	0.0100000	1930	1	1

Table 6.82: Computing times and iteration numbers for the unpreconditioned matrix

	Multigrid			
N	time	iter	inner restarts	outer restarts
13 + 8	0.0000000	14	1	1
49 + 16	0.0000000	16	1	1
201 + 24	0.0000000	53	1	1
801 + 32	0.0000000	100	1	1
3161 + 40	0.0000000	18	1	1
12497 + 48	0.0100000	63	1	1
49609 + 56	0.0200000	16	1	1

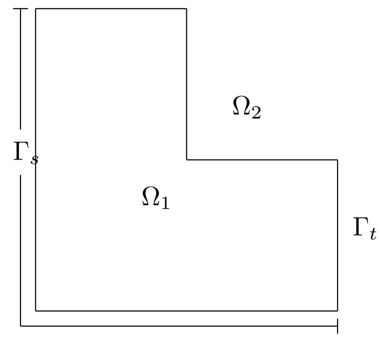
Table 6.83: Computing times and iteration numbers using the Multigrid preconditioner

	BPX			
N	time	iter	inner restarts	outer restarts
13 + 8	0.0000000	14	1	1
49 + 16	0.0000000	42	1	1
201 + 24	0.0000000	69	1	1
801 + 32	0.0000000	52	1	1
3161 + 40	0.0000000	174	1	1
12497 + 48	0.0000000	53	1	1
49609 + 56	0.0200000	136	1	1

Table 6.84: Computing times and iteration numbers using the BPX preconditioner

Example 6.22. *[12]*

$$\begin{aligned}
\Delta u_i &= 0,\ (i=1,2)\\
u_0 &= r^{2/3}\sin\frac{2}{3}(\varphi-\frac{\pi}{2})\\
[u] &= u_1-u_2=u_0|_{\Gamma_t}\\
[\frac{\partial u}{\partial n}] &= \frac{\partial}{\partial n}u_0|_{\Gamma_t}\\
[u] &= u_1-u_2\le u_0|_{\Gamma_s}\\
\frac{\partial u_1}{\partial n} &= \frac{\partial}{\partial n}(u_2+u_0)|_{\Gamma_s}\le 0
\end{aligned}$$



$$\Gamma_s = \overline{(-0.25,-0.25)(0.25,-0.25)}\cup\overline{(-0.25,-0.25)(-0.25,0.25)}$$

coup2/ex6h4in

```
! mixed fem-bem coupling with Signorini condition
open(1) 'test.h' ; open(2) 'ex6h4aina.dat'
geometry('L-Shape',styp=5,dim=(/0.25,.25/)); #ti
#pxg 2 1 2 'Sg'
 0 2 -0.25 -0.25  0.25 -0.25  0.0 -1.0 0
 0 2 -0.25 -0.25 -0.25  0.25 -1.0  0.0 0

problem('Laplace',nickname='SYMMIXCSIGa',pnum=1)
problem('Laplace',nickname='SYMMIXCSIGb',pnum=2)
R=1 ! right hand side
Q=8
EPS=1.0d-8; EPS2=1.0d-9
RHO=2.5
J=4;H=0.0625
do I=1,10
 switch_problem(pnum=1)
 mesh('uniform',n=J,p=0,elements='rectangles',spline='u')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0)
 lft 16 R 0 R
 switch_problem(pnum=2)
 genspl
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0)
 lft 16 R 0 R

 clear('u'); clear('p'); clear('D'); clear('S') ! initial values
 UZIT=0; INTIT=0; CONTIT=0

#time T1
! main loop
 switch_problem(pnum=1)
 eval('Naux=Rhs(N)','no') ! save N-part of right hand side
 switch_problem(pnum=2)
 eval('RS=Rhs(S)','no')
 do
  eval('ua=u','no'); eval('pa=p','no'); eval('Sa=S','no') ! save old solution
  switch_problem(pnum=1)
  eval('Rhs(N)=Naux-Matrix(D)*S','no')
```

```
  defprec(mode='INVCG',spline='p',name='Pp',mat='I+HDiv')
  defprec(mode='INVCG',spline='u',name='Pu',mat='Mu')

!solve(eps=EPS2,mdi='x=0',mit='CG',mdc='p.Pp.p:u.Pu.u',abrflag=1,quiet=1)
  solve(eps=EPS2,mdi='x=0',mit='CG',abrflag=1,quiet=1)
  INTIT=Max(INTIT,ITER)

  switch_problem(pnum=2)
  eval('Rhs(S)=Matrix(W)*S+RHO*Matrix(D,trans)*N','no')

! solve variational inequality  W Plambda  >= vari
  init_defect('S')
  solve(eps=EPS2,mdi='unchanged',mit='POLYAK',abrflag=1,quiet=1,cmode=0)
  CONTIT=Max(CONTIT,ITER)

  norm('uu','Euklid','u','ua')
  norm('pp','Euklid','p','pa')
  norm('SS','Euklid','S','Sa')
  norm('nu','Euklid','u'); norm('np','Euklid','p'); norm('nS','Euklid','S')

  UZIT=UZIT+1
  DOFT=DOFU+DOFP+DOFS
  if (UZIT>DOFT); then
   exit
  fi
  ee=Sqrt(uu*uu+pp*pp+SS*SS)
  ne=Sqrt(nu*nu+np*np+nS*nS)
  qe=ee/ne
  write(0) UZIT,ee,qe
  if (qe<=EPS); then
   exit
  fi
 continue
 #time T2; T=T2-T1
 switch_problem(pnum=1)
 eval('Rhs(N)=Naux','no')
 #ju.
 switch_problem(pnum=2)
 eval('Rhs(S)=RS','no')

 checksig EPS; #rno.
 open(1) 'ex6h4u'//I; #taf. 'u' ; #px. 'u' ; #rci. 'u' ; #cx. 'u'
 open(1) 'ex6h4p'//I; #taf. 'p' ; #px. 'p' ; #rci. 'p' ; #cx. 'p'
 open(1) 'ex6h4s'//I; #taf. 'S' ; #px. 'S' ; #rci. 'S' ; #cx. 'S'
 DOF=DOFU+DOFP+DOFS
 write(2) DOF,DOFU,DOFP,DOFS,JU:12,ITER,T,UZIT,INTIT,CONTIT
 J=J*2 ; H=H/2
continue
end
```

It_{Uzawa}: *Outer iterations of the Uzawa* $\varrho = 2.5$
It_{Int}: *Iterations of the inner solver (GMRES)*
It_{Cont}: *Iterations of the contact solver (Polyak)*
$J(q_h) = \tilde{\Psi}(q_h)$
$\delta J := \sqrt{2|J(q_{ex}) - J(q_h)|}$
α: *Convergence rate*

$\dim H_h$	$\dim L_h$	$\dim H^{1/2}_{s,\tilde{h}}$	It_{Uzawa}	It_{Int}	It_{Cont}	$J(q_h)$	δJ	α	time (s)
32	12	3	23	30	3	-0.857069788	0.3803256		0.05000
112	48	7	22	100	5	-0.820770002	0.2684176	-0.275	0.09000
416	192	15	21	221	9	-0.802992311	0.1910304	-0.258	0.35000
1600	768	31	21	565	13	-0.794067857	0.1365420	-0.249	2.91000
6272	3072	63	21	1551	19	-0.789501534	0.0975247	-0.246	29.5900
24832	12288	127	21	4336	29	-0.787172692	0.0696662	-0.244	356.060
98816	49152	255	32	13229	42	-0.785997790	0.0500358	-0.240	7211.23

Table 6.85: Iteration numbers and cpu-times for the Uzawa algorithm.

7 Boundary Element Methods (3D)

7.1 Convergence

7.1.1 Laplace

The Dirichlet problem of the Laplacian on a screen can be formulated as first kind integral equation using the single layer potential.

$$V\psi(x) := \frac{1}{4\pi}\int_\Gamma \frac{\psi(y)}{|x-y|}\, ds_y = f(x) \quad (x \in \Gamma) \tag{17}$$

The Neumann problem of the Laplacian on a screen can be formulated as first kind integral equation using the hyper singular operator.

$$W\psi(x) := -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma \psi(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\, ds_y = g(x) \quad (x \in \Gamma) \tag{18}$$

Example 7.1. *For the Dirichlet problem of the Laplacian with boundary data $f(x) = 1$ on the L-Shape screen Γ we know the energy norm of the exact solution by extrapolation*

$$\|\psi\|_V = 2.878293$$

bem3/ex1h4in

```
! Dirichlet problem on L-shape, h(4)-version
open(1) 'test' ; open(2) 'ex1h4in.dat'
geometry('L-Shape',gm='Ng',bmode=(/2,3/)) ; #ti
problem('Laplace',nickname='VIGLSCR')
R=3
EPS=1.0d-10
J=2
do I=1,6
 mesh('uniform',n=J,p=0,elements='rectangles',spline='N')
 matrix('analytic'); TM=SEC
 lft 16 R 0 R ; TL=SEC
 solve(eps=EPS,mdi='x=0',mdc='no',mit='CG'); TS=SEC; #rno.
 #hno. 2.878293
 write(2) DOF, I, ENO, ENOERR, TM, TL, TS, ITER, COND
 J=J*2
continue
end
```

bem3/ex1p4in

```
! Dirichlet problem on L-shape, p(4)-version, Laplace
open(1) 'test' ; open(2) 'ex1p4in.dat'
geometry('L-Shape',gm='Ng',bmode=(/2,3/)) ; #ti
problem('Laplace',nickname='VIGLSCR')
R=3
EPS=1.0d-10
do I=0,10
 mesh('uniform',n=2,p=I,elements='rectangles',spline='N')
 matrix('analytic'); TM=SEC
 lft 16 R 0 R ; TL=SEC
```

```
 solve(eps=EPS,mdi='x=0',mdc='diag',mit='GAUSS'); TS=SEC; #rno.
 #hno. 2.878293
 write(2) DOF, I, ENO, ENOERR, TM, TL, TS, ITER, COND
continue
end
```

bem3/ex1a60hin

```
! Dirichlet problem, L-Shape, hierarchical error estimator
open(1) 'sh06b1' ; open(2) 'ex1a60hin.dat'
geometry('L-Shape',bmode=(/2,3/),gm='Ng') ; #ti
problem('Laplace',nickname='VIGLSCR')
mesh('uniform',n=4,p=0,elements='rectangles',spline='N')
do I=2,35
 matrix; TM=SEC
 lft 16 3 16 ; TL=SEC
 solve(eps=1.0d-10,mit='CG',mdc='diag',mdi='x=0'); TS=SEC; #rno.
 #taf. 'N' ; #px. 'N' ; #cx. 'N'
 !#kap.
 #hno. 2.878292605
 adap(pxy=0.60,gq=4,tlocfs=3,spline='N')
 QUOT=ERREST/ENOERR
 write(2) DOF, I, ENO, ENOERR, ERREST, QUOT, TM, TL, TS, COND, ITER
 #res. 'N' ; #ref. 'N'
 refine(spline='N')
 open(1) 'sh06b'//I
continue
end
```

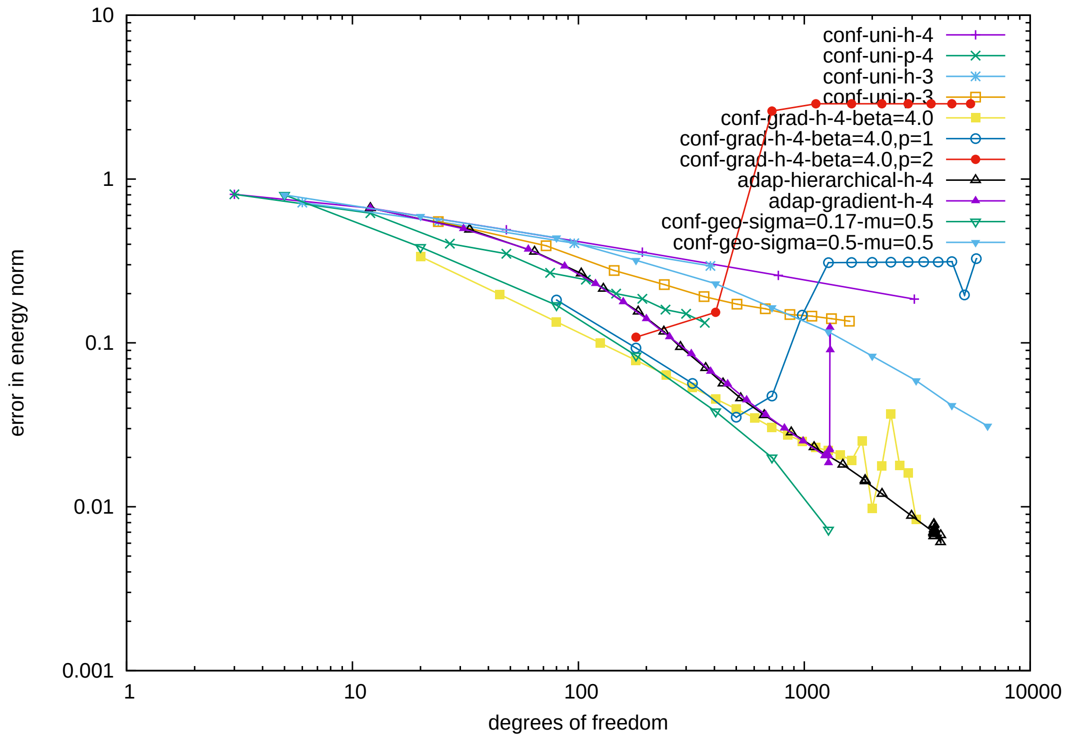


Figure 7.127: Dirichlet problem on the L-Shape

N	E_N	η_N	η_N/E_N	α_N
12	0.6646280	0.2447211	0.3682076	
33	0.4916944	0.1779529	0.3619177	0.298
64	0.3603567	0.1303536	0.3617349	0.469
103	0.2650374	0.0956805	0.3610073	0.646
129	0.2141951	0.0780505	0.3643898	0.946
184	0.1552320	0.0564633	0.3637349	0.907
239	0.1173793	0.0434747	0.3703781	1.069
283	0.0944040	0.0361373	0.3827946	1.289
366	0.0702523	0.0277145	0.3944991	1.149
437	0.0565684	0.0236545	0.4181565	1.222
523	0.0459513	0.0201218	0.4378939	1.157
665	0.0362324	0.0166638	0.4599158	0.989
876	0.0284764	0.0130216	0.4572748	0.874
1105	0.0230880	0.0109869	0.4758702	0.903
1483	0.0180878	0.0085311	0.4716459	0.830
1864	0.0145172	0.0070712	0.4870884	0.962
1867	0.0142886	0.0069917	0.4893194	9.870
2195	0.0120480	0.0060551	0.5025832	1.054
2976	0.0088536	0.0047046	0.5313733	1.012
3734	0.0067934	0.0039495	0.5813747	1.167
5107	0.0044725	0.0030400	0.6797046	1.335

Table 7.86: Errors and error indicators for hierarchical error estimator (Dirichlet problem on L-Shape), adaptive ($\theta = 0.60$)

N	E_N	η_N	η_N/E_N	α_N
12	0.6646280	0.9558277	1.4381393	
31	0.4975515	0.7302504	1.4676880	0.305
60	0.3728701	0.5445751	1.4604957	0.437
87	0.2938917	0.4308739	1.4660978	0.641
119	0.2295661	0.3335543	1.4529771	0.789
158	0.1778675	0.2571996	1.4460181	0.900
200	0.1401281	0.2005923	1.4314923	1.012
253	0.1086874	0.1537018	1.4141645	1.081
316	0.0858641	0.1197785	1.3949786	1.060
385	0.0669581	0.0931711	1.3914838	1.259
458	0.0560811	0.0762683	1.3599652	1.021
556	0.0448273	0.0615713	1.3735235	1.155
674	0.0364714	0.0494294	1.3552911	1.072
817	0.0301795	0.0404624	1.3407256	0.984
988	0.0251823	0.0332756	1.3213885	0.953
1233	0.0205134	0.0280460	1.3672068	0.926
1237	0.0204740	0.0275575	1.3459789	0.594
1241	0.0204542	0.0273563	1.3374390	0.300
1247	0.0204241	0.0275424	1.3485236	0.305
1251	0.0204192	0.0273045	1.3371971	0.075
1257	0.0203936	0.0270187	1.3248638	0.262
1408	0.0184879	0.0244983	1.3250991	0.865
1410	0.0184786	0.0243885	1.3198222	0.354
1412	0.0184740	0.0243565	1.3184206	0.176
1564	0.0168404	0.0230749	1.3702120	0.906
1574	0.0168108	0.0226557	1.3476902	0.276
1582	0.0167966	0.0225195	1.3407204	0.167
1598	0.0167530	0.0227014	1.3550631	0.258
1608	0.0167332	0.0223510	1.3357241	0.190

Table 7.87: Errors and error indicators for gradient error estimator (Dirichlet problem on L-Shape), adaptive ($\theta = 0.70$)

Example 7.2. *If the screen Γ is the square $[-1,1]^2$ and $f(x) = 1$ we know the energy norm of the exact solution by extrapolation*

$$\|\psi\|_V = 3.0361798$$

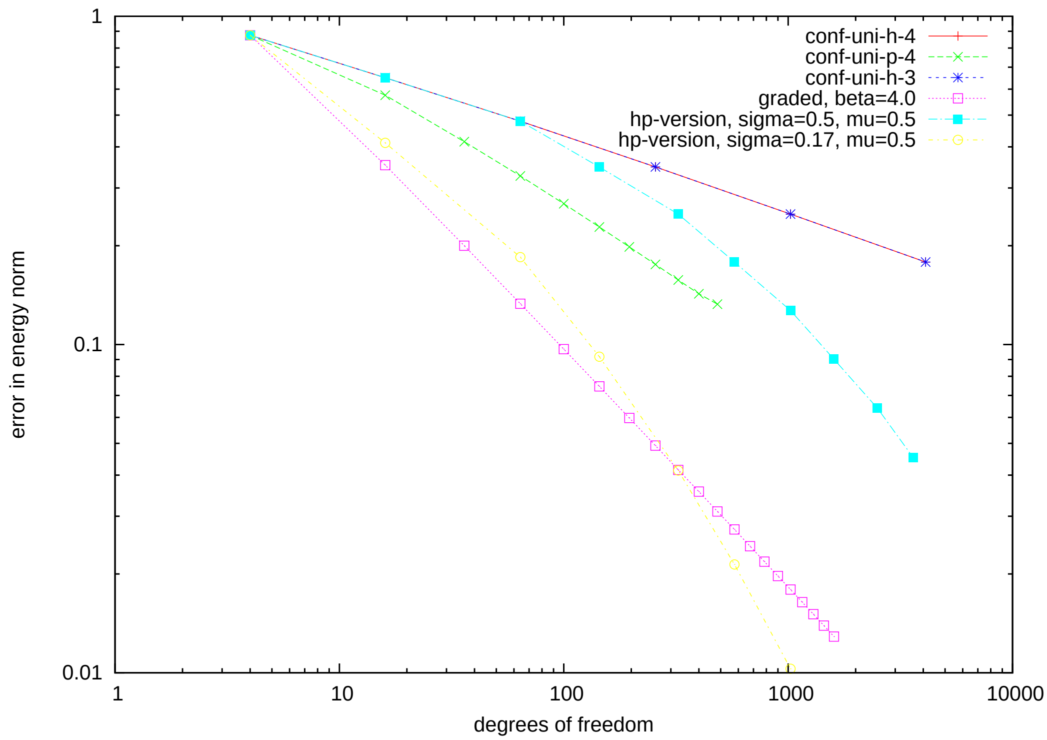


Figure 7.128: Dirichlet problem on the square $[-1,1]^2$.

Example 7.3. *For the Neumann problem of the Laplacian with boundary data $g(x) = 1$ on the L-Shape screen Γ we know the energy norm of the exact solution by extrapolation*

$$\|u\|_W = 1.4310$$

The following script deals with the residual error indicator

$$\eta_N^2 = \sum_i \left(h_i^{1/2} \|g - W u_N\|_{L^2(\Gamma_i)} \right)^2$$

bem3/ex3a90rin

```
! Neumann problem, L-Shape, stirred by Residual error estimator
open(1) 'sn3r90a1' ; open(2) 'ex3a90rin.dat'
write(2) '# dof i enorm enoerr eta mu enoerr/eta enoerr/mu'
geometry('L-Shape') ; #ti
#pro 0 1 1 ; #pro. ; #pol 3
THETA=0.90
mesh('uniform',n=4,p=1,elements='triangles') ; #g.
do I=2,15
matrix('analytic')
lft  0 0
rlgs 1.0d-10 2 1 0 1 1 ; #cx.
#hno. 1.4310         ! compute error in energy norm
resh THETA 2 4 3 0.0 ! compute hierarchical error estimator
ERRESTH=ERREST; QUOTH=ENOERR/ERREST
resh THETA 2 4 0 0.5 ! compute residual error estimator
ERRESTR=ERREST; QUOTR=ENOERR/ERREST
write(2) DOF, I, ENO, ENOERR, ERRESTR, ERRESTH, QUOTR, QUOTH
#res. ; #ref.
refine
open(1) 'sn3r90a'//I
#pro. ; #g.
continue
end
```

N	E_N	η_N	μ_N	η_N/E_N	μ_N/E_N	α_N
8	0.7436729	1.3146346	0.4543120	0.5656879	1.6369212	
16	0.6577310	1.1660701	0.4233171	0.5640579	1.5537548	0.177
28	0.5791694	0.9628661	0.3583279	0.6015056	1.6163111	0.227
51	0.5020311	0.8525550	0.3108206	0.5888548	1.6151797	0.238
88	0.3983020	0.7168153	0.2593030	0.5556550	1.5360488	0.424
144	0.3371978	0.6020480	0.2224468	0.5600845	1.5158581	0.338
231	0.2525063	0.5107048	0.1885293	0.4944271	1.3393477	0.612
377	0.1903463	0.4226601	0.1566249	0.4503530	1.2153005	0.577
614	0.1435580	0.3432571	0.1274863	0.4182229	1.1260664	0.578

Table 7.88: Errors and error indicators for residual error estimator (Neumann problem on L-Shape), adaptive ($\theta = 0.90$)

N	E_N	η_N	μ_N	E_N/η_N	E_N/μ_N	α_N	τ_{Gal}	τ_η	τ_μ
12	0.6968136	1.2371191	0.4427737	0.56326	1.57375		0.02000	0.12000	0.13000
31	0.5816609	0.9437129	0.3463353	0.61635	1.67947	0.190	0.05000	0.51000	0.49000
70	0.4503962	0.7735382	0.2905099	0.58225	1.55036	0.314	0.25000	2.13000	2.00000
163	0.3180149	0.5879044	0.2145841	0.54093	1.48201	0.412	1.13000	9.82000	9.16000
360	0.2190819	0.4412660	0.1630947	0.49648	1.34328	0.470	5.28000	45.3800	42.2000
762	0.1474977	0.3260425	0.1225931	0.45239	1.20315	0.528	23.2500	198.840	184.130
1598	0.0986230	0.2391711	0.0916457	0.41235	1.07613	0.544	100.780	847.730	786.130
3393	0.0871337	0.1741430	0.0672880	0.50036	1.29494	0.164	439.650	3720.67	3477.28

Table 7.89: Errors and error indicators for residual and hierarchical error estimator (Neumann problem on L-Shape), adaptive ($\theta = 0.80$, residual)

N	E_N	η_N	μ_N	E_N/η_N	E_N/μ_N	α_N	τ_{Gal}	τ_η	τ_μ
33	0.5794817	0.9423710	0.3471134	0.61492	1.66943		0	0	0
161	0.3925993	0.6596814	0.2419290	0.59513	1.62279	0.246	0	0	0
705	0.2708639	0.4609843	0.1681837	0.58758	1.61052	0.251	0	0	0
2945	0.1884173	0.3236354	0.1172190	0.58219	1.60740	0.254	0	0	0

Table 7.90: Errors and error indicators for residual and hierarchical error estimator (Neumann problem on L-Shape), uniform h(3) version

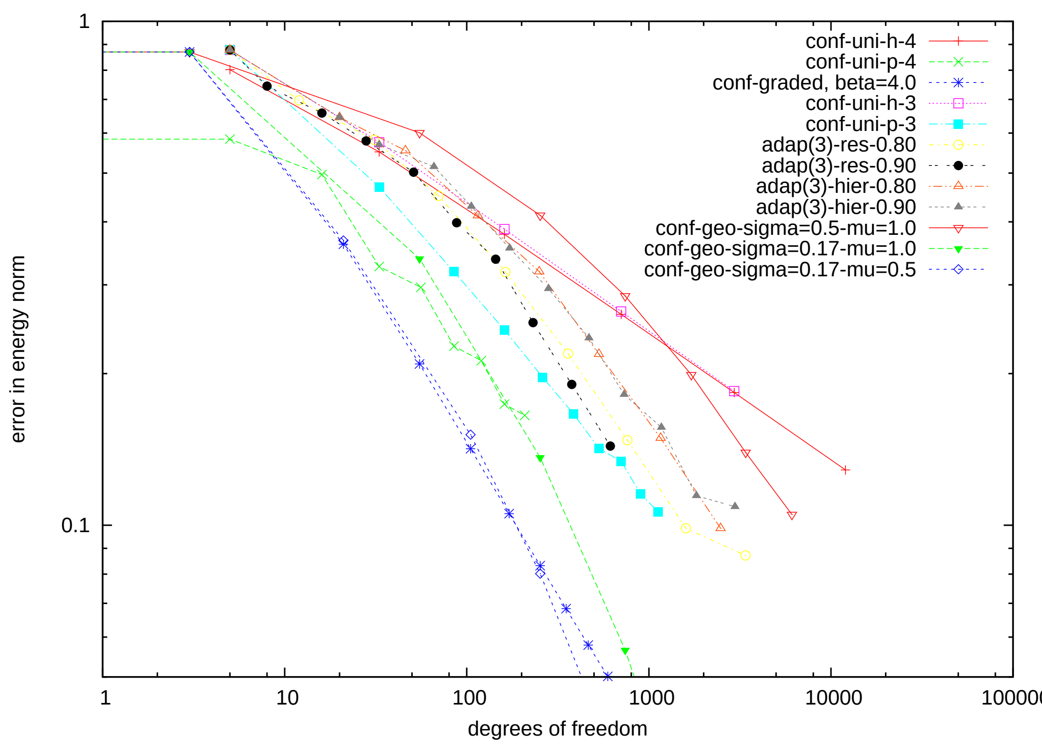


Figure 7.129: Neumann problem on the L-Shape.

N	E_N	η_N	η_N/E_N	α_N
5	0.8773174	0.6245823	1.4046467	
20	0.6448391	0.4041176	1.5956719	0.222
33	0.5691210	0.3374797	1.6863858	0.249
66	0.5146788	0.2958922	1.7394133	0.145
106	0.4292611	0.2553422	1.6811212	0.383
172	0.3545172	0.2267893	1.5632012	0.395
281	0.2948426	0.1885937	1.5633742	0.375
469	0.2351807	0.1606633	1.4638104	0.441
731	0.1818113	0.1347888	1.3488606	0.580
1172	0.1563767	0.1132969	1.3802375	0.319
1824	0.1144709	0.0942434	1.2146297	0.705
2970	0.1088389	0.0788659	1.3800506	0.103

Table 7.91: Errors and error indicators for hierarchical error estimator (Neumann problem on L-Shape), adaptive ($\theta = 0.90$)

N	E_N	η_N	μ_N	E_N/η_N	E_N/μ_N	α_N	τ_{Gal}	τ_η	τ_μ
20	0.6448391	1.1641380	0.4062413	0.55392	1.58733		0.02000	0.24000	0.20000
46	0.5538872	0.9256691	0.3360857	0.59836	1.64805	0.183	0.09000	0.76000	0.73000
115	0.4116299	0.7580858	0.2512338	0.54299	1.63843	0.324	0.44000	3.82000	3.43000
250	0.3187360	0.5808173	0.2035831	0.54877	1.56563	0.329	1.91000	16.4900	15.4500
532	0.2184784	0.4308609	0.1542260	0.50707	1.41661	0.500	8.79000	72.9600	68.0900
1161	0.1489778	0.3192171	0.1157990	0.46670	1.28652	0.491	40.5500	340.710	316.490
2478	0.0986286	0.2313451	0.0850653	0.42633	1.15945	0.544	182.620	1522.11	1402.59

Table 7.92: Errors and error indicators for hierarchical error estimator (Neumann problem on L-Shape), adaptive ($\theta = 0.80$)

Example 7.4. *For the Neumann problem of the Laplacian with boundary data $g(x) = |x - (0, 1.01)|^{-1}$ on the Square screen $\Gamma = [-1, 1]^2$ we know the energy norm of the exact solution by extrapolation*

$$\|u\|_W = 2.511$$

N	E_N	η_N	μ_N	E_N/η_N	E_N/μ_N	α_N	τ_{Gal}	τ_η	τ_μ
22	1.2072099	2.2541222	0.7637518	0.53556	1.58063		0.06000	0.35000	0.43000
48	0.9097344	1.6535456	0.5842659	0.55017	1.55706	0.363	0.14000	0.79000	0.93000
104	0.6769054	1.2157873	0.4378091	0.55676	1.54612	0.382	0.37000	3.21000	2.98000
213	0.4911718	0.8713807	0.3197725	0.56367	1.53600	0.447	1.53000	12.9400	12.0200
445	0.3587789	0.6317357	0.2324232	0.56793	1.54364	0.426	6.30000	53.9600	50.1400
922	0.2604540	0.4540298	0.1672535	0.57365	1.55724	0.440	26.5900	229.870	211.490
1892	0.1915644	0.3250785	0.1202900	0.58929	1.59252	0.427	112.520	959.900	886.820

Table 7.93: Errors and error indicators for residual and hierarchical error estimator (Neumann problem on Square), adaptive ($\theta = 0.80$, residual)

N	E_N	η_N	μ_N	E_N/η_N	E_N/μ_N	α_N	τ_{Gal}	τ_η	τ_μ
49	1.1439843	2.1480329	0.7094919	0.53257	1.61240		0	0	0
225	0.8340395	1.5157682	0.5289063	0.55024	1.57691	0.207	0	0	0
961	0.5947405	1.0548148	0.3807662	0.56383	1.56196	0.233	0	0	0
3969	0.4184993	0.7322123	0.2669809	0.57155	1.56753	0.248	0	0	0

Table 7.94: Errors and error indicators for residual and hierarchical error estimator (Neumann problem on Square), uniform h(3) version

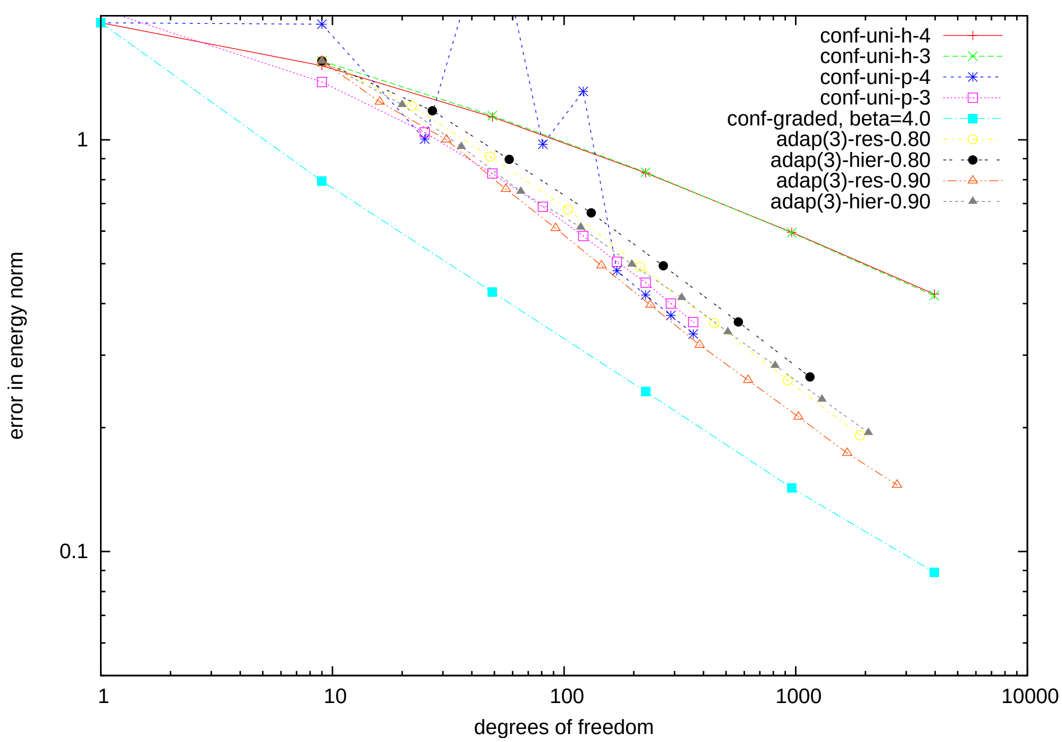


Figure 7.130: Neumann problem on the Square.

N	E_N	η_N	μ_N	E_N/η_N	E_N/μ_N	α_N	τ_{Gal}	τ_η	τ_μ
27	1.1759280	2.2276479	0.7304770	0.52788	1.60981		0.02000	0.25000	0.25000
58	0.8963254	1.6612351	0.5637360	0.53955	1.58997	0.355	0.11000	0.97000	0.98000
131	0.6644998	1.2212580	0.4228734	0.54411	1.57139	0.367	0.51000	4.46000	4.29000
268	0.4942562	0.8997397	0.3157851	0.54933	1.56517	0.414	2.05000	17.6200	16.9400
565	0.3611764	0.6547454	0.2295822	0.55163	1.57319	0.421	8.91000	76.7600	72.8600
1152	0.2655666	0.4726521	0.1677682	0.56186	1.58294	0.432	36.8100	315.540	299.470

Table 7.95: Errors and error indicators for hierarchical error estimator (Neumann problem on Square), adaptive ($\theta = 0.80$)

Example 7.5. *Here we investigate the Dirichlet problem of the Laplacian on a polyhedral domain* Ω*.*

$$V\psi = (I+K)f \text{ on } \Gamma = \partial\Omega$$

We consider the L-block $\Omega = (-1,1)^3 \setminus ((0,1)^2 \times (-1,1))$ *with* $u(\varrho,\varphi,z) = \varrho^{2/3}\cos\frac{2}{3}(2\pi-\varphi)$ *using cylindrical coordinates, i.e. we have boundary data* $f = u|_\Gamma$*. We know the exact energy norm of the solution by extrapolation*

$$\|\psi\|_{H^{-1/2}(\Gamma)} \approx 1.972$$

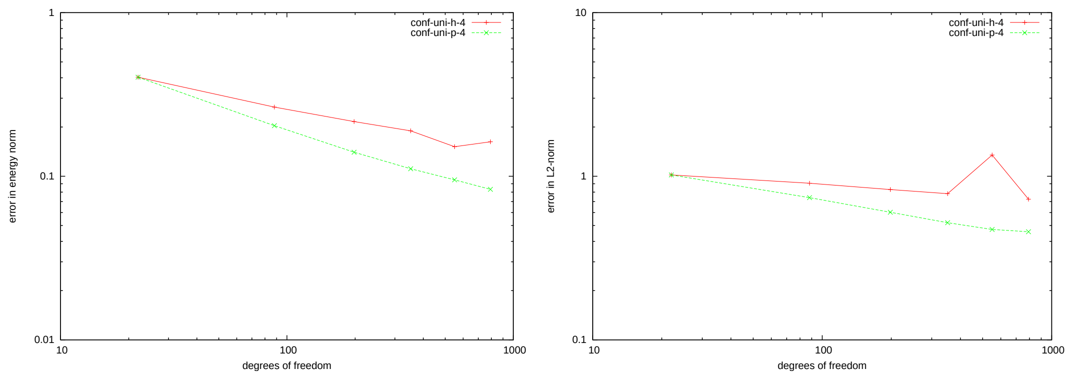


Figure 7.131: Dirichlet problem on L-Block.

Example 7.6. *Here we investigate the Neumann problem of the Laplacian on a polyhedral domain* Ω*. We consider the L-block* $\Omega = (-1,1)^3 \setminus ((0,1)^2 \times (-1,1))$ *with* $v(\varrho,\varphi,z) = \varrho^{2/3}\sin\frac{2}{3}(2\pi-\varphi)$ *using cylindrical coordinates, i.e. we have boundary data* $g(x) = \vec{n}_x \cdot \nabla_x v$*. We know the exact energy norm of the solution by extrapolation*

$$\|v\|_{H^{1/2}(\Gamma)} \approx 2.419339$$

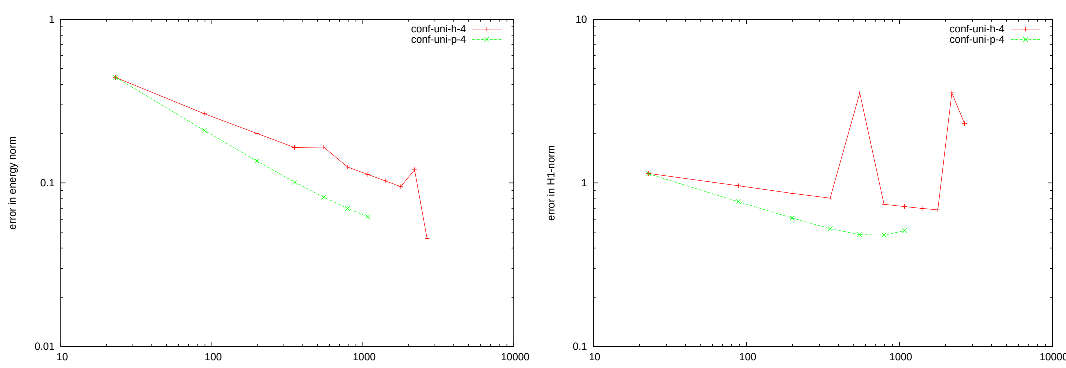


Figure 7.132: Neumann problem on L-Block.

7.1.2 Lamé

Example 7.7. *For the Dirichlet problem of the Lamé equation with boundary data* $f(x,y,z) = (-y,x,0)$ *on the L-Shape screen* Γ *and E-module* $E = 2000$ *and Poisson number* $\sigma = 0.3$ *we know the energy norm of the exact solution by extrapolation*

$$\|\psi\|_V = 103.4091$$

bem3/ex21h4in

```
! Dirichlet problem on L-shape, h(4)-version, Lame
open(1) 'test' ; open(2) 'ex21h4in.dat'
geometry('L-Shape',bmode=(/2,3/),gm='Ng') ; #ti
problem('Lame',nickname='VIGLSCR')
#ep 2000. 0.3
J=2
do I=1,6
 mesh('uniform',n=J,p=0,elements='rectangles',spline='N')
 matrix('analytic'); TM=SEC
 lft 16 0 16 ; TL=SEC
 solve(eps=1.0d-10,mit='CG',mdc='diag',mdi='x=0'); TS=SEC; #rno.
 #hno. 103.4091
 write(2) DOF, I, ENO, ENOERR, COND, TM, TL, TS
 J=J*2
continue
end
```

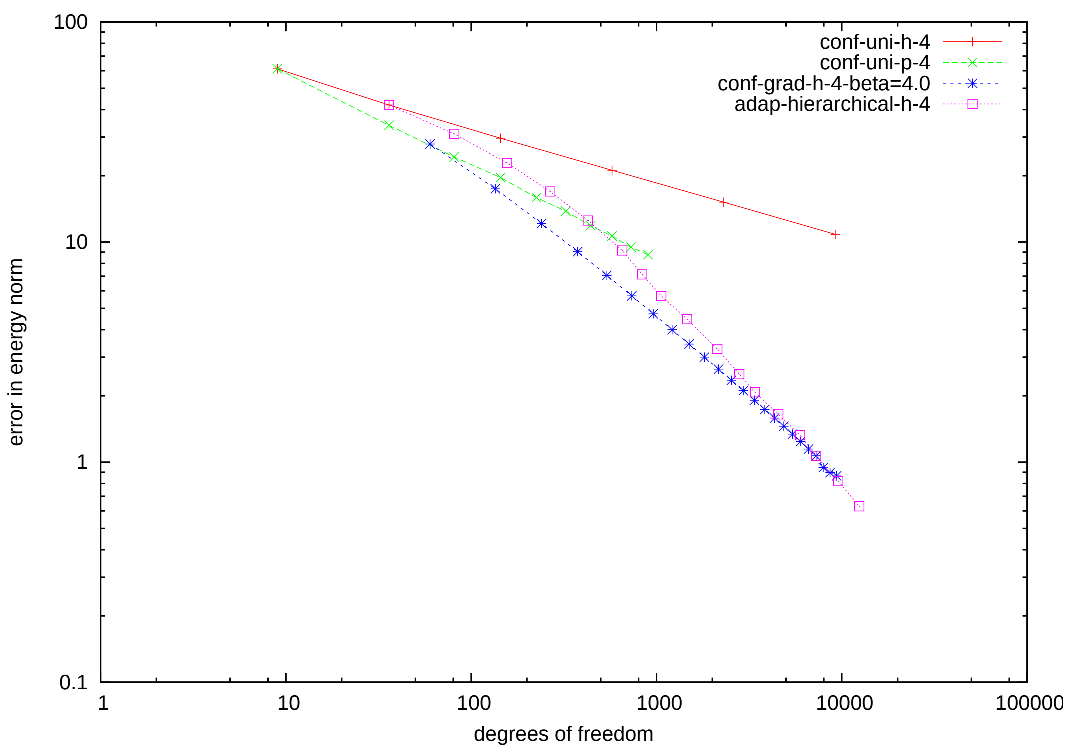


Figure 7.133: Dirichlet problem (Lamè) on the L-Shape

Example 7.8. *For the Neumann problem of the Lamé equation with boundary data* $f(x, y, z) = (-y, x, 0)$ *on the L-Shape screen* Γ *and E-module* $E = 2000$ *and Poisson number* $\sigma = 0.3$ *we know the energy norm of the exact solution by extrapolation*

$$\|\psi\|_W = 0.0331969$$

bem3/ex22h4in

```
! Neumann problem on L-Shape, h(4)-version, Lame
open(1) 'test' ; open(2) 'ex22h4in.dat'
geometry('L-Shape',bmode=(/2,3/),gm='Dg') ; #ti
problem('Lame',nickname='WIGLSCR')
#ep 2000. 0.3
J=4
do I=1,5
 mesh('uniform',n=J,p=1,elements='rectangles',spline='D')
 matrix('analytic'); TM=SEC
 lft 16 0 16 ;TL=SEC
 solve(eps=1.0d-10,mit='CG',mdi='x=0',mdc='diag'); TS=SEC; #rno.
 #hno. 0.0331969
 write(2) DOF, I, ENO, ENOERR, COND, TM, TL, TS
 J=J*2
continue
end
```

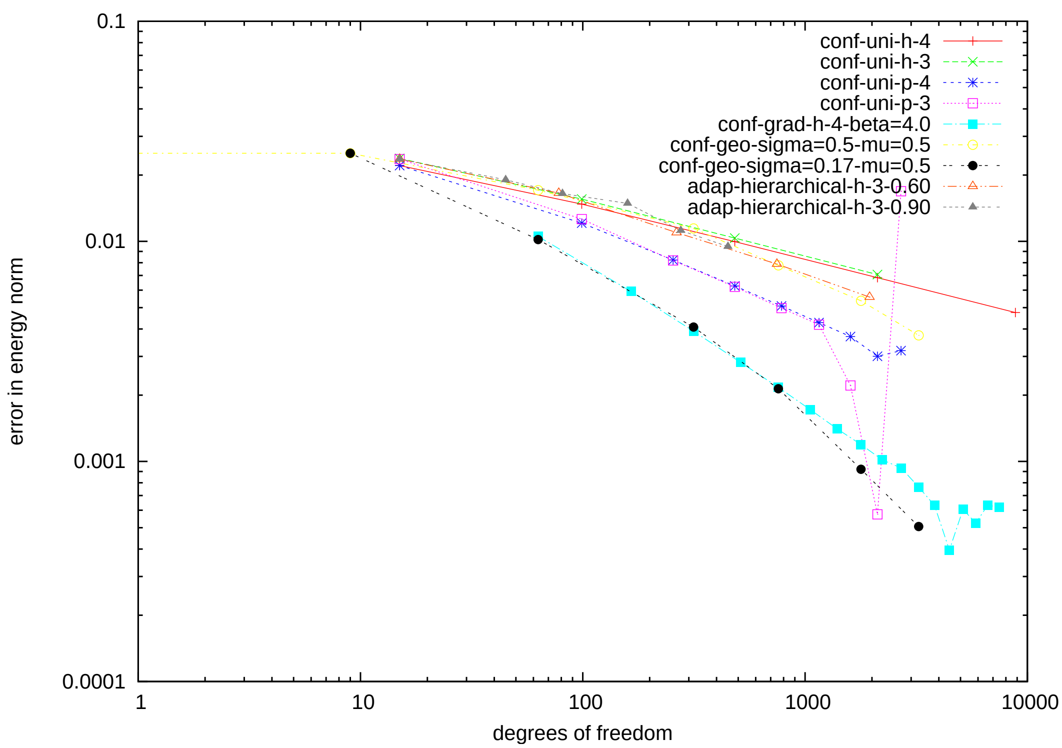


Figure 7.134: Neumann problem (Lamè) on the L-Shape

Example 7.9. *For the Dirichlet problem of the Lamé equation with boundary data* $f(x, y, z) = (-y, x, 0)$ *on the Square screen* Γ *and E-module* $E = 2000$ *and Poisson number* $\sigma = 0.3$ *we know the energy norm of the exact solution by extrapolation*

$$\|\psi\|_V = 115.0355908$$

bem3/ex25h4in

```
! Dirichlet problem on Square, h(4)-version, Lame
open(1) 'test' ; open(2) 'ex25h4in.dat'
geometry('Square') ; #ti
problem('Lame','Dirichlet','1kind','PC') ; #pro. ; #pol 1
#ep 2000. 0.3
J=2
do I=1,7
mesh('uniform',n=J,p=0,elements='rectangles') ; #g.
matrix('semi-analytic',ijn=7)
lft 16 0 ; #l.
rlgs 1.0d-10 2 1 0 1 1
#hno. 115.0355908
write(2) DOF, I, ENO, ENOERR, COND
#rno. ; #cx.
J=J*2
continue
end
```

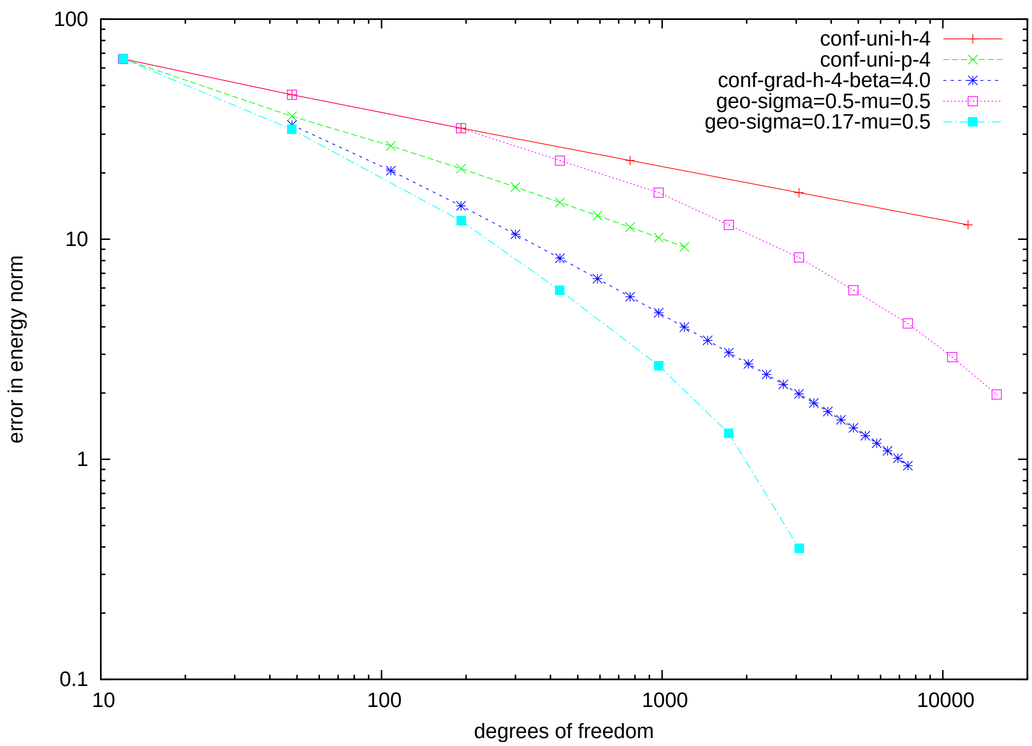


Figure 7.135: Dirichlet problem (Lamè) on the Square

Example 7.10. *For the Neumann problem of the Lamé equation with boundary data* $f(x,y,z) = (-y,x,0)$ *on the Square screen* Γ *and E-module* $E = 2000$ *and Poisson number* $\sigma = 0.3$ *we know the energy norm of the exact solution by extrapolation*

$$\|\psi\|_W = 0.04005011548$$

bem3/ex26h4in

```
! Neumann problem on Square, h(4)-version, Lame
open(1) 'test' ; open(2) 'ex26h4in.dat'
geometry('Square') ; #ti
problem('Lame','Neumann','1kind','C0') ; #pro. ; #pol 3
#ep 2000. 0.3
J=4
do I=1,6
mesh('uniform',n=J,p=1,elements='rectangles') ; #g.
matrix('semi-analytic',ijn=7)
lft 16 0 ; #l.
rlgs 1.0d-10 2 1 0 1 1
 rlgs 1.0d-10 0 0 -3 1 1
#hno. 0.04005011548
write(2) DOF, I, ENO, ENOERR, COND
#rno. ; #cx.
J=J*2
continue
end
```

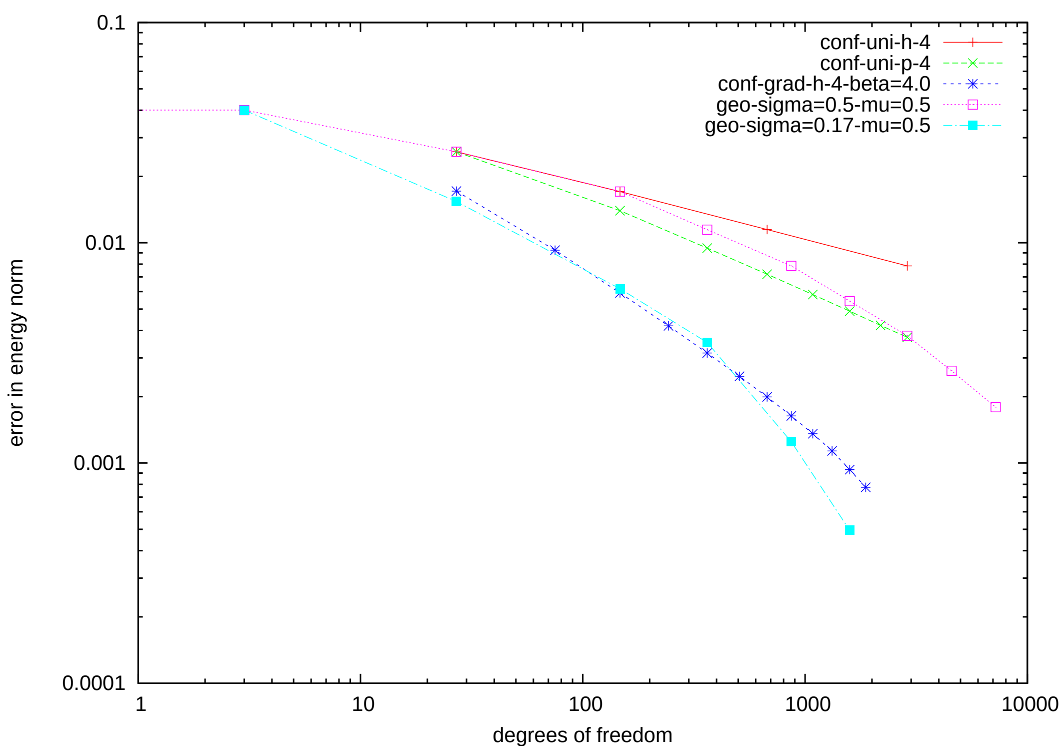


Figure 7.136: Neumann problem (Lamè) on the Square

7.1.3 Helmholtz

Example 7.11. *Here we investigate the Dirichlet problem of the Helmholtz equation on a polyhedral domain* Ω*.*

$$V_k\psi = (I + K_k)f \ \text{ on } \Gamma = \partial\Omega$$

We consider the L-block $\Omega = (-1,1)^3 \backslash ((0,1)^2 \times (-1,1))$ *with* $u(\varrho, \varphi, z) = \tilde{J}_{2/3}(kr) \sin \frac{2}{3}(2\pi - \varphi)$ *using cylindrical coordinates, i.e. we have boundary data* $f = u|_\Gamma$*.*

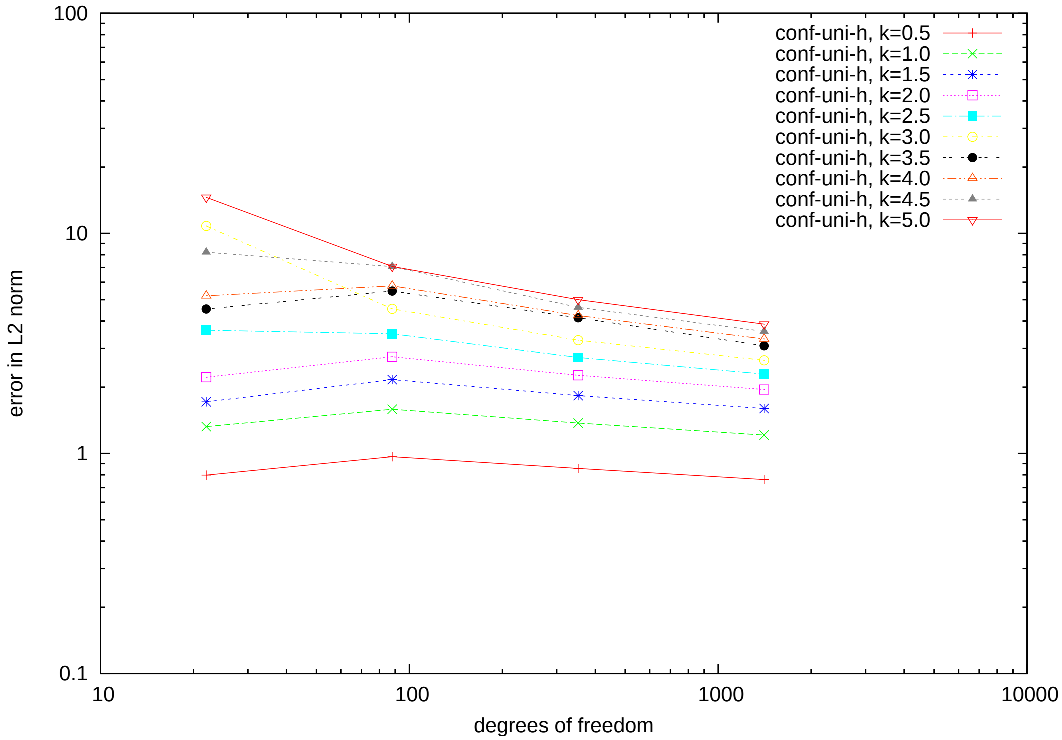


Figure 7.137: Dirichlet problem on L-Block.

Example 7.12. *Here we investigate the Neumann problem of the Helmholtz equation on a polyhedral domain* Ω

$$W_k \psi = (I - K_k')f \text{ on } \Gamma = \partial\Omega$$

for different wavenumbers k.

We consider the L-block $\Omega = (-1,1)^3 \backslash ((0,1)^2 \times (-1,1))$ *with* $u(\varrho, \varphi, z) = \tilde{J}_{2/3}(kr) \cos \frac{2}{3}(2\pi - \varphi)$ *using cylindrical coordinates, i.e. we have boundary data* $f = u|_\Gamma$.

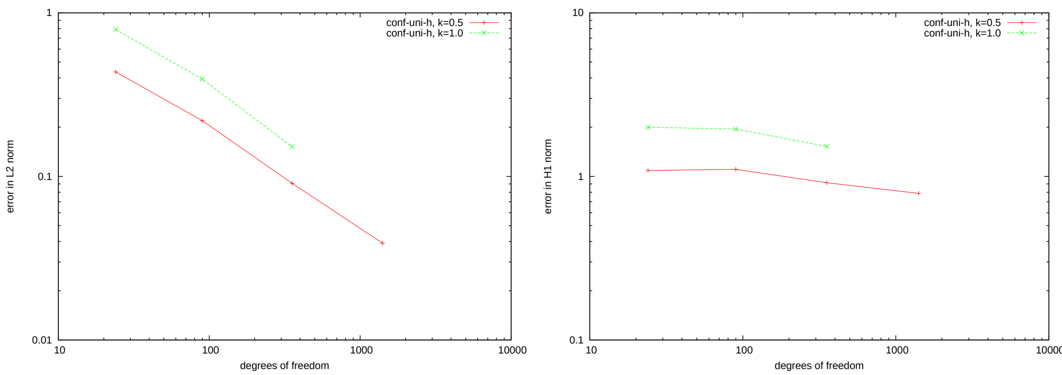


Figure 7.138: Neumann problem on L-Block.

7.1.4 Maxwell

Example 7.13. *We prescribe the exact solution*

$$\mathbf{J} = \frac{1}{8}\left(\begin{array}{c} 0 \\ (1-x_1)(1-x_2)n_3 \\ -(1-x_1)(1-x_2)n_2 \end{array}\right), \qquad M = \frac{1}{8\alpha^2}(x_1-1)n_3$$

on $\Sigma = \partial[-2,2]^3 = \cup_{k=1}^{6}\Sigma_k$ *and compute*

$$\mathbf{E}_T = \sum_{k=1}^{6}\left(\int_{\Sigma_k} G_\alpha(|x-y|)(\mathbf{J}(y))^t \, ds_y + \operatorname{grad}_T \int_{\Sigma_k} G_\alpha(|x-y|)M(y)\, ds_y\right).$$

7.2 Solvers

Here we investigate the performance of different solvers and preconditioners for h- and p-version for the 3d-BEM.

Example 7.14. *Using the configuration of example 7.2 with the uniform h-version with rectangles we apply the multigrid-algorithm with V-cycle and one pre- and one post-smoothing step (Richardson). Also we apply the BPX-algorithm and the hierarchical decomposition. The iteration stops if the last relative change of the iterate is less than* 10^{-10}.

bem3/ex11in

```
! h-version, multigrid
open(1) 'test.mcg'
open(2) 'ex11mcgin.dat' ; open(3) 'ex11bpxin.dat'
open(4) 'ex11hcgin.dat' ; open(5) 'ex11cgin.dat' ; open(6) 'ex11h2cgin.dat'
open(7) 'ex11mcg.tex' ; open(8) 'ex11bpx.tex'
open(9) 'ex11hcg.tex' ; open(10) 'ex11cg.tex' ; open(11) 'ex11h2cg.tex'
geometry('Square') ; #ti
problem('Laplace') ; #pro. ; #pol 3
J=8
do I=1,4
mesh('uniform',n=J,p=0,elements='rectangles') ; #g.
mat0
lft 4 0
mcg 1.0d-10 1 1 1 I 0.5 2 2 200 0 2 0 0 1 ; T=SEC
#hno. 3.0361798
write(2) DOF,I,LMIN,LMAX,COND,T,ITER
write(7) DOF,'&',LMIN:7,'&',LMAX:6,'&',COND:6,'&',ITER,'&',T:6//'\\'
mbpx 1.0d-10 I 300 0 2 2 1 0 1 ; T=SEC
#hno. 3.0361798
write(3) DOF,I,LMIN,LMAX,COND,T,ITER
write(8) DOF,'&',LMIN:6,'&',LMAX:6,'&',COND:6,'&',ITER,'&',T:6//'\\'
ahcg 1.0d-10 I 200 0 2 1 0 0 1 ; T=SEC
#hno. 3.0361798
write(4) DOF,I,LMIN,LMAX,COND,T,ITER
write(9) DOF,'&',LMIN:6,'&',LMAX:6,'&',COND:6,'&',ITER,'&',T:6//'\\'
rlgs 1.0d-10 -1 0 0 1 1 ; T=SEC
#hno. 3.0361798
write(5) DOF,I,LMIN,LMAX,COND,T,ITER
write(10) DOF,'&',LMIN:8,'&',LMAX:6,'&',COND:6,'&',ITER,'&',T:6,'\\'
ahcg 1.0d-10 1 200 0 2 1 0 0 1 ; T=SEC
#hno. 3.0361798
write(6) DOF,I,LMIN,LMAX,COND,T,ITER
write(11) DOF,'&',LMIN:6,'&',LMAX:6,'&',COND:6,'&',ITER,'&',T:6,'\\'
J=J*2
continue
end
```

	Multigrid				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
64	0.47895	1.0000	2.0879	10	0.0000
256	0.25218	1.0000	3.9655	15	0.0100
1024	0.12940	1.0000	7.7280	19	0.2600
4096	0.06464	1.0000	15.471	22	4.8000

Table 7.96: Conjugate Gradients with multigrid-preconditioner

	BPX				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
64	0.1373	0.4911	3.5760	16	0.0000
256	0.0664	0.3228	4.8591	20	0.0100
1024	0.0329	0.1940	5.8879	22	0.1200
4096	0.0163	0.1119	6.8769	24	1.6600

Table 7.97: Conjugate Gradients with BPX-preconditioner

	Hierarchical				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
64	0.4592	4.1228	8.9785	11	0.0000
256	0.3642	4.1776	11.471	21	0.0200
1024	0.2945	4.2130	14.307	27	0.5400
4096	0.2418	4.2342	17.509	33	11.690

Table 7.98: Conjugate Gradients with Hierarchical-preconditioner

	2-Hierarchical				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
64	0.4592	4.1228	8.9785	11	0.0000
256	0.3841	8.1829	21.303	24	0.0200
1024	0.3754	16.335	43.508	32	0.5200
4096	0.3766	32.653	86.699	44	11.450

Table 7.99: Conjugate Gradients with 2-level hierarchical-preconditioner

	CG				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
64	0.001525	0.0301	19.745	12	0.0000
256	0.000173	0.0075	43.476	29	0.0000
1024	.218E-04	0.0019	86.612	43	0.1500
4096	.273E-05	0.0005	172.48	61	3.4000

Table 7.100: Conjugate Gradients without preconditioner

		A_N			$\text{diag}^{-1} A_N$		
N	n	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ
4	1	0.1191994	0.4732010	3.9698255	0.5038005	2.0000000	3.9698255
16	2	0.0118974	0.1197746	10.067320	0.4022772	4.0498527	10.067320
64	3	0.0005752	0.0807153	140.32784	0.3683171	5.7548037	15.624588
144	4	.9142E-04	0.0756208	827.22370	0.3410913	7.2433129	21.235702
324	5	.1145E-04	0.0746848	6520.0717	0.2414518	8.5944699	35.594965
576	6	.1432E-05	0.0744755	.5202E+05	0.2165290	9.8461611	45.472715
1024	7	.1782E-06	0.0744258	.4176E+06	0.2050443	11.020714	53.747953

Table 7.101: Geometric mesh $\sigma = 0.5, \mu = 0.5$, Dirichlet on square, diagonal preconditioner

		A_N			$\text{diag}^{-1} A_N$		
N	n	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ
1	2	0.4625016	0.4625016	1.0000000	1.0000000	1.0000000	1.0000000
9	3	0.1721184	0.2612885	1.5180746	0.7442931	1.1298925	1.5180746
49	4	0.0018364	0.3267425	177.92454	0.0608815	2.9135452	47.856016
121	5	0.0010078	0.3215332	319.04545	0.0403745	3.2573790	80.679055
289	6	0.0001670	0.3230407	1934.7695	0.0364428	3.5027871	96.117383
529	7	0.0001069	0.3241933	3032.1296	0.0355647	3.7097124	104.30878

Table 7.102: Geometric mesh $\sigma = 0.5, \mu = 0.5$, Neumann on square, diagonal preconditioner

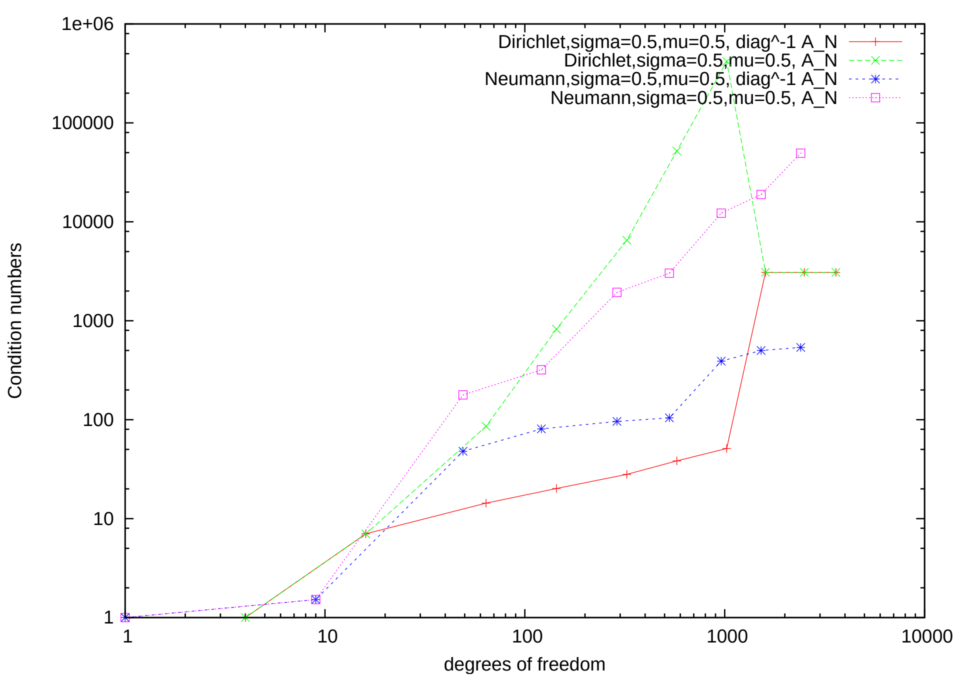


Figure 7.139: Geometric mesh

Example 7.15. *Multiplicative Schwarz algorithm for the Dirichlet problem on the square [17].*

bem3/ex12in

```
! p-Version,  Laplace, Dirichlet, Multiplicativ Schwarz
open(1) 'test.mspv'
! Choosing the square geometry [-1,1]^2
geometry('Square') ; #ti

P=10; M=2; Q=6

problem('Laplace',0) ; #pro. ; #pol 1

K=2
do J=1,5
inquire(file='mspv'//J//'.dat',T)
if (T.eq.0); then
open(2) 'mspv'//J//'.dat'
write(2) '# Dirichlet, p-version with coarse grid, n=',K
write(2) '# p rate iter enorm time dof'
do I=1,P
mesh('uniform',n=K,p=I,elements='rectangles') ; #g.
matrix('numeric',sigma=0.17,ijn=6,gqna=16,gqnb=16)
lft 16 0
msm 1.0d-10 200 1 2 0 1 1.0 1 ; S=SEC ! Solver
#hno. 3.0361798
#rno.
msm 1.0d-10   0 1 2 0 1 1.0 1  ! Contraction rate
write(2) I , LMAX,ITER,ENO,S,DOF,ENOERR
continue
write(2) ' '
fi
K=K*2
continue
end
```

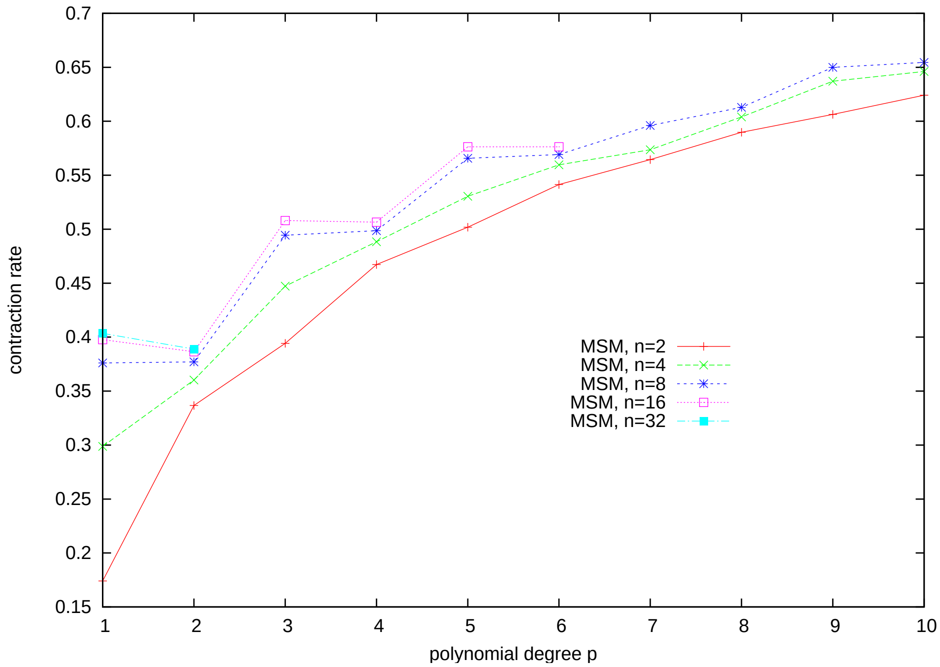


Figure 7.140: Weakly singular integral equation, p-version, contraction rate

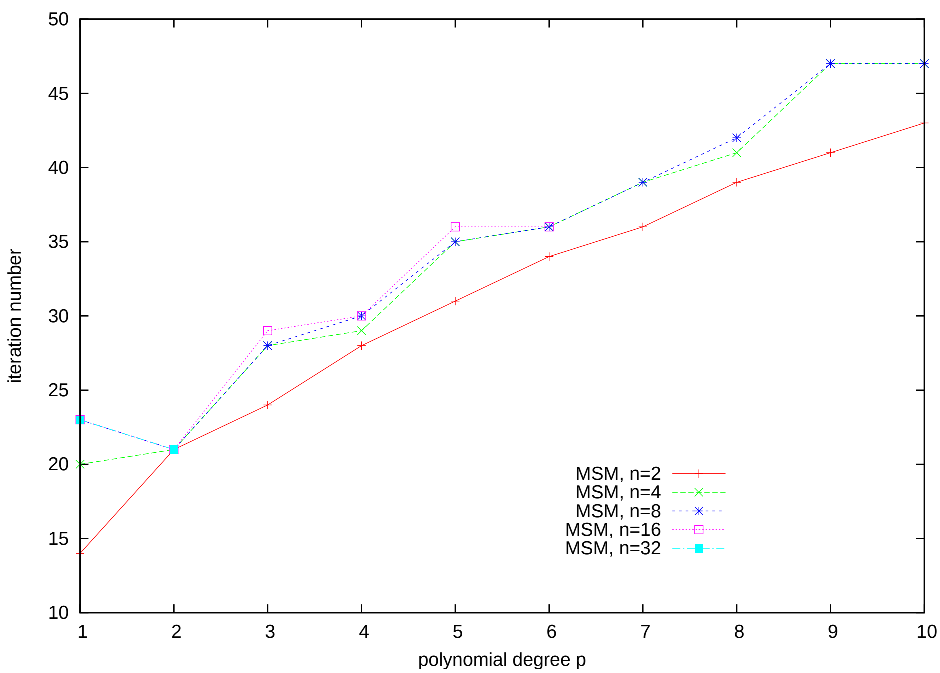


Figure 7.141: Weakly singular integral equation, p-version, iteration numbers

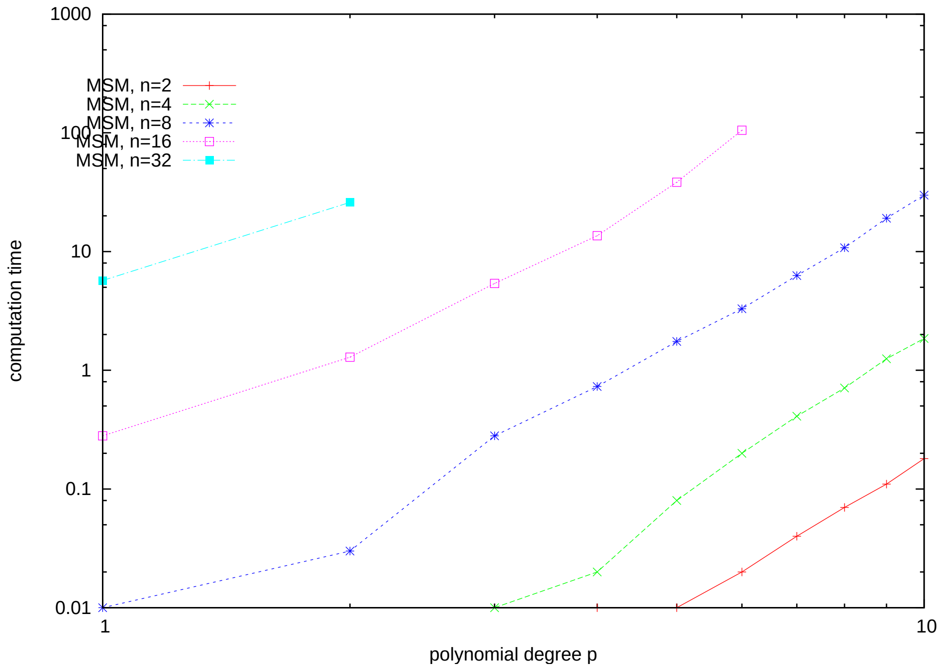


Figure 7.142: Weakly singular integral equation, p-version, computing time

Example 7.16. *Additive Schwarz algorithm for the Dirichlet problem on the square.*

bem3/ex13in

```
! p-Version,  Laplace, Dirichlet, Additive Schwarz
open(1) 'test'
! Choosing the square geometry [-1,1]^2
geometry('Square') ; #ti

P=10; M=2; Q=6

problem('Laplace',0) ; #pro. ; #pol 1

K=2
do J=1,4
inquire(file='aspv'//J//'.dat',T)
if (T.eq.0); then
open(2) 'aspv'//J//'.dat'
write(2) '# Dirichlet, p-version with coarse grid, n=',K
write(2) '# p rate iter enorm time dof'
do I=1,P
#rc K I 0 ; #g.
matrix('numeric',sigma=0.17,ijn=6,gqna=16,gqnb=16,mu=1.0)
lft 16 0
asmcg 1.0d-10 200 1 2 0 1 - 2 0 ; S=SEC ! Solver
#hno. 3.0361798
#rno.
asmcg 1.0d-10 200 1 2 0 1 -3 0 0  ! Condition number
write(2) I , LMIN,LMAX,COND,ITER,ENO,ENOERR,S,DOF
continue
write(2) ' '
fi
K=K*2
continue
end
```

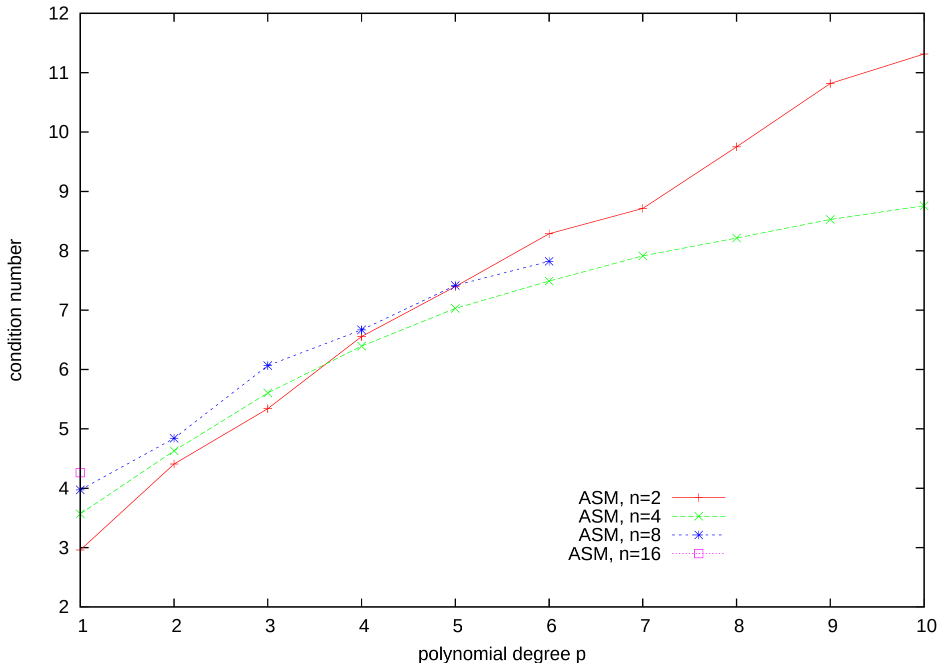


Figure 7.143: Weakly singular integral equation, p-version, condition number

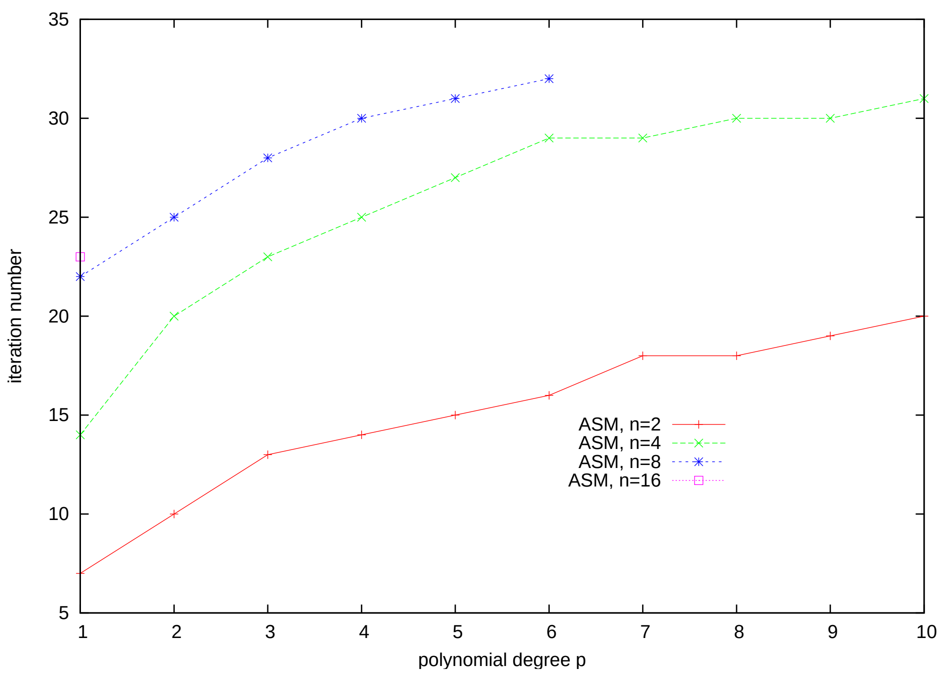


Figure 7.144: Weakly singular integral equation, p-version, iteration numbers

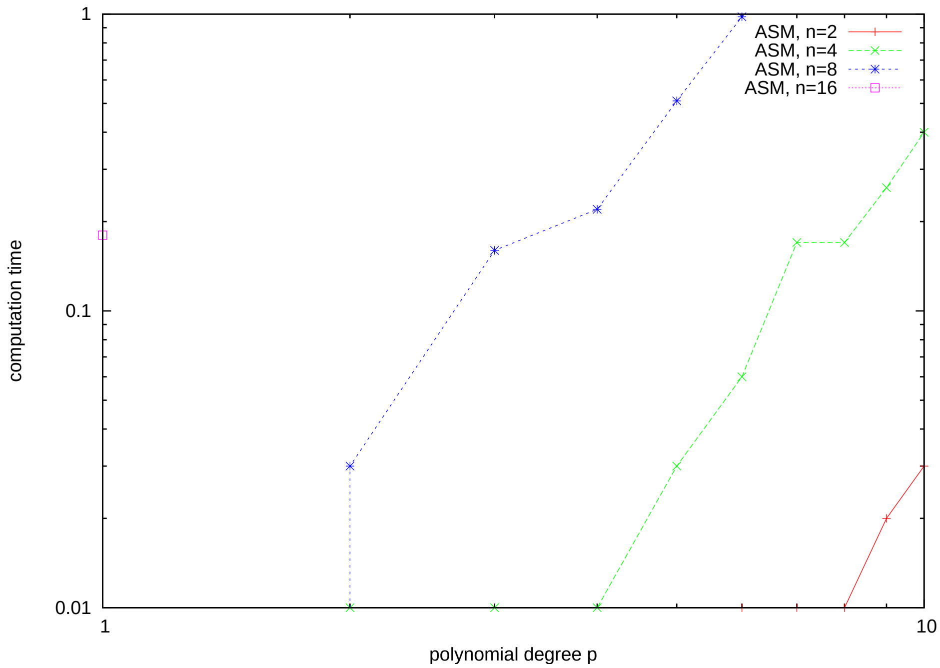


Figure 7.145: Weakly singular integral equation, p-version, computing time

8 Contact Problems (3D)

8.1 Signorini Problems (Lamé)

Example 8.1. *This example solves the Hertz contact problem, i.e. an elastic body under constant load pressed against a rigid obstacle, using Finite elements.*

Find $\mathbf{u} \in K := \{\mathbf{v} \in H^1(\Omega) : \mathbf{v}_n \leq g|_{\Gamma_S}\}$ *such that*

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq \mathbf{l}(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K$$

which is equivalent to find $\mathbf{u}$ *such that*

$$J(\mathbf{u}) = \min_{\mathbf{v} \in K} J(\mathbf{v}) \text{ with } J(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - \mathbf{l}(\mathbf{v}), \quad a(\mathbf{u}, \mathbf{v}) = \int_\Omega \varepsilon(\mathbf{u}) : C : \varepsilon(\mathbf{v}) \, dx$$

cont3/ex42h4in

```
! Lame, Signorini-problem, FEM, 3d Hertz contact problem
open(1) 'test.h' ; open(2) 'ex42h4in.dat'
geometry('Half-Sphere',bmode=(/3,3/),dim=(/8.0,8.0,8.0/)); #ti
problem('Lame',nickname='FEMSIG')
R=25  ! rhs
#ep 2000.0 0.3
#pxbd 8 2 3 'ubd'
  1  4  8.  0. 8.  5.65685424949  5.65685424949 8.  0.  8. 8. 0. 0. 8. -1
  1  4  0.  8. 8. -5.65685424949  5.65685424949 8. -8.  0. 8. 0. 0. 8. -1
  1  4 -8.  0. 8. -5.65685424949 -5.65685424949 8.  0. -8. 8. 0. 0. 8. -1
  1  4  0. -8. 8.  5.65685424949 -5.65685424949 8.  8.  0. 8. 0. 0. 8. -1
  2  4   8.  0. 8.  0.  8. 8.  0. 0. 0.  4.61880215352  4.61880215352 3.38119784648 -3 ! Signorini
  2  4   0.  8. 8. -8.  0. 8.  0. 0. 0. -4.61880215352  4.61880215352 3.38119784648 -3 ! Signorini
  2  4  -8.  0. 8.  0. -8. 8.  0. 0. 0. -4.61880215352 -4.61880215352 3.38119784648 -3 ! Signorini
  2  4   0. -8. 8.  8.  0. 8.  0. 0. 0.  4.61880215352 -4.61880215352 3.38119784648 -3 ! Signorini

#pxg 1 2 3 'obsg'
0 4 -8. -8. 0.  8. -8. 0.  8. 8. 0.  -8. 8. 0.  0. 0. 1. 0

#cmode 1
J=4;H=2.
do I=1,10
 mesh('uniform',n=2,p=1,spline='obs',genspl='no')
 mesh('uniform',n=J,p=1,elements='tetrahedral')
 compobstacle('uo','obs','z')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0)
 lft 16 R 0 R
 compdefect(obs='uo',to='u',mode='Sig')
 solve(eps=1.0d-8,mit='POLYAK'); T=SEC
 checksig 1.0e-4; #rno.
 computestress('u','sigma')
 open(1) 'ex42h4in'//I
 #taf. 'u'; #pnod. 'u'; #cx. 'u'
 #taf. 'sigma'; #pnod. 'sigma'; #cx. 'sigma'
 #obs.
 close(1)
 #ju.
```

```
 write(2) DOF,ITER,JU:12,T
 J=J*2; H=H/2
continue
end
```

N	$J(u_N)$	$\|u-u_N\|_{H^1(\Omega)}$	α_N	Iter	τ
255	-.541E+05	231.59077		57	0.0040010
1467	-.335E+05	111.67581	-0.416853	133	0.0200010
9843	-.299E+05	72.704099	-0.225476	408	0.4240260
71907	-.286E+05	50.826037	-0.180019	1222	15.240952
549315	-.280E+05	36.451145	-0.163496	3720	383.97200

Table 8.103: Convergence rate α_N of the h-version (FEM)

9 Finite Element Methods (3D)

9.1 Convergence

9.1.1 Laplace

First we deal with the homogenous Dirichlet problem of the Laplacian on the Cube, i.e. $\Omega = [-1, 1]^3$.

$$-\Delta u = f \text{ in } \Omega, \tag{19}$$
$$u = 0 \text{ on } \Gamma = \partial\Omega \tag{20}$$

Example 9.1. *This example solves the homogenous Dirichlet problem with*

$$f(x, y, z) = 3\pi^2 \sin(\pi x) \sin(\pi y) \sin(\pi z)$$

using the uniform h-version with cubes. The exact energy norm is known to be

$$\|u\|_E = \pi\sqrt{3} = 5.441398093$$

fem3/ex1h8in

```
! FEM(3D)-problem on the Cube, h-version(8)
open(1) 'test' ; open(2) 'ex1h8in.dat'
geometry('Cube') ; #ti
problem('Laplace',nickname='FEMHD')
R=8 ; J=4 ; Q=8
do I=1,8
 mesh('uniform',n=J,p=1,elements='hexahedrals')
 matrix; TM=SEC;TMW=WSEC
 lft Q R - R ;TL=SEC;TLW=WSEC
 solve(eps=1.0d-10,mdi='x=1',mdc='diag',mit='CG') ; TS=SEC;TSW=WSEC ; #rno.
 #hno. 5.441398093
 #cx. 'u'
 #err. Q R 'L2'  0 'u' ; E[0]=ERR
 #err. Q R 'H10' 0 'u' ; E[1]=ERR
 #no. 'L2'  'u'
 #no. 'H10' 'u'
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], TM,TL,TS,TMW,TLW,TSW
J=J*2
continue
end
```

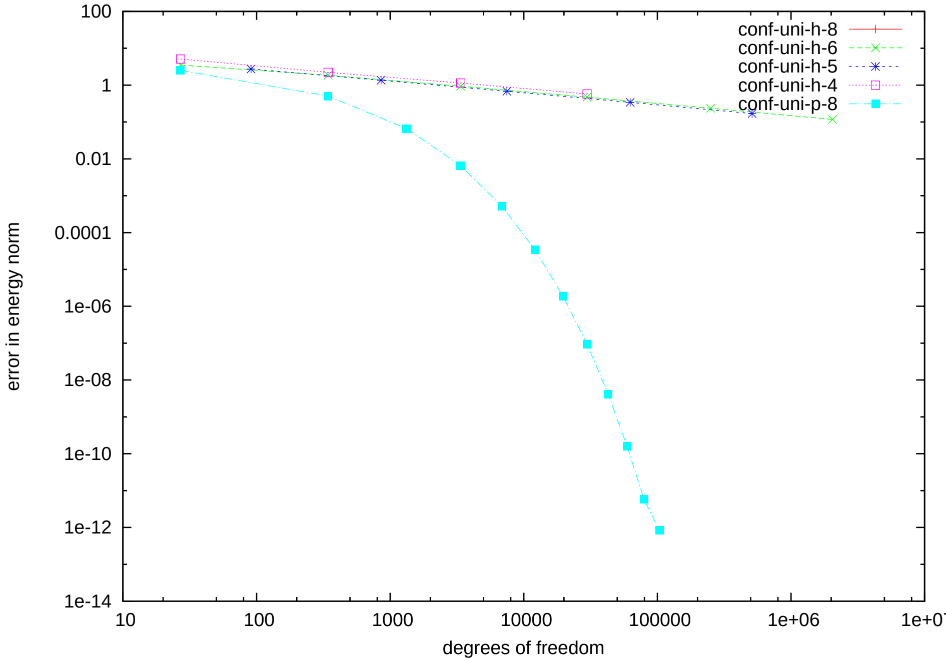


Figure 9.146: Homogenous Dirichlet problem, $f(x, y) = 3\pi^2 \sin(\pi x) \sin(\pi y) \sin(\pi z)$.

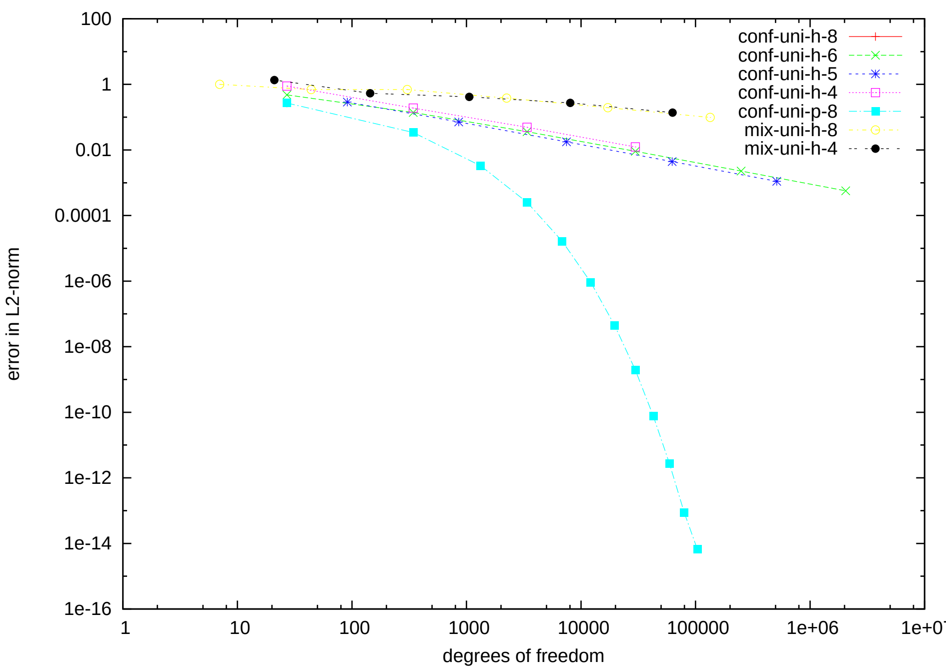


Figure 9.147: Homogenous Dirichlet problem, $f(x, y) = 3\pi^2 \sin(\pi x) \sin(\pi y) \sin(\pi z)$. Error in L^2-norm.

Example 9.2. *Let* $\Omega = [-1,1]^3 \backslash [0,1]^2 \times [-1,1]$ *(L-Block),* $\Gamma_D = \Gamma$ *and* $\Gamma_N = \partial\Omega \backslash \Gamma_D$*. This example deals with a non-linear Laplace problem with mixed boundary conditions*

$$-\operatorname{div}(\varrho(|\nabla u|)\nabla u) = f \text{ in } \Omega$$
$$u = u_0 \text{ on } \Gamma_D$$
$$\varrho(|\nabla u|)\frac{\partial u}{\partial n} = t_0 \text{ on } \Gamma_N$$

The variational formulation is: Find $u \in H^1_D(\Omega) := \{v \in H^1(\Omega) : v|_{\Gamma_D} = u_0\}$ *such that*

$$\int_\Omega \varrho(|\nabla u|)\nabla u \nabla v \, dx = \int_\Omega f v \, dx + \int_{\Gamma_N} t_0 v \, ds \quad \forall v \in H^1_{D,0}(\Omega)$$

Let $\tilde{\varrho}(x) = \varrho(|x|)I + \varrho'(|x|)\frac{xx^t}{|x|}$*. This gives rise to the following Newton scheme: Let* $u^{(0)} \in H^1_D(\Omega)$*. For* $n = 1, 2, \ldots$ *until a stopping criterion is satisfied: Find* $\delta \in H^1_{D,0}(\Omega)$ *such that*

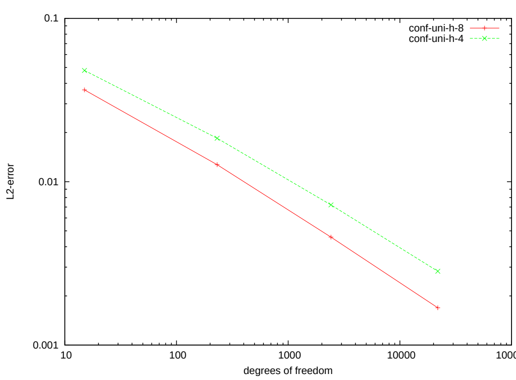
$$\int_\Omega (\tilde{\varrho}(\nabla u^{(n-1)})\nabla\delta)\nabla v \, dx = \int_\Omega f v \, dx + \int_{\Gamma_N} t_0 v \, ds - \int_\Omega \varrho(|\nabla u^{(n-1)}|)\nabla u^{(n-1)}\nabla v \, dx \quad \forall v \in H^1_{D,0}(\Omega)$$

and $u^{(n)} = u^{(n-1)} + \delta$*.*

Here we choose $\varrho(t) = \frac{1}{6}(1 + \frac{5}{1+5t})$ *and* $u(r,\varphi,z) = r^{2/3}\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$ *and we stop if* $\|\delta\|_{H^1(\Omega)} \leq 10^{-8}$*. We have* $u_0(r,\varphi,z) = r^{2/3}\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$*.*

DOF	$\|u - u_N\|_{L^2(\Omega)}$	α_{L^2}	$\|u - u_N\|_{H^1(\Omega)}$	α_{H^1}	It_{Newton}
15	0.0364745	—	0.3012205	—	5
231	0.0127185	-0.385301	0.1930157	-0.162769	6
2415	0.0045775	-0.435402	0.1230563	-0.191786	6
21855	0.0016917	-0.451902	0.0781322	-0.206217	6

Table 9.104: Non-Linear FEM, h(8), convergence rates and Newton steps



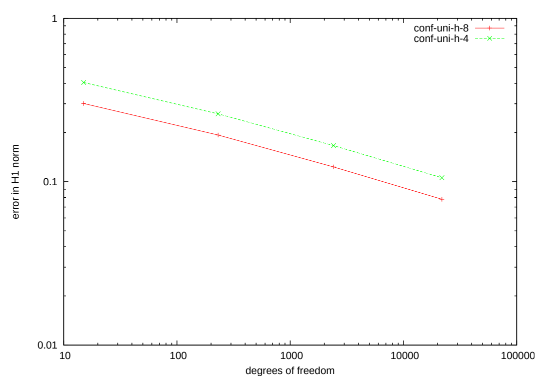


Figure 9.148: $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|u - u_n\|_{H^1(\Omega)}$ (right) — Non-linear.

Example 9.3. *Let* $\Omega = [-1,1]^3 \backslash [0,1]^2 \times [-1,1]$ *(L-Block),* $\Gamma = \partial\Omega$*. This example solves a Dirichlet problem for the Laplacian.*

$$\begin{aligned} -\Delta u &= 0 \text{ in } \Omega \\ u &= \varrho^{2/3} \sin(\frac{2}{3}(\varphi - \frac{\pi}{2})) \text{ on } \Gamma \end{aligned}$$

with the exact solution

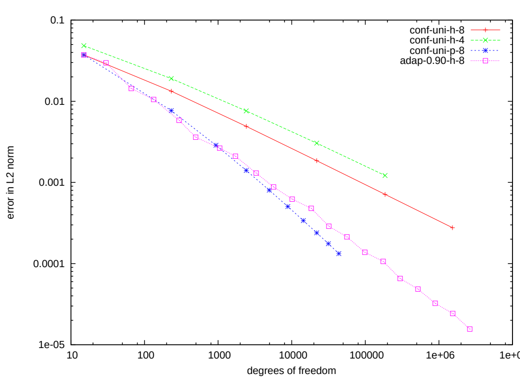
$$u(\varrho, \varphi, z) = \varrho^{2/3} \sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$$

using cylindrical coordinates (ϱ, φ, z)*.*

We apply the residual error indicator on hexahedral meshes with hanging nodes.

fem3/ex30h8a90in

```
! Dirichlet-FEM on the L-Block, adaptive, hanging nodes, theta=0.90
open(1) 'test.h' ; open(2) 'ex30h8a90in.dat'
geometry('L-Block') ; #ti
problem('Laplace',nickname='FEMNHD')
R=1    ! right hand side
Q=8
J=4;H=0.0625
mesh('uniform',n=J,p=1,elements='cubes',mode='hanging')
do I=1,30
! checkcont('u',quiet=1)
 approx 0 R 'u_bd' 'u0'
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
 lft 16 R 0 -1
 solve(eps=1.0d-10,mdi='x=0',mit='CG',abrflag=1,quiet=1);T=SEC; #rno.
 #taf. 'u'; #hrc. 'u'; #hrc. 'u_ex'
 extend('u','u_bd','u_ex')
 #err. Q R 'L2' 0 'u_ex' 'u' ; E[0]=ERR ! FEM
 #err. Q R 'H1' 0 'u_ex' 'u' ; E[1]=ERR
! resh 0.90 2 4 0
 adap(0.90,2,4,'RESID')
 write(2) DOF,ITER,E[0],E[1],ERREST,T
 refine(spline='u')
continue
end
```



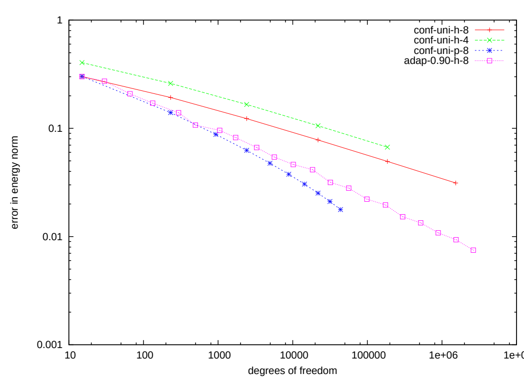


Figure 9.149: $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|u - u_n\|_{H^1(\Omega)}$ (right).

Example 9.4. *Here we investigate the Discontinuous Galerkin method (cf. Example 5.9) for the model problem*

$$\begin{aligned} -\Delta u &= f \ \text{in } \Omega, \\ u &= g \ \text{on } \Gamma. \end{aligned}$$

Find $u_h \in V_h$ *such that*

$$A_h(u_h, v) = F_h(v) \qquad \forall v \in V_h$$

where

$$\begin{aligned} A_h(u,v) &= \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, dx - \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa (\{\{\nabla_h v\}\} \cdot [[u]] + \{\{\nabla_h u\}\} \cdot [[v]]) ds \\ &+ \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa c[[u]] \cdot [[v]] ds, \\ F_h(v) &= \int_\Omega f v \, dx - \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa g \nabla_h v \cdot \mathbf{n} \, ds + \sum_{\kappa \in \mathcal{E}(\mathcal{T}_h)} \int_\kappa c g v \, ds \end{aligned}$$

with

$$\{\{v\}\} = \frac{1}{2}(v^+ + v^-), \quad \{\{\mathbf{q}\}\} = \frac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-), \quad [[v]] = v^+ \mathbf{n}_{K^+} + v^- \mathbf{n}_{K^-}, \quad [[\mathbf{q}]] = \mathbf{q}^+ \cdot \mathbf{n}_{K^+} + \mathbf{q}^- \cdot \mathbf{n}_{K^-}.$$

On a boundary face $\kappa \in \mathcal{E}_B(\mathcal{T}_h)$ *we set* $\{\{v\}\} = v$, $\{\{\mathbf{q}\}\} = \mathbf{q}$ *and* $[[v]] = v\mathbf{n}$.

Let h_K, k_K *denote diameter and polynomial degree of element* $K \in \mathcal{T}_h$.

We have $c = \gamma k^2 h^{-1}$ *with*

$$h(x) = \begin{cases} \min(h_K, h_{K'}), & x \in \kappa \in \mathcal{E}_I(\mathcal{T}_h), \kappa = \partial K \cap \partial K' \\ h_K, & x \in \kappa \in \mathcal{E}_B(\mathcal{T}_h), \kappa = \partial K \cap \Gamma, \end{cases}$$

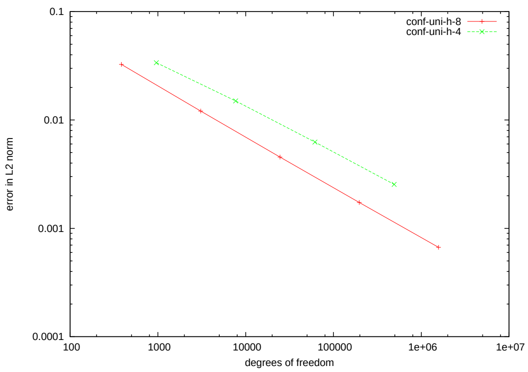
$$k(x) = \begin{cases} \min(k_K, k_{K'}), & x \in \kappa \in \mathcal{E}_I(\mathcal{T}_h), \kappa = \partial K \cap \partial K' \\ k_K, & x \in \kappa \in \mathcal{E}_B(\mathcal{T}_h), \kappa = \partial K \cap \Gamma, \end{cases}$$

In our example we have the exact solution $u(r, \varphi) = r^{2/3} \sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$ *and we choose* $\gamma = 100$.

fem3/ex43h8in

```
! FEM(3D)-problem on the L-Block, h-version(8)
open(1) 'test' ; open(2) 'ex43h8in.dat'
geometry('L-Block'); #ti
problem('Laplace',nickname='DGFEMNHD')
#dg 100
EPS=1.0e-15
R=1; J=4
do I=1,8
 mesh('uniform',n=J,p=1,elements='hexahedral')
 matrix
 lft 8 R 0 R
 solve(eps=EPS,mdi='x=1',mdc='no',mit='CG'); T=SEC; #rno.
 #hno. 3.847649490
 #cx. 'u'
 #err. 8 R 'L2' 0 'u' ; E[0]=ERR
 #err. 8 R 'H10'0 'u' ; E[1]=ERR
```

```
 #no. 'L2'
 #no. 'H10'
 write(2) DOF, I, ENO, ENOERR, E[0], E[1], COND, T, ITER
J=J*2
continue
end
```



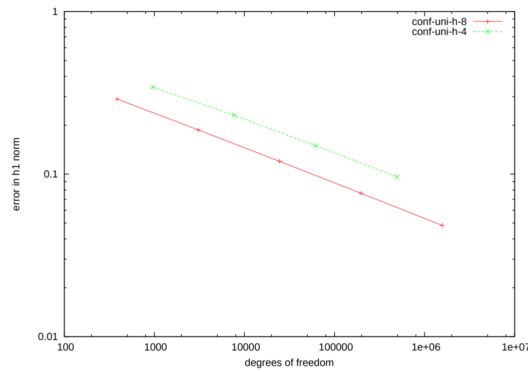


Figure 9.150: Laplace-3d DG: $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|u - u_n\|_{H^1(\Omega)}$ (right).

Example 9.5. *In this example we investigate the homogenous Dirichlet problem of the Laplacian on the Cube* $[-1,1]^3$.

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma = \partial\Omega \end{aligned}$$

We are expecially interested in the point-wise convergence, using a postprocessing scheme

$$\begin{aligned}
G(x,y) &:= \frac{1}{4\pi}\frac{1}{|x-y|} \\
\psi(x,\varrho) &= \begin{cases} 1 & \text{for } |x| \le \varrho/2 \\ 0 & \text{for } |x| \ge \varrho \end{cases} \\
&= \frac{1}{2}\left(1 - \chi(4\frac{x}{\varrho} - 3)\right) \\
\chi(t) &= \begin{cases} -1 & \text{for } t \le -1 \\ 2.4609375t - 3.28125t^3 + 2.953125t^5 - 1.40625t^7 + 0.2734375t^9 & \text{for} -1 < t < 1 \\ +1 & \text{for } t >= 1 \end{cases} \\
H(x) &:= G(x,x_0)\psi(x-x_0) \\
\Psi &:= -\int_{\omega(x_0,\varrho)} f(x)H(x)\,dx \\
\tilde{u}_N(x_0) &:= \int_{\omega^*} u_N(x)\Delta H(x)\,dx - \Psi \\
|u(x_0) - \tilde{u}_N(x_0)| &\le C\,N^{\alpha}
\end{aligned}$$

In our example we have $f(x) = 3\pi^2 \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)$ *and a ll computations are done using uniform meshes with hexahedrals and piecewise linear polynomials. The singular integral* Ψ *is computed using a composite quadrature scheme with geometrical refinement towards the singular point.*

All smooth integrals are done using a tensor product rule with a 24 point 1d-Gauss-Quadrature formula.

In Table 9.105 we present the point-evaluation $u_N(A)$*, the point-wise error* $\delta u(A) = u(A) - u_N(A)$*, the convergence rate* α_A*, the post-processed value* $\tilde{u}_N(A)$*, the error* $|u - \tilde{u}_N(A)|$ *and the rate* α.

Table 9.106 gives the corresponding values in point B*, nearer to the boundary of the L-Shape.*

$u(A)$ *and* $u(B)$ *have been determined by using Aitken extrapolation.*

DOF	$u_N(A)$	$\delta u(A)$	α_A	$\tilde{u}_N(A)$	$\lvert u - \tilde{u}_N(A)\rvert$	α
27	-1.411189244	0.4123091		-1.862161062	0.8632809	
343	-1.077903787	0.0790236	-0.650	-1.009176204	0.0102960	-1.742
3375	-1.011685992	0.0128058	-0.796	-0.996901937	0.0019782	-0.721
29791	-0.999341870	0.0004617	-1.526	-0.994374230	0.0045059	0.3780
250047	-0.998061394	0.0008188	0.2693	-0.998032325	0.0008478	-0.785
2048383	-0.998628765	0.0002514	-0.561	-0.998681607	0.0001986	-0.690

Table 9.105: Pointwise error and postprocessed pointwise error in $A = (-0.513, 0.507, 0.497)$, $\varrho = 0.025$

DOF	$u_N(B)$	$\delta u(B)$	α_B	$\tilde{u}_N(B)$	$\lvert u - \tilde{u}_N(B)\rvert$	α
27	-0.014917819	0.0199580		-0.010472084	0.0244038	
343	-0.031610995	0.0032648	-0.712	-0.029793485	0.0050823	-0.617
3375	-0.034541147	0.0003347	-0.996	-0.034501674	0.0003742	-1.141
29791	-0.034782685	.9315E-04	-0.587	-0.034811665	.6417E-04	-0.810
250047	-0.034812904	.6293E-04	-0.184	-0.034875595	.2392E-06	-2.628
2048383	-0.034871070	.4764E-05	-1.227	-0.034869122	.6713E-05	1.5854

Table 9.106: Pointwise error and postprocessed pointwise error in $B = (0.114, -0.9521, 0.231)$, $\varrho = 0.025$

Example 9.6. *Here we rewrite the PDE from Example 9.1 as a L^2-Least-Squares Problem with (possible) mixed boundary conditions. We first obtain the first order system*

$$\begin{aligned} -\operatorname{div} p &= f(x) \ \ in\ \Omega \\ p &= \nabla u \ \ in\ \Omega \\ u(x) &= u_D(x), \quad x \in \Gamma_D \\ p(x) \cdot n &= p_N(x), \quad x \in \Gamma_N \end{aligned}$$

Introducing the spaces

$$\begin{aligned} H^1_D(\Omega) &:= \{u \in H^1(\Omega)\,:\, u|_{\Gamma_D} = u_D\} \\ H_N(\operatorname{div};\Omega) &:= \{p \in H(\operatorname{div};\Omega)\,:\, p\cdot n|_{\Gamma_N} = u_N\} \end{aligned}$$

we obtain the Least-Squares minimization problem

$$(u,p) = \underset{(v,q)\in H^1_D(\Omega)\times H_N(\operatorname{div};\Omega)}{\operatorname{minarg}} \mathcal{F}(v,q;f)$$

with

$$\mathcal{F}(v,q;f) := \| -\operatorname{div} q - f\|^2_{L^2(\Omega)} + \|q - \nabla v\|^2_{L^2(\Omega)}$$

The variational formulation now reads: Find $(u,p) \in H^1_D(\Omega) \times H_N(\operatorname{div};\Omega)$ s.t.

$$\begin{aligned} (\nabla u, \nabla v) - (p, \nabla v) &= 0 \\ -(\nabla u, q) + (p,q) + (\operatorname{div} p, \operatorname{div} q) &= (f, \operatorname{div} q) \end{aligned}$$

for all $(v,q) \in H^1_{D,0}(\Omega) \times H_{N,0}(\operatorname{div};\Omega)$.

fem3/ex100h8in

```
! Laplace, Least-Squares-L2-formulation with homogenuous Dirichlet data
open(1) 'test'; open(2) 'ex100h8in.dat'
geometry('Cube'); #ti
problem('Laplace',nickname='LS2HD')
EPS=1.0e-15
R=8; J=2
do I=1,10
 mesh('uniform',n=J,p=1,elements='hexahedral')
 matrix; TM=SEC; WM=WSEC
 show('matrix')
 lft 8 R 0 R; TL=SEC; WL=WSEC
 solve(eps=EPS,mdi='x=1',mdc='diag',mit='CG'); TS=SEC; WS=WSEC
 #rno.
! open(1) 'ex100h4in'//I
 #hno. 3.847649490
! #taf. 'u'; #px. 'u'; #cx. 'u'
! #taf. 'p'; #px. 'p'; #cx. 'p'
 #err. 8 R 'L2'   0 'u' ; E[0]=ERR
 #err. 8 R 'H10'  0 'u' ; E[1]=ERR
 #err. 8 R 'L2'   0 'p' 'p' ; E[2]=ERR
 #err. 8 R 'HDIV' 0 'p' 'p' ; E[3]=ERR
 #no. 'L2' 'u'
 #no. 'H10' 'u'
 write(2) DOF, DOFU, DOFP, I, ENO, ENOERR, E[0], E[1], E[2], E[3], COND, ITER, TM, TL, TS, WM, WL,
 J=J*2
continue
end
```

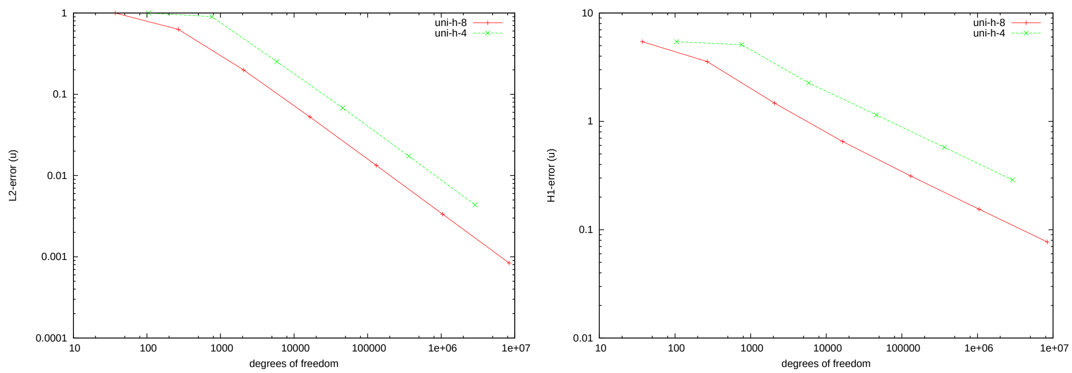


Figure 9.151: Laplace (L2-Least Squares): $\|u - u_n\|_{L^2(\Omega)}$ (left) and $|u - u_n|_{H^1(\Omega)}$ (right).

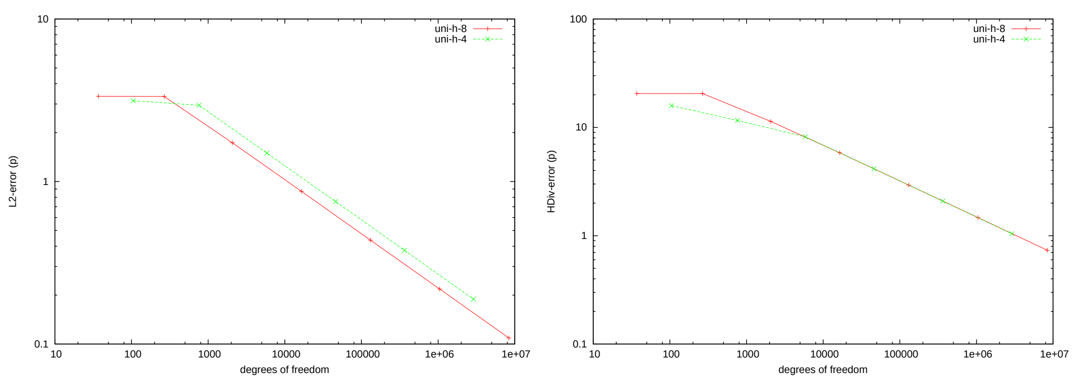


Figure 9.152: Laplace (L2-Least Squares): $\|p - p_n\|_{L^2(\Omega)}$ (left) and $|p - p_n|_{H(\mathrm{div};\Omega)}$ (right).

9.1.2 Lamé

Example 9.7. *3d-Stein-Benchmark, elastic case. E-module* $E = 206900.0$, $\nu = 0.29$. $\mathcal{T} = (0, 0, 450.0)^T$.

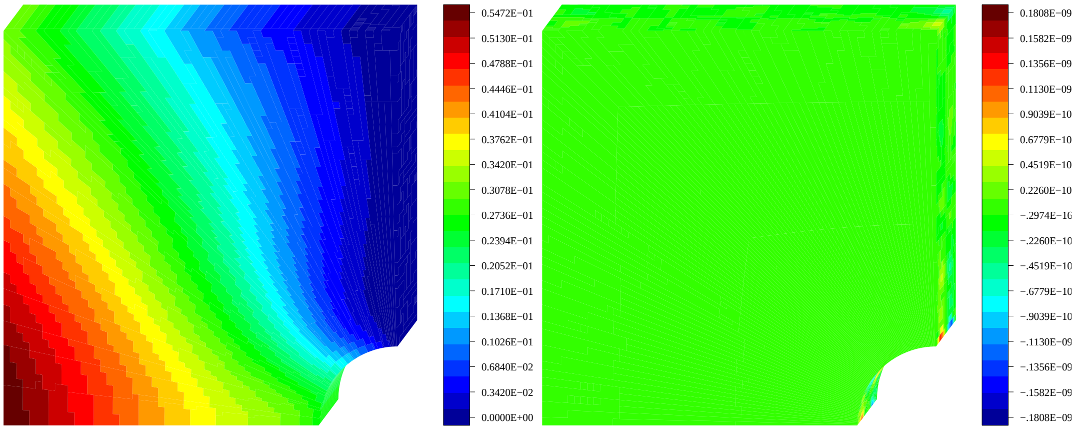


Figure 9.153: Deformation x and y-components.

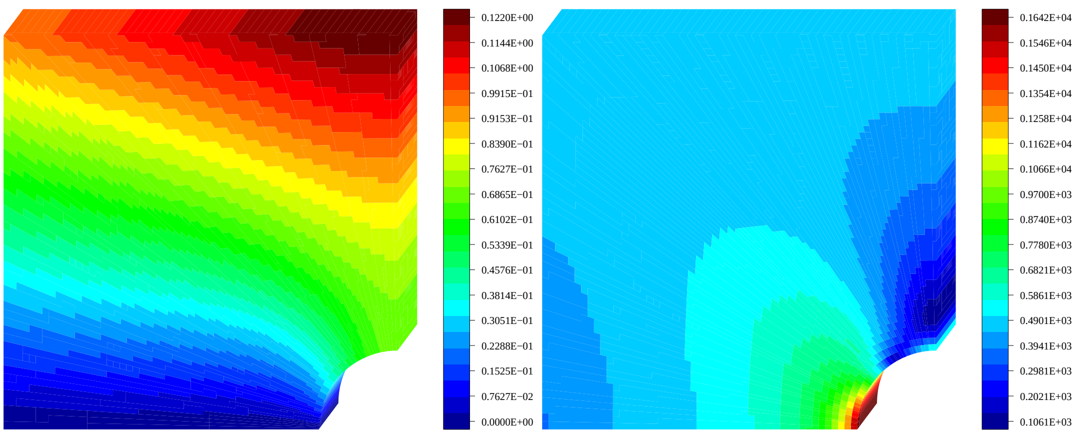


Figure 9.154: Deformation z-component and stress.

9.1.3 Helmholtz

Example 9.8. *Let* $\Omega = [-1,1]^3 \backslash [0,1]^2 \times [-1,1]$ *(L-Block),* $\Gamma = \partial\Omega$*. This example solves an inhomogenous Dirichlet problem for the Helmholtz equation.*

$$\begin{aligned} -\Delta u - k^2 u &= 0 \text{ in } \Omega \\ u &= \tilde{J}_{2/3}(k\varrho)\sin(\frac{2}{3}(\varphi - \frac{\pi}{2})) \text{ on } \Gamma \end{aligned}$$

with the exact solution

$$u(\varrho, \varphi, z) = \tilde{J}_{2/3}(k\varrho)\sin(\frac{2}{3}(\varphi - \frac{\pi}{2}))$$

using cylindrical coordinates (ϱ, φ, z)*.* $\tilde{J}_{2/3}(x) = \Gamma(2/3+1)J_{2/3}(x)$ *is a rescaled Bessel function.*

fem3/ex40h8in

```
! Dirichlet-FEM on the L-Block, uniform mesh h(8), Helmholtz
open(1) 'test.h'
geometry('L-Block') ; #ti
problem('Helmholtz',nickname='FEMNHD')
R=1    ! right hand side
do K=1,10
 KW=Real(K)/2.0; #kw KW
 J=4;H=0.0625; Q=4
 open(2) 'ex40h8k'//KW:3//'in.dat'
 do I=1,5
  mesh('uniform',n=J,p=1,elements='cubes')
  approx 0 R 'u_bd' 'u0'
  matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
  lft Q R 0 R
  solve(eps=1.0d-10,mdi='x=0',mit='CG',abrflag=1,quiet=1);T=SEC; #rno.
  extend('u','u_bd','u_ex')
  #err. Q R 'L2' 0 'u_ex' 'u' ; E[0]=ERR ! FEM
  #err. Q R 'H1' 0 'u_ex' 'u' ; E[1]=ERR
  norm('NO','H1','u_ex')
  write(2) DOF,ITER,E[0],E[1],T,NO
  J=J*2 ; H=H/2
 continue
 close(2)
continue
end
```

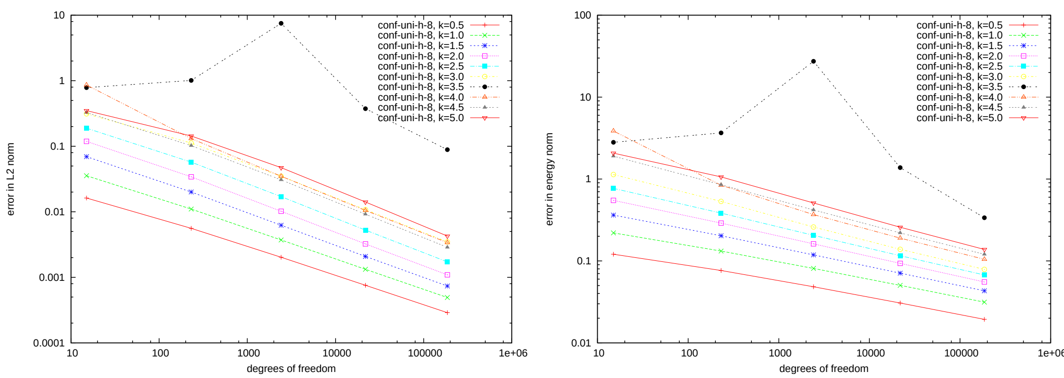


Figure 9.155: Helmholtz (3d-FEM): $\|u - u_n\|_{L^2(\Omega)}$ (left) and $\|u - u_n\|_{H^1(\Omega)}$ (right).

9.1.4 Maxwell

In our next example we deal with Maxwell's equation on the Cube.

Example 9.9. *This solves the homogenous Dirichlet problem with Nedelec-elements, $\beta = i\cdot 1$ and*

$$\begin{aligned} f(x,y,z) &= \begin{pmatrix} 4-2(y^2+z^2)+i(1-y^2)(1-z^2) \\ 4-2(z^2+x^2)+i(1-z^2)(1-x^2) \\ 4-2(x^2+y^2)+i(1-x^2)(1-y^2) \end{pmatrix} \\ u(x,y,z) &= ((1-y^2)(1-z^2),(1-x^2)(1-z^2),(1-x^2)(1-y^2)) \end{aligned}$$

fem3/ex2h8in

```
! Maxwell FEM(3D)-problem on the Cube, h-version(8)
open(1) 'test' ; open(2) 'ex2h8in.dat'
geometry('Cube') ; #ti
problem('Maxwell',nickname='FEMHD')
#maxw 0 0 1.0
#maxw.
EPS=1.0d-10
R=1 ; J=2
do I=1,6
 mesh('uniform',n=J,p=1,elements='hexahedrals'); #g.
 matrix
 lft 8 R
 solve(eps=EPS,mdi='x=0',mdc='no',mit='GMRES'); T=SEC; #rno.
 #hno.
 #err. 4 R 'L2'   0 'u' ; E[0]=ERR
 #err. 4 R 'Hrot' 0 'u' ; E[1]=ERR
 write(2) DOF, I, ENO, E[0], E[1], T
 J=J*2
continue
end
```

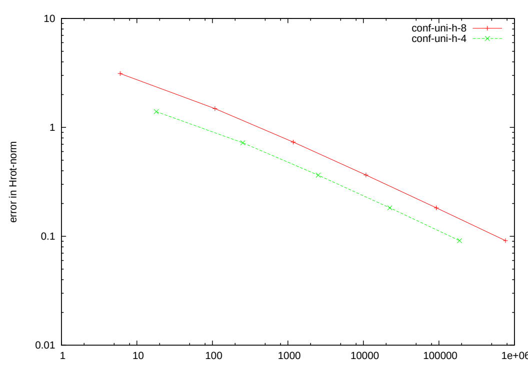


Figure 9.156: Homogenous Maxwell problem.

9.1.5 Stokes

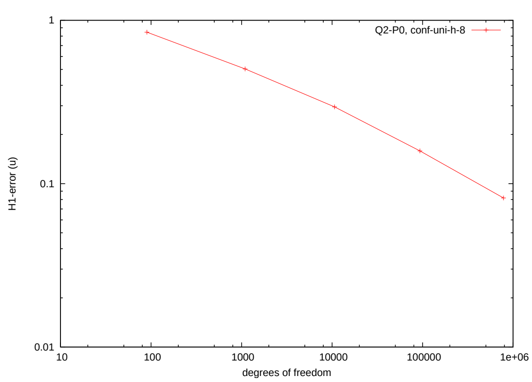
Example 9.10. *Here we investigate the Stokes problem (cf. Example 5.27)*

$$\begin{aligned} -\nu\Delta\vec{u} + \nabla p &= f \ \ in\ \Omega \\ \operatorname{div}\vec{u} &= 0 \ \ in\ \Omega \\ \vec{u} &= u_0 \ \ on\ \partial\Omega \end{aligned}$$

In this example we choose $\Omega = [-1,1]^3$, $\nu = 1$ *and* $\vec{u} = \frac{1}{\nu}(\frac{1}{r} + (x_0-\bar{x}_0)^2/r^3, (x_0-\bar{x}_0)(x_1-\bar{x}_1)/r^3), (x_0-\bar{x}_0)(x_2-\bar{x}_2)/r^3)$ *and* $p = -2(x_0-\bar{x}_0)/r^3$ *with* $r = |x-\bar{x}|$ *and* $\bar{x} = (0,0,1.5)$, *such that* $f \equiv 0$.

fem3/ex60h8in

```
! Stokes on Cube
open(1) 'test.h'; open(2) 'ex60h8in.dat'; #ti
problem('Stokes',nickname='FEMNHD')
geometry('Cube')
NU=1.; #stokes NU
R=10
J=2
do I=0,8
 mesh('uniform',n=J,p=2,elements='hexahedrals',spline='u',gm='ug',genspl='no')
 mesh('global',n=1,spline='xi',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix
 lft 16 R 0 R
 solve(eps=1.e-10,mdc='no',mit='MINRES',quiet=0,restart=400); T=SEC
 #rno.
 extend('u','u_bd','u_ex')
 #err. 16 R 'L2' 0 'u_ex' 'u'; E[1]=ERR
 #err. 16 R 'H1' 0 'u_ex' 'u'; E[2]=ERR
 #err. 16 R 'L2' 0 'p' 'p'; E[3]=ERR

 write(2) DOF,E[1],E[2],E[3],T,ITER
 J=J*2
continue
end
```



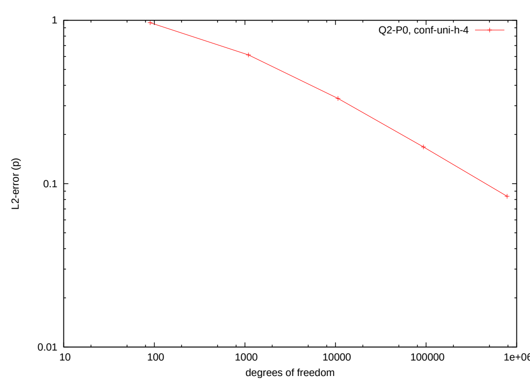


Figure 9.157: Stokes (3d-FEM): $\|u-u_n\|_{H^1(\Omega)}$ (left) and $\|p-p_n\|_{L^2(\Omega)}$ (right).

9.1.6 Transport

Example 9.11. *Here we investigate the Transport problem*

$$\beta \cdot \nabla u + c \cdot u = f \text{ in } \Omega$$
$$u = u_0 \text{ on } \Gamma_{inflow}$$

The variational formulation, using the Galerkin method, is:
Find $u_h \in V^h := \{u^h \in P_k \,|\, u^h = u_0 \text{ on } \Gamma_{inflow}\}$*, such that*

$$(u^h_\beta + cu^h, v^h) = (f, v^h), \quad \forall v^h \in V^h_0 = \{v^h \in P_k \,|\, v^h = 0 \text{ on } \Gamma_{inflow}\}$$

In this example we use $\beta = (1, \tan 35°, 0)$, $c = 1$, $u(x, y, z) = \exp(x)\sin(x)\sin(y)\sin(z)$*, see [4].*

fem3/ex99h8in

```
! Convection on Cube with hexahedrals
open(1) 'test.h'; open(2) 'ex99h8in.dat'
problem('Convection',nickname='PGCSBD'); #ti
#pxg 1 3 3 'ug'
0 8  0. 0. 0.  1. 0. 0.  1. 1. 0.  0. 1. 0.  0. 0. 1.  1. 0. 1.  1. 1. 1.  0. 1. 1. 0
#pxbd 2 2 3 'ubd'
0 4 0. 0. 0.  1. 0. 0.  1. 0. 1.  0. 0. 1. -2
0 4 0. 0. 0.  0. 1. 0.  0. 1. 1.  0. 0. 1. -2

Q=6; R=0; J=2
do I=0,8
 mesh('uniform',n=J,p=1,elements='hexahedrals',spline='u',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix; TM=SEC
 lft Q R 0 R
 solve(eps=1.e-10,mdc='no',mit='CGNR'); TS=SEC; #rno.
 extend('u','u_bd','u_ex','Dirichlet')
 open(1) 'ex99h8_'//I; #taf. 'u_ex'; #pnod. 'u_ex'; #cx. 'u_ex'; close(1)
 #err. Q R 'L2' 0 'u_ex' 'u'; E[1]=ERR
 #err. Q R 'H1' 0 'u_ex' 'u'; E[2]=ERR
 write(2) DOF,E[1],E[2],ITER,TS,TM
 J=J*2
continue
end
```

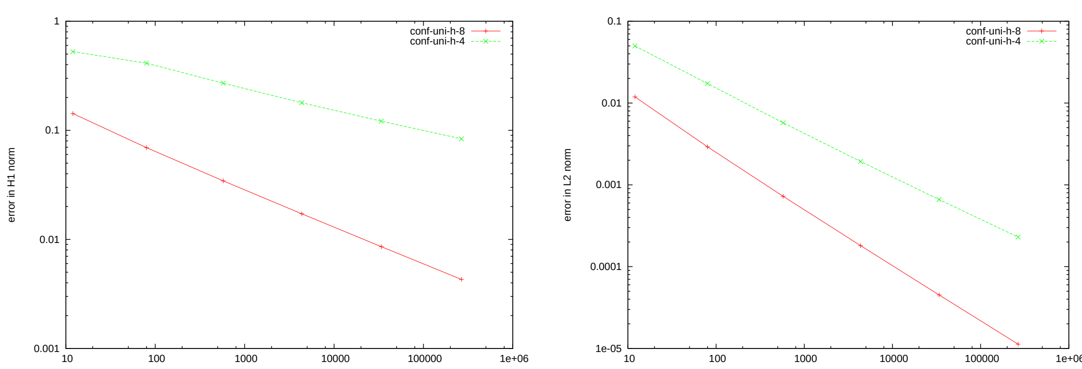


Figure 9.158: Transport(Galerkin): Error in $H^1(\Omega)$-norm (left) and in $L^2(\Omega)$-norm (right)

Example 9.12. *Similarly to Example 9.11 we investigate here the Transport Problem*

$$\beta \cdot \nabla u + c \cdot u = f \;\; in \; \Omega$$
$$u = u_0 \;\; on \; \Gamma_{inflow}$$

The variational formulation, using the SUPG method, reads:
Find $u_h \in V^h := \{u^h \in P_k \,|\, u^h = u_0 \; on \; \Gamma_{inflow}\}$*, such that*

$$(u^h_\beta + cu^h, v^h) + \sum_\Delta (u^h_\beta + cu^h, \frac{\xi_\Delta h_\Delta}{2|\beta_\Delta|} v^h_\beta)_\Delta = (f, v^h) + \sum_\Delta (f, \frac{\xi_\Delta h_\Delta}{2|\beta_\Delta|} v^h_\beta)_\Delta, \quad \forall v^h \in V^h_0$$

with $V^h_0 = \{v^h \in P_k \,|\, v^h = 0 \; on \; \Gamma_{inflow}\}$*. In this example we use* $\beta = (1, \tan 35°, 0)$*,* $c = 1$*,* $u(x, y, z) = \exp(x) \sin(x) \sin(y) \sin(z)$*, see [4].*

fem3/ex97h8in

```
! Convection on Cube with hexahedrals, SUPG
open(1) 'test.h'; open(2) 'ex97h8in.dat'
problem('Convection',nickname='SUPGCSBD'); #ti
#pxg 1 3 3 'ug'
0 8  0. 0. 0.  1. 0. 0.  1. 1. 0.  0. 1. 0.  0. 0. 1.  1. 0. 1.  1. 1. 1.  0. 1. 1. 0
#pxbd 2 2 3 'ubd'
0 4 0. 0. 0.  1. 0. 0.  1. 0. 1.  0. 0. 1. -2
0 4 0. 0. 0.  0. 1. 0.  0. 1. 1.  0. 0. 1. -2

Q=6; R=0; J=2
do I=0,8
 mesh('uniform',n=J,p=1,elements='hexahedrals',spline='u',gm='ug')
 approx 0 R 'u_bd' 'u0'
 matrix; TM=SEC
 lft Q R 0 R
 solve(eps=1.e-10,mdc='no',mit='CGNR'); TS=SEC; #rno.
 extend('u','u_bd','u_ex','Dirichlet')
 open(1) 'ex97h8_'//I; #taf. 'u_ex'; #pnod. 'u_ex'; #cx. 'u_ex'; close(1)
 #err. Q R 'L2' 0 'u_ex' 'u'; E[1]=ERR
 #err. Q R 'H1' 0 'u_ex' 'u'; E[2]=ERR
 write(2) DOF,E[1],E[2],ITER,TS,TM
 J=J*2
continue
end
```

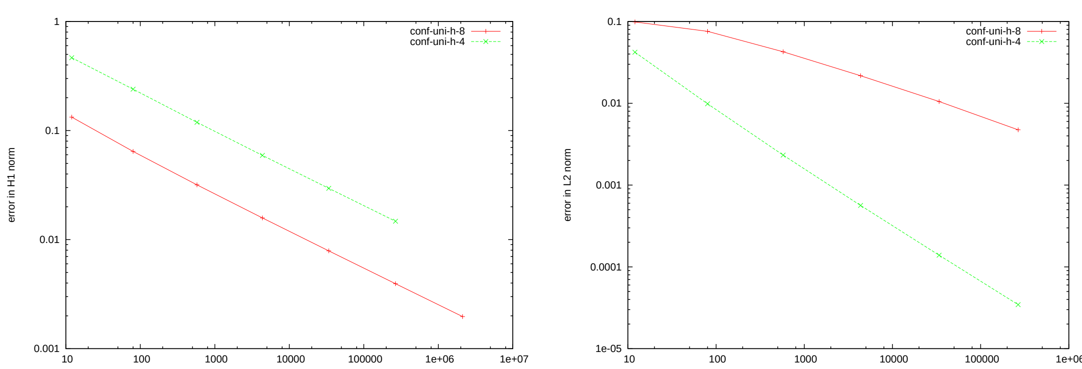


Figure 9.159: Transport(SUPG): Error in $H^1(\Omega)$-norm (left) and in $L^2(\Omega)$-norm (right)

9.2 Solvers

Here we investigate the performance of different solvers and preconditioners for h- and p-version for the 3d-FEM.

Example 9.13. *Using the configuration of example 9.1 for the Poisson-equation with the uniform h-version with hexahedrals we apply the multigrid-algorithm with V-cycle and one pre- and one post-smoothing step (damped Jacobi* $\omega = 0.5$*). The iteration stops if the last relative change of the iterate is less than* 10^{-10}.

fem3/ex3mcgin

```
! h-version, multigrid, Laplace
open(1) 'test.mcg' ; open(2) 'ex3mcgin.dat' ; open(3) 'ex3mcg.tex'
geometry('Cube'); #ti
problem('Laplace',nickname='FEMHD')
EPS=1.0d-10
R=8; J=4
do I=0,8
#time T0
mesh('uniform',n=J,p=1,elements='hexahedrals')
#time T1
matrix
lft 4 R 0 R
defprec(mode='MG',spline='u',name='Pu',mat='A',mtop=I,hpmodus=0,stp=2,mdc=0,&
 & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)

solve(eps=EPS,mdi='x=1',mdc='u.Pu.u',mit='CG'); T=SEC
#rno.
#hno. 5.441398093
write(2) DOF,I,LMIN,LMAX,COND,T,ITER,T1-T0
write(3) DOF//'&'//LMIN:7//'&'//LMAX:6//'&'//COND:6//'&'//ITER//'&'//T:6//'\\'
J=J*2
continue
end
```

	Multigrid				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
27	1.00000	1.0000	1.0000	2	0.0080
343	0.47942	1.0000	2.0858	12	0.0400
3375	0.47896	1.0000	2.0879	13	0.1440
29791	0.48555	1.0000	2.0595	13	1.6041
250047	0.50593	1.0000	1.9765	13	17.317
2048383	0.54241	0.9999	1.8434	13	198.88

Table 9.107: Conjugate Gradients with multigrid-preconditioner

	BPX				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
27	0.71548	1.6179	2.2612	6	0.0000
343	0.37132	2.0868	5.6200	16	0.0280
3375	0.18699	2.2126	11.833	27	0.0920
29791	0.10075	2.2464	22.298	41	1.1161
250047	0.06264	2.2553	36.003	57	13.097
2048383	0.03140	2.2575	71.904	80	175.33

Table 9.108: Conjugate Gradients with BPX-preconditioner

	CG				
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	#it	t (sec)
27	0.71548	1.6179	2.2612	6	0.0000
343	0.10846	0.9404	8.6701	15	0.0000
3375	0.01423	0.4921	34.590	28	0.0080
29791	0.00180	0.2490	138.34	57	0.2520
250047	0.00023	0.1249	553.35	113	5.0163
2048383	.28E-04	0.0625	2213.4	217	87.941

Table 9.109: Conjugate Gradients without preconditioner

10 FEM-BEM coupling (3D)

10.1 Convergence

10.1.1 Laplace

Example 10.1. ***Symmetric coupling (3D)*** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ, and $\Omega_c := \mathbb{R}^3 \setminus \overline{\Omega}$. We consider the model transmission problem of finding $u_1 \in H^1(\Omega)$, $u_2 \in H^1_{loc}(\overline{\Omega}_c)$ such that*

$$-\Delta u_1 = f \quad \text{in } \Omega \tag{21}$$
$$\Delta u_2 = 0 \quad \text{in } \Omega_c \tag{22}$$
$$u_1 = u_2 + u_0 \quad \text{on } \Gamma \tag{23}$$
$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} + t_0 \quad \text{on } \Gamma \tag{24}$$
$$u_2(x) = O(|x|^{-1}), \quad |x| \to \infty \tag{25}$$

We prescribe the transmission conditions (23) and (24) on the cube $\Omega = [-1,1]^3$ by

$$u_0(x,y,z) = u(x,y,z) \tag{26}$$
$$t_0(x,y,z) = \frac{\partial u}{\partial n}, \tag{27}$$

using

$$u(x,y,z) = x^2 + y^2 + z^2$$
$$f(x,y,z) = -\Delta u = -6$$

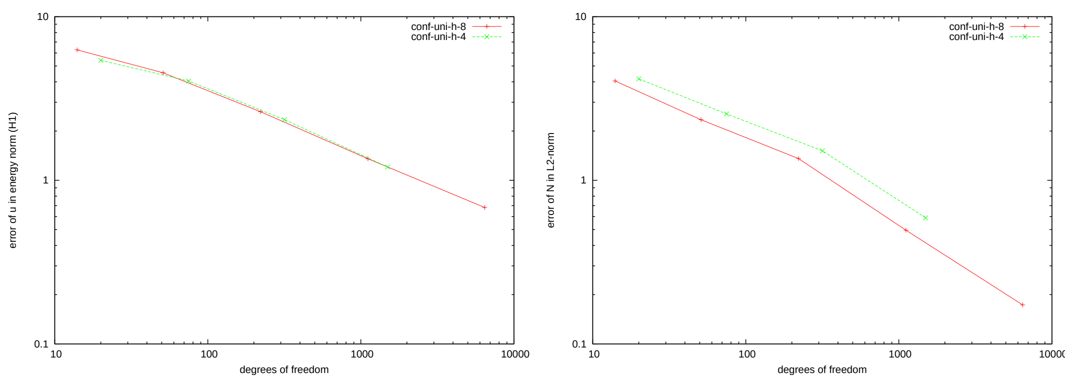


Figure 10.160: Error of u in energy-norm (left) and error of ϕ in L_2-norm.

Example 10.2. ***Symmetric coupling (3D)*** *In this example we investigate the same problem as in Example 10.1 on the L-Block* $[-1,1]^3 \backslash [0,1]^2 \times [-1,1]$ *with a singular solution. I.e. we prescribe the jump as follows:*

$$u_0 = \varrho^{2/3} \sin(\frac{2}{3}(\varphi - \frac{\pi}{2})), \quad \varrho = \sqrt{x^2 + y^2} \tag{28}$$

$$t_0 = \frac{\partial u}{\partial n} u_0 \tag{29}$$

coup3/ex28h8in

```
! Laplace, Symmetric FEM-BEM, L-Block, Interpolated
open(1) 'test.h' ; open(2) 'ex28h8in.dat'
geometry('L-Block') ; #ti

problem('Laplace',nickname='SYMCIRHS')
R=1    ! right hand side
EPS=1.0d-8
J=1;H=0.0625
#pxg. 'ug'
do I=1,12
 mesh('uniform',n=J,p=1,elements='Hexahedrals')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
 approx 0 R 'Ht0' 't0'
 approx 0 R 'Hu0' 'u0'
 lft 16 R 0 R 0
defprec(mode='MG',spline='ui',name='Pui',mat='Ai', &
& nu1=1,nu2=1,mu=1,mtop=I,omega=0.5,mdc=0,mds=0,hpmodus=0,stp=2)
!defprec(mode='INVCG',spline='ui',name='Pui',mat='I+Ai')

defprec(mode='DIAG',spline='D',name='PD',mat='W')

defprec(mode='MG',spline='N',name='PN',mat='V', &
& nu1=1,nu2=1,mu=1,mtop=I,omega=0.5,mdc=2,mds=2,hpmodus=0,stp=2)
!defprec(mode='INV',spline='N',name='PN',mat='V')
!solve(eps=EPS,mit='MINRES',mdi='x=0',mdc='no',mnum=B); T=SEC
 solve(eps=EPS,mit='MINRES',mdi='x=0'); T=SEC

 #rno.
 #err. 8 R 'L2' 0 'N' ; E[1]=ERR
 #err. 8 R 'L2' 0 'D' ; E[2]=ERR
 #err. 8 R 'H1' 0 'D' ; E[3]=ERR
 #err. 8 R 'L2' 0 'u' ; E[4]=ERR
 #err. 8 R 'H1' 0 'u' ; E[5]=ERR
 write(2) DOF,ITER,E[1],E[2],E[3],E[4],E[5],T
 #no. 'L2' 'u'
 #no. 'H1' 'u'
 J=J*2 ; H=H/2
continue
end
```

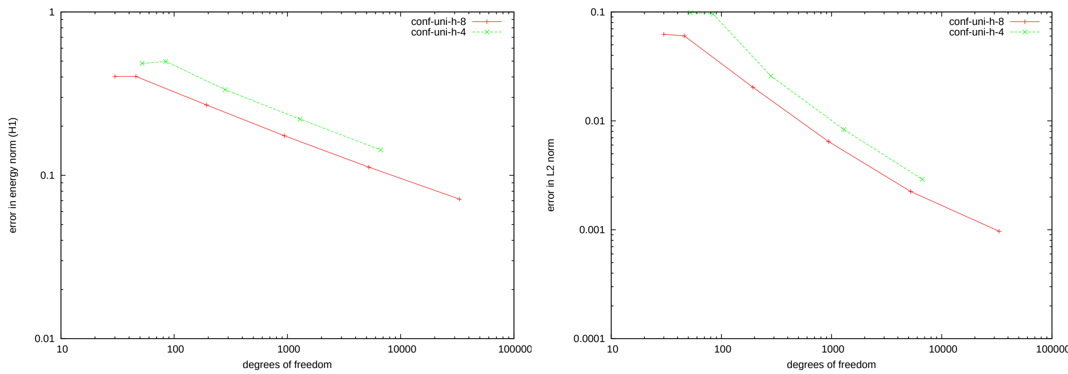


Figure 10.161: Error of u in energy-norm (left) and error of u in L_2-norm.

10.1.2 Maxwell

Example 10.3. ***Eddy current*** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ, and $\Omega_c := \mathbb{R}^3 \setminus \overline{\Omega}$.*

$\Phi(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi}\frac{1}{\|\mathbf{x}-\mathbf{y}\|}$ *($\mathbf{x} \neq \mathbf{y}$)*

$$V(u)(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y})u(\mathbf{x})\,dS(\mathbf{y}), \quad \mathbf{x} \notin \Gamma$$

$$\mathbf{K}(\boldsymbol{\lambda}) := \operatorname{curl}\mathbf{V}(\mathbf{n}\times\boldsymbol{\lambda}).$$

$$\mathbf{V}(\boldsymbol{\lambda})(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{y})\,dS(\mathbf{y}), \quad \mathbf{x} \notin \Gamma,$$

$$\begin{aligned} \mathcal{V}(\boldsymbol{\lambda}) &:= \gamma_t^+\mathbf{V}(\boldsymbol{\lambda}),\\ \mathcal{K}(\boldsymbol{\lambda}) &:= \gamma_t^+\mathbf{K}(\boldsymbol{\lambda}),\\ \tilde{\mathcal{K}}(\boldsymbol{\lambda}) &:= \gamma_N^+\mathbf{V}(\boldsymbol{\lambda}) = (\gamma_t^\times)^+\mathbf{K}(\boldsymbol{\lambda}\times\mathbf{n}),\\ \mathcal{W}(\boldsymbol{\lambda}) &:= \gamma_N^+\mathbf{K}(\boldsymbol{\lambda}). \end{aligned} \tag{30}$$

Find $\mathbf{u}_h \in \mathcal{ND}_1(\mathcal{T}_h)$, $\varphi_h \in \mathcal{S}_1(\mathcal{K}_h)$ such that

$$\begin{aligned} &(\mu^{-1}\operatorname{curl}\mathbf{u}_h, \operatorname{curl}\mathbf{v}_h)_\Omega + i\omega(\sigma\mathbf{u}_h,\mathbf{v}_h)_\Omega\\ &-\langle\mathcal{W}\gamma_t\mathbf{u}_h,\gamma_t\mathbf{v}_h\rangle_\Gamma + \langle\tilde{\mathcal{K}}\operatorname{curl}_\Gamma\varphi_h,\gamma_t\mathbf{v}_h\rangle_\Gamma = -i\omega(\mathbf{J}_0,\mathbf{v}_h)_\Omega,\\ &\langle(I-\mathcal{K})\gamma_t\mathbf{u}_h,\operatorname{curl}_\Gamma\tau_h\rangle_\Gamma + \langle\mathcal{V}\operatorname{curl}_\Gamma\varphi_h,\operatorname{curl}_\Gamma\tau_h\rangle_\Gamma + \mathcal{P}(\varphi_h,\tau_h) = 0 \end{aligned} \tag{31}$$

for all $\mathbf{v}_h \in \mathcal{ND}_1(\mathcal{T}_h)$, $\tau_h \in \mathcal{S}_1(\mathcal{K}_h)$.

We use the following prescribed solution of the problem, satifying the given assumptions.

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \operatorname{curl}\int_{\Omega_C}\frac{1}{|\mathbf{x}-\mathbf{y}|}\boldsymbol{\rho}(\mathbf{y})\,d\mathbf{y}, \quad \mathbf{x}\in\Omega_C = [-1,1]^3\\ \boldsymbol{\lambda} &= \operatorname{curl}\mathbf{u}\times\mathbf{n} \qquad \text{on } \Gamma = \partial\Omega \end{aligned}$$

with

$$\boldsymbol{\rho}(\mathbf{x}) = \rho(\mathbf{x}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathit{in}\ \Omega_C.$$

$$\rho(\mathbf{x}) = \big((1-x_1^2)(1-x_2^2)(1-x_3^2)\big)^2 x_1 x_2 x_3.$$

coup3/ex65h8in

```
! Eddy Current, symmetric coupling, h-version on cube
open(1) 'test.h' ; open(2) 'ex65h8in.dat'
write(2) 'j  DOF  L2(u)   Hrot(u)   L2(D)     L2(N)    H-1/2(N)  E-Norm-Error'
geometry('Cube',0); #ti
#maxw 0 0 1.   !mu,sigma,omega
#maxw.
problem('Maxwell',nickname='SYMC')
R=65    ! rhs
J=2
EPS=1.0d-8
do I=1,40
mesh('uniform',n=J,p=1,elements='hexahedral')
matrix('analytic',ijn=6,sigma=0.17,mu=1.0); TM=SEC
lft 16 R 0 R
defprec(mode='INVCG',spline='u',name='Pu',mat='I+CC')
defprec(mode='INV',spline='N',name='PN',mat='V')
solve(eps=EPS,mdc='u.Pu.u:N.PN.N',mit='GMRES',restart=1000)
!ddsolv(EPS,1,-1,1000,1,mode=11);
IT0=ITER; TD0=SEC
#rno.
#err. 8 R 'L2' 0 'N' ; E[1]=ERR   ! Hutfunktionen
#err. 8 R 'L2' 0 'D' ; E[2]=ERR   ! TND_Elemente
#err. 8 R 'L2' 0 'u' ; E[4]=ERR   ! Nedelec-Elemente
#err. 8 R 5 0 'u'   ; E[5]=ERR  ! Hrot-Norm
#no. 'V+PP' 'N'; E[6]=NORM !Energienorm von lambda
ERRORN=(0.00504793306**2-NORM**2)**(0.5)
ENOERROR=(ERRORN**2+E[5]**2)**(0.5)
write(2)J, DOF,E[4],E[5],E[2],E[1],ERRORN,ENOERROR,E[6],IT0,TD0,TM
J=J+1
continue
end
```

N	$\|u-u_h\|_{H(\text{curl},\Omega)}$	$It_{Inv,Inv}$	$\tau_{Inv,Inv}$
80	0.3098332	5	0.0100000
200	0.3036590	8	0.0400000
398	0.2354476	12	0.1600000
692	0.1899109	13	0.3800000
1100	0.1593494	13	0.8900000
1640	0.1374574	14	1.7600000
2330	0.1209329	14	3.6200000
3188	0.1079784	15	6.4000000
4232	0.0975331	15	8.8800000
5480	0.0889259	16	13.370000
6950	0.0817088	16	20.570000
8660	0.0755697	16	28.910000

Table 10.110: Error and iteration numbers for block preconditioners

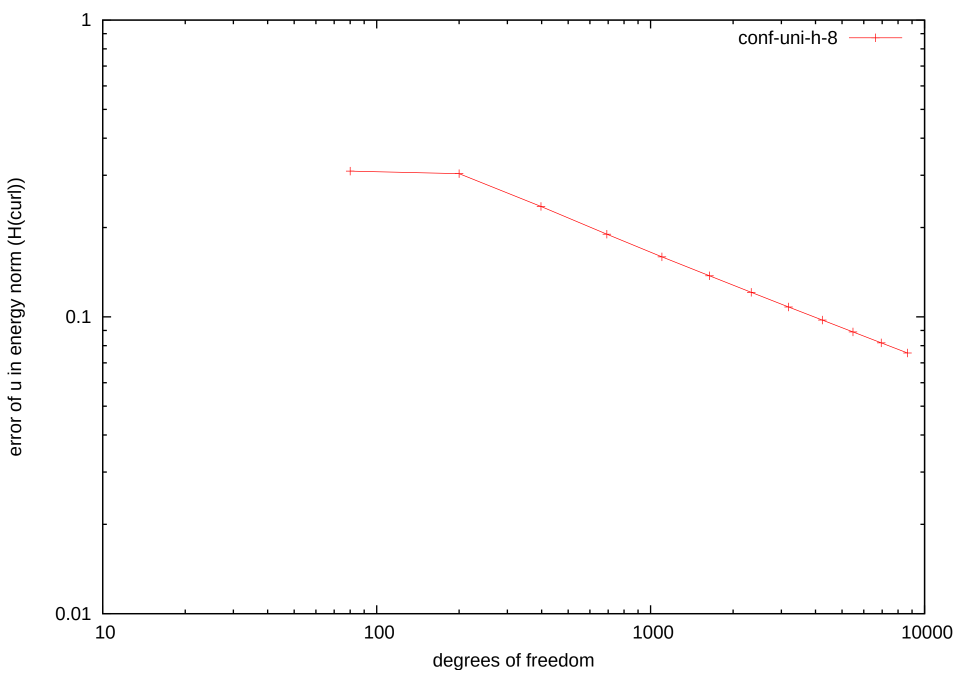


Figure 10.162: Error of u in $H(\text{curl}, \Omega)$-norm.

Example 10.4. *Here we now use the Least-Squares method for the Eddy-Current problem. The model problem itself is described in the previous example.*

coup3/ex66h8in

```
! Eddy-current, Least-Squares on Cube
open(1) 'test.h' ; open(2) 'ex66ivrth8in.dat'
geometry('Cube',0) ; #ti
#maxw 0 0 1.   !mu,sigma,omega
#maxw.
problem('Maxwell',nickname='LSC')
R=65   ! right hand side
EPS=1.0d-8
J=2;H=0.0625
do I=1,8
 mesh('uniform',n=J,p=1,elements='hexahedral')
 matrix('analytic',ijn=6,sigma=0.17,mu=1.0,gqna=14,gqnb=16)
 lft 16 R 0 R
 defprec(mode='ID',spline='p',name='Pp')

!defprec(mode='MG',spline='u',name='Pu',mat='I+Acc',mtop=I,hpmodus=0,stp=2,mdc=0,&
! & nu1=1,nu2=1,mu=1,omega=0.5,mds=0)
!defprec(mode='MG',spline='N',name='PN',mat='VP',mtop=I,hpmodus=0,stp=2,mdc=2, &
! & nu1=1,nu2=1,mu=1,omega=0.5,mds=2)
 defprec(mode='INVCG',spline='u',name='Pu',mat='I+Acc')
 defprec(mode='INV',spline='N',name='PN',mat='VP')

 lsqsolve(eps=EPS,mdi='x=1',mdc='scp',scp='Pp:0.5*Pu:PN',mit='CG',mnum=1000); T=SEC

 !#err. 24 R 'L2' 0 'N' ; E[1]=ERR ! Neumann-Rand
 ! #err. 24 R 'E'  0 'N' ; E[2]=ERR
 #err. 8 R 'L2' 0 'u' ; E[1]=ERR ! FEM
 ! #err. 16 R 'H1' 0 'u' ; E[4]=ERR
 #err. 8 R 'Hcurl' 0 'u' ; E[2]=ERR
 #err. 8 R 'L2' 0 'p' ; E[3]=ERR
 write(2) DOF,DOFP,DOFU,DOFN,E[1],E[2],E[3],ITER,COND,T
 J=J*2 ; H=H/2
continue
end
```

DOF	$\|\mathbf{u}-\mathbf{u}_N\|_{L^2}$	$\alpha_{\mathbf{u},L^2}$	$\|\mathbf{u}-\mathbf{u}_N\|_{\mathbf{H}(\text{curl})}$	$\alpha_{\mathbf{u},\mathbf{H}(curl)}$	$\|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_N\|_{L^2}$	$\alpha_{\boldsymbol{\vartheta},L^2}$	κ	It
134	0.0529507	—	0.3164327	—	0.3339011	—	31.2131	20
698	0.0459790	0.106	0.2839509	0.082	0.2349417	0.265	48.7913	55
4274	0.0188263	0.651	0.1374360	0.529	0.0928535	0.677	91.2165	79
29282	0.0074406	0.672	0.0641257	0.551	0.0400396	0.608	189.885	112

Table 10.111: Eddy current Least-Squares FEM-BEM coupling, convergence rates and condition numbers. $\boldsymbol{\vartheta}_N$ with Nedelec elements.

10.2 Solvers

Example 10.5. *The current example illustrates the action of various preconditioners for the symmetric coupling problem of Example 10.1 using the two block structure of the corresponding matrix by merging the Galerkin matrix of the hyper singular operator into the fem matrix.*

	unpreconditioned			
N	λ_1	λ_2	λ_3	λ_4
125 + 96	-0.957512	-0.021250809	0.1972928	3.40915
729 + 384	-0.230746	-0.002613525	0.0342006	1.90032
4913 + 1536	-0.055180	-0.000325145	0.0051586	0.98568

Table 10.112: Extreme eigenvalues of the unpreconditioned matrix

	Multigrid + 1 Mass-matrix			
N	λ_1	λ_2	λ_3	λ_4
125 + 96	-2.669257	-0.848210357	1.0014839	4.13017
729 + 384	-2.680533	-0.784481305	0.9229813	4.27892
4913 + 1536	-2.683024	-0.457425751	0.8903303	4.32527

Table 10.113: Extreme eigenvalues of the Multigrid preconditioner (2 block)

	BPX			
N	λ_1	λ_2	λ_3	λ_4
125 + 96	-4.854175	-0.563401547	0.6292511	7.68674
729 + 384	-5.096383	-0.267506880	0.3321810	8.23001
4913 + 1536	-5.156541	-0.131355377	0.1664291	8.36934

Table 10.114: Extreme eigenvalues of the BPX preconditioner (2 block)

	Hierarchical			
N	λ_1	λ_2	λ_3	λ_4
125 + 96	-12.77249	-0.359103365	0.2424615	3.41675
729 + 384	-12.47322	-0.344804265	0.3321810	8.23001

Table 10.115: Extreme eigenvalues of the Hierarchical preconditioner (2 block)

A Appendix

In this appendix we will present details which have been skipped in the previous sections, where we have concentrated on the numerical experiments. E.g. we show the use of the program *maigraf* for the production of images and plots.

Example A.1. ***graf/ex1in***

```
! demonstrates the form functions on a quadrilateral
open(1) 'test'
#bmode 2 3
geometry('Square')
open(2) 'ex1in.fig'
#pmode 4
#taf 'C0' 1 1 1 3 'u' 2 3
mesh('uniform',n=1,p=2,spline='u',gm='ug')
#scale 2.0
V[0]=0.5;V[1]=2.0;V[2]=0.666666666666666666666
#view V[0] V[1] V[2] 0.0 0.0 0.0
#mesh 4.0 4.0 3
#c 0
 0 0 1.0

#plot 'u' - 4.0 4.0 10 - - 0
#mesh 10.0 4.0 3 - - 'u'
#c 0
 0 1 1.0

#plot 'u' - 10.0 4.0 10 - - 0
#mesh 16.0 4.0 3 - - 'u'
#c 0
 0 2 1.0

#plot 'u' - 16.0 4.0 10 - - 0
#mesh 4.0 8.0 3 - - 'u'
#c 0
 1 0 1.0

#plot 'u' - 4.0 8.0 10 - - 0
#mesh 10.0 8.0 3 - - 'u'
#c 0
 1 1 1.0

#plot 'u' - 10.0 8.0 10 - - 0
#mesh 16.0 8.0 3 - - 'u'
#c 0
 1 2 1.0

#plot 'u' - 16.0 8.0 10 - - 0
#mesh 4.0 12.0 3 - - 'u'
#c 0
 2 0 1.0

#plot 'u' - 4.0 12.0 10 - - 0
#mesh 10.0 12.0 3 - - 'u'
```

```
#c 0
 2 1 1.0

#plot 'u' - 10.0 12.0 10 - - 0
#mesh 16.0 12.0 3 - - 'u'
#c 0
 2 2 1.0

#plot 'u' - 16.0 12.0 10 - - 0
#e
```

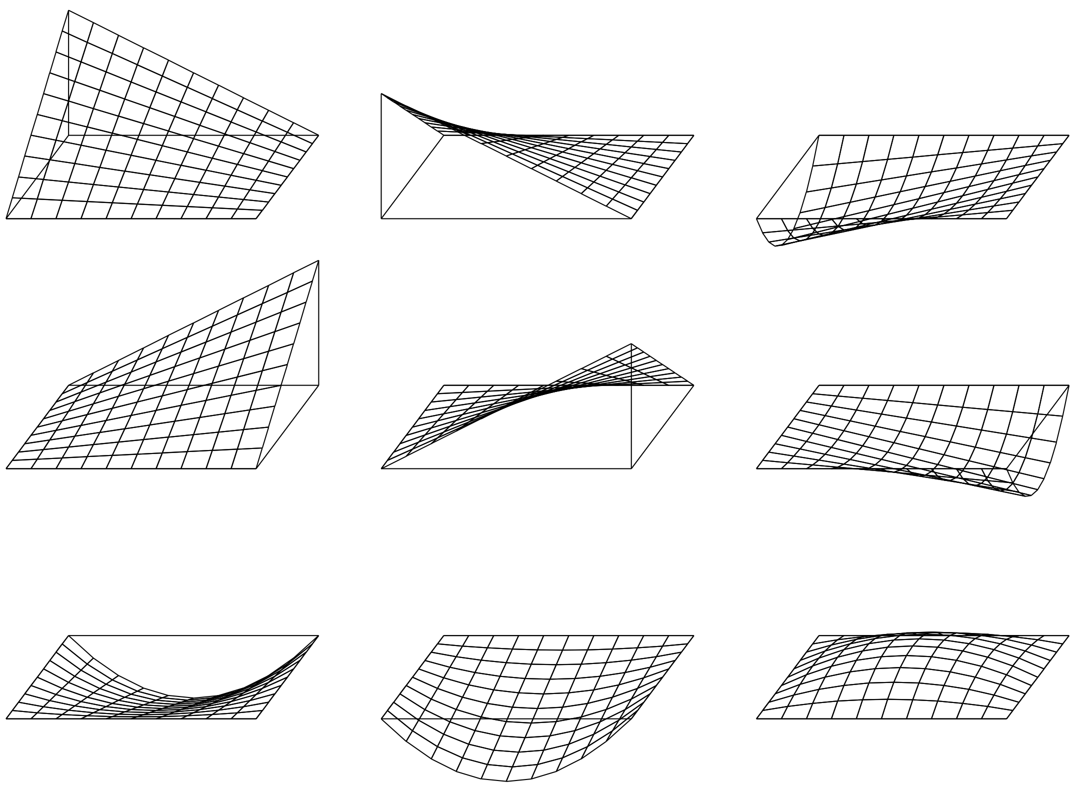


Figure 1.163: Base functions on quadrilaterals (continuous)

Example A.2. ***graf/ex2in***

```
! demonstrates the form functions on a triangle
open(1) 'test'
#bmode 2 3
geometry('Triangle')
open(2) 'ex2in.fig'
#pmode 4
#taf 'C0' 1 1 1 3 'u' 2 3
mesh('uniform',n=1,p=3,spline='u',gm='ug')
#scale 3.0
V[0]=0.5;V[1]=2.0;V[2]=0.66666666666666666666
#view V[0] V[1] V[2] 0.0 0.0 0.0
#mesh 4.0 4.0
#c 0
 0 0 1.0
```

```
#plot - - 4.0 4.0 10 - - 0
#mesh 10.0 4.0
#c 0
 0 1 1.0

#plot - - 10.0 4.0 10 - - 0
#mesh 16.0 4.0
#c 0
 0 2 1.0

#plot - - 16.0 4.0 10 - - 0
#mesh 22.0 4.0
#c 0
 0 3 1.0

#plot - - 22.0 4.0 10 - - 0
#mesh 4.0 8.0
#c 0
 1 0 1.0

#plot - - 4.0 8.0 10 - - 0
#mesh 10.0 8.0
#c 0
 1 1 1.0

#plot - - 10.0 8.0 10 - - 0
#mesh 16.0 8.0
#c 0
 1 2 1.0

#plot - - 16.0 8.0 10 - - 0
#mesh 4.0 12.0
#c 0
 2 0 1.0

#plot - - 4.0 12.0 10 - - 0
#mesh 10.0 12.0
#c 0
 2 1 1.0

#plot - - 10.0 12.0 10 - - 0
#mesh 4.0 16.0
#c 0
 3 0 1.0

#plot - - 4.0 16.0 10 - - 0
#e
```

Example A.3. *In this example we start with a 1d mesh, which will be rotated to create a 3d-surface mesh.*

graf/ex8in

```
! generation of a rotated 3d-surface mesh
```

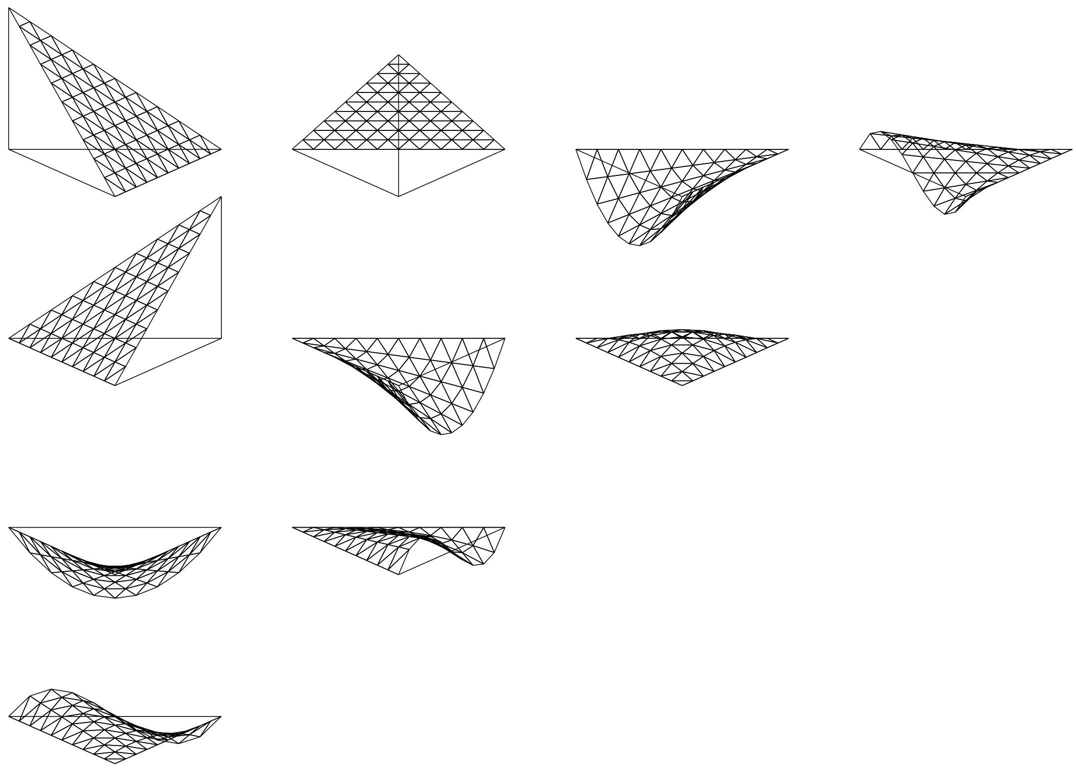


Figure 1.164: Base functions on triangles (continuous)

```
#bmode 2 3
! Circle geometry
#pxg 4 1 2 'geom1D'
1 3  3.  0.  2.707  0.707   2.  1.  0.707  0.707 0
1 3  3.  0.  2.707 -0.707   2. -1.  0.707 -0.707 0
1 3  2. -1.  1.293 -0.707   1.  0. -0.707 -0.707 0
1 3  1.  0.  1.293  0.707   2.  1. -0.707  0.707 0

#taf 'C0' 1 1 1 3 'spline12' 1 2
#taf 'C0' 1 1 1 3 'spline23' 2 3
! generate the 1d-mesh
mesh('uniform',n=8,p=1,spline='spline12',gm='geom1D')
! rotate the mesh
rotmesh('uniform',n=32,p=1,elements='triangles',spline='spline23',orig='spline12')
open(2) 'ex8in.fig'
#pmode 4
#scale 4.0
V[0]=0.5; V[1]=2.0; V[2]=0.666666666666666666
#view V[0] V[1] V[2] 0.0 0.0 0.0
#mesh 4.0 4.0 3 1.0 1 'spline23'
close(2)
end
```

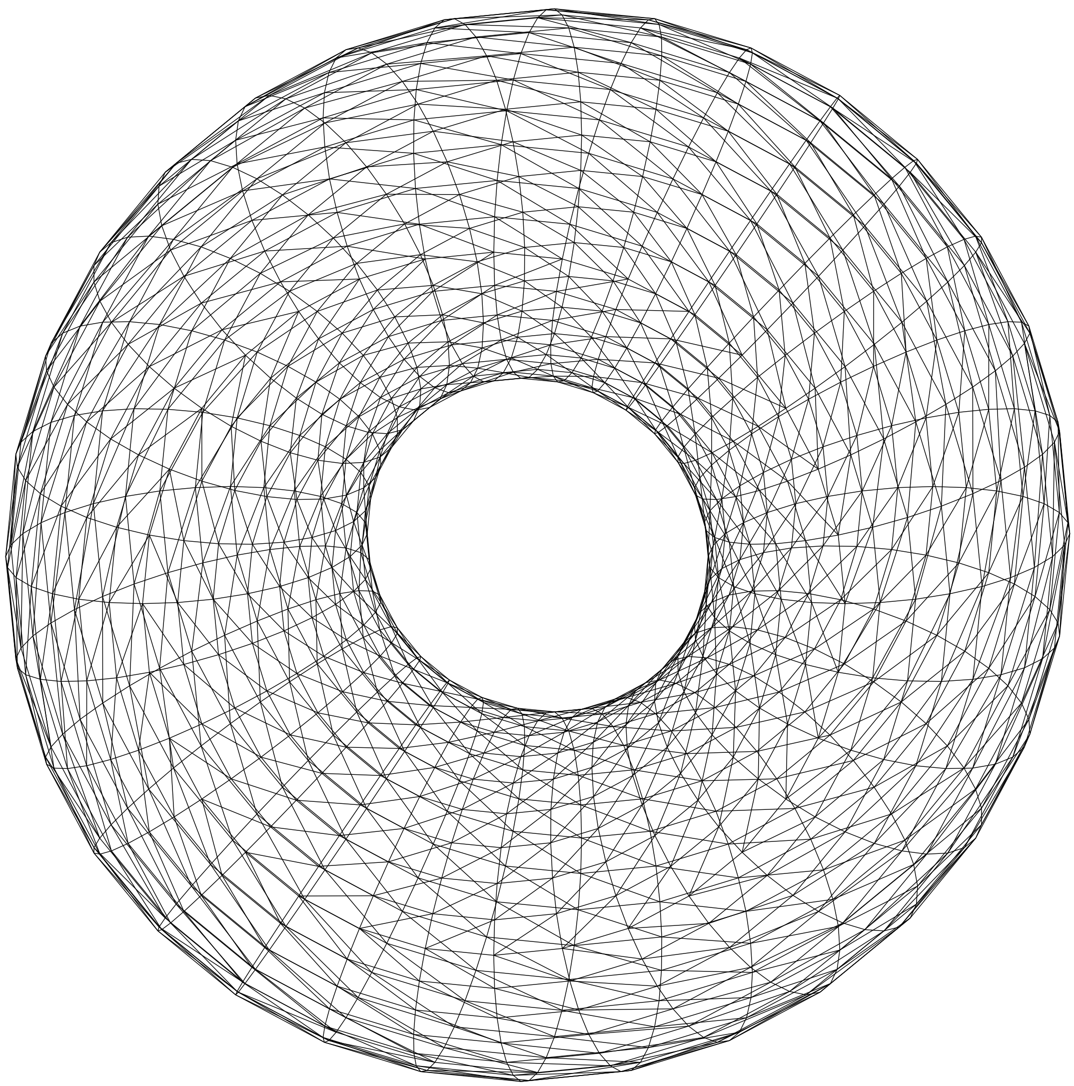


Figure 1.165: Rotated surface mesh (tyre) with triangles

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