# PORTFOLIO OPTIMISATION – THE MARKOWITZ APPROACH

We consider what happens when we need to decide the investment to be made in each of the assets that can be present in a portfolio, where that portfolio is derived based upon standard Markowitz mean-variance criteria.

Let:

| | |
|---|---|
| $N$ | be the number of assets (e.g. stocks) available |
| $\mu_i$ | be the expected return of asset i |
| $\rho_{ij}$ | be the correlation between the **returns** for assets i and j ($-1 \le \rho_{ij} \le +1$) |
| $s_i$ | be the standard deviation in **return** for asset i |

Then the decision variables are:

$w_i$ the proportion of the total investment associated with (invested in) asset i ($0 \le w_i \le 1$)

## SIMPLE EXAMPLE

Suppose N=2, so we have two assets available in which we can invest. Then the Markowitz approach says that the return we get from investing a proportion $w_1$ of our wealth in asset 1 and a proportion $w_2$ of our wealth in asset 2 is

$$\sum_{i=1}^{N} w_i\mu_i = w_1\mu_1 + w_2\mu_2$$

where it must be true that

$$w_1 + w_2 = 1$$

The **risk (variance)** associated with this investment is given by

$$\sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \rho_{ij} s_i s_j$$

$$\begin{aligned}
= \quad & w_1 w_1 \rho_{11} s_1 s_1 + \\
& w_1 w_2 \rho_{12} s_1 s_2 + \\
& w_2 w_1 \rho_{21} s_2 s_1 + \\
& w_2 w_2 \rho_{22} s_2 s_2
\end{aligned}$$

$$= \quad w_1 w_1 s_1 s_1 + 2 w_1 w_2 \rho_{12} s_1 s_2 + w_2 w_2 s_2 s_2$$

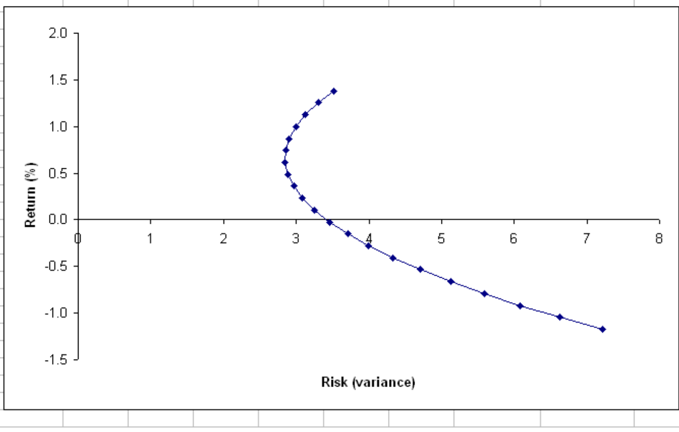
$$= \quad (w_1)^2 (s_1)^2 + 2 w_1 w_2 \rho_{12} s_1 s_2 + (w_2)^2 (s_2)^2$$

Suppose I take some data for two assets and vary $w_1$ and $w_2$ and plot the return that I get from my portfolio (y-axis) against the risk (variance) associated with that portfolio (x-axis) what do you think the plot will look like?

Below we show the spreadsheet considered in class.

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Stock prices | | | | | | | Stock 1 | Stock 2 | Return 1 (%) | Return 2 (%) |
| 2 | A | B | C | D | E | Period | | A | C | A | C |
| 3 | 916 | 630.5 | 440 | 642 | 740 | 0 | | 916 | 440 | | |
| 4 | 932 | 639.5 | 440.5 | 642 | 755 | 1 | | 932 | 440.5 | 1.73 | 0.11 |
| 5 | 910.5 | 644.5 | 443 | 634 | 761 | 2 | | 910.5 | 443 | -2.33 | 0.57 |
| 6 | 872 | 626.5 | 446 | 642.5 | 712 | 3 | | 872 | 446 | -4.32 | 0.67 |
| 7 | 874 | 637 | 465 | 617.5 | 675 | T=4 | | 874 | 465 | 0.23 | 4.17 |
| 8 | | | | | | | | | Average | -1.17 | 1.38 |
| 9 | Standard deviation STDEV (2dp) PRICES | | | | | | | Standard deviation STDEV (2dp) RETURNS | | | |
| 10 | 26.67 | 7.16 | 10.39 | 10.72 | 35.44 | | | 2.69 | 1.88 | | |
| 11 | | | | | | | | | | | |
| 12 | Correlation CORREL (2dp) PRICES | | | | | | | Correlation CORREL (2dp) RETURNS | | | |
| 13 | A | B | C | D | E | | | A | C | | |
| 14 | 1.00 | 0.46 | -0.72 | 0.50 | 0.85 | A | | 1 | 0.22 | A | |
| 15 | 0.46 | 1.00 | 0.03 | -0.36 | 0.40 | B | | 0.22 | 1 | C | |
| 16 | -0.72 | 0.03 | 1.00 | -0.93 | -0.90 | C | | | | | |
| 17 | 0.50 | -0.36 | -0.93 | 1.00 | 0.69 | D | | | | | |
| 18 | 0.85 | 0.40 | -0.90 | 0.69 | 1.00 | E | | | | | |

| | L | M | N | O | P |
|---|---|---|---|---|---|
| 1 | | Stock 1 weight | Stock 2 weight | Return | Risk |
| 2 | | 0 | 1 | 1.38 | 3.52 |
| 3 | | 0.05 | 0.95 | 1.25 | 3.30 |
| 4 | | 0.1 | 0.9 | 1.13 | 3.13 |
| 5 | | 0.15 | 0.85 | 1.00 | 2.99 |
| 6 | | 0.2 | 0.8 | 0.87 | 2.90 |
| 7 | | 0.25 | 0.75 | 0.74 | 2.86 |
| 8 | | 0.3 | 0.7 | 0.62 | 2.85 |
| 9 | | 0.35 | 0.65 | 0.49 | 2.89 |
| 10 | | 0.4 | 0.6 | 0.36 | 2.97 |
| 11 | | 0.45 | 0.55 | 0.23 | 3.09 |
| 12 | | 0.5 | 0.5 | 0.10 | 3.25 |
| 13 | | 0.55 | 0.45 | -0.02 | 3.46 |
| 14 | | 0.6 | 0.4 | -0.15 | 3.71 |
| 15 | | 0.65 | 0.35 | -0.28 | 4.00 |
| 16 | | 0.7 | 0.3 | -0.41 | 4.33 |
| 17 | | 0.75 | 0.25 | -0.53 | 4.71 |
| 18 | | 0.8 | 0.2 | -0.66 | 5.12 |
| 19 | | 0.85 | 0.15 | -0.79 | 5.58 |
| 20 | | 0.9 | 0.1 | -0.92 | 6.09 |
| 21 | | 0.95 | 0.05 | -1.05 | 6.63 |
| 22 | | 1 | 0 | -1.17 | 7.22 |
| 23 | | | | min | 2.85 |




Note that some points on the trade-off curve between risk (variance) and return above are efficient, some are not.

**OPTIMISATION**

If we had just N=2 assets as above then it is a simple matter to consider possible investment portfolios simply by enumerating choices for $w_1$ and $w_2$ (where $w_2$=$w_1$-1 in the two asset case).

Of course we almost always have many more than two assets in which we could invest and so the approach considered above becomes infeasible. We need to move from enumerating choices to making a choice via **optimisation**.

Let:

R be the desired expected return from the portfolio chosen

Using the standard **Markowitz mean-variance approach** we have that the unconstrained portfolio optimisation problem is:

$$\text{minimise} \quad \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \rho_{ij} s_i s_j \qquad (1)$$

subject to

$$\sum_{i=1}^{N} w_i \mu_i = R \qquad (2)$$

$$\sum_{i=1}^{N} w_i = 1 \qquad (3)$$

$$0 \leq w_i \leq 1 \qquad i=1,...,N \qquad (4)$$

Equation (1) minimises the total variance (**risk**) associated with the portfolio whilst equation (2) ensures that the portfolio has an expected **return** of R. Equation (3) ensures that the proportions add to one.

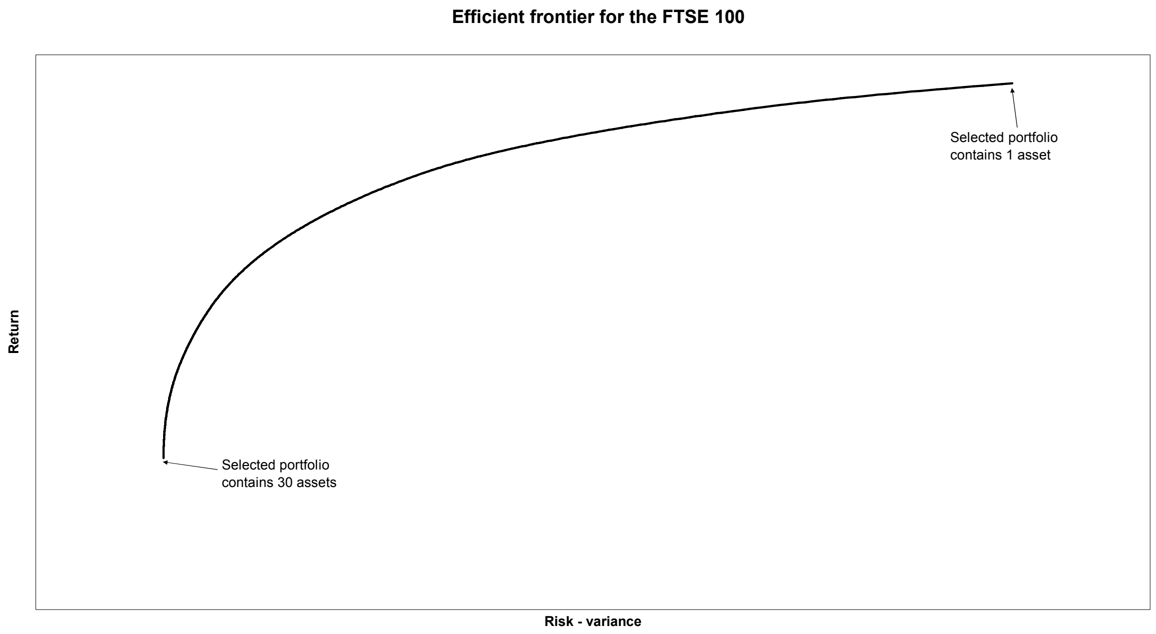
This formulation (equations (1)-(4)) is a simple nonlinear programming problem.

Usually nonlinear problems are difficult to solve but in this case because the objective is **quadratic**, computationally effective algorithms exist so that there is (in practice) little difficulty in calculating the optimal solution for any particular data set.

Note here that the above formulation (equations (1)-(4)) can be expressed in terms of $\sigma_{ij}$ the **covariance between the returns** associated with assets i and j since $\sigma_{ij}=\rho_{ij}s_is_j$.

The point of the above optimisation problem is to construct an *efficient frontier*, (unconstrained efficient frontier, UEF) a smooth non-decreasing curve that gives the best possible tradeoff of **risk** against **return**, i.e. the curve represents the set of **Pareto-optimal** (**non-dominated**) portfolios.

One such efficient frontier is shown below for assets (shares) drawn from the UK FTSE (Financial Times Stock Exchange) index of 100 top companies.


Efficient frontier for the FTSE 100
Return
Risk - variance
Selected portfolio
contains 1 asset
Selected portfolio
contains 30 assets