PORTFOLIO OPTIMISATION – THE MARKOWITZ APPROACH

We consider what happens when we need to decide the investment to be made in each of the assets that can be present in a portfolio, where that portfolio is derived based upon standard Markowitz mean-variance criteria.

One thing to note before we proceed - we are going to use mathematics. Although there are some areas of finance where the amount of mathematics involved is not high (e.g. corporate finance, mergers and acquisitions) the plain fact is that to do modern quantitative finance, also sometimes referred to as financial engineering, you need to cope with a certain level of mathematics. Indeed the level of mathematics involved these days is in some areas extremely high.

To proceed with Markowitz mean-variance portfolio optimisation we need some notation, let:

N	be the number of assets (e.g. stocks) available
μ_i	be the expected (average, mean) return of asset i
ρ_{ij}	be the correlation between the **returns** for assets i and j ($-1 \le \rho_{ij} \le +1$)
s_i	be the standard deviation in **return** for asset i

Then the decision variables are:

w_i the proportion of the total investment associated with (invested in) asset i ($0 \le w_i \le 1$)

Reflect for a moment – are you surprised that correlation makes an appearance here?

Note here that we have used the word "asset" above. The framework we use is completely general – provided we have a **price history** for an asset it can be included, so we could consider making up a portfolio from stocks, commodities (e.g. oil, metals, foods), and bonds.

SIMPLE EXAMPLE

Suppose N=2, so we have two assets available in which we can invest. Then the Markowitz approach says that the return we get from investing a proportion w_1 of our wealth in asset 1 and a proportion w_2 of our wealth in asset 2 is

$$\sum_{i=1}^{N} w_i\mu_i = w_1\mu_1 + w_2\mu_2$$

where it must be true that

$$w_1 + w_2 = 1$$

which states that we invest all of the money we have available. In practice we can introduce a risk-free asset if we wish to avoid putting all of our investment into risky assets.

The **risk (variance)** associated with this investment is given by

$$\sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \rho_{ij} s_i s_j$$

$$= \quad w_1 w_1 \rho_{11} s_1 s_1 + w_1 w_2 \rho_{12} s_1 s_2 + w_2 w_1 \rho_{21} s_2 s_1 + w_2 w_2 \rho_{22} s_2 s_2$$

$$= \quad w_1 w_1 s_1 s_1 + 2 w_1 w_2 \rho_{12} s_1 s_2 + w_2 w_2 s_2 s_2$$

$$= \quad (w_1)^2 (s_1)^2 + 2 w_1 w_2 \rho_{12} s_1 s_2 + (w_2)^2 (s_2)^2$$

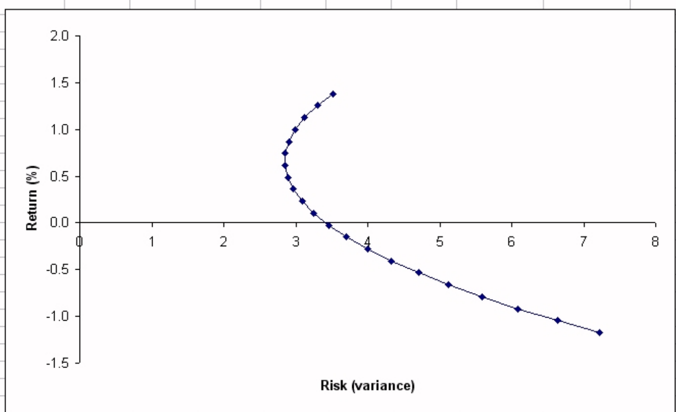
Suppose I take some data for two assets and vary w_1 and w_2 and plot the return that I get from my portfolio (y-axis) against the risk (variance) associated with that portfolio (x-axis) what do you think the plot will look like?

Note here that the standard presentation we make in terms of risk and return is that return is plotted on the vertical axis and risk on the horizontal axis.

Below we show the spreadsheet considered in class.

	A	B	C	D	E	F	G	H	I	J	K	L
1	Stock prices							Stock 1	Stock 2	Return 1 (%)	Return 2 (%)	
2	A	B	C	D	E	Period		A	C	A	C	
3	916	630.5	440	642	740	0		916	440			
4	932	639.5	440.5	642	755	1		932	440.5	1.73	0.11	
5	910.5	644.5	443	634	761	2		910.5	443	-2.33	0.57	
6	872	626.5	446	642.5	712	3		872	446	-4.32	0.67	
7	874	637	465	617.5	675	T=4		874	465	0.23	4.17	
8									Average	-1.17	1.38	
9								Standard deviation STDEV (2dp) RETURNS				
10								2.69	1.88			
11												
12								Correlation CORREL (2dp) RETURNS				
13								A	C			
14								1	0.22	A		
15								0.22	1	C		

	M	N	O	P
1	Stock 1 weight	Stock 2 weight	Return	Risk
2	0	1	1.38	3.52
3	0.05	0.95	1.25	3.30
4	0.1	0.9	1.13	3.13
5	0.15	0.85	1.00	2.99
6	0.2	0.8	0.87	2.90
7	0.25	0.75	0.74	2.86
8	0.3	0.7	0.62	2.85
9	0.35	0.65	0.49	2.89
10	0.4	0.6	0.36	2.97
11	0.45	0.55	0.23	3.09
12	0.5	0.5	0.10	3.25
13	0.55	0.45	-0.02	3.46
14	0.6	0.4	-0.15	3.71
15	0.65	0.35	-0.28	4.00
16	0.7	0.3	-0.41	4.33
17	0.75	0.25	-0.53	4.71
18	0.8	0.2	-0.66	5.12
19	0.85	0.15	-0.79	5.58
20	0.9	0.1	-0.92	6.09
21	0.95	0.05	-1.05	6.63
22	1	0	-1.17	7.22
23			min	2.85



Note that some points on the trade-off curve between risk (variance) and return above are **efficient**, some are not. Points on this curve which are inefficient are **dominated** by other points.

OPTIMISATION

If we had just N=2 assets as above then it is a simple matter to consider possible investment portfolios simply by enumerating choices for w_1 and w_2 (where w_2=w_1-1 in the two asset case).

Of course we almost always have many more than two assets in which we could invest and so the approach considered above becomes infeasible. We need to move from enumerating choices to making a choice via **optimisation**.

Let:

R be the desired expected return from the portfolio chosen

Using the standard **Markowitz mean-variance approach** we have that the unconstrained portfolio optimisation problem is:

minimise $$\sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \rho_{ij} s_i s_j \quad (1)$$

subject to

$$\sum_{i=1}^{N} w_i \mu_i = R \quad (2)$$

$$\sum_{i=1}^{N} w_i = 1 \quad (3)$$

$$0 \le w_i \le 1 \qquad i=1,\ldots,N \quad (4)$$

Equation (1) minimises the total variance (**risk**) associated with the portfolio whilst equation (2) ensures that the portfolio has an expected **return** of R. Equation (3) ensures that the proportions add to one.

This formulation (equations (1)-(4)) is a simple nonlinear programming problem.

Usually nonlinear problems are difficult to solve but in this case because the objective is **quadratic**, computationally effective algorithms exist so that there is (in practice) little difficulty in calculating the optimal solution for any particular data set.

Note here that above we have, for a given return, found the minimum risk portfolio. Logically we could have specified the risk we were prepared to take and found the maximum return portfolio that had this specified risk. Whilst this is a logical equivalent the way presented above is the way we proceed in numeric practice. Numerically finding a minimum risk portfolio that has a specified return is much easier than finding a maximum return portfolio that has a specified risk.

Note here that the above formulation (equations (1)-(4)) can be expressed in terms of σ_{ij} the **covariance between the returns** associated with assets i and j since $\sigma_{ij}=\rho_{ij}s_is_j$.

The point of the above optimisation problem is to construct an ***efficient frontier***, (**unconstrained efficient frontier, UEF**) a smooth non-decreasing curve that gives the best possible tradeoff of **risk** against **return**, i.e. the curve represents the set of **Pareto-optimal** (**non-dominated**) portfolios.

One such efficient frontier is shown below for assets (shares) drawn from the UK FTSE (Financial Times Stock Exchange) index of 100 top companies.