What are these?

Dow Jones

FTSE All Share

S&P500

They are all stock market indices, which, in an easy to digest form, tell you how the stocks (companies) represented in the index have changed in value over time.

If you want you can invest your money in index tracking funds ("trackers") which aim to reproduce the performance of the index over time, perhaps by investing in all of the stocks that make up the index.

In essence the index tracking problem is a ***decision problem***, namely decide the ***subset*** of stocks to choose so as to (perfectly) reproduce the performance of the index over time. We call the subset of stocks we choose a **tracking portfolio (TP)**.

Question

How are index values arrived at (i.e. what formula is used)?

Question

It is estimated that approximately 10^F dollars are invested in index tracking funds in the USA alone - what is the value of F?

Questions

Where does the money to invest in index tracking funds come from?

How else might people choose to invest in stock (equity) markets if not via index tracking funds?

Why do people choose to invest in index tracking funds?

INDEX TRACKING

Suppose that we observe over time 0,1,2,...,T the value of N stocks, as well as the value of the index we want to track. Further suppose that we are interested in deciding the best set of K stocks to hold (where K<N), as well as their appropriate quantities. In index tracking we want to answer the question:

*"what will be the best set of K stocks to hold, as well as their appropriate quantities, so as to best track the index in the **future** (from time T onward)?"*

Our basic approach in index tracking is a historical look-back approach. To ask the historical question:

*"what would have been the best set of K stocks to have held, as well as their appropriate quantities, so as to have best tracked the index in the **past** (i.e. over the time period [0,T])?"*

and then hold the stocks that answer this question into the future.

NOTATION

Let:

N be the total number of distinct stocks (companies) in which we can invest

K be the desired number of distinct stocks in the TP

ε_i be the minimum proportion, and

δ_i be the maximum proportion, of the TP that must be held in stock i if any of stock i is held

X_i be the number of units of stock i in the current TP

V_{it} be the value of one unit of stock i at time t

T be such that we have observed historical values for stocks and the index over the time period 0,1,2,...,T. The time T represents a decision point, a time at which we **may** switch from our current TP [X_i] to a new TP.

I_t be the value of the index at time t

C be the total value of the current TP $[X_i]$ at time T plus any **cash change** in the portfolio (either new cash available for investment or cash being withdrawn at time T), i.e.

$$C = \sum_{i=1}^{N} V_{iT}X_i + \text{cash change}$$

γ be the limit on the proportion of C that can be consumed by transaction cost

Then our decision variables are:

x_i the number of units of stock i that we choose to hold in the new TP

z_i = 1 if any of stock i is held in the new TP
= 0 otherwise

Without significant loss of generality we allow $[x_i]$ to take fractional values.

CONSTRAINTS

The constraints associated with the index tracking problem are:

$$\sum_{i=1}^{N} z_i = K$$

$$\varepsilon_i z_i \le V_{iT} x_i / C \le \delta_i z_i \quad i=1,\ldots,N$$

$$C_{trans} = \sum_{i=1}^{N} \text{transaction cost}(X_i \rightarrow x_i \text{ at time T})$$

$$C_{trans} \le \gamma C$$

$$\sum_{i=1}^{N} V_{iT} x_i = C - C_{trans}$$

$$x_i \ge 0 \quad i=1,\ldots,N$$

$$z_i \in [0,1] \quad i=1,\ldots,N$$

OBJECTIVES - TRACKING

In time period t we get a return associated with the index, $R_t = \log_e(I_t/I_{t-1})$, where we define return using continuous time.

If, in each and every time period, the return associated with the TP:

$$r_t = \log_e[[\sum_{i=1}^{N} V_{it}x_i]/[\sum_{i=1}^{N} V_{it-1}x_i]]$$

was EXACTLY equal to R_t then this might seem ideal.

A possible objective in terms of index tracking is therefore:

$$\text{minimise} \sum_{t=1}^{T} (r_t - R_t)^2/T$$

i.e. minimise average squared error

Here we have a formulation of the index tracking problem as a mixed-integer nonlinear problem.

To get some insight into how we might try and solve this formulation let us assume that:

- we have zero transaction cost, i.e. buying and selling stocks costs us nothing
- we have no limits on the proportion of the tracking portfolio in each stock (i.e. ε_i=0 and δ_i=1

The index tracking problem then becomes:

$$\text{minimise} \sum_{t=1}^{T} (r_t - R_t)^2/T$$

subject to

$$\sum_{i=1}^{N} z_i = K$$

$$V_{iT}x_i/C \le z_i \quad i=1,...,N$$

$$\sum_{i=1}^{N} V_{iT}x_i = C$$

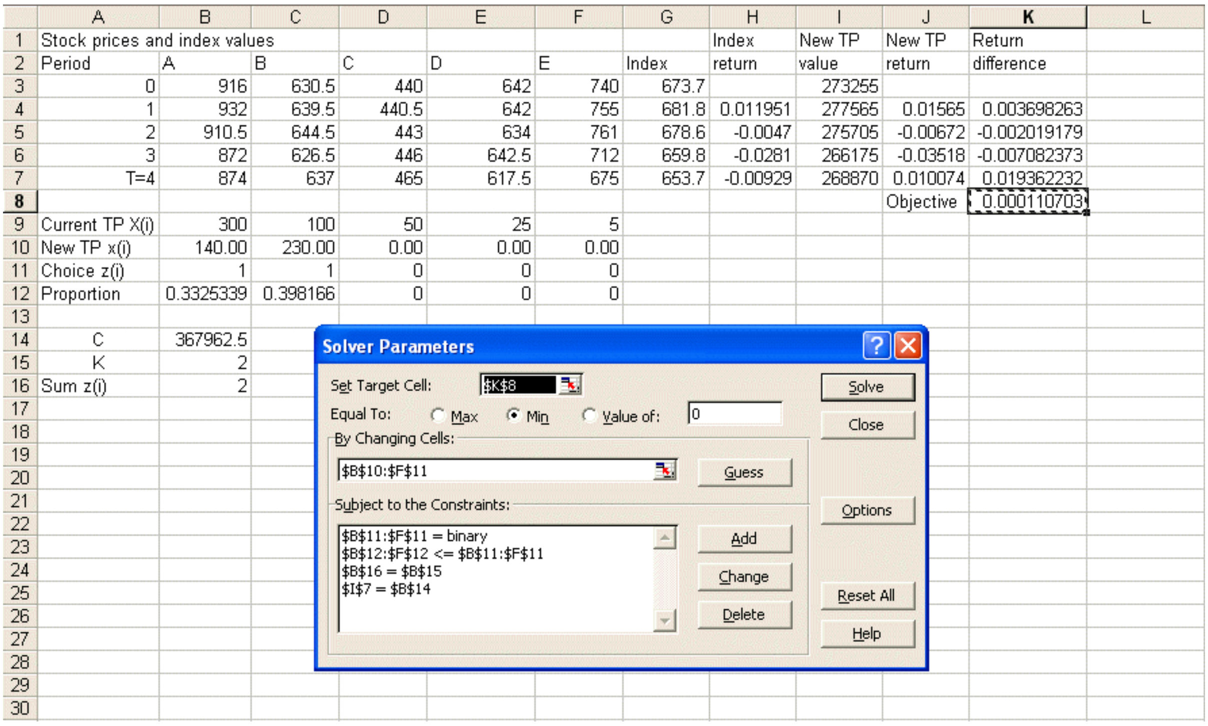
$$x_i \ge 0 \quad i=1,...,N$$

$$z_i \in [0,1] \quad i=1,...,N$$

The example below is the one we will deal with in class

	A	B	C	D	E	F	G	H	I	J	K
1	Stock prices and index values							Index	New TP	New TP	Return
2	Period	A	B	C	D	E	Index	return	value	return	difference
3	0	916	630.5	440	642	740	673.7		273255		
4	1	932	639.5	440.5	642	755	681.8	0.011951	277565	0.01565	0.003698263
5	2	910.5	644.5	443	634	761	678.6	-0.0047	275705	-0.00672	-0.002019179
6	3	872	626.5	446	642.5	712	659.8	-0.0281	266175	-0.03518	-0.007082373
7	T=4	874	637	465	617.5	675	653.7	-0.00929	268870	0.010074	0.019362232
8										Objective	0.000110703
9	Current TP X(i)	300	100	50	25	5					
10	New TP x(i)	140.00	230.00	0.00	0.00	0.00					
11	Choice z(i)	1	1	0	0	0					
12	Proportion	0.3325339	0.398166	0	0	0					
13											
14	C	367962.5									
15	K	2									
16	Sum z(i)	2									

with the Solver model for this example being shown below



	A	B	C	D	E	F	G	H	I	J	K	L
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